Projective parabolic geometries

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ESI Wien, September 2012

Based partly on:

- "Hamiltonian 2-forms in Kähler geometry", with Vestislav Apostolov (UQAM), Paul Gauduchon (Ecole Polytechnique) and Christina Tønnesen-Friedman (Union College);
- "Hamiltonian 2-vectors in H-projective geometry", informal notes, August 2011;
- work with Vladimir Matveev and Stefan Rosemann (Jena);
- work with Aleksandra Borowka (Bath);

Some motivation

Projective geometry and conformal geometry both play an important role in riemannian geometry. In *complex geometry*, conformal hermitian structures have provided insight, but the impact has been limited. This raises the question:

> *How does projective geometry illuminate complex (hermitian and Kähler) geometry?*
>
> *Contention: projective geometry, with a kählerian interpretation, is more deeply embedded in Kähler geometry than conformal geometry is, and has interesting links with other special geometric structures.*

This case cannot be made in the usual context of (holomorphic) complex projective geometry, because holomorphic unitary connections are flat. **H-projective geometry** instead concerns aspects of complex projective geometry which are *not* holomorphic.

Irony: the "H" originally stood for "holomorphic"!

The literature (name dropping)

Projective geometry: too many names to mention!

H-projective geometry: large Japanese and former soviet schools (T. Otsuki, Y. Tashiro, S. Ishihara, S. Tachibana, Y. Yoshimatsu, J. Mikes, V. Domashev, N. Sinjukov,...).

Projective and H-projective metrics: recent works by R. Bryant, M. Dunajski, M. Eastwood, V. Kiosak, V. Matveev, A. Federova, S. Rosemann,...

Quaternionic geometries: S. Salamon, A. Swann, M. Eastwood,...

Parabolic geometries: A. Cap, J. Slovak, V. Soucek, R. Baston, T. Diemer, M. Eastwood, S. Gindikin, R. Gover, M. Hammerl, P. Somberg,...

H-projective case: S. Armstrong, A. Cap, J. Hrdina,...

Projective structures

Let D be a torsion-free connection on an n-manifold M (e.g., $D = \nabla^g$ for a riemannian metric g on M).

- A curve c in M is a *geodesic* wrt. D iff for all T tangent to c, $D_T T \in \mathrm{span}\{T\}$.
- Torsion-free connections D and $\tilde{D}$ have the same geodesics iff $\exists\, \gamma \in \Omega^1(M)$, a 1-form, with

  $$\tilde{D}_X - D_X = [\![X, \gamma]\!]^r \in C^\infty(M, \mathfrak{gl}(TM)),$$

  where $$[\![X, \gamma]\!]^r(Y) := \gamma(X)Y + \gamma(Y)X.$$

  Then D and $\tilde{D}$ are said to be *projectively equivalent*.
  We write $\tilde{D} = D + \gamma$ for short (instead of $\tilde{D} = D + [\![\cdot, \gamma]\!]^r$).
- A *projective structure* on M^n ($n > 1$) is a projective class $\Pi^r = [D]$ of torsion-free connections.

H-projective structures

Let (M,J) be a complex manifold of real dimension $n=2m$ and let D be a torsion-free connection on M (a smooth n-manifold) with $DJ=0$ (e.g., $D=\nabla^g$ for a Kähler metric g on M).

- A curve c is an *H-planar geodesic* wrt. D iff for all T tangent to c, $D_T T \in \mathrm{span}\{T, JT\}$.
- Torsion-free complex connections D and $\tilde{D}$ have the same H-planar geodesics iff $\exists\, \gamma \in \Omega^1(M)$, a (real) 1-form, with

  $$\tilde{D}_X - D_X = [\![X,\gamma]\!]^c \in C^\infty(M, \mathfrak{gl}(TM,J)),$$
  $$[\![X,\gamma]\!]^c(Y) := \tfrac{1}{2}\big(\gamma(X)Y + \gamma(Y)X - \gamma(JX)JY - \gamma(JY)JX\big).$$

  Then D and $\tilde{D}$ are said to be *H-projectively equivalent*. We write $\tilde{D} = D + \gamma$ for short.
- An *H-projective structure* on M^{2m} $(m>1)$ is an H-projective class $\Pi^c = [D]$ of torsion-free complex connections.

Quaternionic structures

Let (M, Q) be a quaternionic manifold of real dimension $n = 4\ell$ (thus $Q \subset \mathfrak{gl}(TM)$, with fibres isomorphic to $\mathfrak{sp}(1)$, spanned by imaginary quaternions J_1, J_2, J_3) and let D be a torsion-free connection on M preserving Q (e.g., $D = \nabla^g$ for a quaternion Kähler metric g on M).

- A curve c is a *Q-planar geodesic* wrt. D iff for all T tangent to c, $D_T T \in \mathrm{span}\{T, JT : J \in Q\}$.
- **Fact**. Any two torsion-free quaternionic connections D and $\tilde{D}$ have the same Q-planar geodesics: $\exists\, \gamma \in \Omega^1(M)$ with
$$\tilde{D}_X - D_X = [\![X, \gamma]\!]^q \in C^\infty(M, \mathfrak{gl}(TM, Q)),$$
$$[\![X, \gamma]\!]^q(Y) := \tfrac{1}{2}\Big(\gamma(X)Y + \gamma(Y)X - \sum_i \big(\gamma(J_iX)J_iY + \gamma(J_iY)J_iX\big)\Big).$$
- The class of torsion-free quaternionic connections may be denoted analogously by $\Pi^q = [D]$.

Common framework: parabolic geometries

Projective, H-projective and quaternionic classes Π of torsion-free connections are affine spaces modelled on 1-forms.

Torsion-free conformal connections ("Weyl connections") on a conformal manifold (M^n, c) also form such an affine space.

These are *parabolic geometries*, i.e., *Cartan geometries* modelled on a *generalized flag variety* G/P, where G is a semisimple Lie group and P a *parabolic subgroup* of G, i.e., its Lie algebra $\mathfrak{p}$ is parabolic: $\mathfrak{p} = \mathfrak{g}_0 \ltimes \mathfrak{p}^\perp$ with $\mathfrak{g}_0$ reductive and $\mathfrak{p}^\perp$ nilpotent.

In all the examples discussed, G/P is the projectivized highest weight orbit of G in the tangent space (isotropy) V of a bigger generalized flag variety $X = Q/\big((GL_1 \times G) \ltimes V^*\big)$.
The associated compact complex homogeneous manifold $X_{\mathbb{C}}$ is a self-dual hermitian symmetric space $\cong Q_{cpt}/\big(U(1) \times G_{cpt}\big)$. The relevant real form $X_{\mathbb{R}}$ is the Shilov boundary of the noncompact dual of $X_{\mathbb{C}}$ (a bounded symmetric domain of tube type with biholomorphism group $Q_{\mathbb{R}}$); this makes $V_{\mathbb{R}}$ into a Jordan algebra.

The Jordan algebra tables

Big group Q	G	V	H	Geometry
Sp_{2n}	SL_n	$S^2\mathbb{F}^n$	SO_n	Projective
SL_{2n}	$SL_n \times SL_n$	$\mathbb{F}^n \otimes \mathbb{F}^n$	SL_n	H-projective
SO_{4n}	SL_{2n}	$\bigwedge^2\mathbb{F}^{2n}$	Sp_{2n}	Quaternionic
E_7	E_6	$\mathbb{F}^{27}$	F_4	Octonionic
SO_{m+3}	SO_{m+1}	$\mathbb{F}^{m+1}$	SO_m	Conformal

(G, V) is *prehomogeneous* with isotropy H: G acts on $\mathrm{P}(V)$ with an open dense orbit $\cong G/H$.

$Q_\mathbb{R}$	$G_\mathbb{R}$	$V_\mathbb{R}$	$H_\mathbb{R}$	$G_\mathbb{R}/P$
$\mathrm{Sp}_{2n}(\mathbb{R})$	$\mathrm{SL}_n(\mathbb{R})$	$J_n(\mathbb{R}) \cong S^2\mathbb{R}^n$	$\mathrm{SO}(n)$	$\mathbb{R}P^{n-1}$
$\mathrm{SU}(n,n)$	$\mathrm{SL}_n(\mathbb{C})$	$J_n(\mathbb{C}) \subset \mathbb{C}^n \otimes \overline{\mathbb{C}^n}$	$\mathrm{SU}(n)$	$\mathbb{C}P^{n-1}$
$\mathrm{SO}_{2n}(\mathbb{H})$	$\mathrm{SL}_n(\mathbb{H})$	$J_n(\mathbb{H}) \subset \bigwedge^2\mathbb{C}^{2n}$	$\mathrm{Sp}(n)$	$\mathbb{H}P^{n-1}$
$\mathrm{E}_{7(-25)}$	$\mathrm{E}_{6(-26)}$	$J_3(\mathbb{O}) \cong \mathbb{R}^{27}$	$\mathrm{F}_{4(-52)}$	$\mathbb{O}P^2$
$\mathrm{SO}(m+1,2)$	$\mathrm{SO}(m,1)$	$J_m \cong \mathbb{R}^{m,1}$	$\mathrm{SO}(m)$	S^{m-1}

$G_\mathbb{R}$ acts on $\mathrm{P}(V_\mathbb{R})$ with highest weight orbit $\cong G_\mathbb{R}/P$; $V_\mathbb{R}$ has a Jordan algebra structure with automorphism group $H_\mathbb{R}$, and then P is the stabilizer of a primitive idempotent ("rank 1 projection").

The Cartan connection

A Cartan geometry on M, modelled on G/P, where $\dim M = \dim G/P$, is a principal G-bundle over M equipped with

- a principal G-connection and
- a reduction of structure group to $P \leq G$

satisfying the *Cartan condition*: the induced 1-form on M with values in the bundle $\mathfrak{g}(M)/\mathfrak{p}(M)$ associated to $\mathfrak{g}/\mathfrak{p}$ is bundle isomorphism. Thus M inherits the first order geometry of G/P.

By the duality between $\mathfrak{p}^\perp$ and $\mathfrak{g}/\mathfrak{p}$, T^*M is isomorphic to the associated bundle $\mathfrak{p}^\perp(M)$ of nilradicals in $\mathfrak{p}(M)$. Hence $\mathfrak{p}(M) \cong \mathfrak{g}_0(M) \ltimes T^*M$, where $\mathfrak{g}_0(M) \subset \mathfrak{gl}(TM)$.

Our examples have the simplifying feature that $\mathfrak{p}^\perp$ is abelian. There is then an algebraic bracket

$$[\![,]\!] \colon TM \times T^*M \to \mathfrak{g}_0(M) \subseteq \mathfrak{gl}(TM)$$

and the affine structure $D \mapsto D + \gamma \in \Pi$ on compatible connections is given by $D + \gamma := D + [\![\cdot, \gamma]\!]$.

Computing with projective connections

A function F on Π is an *invariant* if it is constant, i.e., $\forall\, D \in \Pi$, $\gamma \in \Omega^1(M)$, $\partial_\gamma F(D) := \frac{d}{dt} F(D + t\gamma)|_{t=0}$ is zero.

For a section s of a vector bundle E associated to the frame bundle, $\partial_\gamma D_X s = [\![X, \gamma]\!] \cdot s$ (the natural action of $\mathfrak{g}_0(M)$ on E).

Variation of the second derivative:

$$\partial_\gamma D^2_{X,Y} s = [\![X, \gamma]\!] \cdot D_Y s + [\![Y, \gamma]\!] \cdot D_X s - D_{[\![X,\gamma]\!] \cdot Y} s + [\![Y, D_X \gamma]\!] \cdot s.$$

Hence the curvature $R^D \in \Omega^2(M, \mathfrak{g}_0(TM))$ of D, given by $D^2_{X,Y} s - D^2_{Y,X} s = R^D_{X,Y} \cdot s$, satisfies

$$\partial_\gamma R^D_{X,Y} = -[\![Id \wedge D\gamma]\!]_{X,Y} := -[\![X, D_Y \gamma]\!] + [\![Y, D_X \gamma]\!].$$

Can write: $R^D = W + [\![Id \wedge r^D]\!]$, where W is invariant ($\partial_\gamma W = 0$), and the *normalized Ricci tensor* $r^D \in \Omega^1(M, T^*M)$ satisfies $\partial_\gamma r^D = -D\gamma$.

Projective hessians

Consequence:

$$\partial_\gamma(D^2_{X,Y}s + [\![Y, r^D_X]\!] \cdot s) = [\![X,\gamma]\!] \cdot D_Y s + [\![Y,\gamma]\!] \cdot D_X s - D_{[\![X,\gamma]\!]\cdot Y} s$$

so $D^2_{X,Y}s + [\![X, r^D_Y]\!] \cdot s$ is algebraic in D.

On densities of weight k (sections of a certain line bundle $\mathcal{O}(k)$) this simplifies to

$$\partial_\gamma(D^2_{X,Y}s + kr^D_X(Y)s) = k\gamma(X)D_Y s + k\gamma(Y)D_X s - D_{[\![X,\gamma]\!]\cdot Y} s.$$

In the projective case this gives a natural hessian operator on sections of $\mathcal{O}(1)$ whose solutions are "affine coordinates". In other cases, we must make a projection, and obtain the first BGG operator of the representation V^* dual to the Jordan algebra. The H-projective case yields functions with J-invariant natural hessian: in Kähler geometry, these are hamiltonians for Killing vector fields!

[Aside: a *Hessian operator* or *Hill's equation* can be used to define projective structures on 1-manifolds, and similarly H-projective structures on Riemann surfaces, also known as *Möbius structures*.]

Compatible metrics

Q: Given a parabolic geometry with torsion-free connections Π, describe the space of compatible metrics g with Levi-Civita connection $\nabla^g \in \Pi$. Is it nonempty?

In our examples, the equation for compatible g *linearizes* for a weighted inverse metric, using the first BGG operator of the Jordan algebra V. In projective geometry, this is the well-known linear first order equation of finite type for h in $S^2 TM \otimes \mathcal{O}(-1)$.

In the H-projective case, we seek compatible Kähler metrics (which are J-invariant) and so can work with the corresponding J-invariant 2-vector $\phi = h(J\cdot, \cdot) \in \wedge^{1,1} TM \otimes \mathcal{O}(-1)$. This satisfies

$$D_X \phi = X \wedge K^D + JX \wedge JK^D$$

for some, hence any, $D \in \Pi^c$; K^D determined by the trace of $D\phi$.

If $D = \nabla^g$ for a Kähler metric g, the equation means that the 2-form dual to ϕ with respect to g is a *hamiltonian 2-form*!

H-projective metrics and hamiltonian 2-forms

The *mobility* of an H-projective structure is the dimension of the space of solutions of the linear equation for compatible Kähler metrics.

- Generically the mobility will be zero, and it remains open to characterize when it is positive, and when an H-projective structure is Kählerian.
- The theory of hamiltonian 2-forms provides local and global classification results for mobility ≥ 2, i.e., of H-projectively equivalent Kähler metrics which are not affinely equivalent.
- Within this classification, the mobility ≥ 3 case can be identified; such metrics are rare, and in the compact case, have constant holomorphic sectional curvature.

The complicated geometry of these metrics can be illuminated via cone constructions, which represent Cartan connections as affine connections on a (generalized) cone manifold, but there is still much to be understood.

Projective structures and Cartan holonomy

$\mathbb{R}P^{2m+1}$ is a circle bundle over $\mathbb{C}P^m$ (the Hopf fibration), given by a choice of complex structure on the fundamental representation $\mathbb{R}^{2m+2}$ of $GL(2m+2, \mathbb{R})$ (yielding the fundamental representation $\mathbb{C}^{m+1}$ of $GL(m+1, \mathbb{C})$).

In general, any H-projective manifold M^{2m} has a circle bundle N^{2m+1} with a projective structure on it, and the projective Cartan connection preserves a complex structure in its fundamental representation.

Conversely, a projective structure on a $(2m+1)$-manifold whose Cartan connection has such a holonomy reduction is locally a circle bundle over an H-projective manifold.

There are results about the interplay of Cartan holonomy with other structures (compatible metrics, quaternionic structures), but much remains unexplored.

Quaternion Kähler metrics and twistor theory

On a quaternionic manifold $(N^{4\ell}, Q)$, the compatible (quaternion Kähler) metrics are given by Q-hermitian sections h of $S^2TM \otimes \mathcal{O}(-1)$ satisfying

$$D_X h = X \odot K^D + \sum_i J_i X \odot J_i K^D$$

for some (hence any) $D \in \Pi^q$ (with K^D a trace of Dh).

This linear equation for compatible quaternion Kähler metrics has an interpretation in terms of the twistor space Z of N, the unit sphere bundle in Q. Z is a complex $2\ell + 1$ manifold with real structure, containing real "twistor lines" (rational curves with normal bundle $\mathcal{O}(1) \otimes C^{2\ell}$): N is the space of such twistor lines.

The Penrose transform associates h with a holomorphic section π of $\wedge^2 TZ \otimes K_Z^{1/(\ell+1)}$, which has maximal rank for h nondegenerate.

The standard theory of quaternion Kähler metrics uses instead the section θ of $T^*Z \otimes K_Z^{-1/(\ell+1)}$ dual to $\pi^{\wedge \ell}$, and the inverse of π on $\ker\theta$. If the metric is hyperkähler, this defines a symplectic foliation of Z over $\mathbb{C}P^1$; if not, it is a contact structure on Z.

Totally complex submanifolds of quaternionic manifolds

Q. Are H-projective structures interesting beyond the realm of Kähler geometry?

Observation. Let $(N^{4\ell}, Q)$ be a quaternionic manifold and $M^{2\ell}$ a *maximal totally complex submanifold*, i.e., each tangent space of M is invariant under some $J \in Q$, but for any $I \in Q$ anticommuting with J, $I(TM)$ is complementary to TM.

Then (M, J) inherits an H-projective structure from (N, Q).

Indeed, we just project the quaternionic connections onto TM (along the complement, which is independent of I), observing that for $X, Y \in TM$, the projection onto TM of $[\![X, \gamma]\!]^q(Y)$ is $[\![X, i^*\gamma]\!]^c(Y)$, where $i\colon M \to N$ is the inclusion.

This prompts a further question: when does an H-projective structure arise this way?

If it does then the quaternionic manifold N is locally a neighbourhood of the zero section in $TM \otimes \mathcal{L}$ for a unitary line bundle $\mathcal{L}$.

A generalized Feix–Kaledin construction

In the early 2000's, B. Feix and D. Kaledin gave independent constructions of hyperkähler metrics on cotangent bundles of real analytic Kähler manifolds. The metrics were defined on a neighbourhood of the zero section. They placed these constructions within a more general context: hypercomplex structures on the tangent bundle of a complex manifold equipped with a real analytic torsion-free hermitian connection whose curvature has type (1,1).

Theorem. Let $(M^{2\ell}, J, \Pi^c)$ be a real analytic H-projective manifold whose H-projective Weyl curvature W has type (1,1). Then there is a natural quaternionic structure Q on a neighbourhood $N^{4\ell}$ of the zero section in $TM \otimes \mathcal{L}$ for a certain unitary line bundle $\mathcal{L}$.

Construction via the twistor space

Idea for proof (following Feix). We construct the twistor space Z of (N, Q).

Flat model. When $M = \mathbb{C}P^\ell$, its complexification is $\mathbb{C}P^\ell \times \mathbb{C}P^\ell$ and the total space of $P(\mathcal{O} \oplus \mathcal{O}(1,-1))$ is birational to $\mathbb{C}P^{2\ell+1}$ by a partial blow down of the zero and infinity sections (inversely, write $\mathbb{C}^{2\ell+2} = \mathbb{C}^{\ell+1} \oplus \mathbb{C}^{\ell+1}$ and blow up two projective ℓ-spaces in $\mathbb{C}P^{2\ell+1}$). This is the twistor space of $\mathbb{H}P^\ell$, and the fibres of $P(\mathcal{O} \oplus \mathcal{O}(1,-1))$ project to twistor lines.

We make the same construction over the complexification M^c of M (a neighbourhood of the diagonal in $M \times \overline{M}$).

M^c has two complementary foliations integrating the $(1,0)$ and $(0,1)$ distributions (which restrict to $T^{1,0}M$ and $T^{0,1}M$ in $TM \otimes \mathbb{C}$ along M).

The blow-down

The analogue of $P(\mathcal{O} \oplus \mathcal{O}(1,-1))$ is obtained by gluing the line bundles $\mathcal{O}(1) \otimes \overline{\mathcal{O}(-1)}$ and $\mathcal{O}(-1) \otimes \overline{\mathcal{O}(1)}$ by inversion on the complement of their zero sections. We then need to blow-down the zero sections along corresponding foliations.

The model for this blow-down is based on the blow-up of $\mathbb{C}^{\ell+1}$ at the origin, which is the total space of $\mathcal{O}(-1)$ over $\mathbb{C}P^{\ell}$. Inversely, we reconstruct $\mathbb{C}^{\ell+1}$ as the dual space to the space of affine sections of $\mathcal{O}(1)$ over $\mathbb{C}P^{\ell}$.

This is where the type (1,1) curvature condition on M enters: it implies that the two foliations of M^c have projectively flat leaves. Hence the hessian equation for affine sections of $\mathcal{O}(1)$ is completely integrable and we can integrate it leafwise to obtain rank $\ell+1$ vector bundles over the leaf spaces.