Relative BGG sequences

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- This talk reports on joint work in progress with Vladimir Souček (Prague).
- On the level of the homogeneous model, a relative BGG sequence is a BGG sequence along the fibers of a natural fibration. We prove that this allows a curved analog, although in general curved cases, there is no fibration present.
- This leads to a machinery relating operators on partially defined differential forms with values in a "relative tractor bundle" to operators defined on certain completely reducible bundles induced by homology spaces.
- The method applies both in regular and in singular infinitesimal character. In the first case, one obtains sub–sequences and sometimes sub–complexes in standard BGG's. For singular infinitesimal character, this leads to the first general construction for curved analogs of standard operators.

Contents

We start with a semisimple Lie algebra $\mathfrak{g}$ and a parabolic subalgebra $\mathfrak{q}$, and we will denote the decomposition of $\mathfrak{g}$ from the corresponding grading by

$$\mathfrak{g} = \mathfrak{q}_- \oplus \mathfrak{q}_0 \oplus \mathfrak{q}_+.$$

Next, we choose a second parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ with $\mathfrak{q} \subset \mathfrak{p}$. (Putting $\mathfrak{p} = \mathfrak{g}$, will recover usual BGG sequences.) Via $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$, this will give rise to a subbundle in the tangent bundle.

The decomposition $\mathfrak{g} = \mathfrak{p}_- \oplus \mathfrak{p}_0 \oplus \mathfrak{p}_+$ has the properties

$$\mathfrak{p}_- \subset \mathfrak{q}_- \qquad \mathfrak{p}_0 \supset \mathfrak{q}_0 \qquad \mathfrak{p}_+ \subset \mathfrak{q}_+$$

Further, we choose an irreducible representation $\mathbb{V}$ of $\mathfrak{p}$. In terms of weights, this corresponds to a $\mathfrak{p}$–dominant integral weight of $\mathfrak{g}$ (which may have singular infinitesimal character for $\mathfrak{g}$).

We will use $\mathbb{V}$ to induce a "relative tractor bundle". Partially defined differential forms with values in this bundle will correspond to the representations $\Lambda^k(\mathfrak{p}/\mathfrak{q})^* \otimes \mathbb{V}$ for $k = 0, \dots, \dim(\mathfrak{p}/\mathfrak{q})$.

Proposition

- Via the Killing form of $\mathfrak{g}$, we get $(\mathfrak{p}/\mathfrak{q})^* \cong \mathfrak{q}_+/\mathfrak{p}_+$.
- The subalgebra $(\mathfrak{q}_- \cap \mathfrak{p}) \subset \mathfrak{p}$ is complementary to $\mathfrak{q} \subset \mathfrak{p}$, so as a $\mathfrak{q}_0$–module, we can identify $\mathfrak{p}/\mathfrak{q}$ with $\mathfrak{q}_- \cap \mathfrak{p}$.

The Lie subalgebras $\mathfrak{q}_- \cap \mathfrak{p}$ and $\mathfrak{q}_+ \subset \mathfrak{p}$ both naturally act on $\mathbb{V}$. Moreover, $\mathfrak{p}_+$ is an ideal in $\mathfrak{p}$ and thus in $\mathfrak{q}_+$, so also $\mathfrak{q}_+/\mathfrak{p}_+$ is a Lie algebra. Since we assumed the $\mathbb{V}$ is an irreducible representation of $\mathfrak{p}$, the nilradical $\mathfrak{p}_+ \subset \mathfrak{p}$ acts trivially on $\mathbb{V}$. Hence the $\mathfrak{q}_+$–action on $\mathbb{V}$ descends to an action of $\mathfrak{q}_+/\mathfrak{p}_+$.

Let us collect what we have obtained:

- We may identify $\Lambda^k(\mathfrak{p}/\mathfrak{q})^* \otimes \mathbb{V}$ as a $\mathfrak{q}$–module with the chain space $C_k(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$, which gives rise to a Lie algebra homology differential ∂^*_ρ.
- As a $\mathfrak{q}_0$–module, $\Lambda^k(\mathfrak{p}/\mathfrak{q})^* \otimes \mathbb{V}$ may also be identified with the cochain space $C^k(\mathfrak{q}_- \cap \mathfrak{p}, \mathbb{V})$ which gives rise to a Lie algebra cohomology differential ∂_ρ.

Together, we obtain the familiar picture

$$\cdots \Lambda^{k-1}(\mathfrak{p}/\mathfrak{q})^* \otimes \mathbb{V} \underset{\partial^*_\rho}{\overset{\partial_\rho}{\rightleftarrows}} \Lambda^k(\mathfrak{p}/\mathfrak{q})^* \otimes \mathbb{V} \underset{\partial^*_\rho}{\overset{\partial_\rho}{\rightleftarrows}} \Lambda^{k+1}(\mathfrak{p}/\mathfrak{q})^* \otimes \mathbb{V} \cdots$$

with the leftward pointing arrows being q–homomorphisms and the rightward pointing ones being only $\mathfrak{q}_0$–homomorphisms.

Now one proceeds as for the usual BGGs. Defining $\Box_\rho := \partial_\rho^*\partial_\rho + \partial_\rho\partial_\rho^*$, one obtains the algebraic Hodge decomposition

$$\Lambda^k(\mathfrak{p}/\mathfrak{q})^* \otimes \mathbb{V} = \operatorname{im}(\partial_\rho^*) \oplus \ker(\Box_\rho) \oplus \operatorname{im}(\partial_\rho),$$

so $\ker(\Box_\rho) \cong H_k(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$. This homology space is a completely reducible $\mathfrak{q}$–module which can be computed by (a slight generalization of) Kostant's version of the Bott–Borel–Weil theorem.

To translate this to geometry, let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, Q). Then we define

- $T_\rho M := \mathcal{G} \times_Q (\mathfrak{p}/\mathfrak{q}) \subset \mathcal{G} \times_Q (\mathfrak{g}/\mathfrak{q}) = TM$.
- $T^*_\rho M := (T_\rho M)^* \cong \mathcal{G} \times_Q (\mathfrak{q}_+/\mathfrak{p}_+)$.
- $\mathcal{V}M := \mathcal{G} \times_Q \mathbb{V}$ ("relative tractor bundle")
- $\mathcal{H}_k(T^*_\rho M, \mathcal{V}M) := \mathcal{G} \times_Q H_k(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$.

The Q–module homomorphism ∂^*_ρ induces bundle maps

$$\partial^*_\rho : \Lambda^k T^*_\rho M \otimes \mathcal{V}M \to \Lambda^{k-1} T^*_\rho M \otimes \mathcal{V}M$$

and thus natural subbundles $\mathrm{im}(\partial^*_\rho) \subset \ker(\partial^*_\rho) \subset \Lambda^k T^*_\rho M \otimes \mathcal{V}M$ such that $\ker(\partial^*_\rho)/\mathrm{im}(\partial^*_\rho) \cong \mathcal{H}_k(T^*_\rho M, \mathcal{V}M)$. In particular, we get a natural projection $\pi_h : \ker(\partial^*_\rho) \to \mathcal{H}_k(T^*_\rho M, \mathcal{V}M)$.

With the $\mathfrak{q}_0$–homomorphism ∂_ρ, the situation is a bit more complicated. The spaces $\Lambda^k(\mathfrak{p}/\mathfrak{q})^* \otimes \mathbb{V}$ carry a $\mathfrak{q}_0$–invariant grading, which induces a $\mathfrak{q}$–invariant filtration.

Then one can interpret the $\mathfrak{q}_0$–equivariant isomorphism $\mathfrak{p}/\mathfrak{q} \to (\mathfrak{q}_- \cap \mathfrak{p})$ as a q–equivariant isomorphism between the associated graded spaces. Using gr(_) to denote associated graded modules, this in gives rise to $\mathfrak{q}$–isomorphisms

$$\mathrm{gr}(\Lambda^k(\mathfrak{p}/\mathfrak{q})^* \otimes \mathbb{V}) \to C^k(\mathfrak{q}_- \cap \mathfrak{p}, \mathrm{gr}(\mathbb{V})),$$

and the Lie algebra cohomology differential makes sense in the latter picture, too.

This now carries over to geometry, providing filtrations of the bundles $\Lambda^k T^*_\rho M \otimes \mathcal{V}M$ by natural smooth subbundles, and bundle maps

$$\partial_\rho : \mathrm{gr}(\Lambda^k T^*_\rho M \otimes \mathcal{V}M) \to \mathrm{gr}(\Lambda^{k+1} T^*_\rho M \otimes \mathcal{V}M).$$

On this level, one thus gets the full algebraic Hodge theory.

We now put $\Omega^k_\rho(M, \mathcal{V}M) := \Gamma(\Lambda^k T^*_\rho M \otimes \mathcal{V}M)$ and are ready to define the class of operators to which we can apply the BGG machinery.

Definition

A family $\mathcal{D}_k : \Omega^k(M, \mathcal{V}M) \to \Omega^{k+1}(M, \mathcal{V}M)$ of operators is called *compressible* iff for any $\varphi \in \Omega^k(M, \mathcal{V}M)$ which is filtration homogeneous of degree $\geq \ell$, the form $\mathcal{D}_k(\varphi)$ is filtration homogeneous of the same degree and

$$\mathrm{gr}_\ell(\mathcal{D}_k(\varphi)) = \partial_\rho \circ \mathrm{gr}_\ell(\varphi).$$

As we shall see in the last part, there is a compressible family consisting of strongly invariant differential operators of first order. All compressible families are then obtained from this one by adding arbitrary operators which strictly increase the degree of filtration homogeneity.

Given a sequence of compressible operators $\mathcal{D}_k$, we now define

$$\Box_k^{\mathcal{D}} := \partial_\rho^* \circ \mathcal{D}_k + \mathcal{D}_{k-1} \circ \partial_\rho^*,$$

which maps $\Omega^k(M, \mathcal{V}M)$ to itself. The following properties are easy to prove.

Proposition

1. $\ker(\Box_k^{\mathcal{D}}) = \ker(\partial_\rho^*) \cap \ker(\partial_\rho^* \circ \mathcal{D}_k)$
2. $\pi_h|_{\ker(\Box_k^{\mathcal{D}})} : \ker(\Box_k^{\mathcal{D}}) \to \Gamma(\mathcal{H}_k(T_\rho^* M, \mathcal{V}M))$ is injective.

For the next step, we can use an analog of the construction of splitting operators via curved Casimirs. The only ingredients needed for this construction are representation theory data related to $\mathrm{im}(\partial_\rho^*) \subset \Lambda^k(\mathfrak{p}/\mathfrak{q})^* \otimes \mathbb{V}$ (eigenvalues of $\Box_\rho$ on composition factors).

The splitting operators

Theorem

There exists an operator $S : \Gamma(\ker(\partial_\rho^*)) \to \Gamma(\ker(\partial_\rho^*))$ with the following properties

1. S is a polynomial in $\partial_\rho^* \circ \mathcal{D}_k$.
2. $\pi_h \circ S = \pi_h$, $S|_{\Gamma(\mathrm{im}(\partial_\rho^*))} = 0$ and $\Box_r^{\mathcal{D}} \circ S = 0$.

Hence S descends to an operator

$$S : \Gamma(\mathcal{H}_k(T_\rho^* M, \mathcal{V}M) \to \ker(\Box_r^{\mathcal{D}}) \subset \Gamma(\ker(\partial_\rho^*)) \subset \Omega_\rho^k(M, \mathcal{V}M)$$

which is inverse to $\pi_h|_{\ker(\Box_r^{\mathcal{D}})}$.

Note that the first part implies that nice properties of $\mathcal{D}_k$ like being (pseudo)differential or (strongly) invariant carry over to S.

The compressed operators

For $\alpha \in \Gamma(\mathcal{H}_k(T^*_\rho M, \mathcal{V}M))$, we can now form $S(\alpha) \in \Omega^k_\rho(M, \mathcal{V}M)$, which is uniquely characterized by $\partial^*_\rho(S(\alpha)) = 0$, $\pi_h(S(\alpha)) = \alpha$ and $\partial^*_\rho(\mathcal{D}_k(S(\alpha))) = 0$. Thus we can define:

The *compressed operator* or *BGG operator*

$$D_k : \Gamma(\mathcal{H}_k(T^*_\rho M, \mathcal{V}M)) \to \Gamma(\mathcal{H}_{k+1}(T^*_\rho M, \mathcal{V}M)).$$

is defined by $D_k(\alpha) := \pi_h(\mathcal{D}_k(S(\alpha)))$.

Evidently, if $\mathcal{D}_k$ is (pseudo)differential and/or (strongly) invariant then D_k inherits this properties.
Moreover, we can now collect the main properties of the BGG machinery:

Theorem

1. π_h restricts to an injection from $\ker(\partial_\rho^*) \cap \ker(\mathcal{D}_k)$ to a subspace of $\ker(D_k)$ ("normal solutions")
2. For each $\varphi \in \Omega^k(M, \mathcal{V}M)$, there is a $\psi \in \Omega^{k-1}(M, \mathcal{V}M)$ such that $\partial_\rho^*(\varphi + \mathcal{D}_{k-1}(\psi)) = 0$.
3. If $\mathcal{D}_k \circ \mathcal{D}_{k-1} = 0$, then $D_k \circ D_{k-1} = 0$ and $\ker(\mathcal{D}_k)/\operatorname{im}(\mathcal{D}_{k-1}) \cong \ker(D_k)/\operatorname{im}(D_{k-1})$ via π_h.

To close the circle, we construct a sequence of strongly invariant differential operators of order one which is compressible. The only ingredient needed to construct this is the fundamental derivative. Recall that the *adjoint tractor bundle* $\mathcal{A}M := \mathcal{G} \times_Q \mathfrak{g}$ is a bundle of filtered Lie algebras with algebraic bracket $\{\ ,\ \}$. The subalgebra $\mathfrak{p} \subset \mathfrak{g}$ gives rise to a smooth subbundle $\mathcal{A}_\rho M := \mathcal{G} \times_Q \mathfrak{p}$ of Lie subalgebras. Of course, we have $\mathcal{A}^0 M = \mathcal{G} \times_Q \mathfrak{q} \subset \mathcal{A}_\rho M$. Sections of $\mathcal{A}M$ can be interpreted as Q–invariant vector fields on $\mathcal{G}$. Using this fields to differentiate the equivariant functions corresponding to sections of any associated bundle E to $\mathcal{G}$ defines the fundamental derivative

$$\Gamma(E) \to \Gamma(\mathcal{A}^* M \otimes E) \qquad (s, \varphi) \mapsto D_s\varphi.$$

Projecting the values, we obtain D^ρ with values in $\mathcal{A}^*_\rho M$.

Now consider the bundles $\Lambda^k \mathcal{A}_\rho^* M \otimes \mathcal{V}M$, so sections are just multilinear alternating maps mapping k entries from $\mathcal{A}_\rho M$ to $\mathcal{V}M$. Since $\mathbb{V}$ is a $\mathfrak{p}$–module, the Lie algebra cohomology differential

$$\partial_{\mathfrak{p}} : \Lambda^k \mathfrak{p}^* \otimes \mathbb{V} \to \Lambda^{k+1} \mathfrak{p}^* \otimes \mathbb{V}$$

induces a tensorial bundle map. Thus for a section φ of $\Lambda^k \mathcal{A}_\rho^* M \otimes \mathcal{V}M$ we may consider:

$$(\tilde{d}^{\mathbb{V}} \varphi)(s_0, \dots, s_k) := \mathrm{Alt}((D^\rho_{s_0} \varphi)(s_1, \dots, s_k)) + (\partial_{\mathfrak{p}} \varphi)(s_0, \dots, s_k),$$

which clearly defines a section of $\Lambda^{k+1} \mathcal{A}_\rho^* M \otimes \mathcal{V}M$.

Of course, $\Omega^k_\rho(M, \mathcal{V}M)$ can be identified with the space of those sections of $\Lambda^k \mathcal{A}^*_\rho M \otimes \mathcal{V}M$ which vanish upon insertion of one section of $\mathcal{A}^0 M \subset \mathcal{A}_\rho M$. Using basic properties of the fundamental derivative, it is now easy to prove

Proposition

If $\varphi \in \Gamma(\Lambda^k \mathcal{A}^*_\rho M \otimes \mathcal{V}M)$ has the property that $i_s \varphi = 0$ for all $s \in \Gamma(\mathcal{A}^0 M)$, then the same is true for $\tilde{d}^{\mathbb{V}} \varphi$. Thus $\tilde{d}^{\mathbb{V}}$ descends to a sequence of operators

$$d_k^{\mathbb{V}} : \Omega^k_\rho(M, \mathcal{V}M) \to \Omega^{k+1}_\rho(M, \mathcal{V}M)$$

and this sequence is compressible.

An alternative interpretation

Let $\bullet : \mathcal{A}_\rho M \times \mathcal{V}M \to \mathcal{V}M$ be the bilinear bundle map induced by the $\mathfrak{p}$–action on $\mathbb{V}$.

The map $\Gamma(\mathcal{A}_\rho M \otimes \mathcal{V}M) \to \Gamma(\mathcal{V}M)$ defined by $\mathcal{D}^\rho_s \sigma + s \bullet \sigma$ descends to a partial connection $\nabla^{\rho,\mathbb{V}} : \Gamma(\mathcal{V}M) \to \Gamma(T^*_\rho M \otimes \mathcal{V}M)$ generalizing normal tractor connections.

Next, define a bracket $[\! | \ , \ | \!]$ on $\Gamma(\mathcal{A}^* M)$ by $[\! | \, s_1, s_2 \, | \!] := D_{s_1} s_2 - D_{s_2} s_1 + \{s_1, s_2\}$. Basic properties of the fundamental derivative easily imply

- If $s_1 \in \Gamma(\mathcal{A}^0 M)$ then $[\! | \, s_1, s_2 \, | \!] \in \Gamma(\mathcal{A}^0 M)$.
- If $s_1, s_2 \in \Gamma(\mathcal{A}_\rho M)$ then $[\! | \, s_1, s_2 \, | \!] \in \Gamma(\mathcal{A}_\rho M)$.

Then writing Π for the natural projection $\mathcal{A}_\rho M \to T_\rho M$, the operator $\tilde{d}^{\mathbb{V}}$ inducing $d^{\mathbb{V}}$ can be equivalently rewritten as

$$\tilde{d}^{\mathbb{V}}\varphi(s_0,\dots,s_k) = \sum_i (-1)^i \nabla^{\rho,\mathbb{V}}_{\Pi(s_i)}\varphi(s_0,\dots,\hat{i},\dots,s_k) \\ + \sum_{i<j}(-1)^{i+j}\varphi([\!|\, s_i,s_j \,|\!],s_0,\dots,\hat{i},\dots,\hat{j},\dots,s_k).$$

Happy $\dim(\overset{1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{2}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ})$, Mike!