Instantons, Quotient Singularities,

and the

Geometrization of Four-Manifolds

Claude LeBrun
Stony Brook University

Vienna, Sept 10, 2012

For Mike Eastwood

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but who, somehow,
never convinced me to
try bungee jumping.

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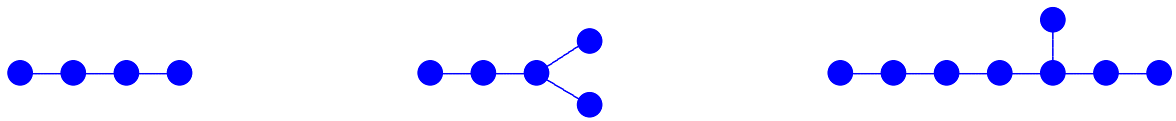

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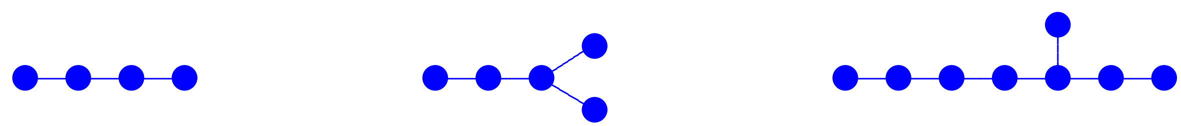

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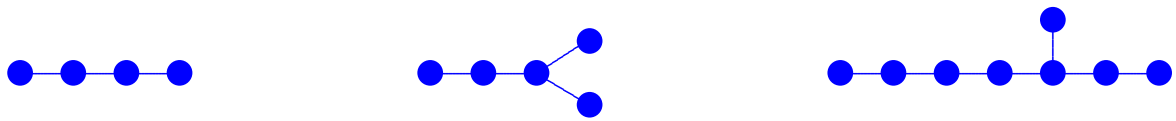

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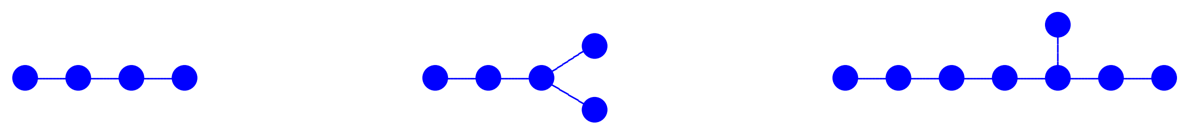

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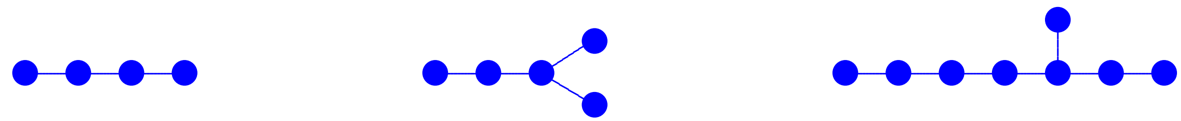

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But it is directly related to representation theory.

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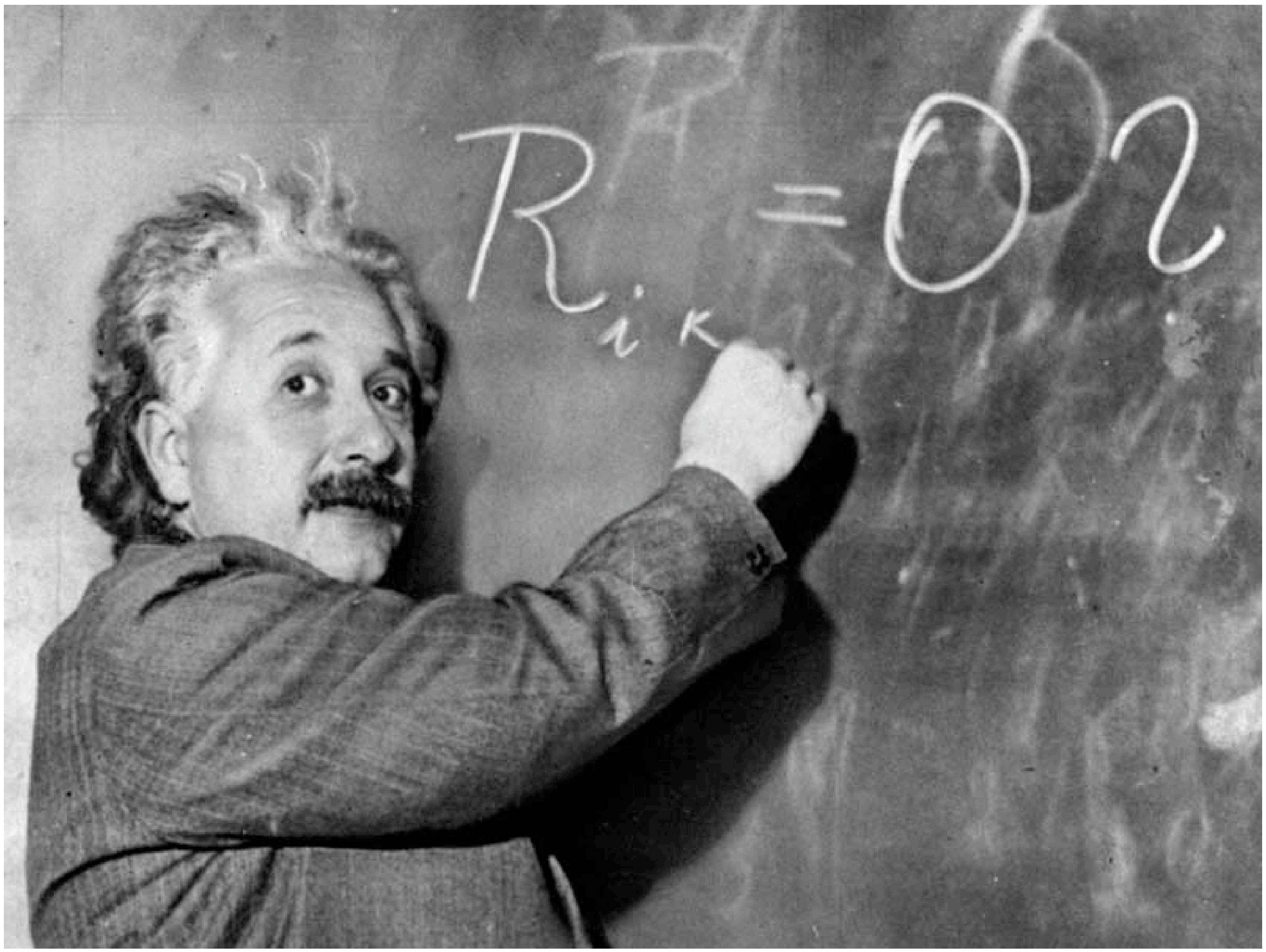

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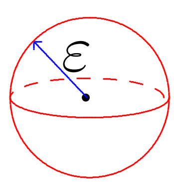

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Try to find Einstein metrics by minimizing?

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is realized by an Einstein metric g_j with $\lambda < 0$.

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Donaldson: moduli spaces of SD connections $\Longrightarrow$ differential topological invariants of M^4.

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$$D^\vartheta : \Gamma(\mathbb{V}_+) \to \Gamma(\mathbb{V}_-)$$

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$$d^{*}(\vartheta - \vartheta_0) = 0$$

imposed to eliminate automorphisms of $L \to M$.

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For complex surfaces, b_1 even $\iff$ Kähler type.

Theorem (L '99). *Let M^4 be underlying smooth manifold of a compact complex surface (M, J) with b_1 even. Then $\mathcal{I}_s(M) \neq 0 \iff (M, J)$ is of general type.*

Recall: $\mathcal{I}_s(M^4) := \inf_g \int_M s_g^2 d\mu_g$

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Conjecture. *For any compact complex surface (M^4, J) with b_1 odd, $\mathcal{I}_s(M) = 0$.*

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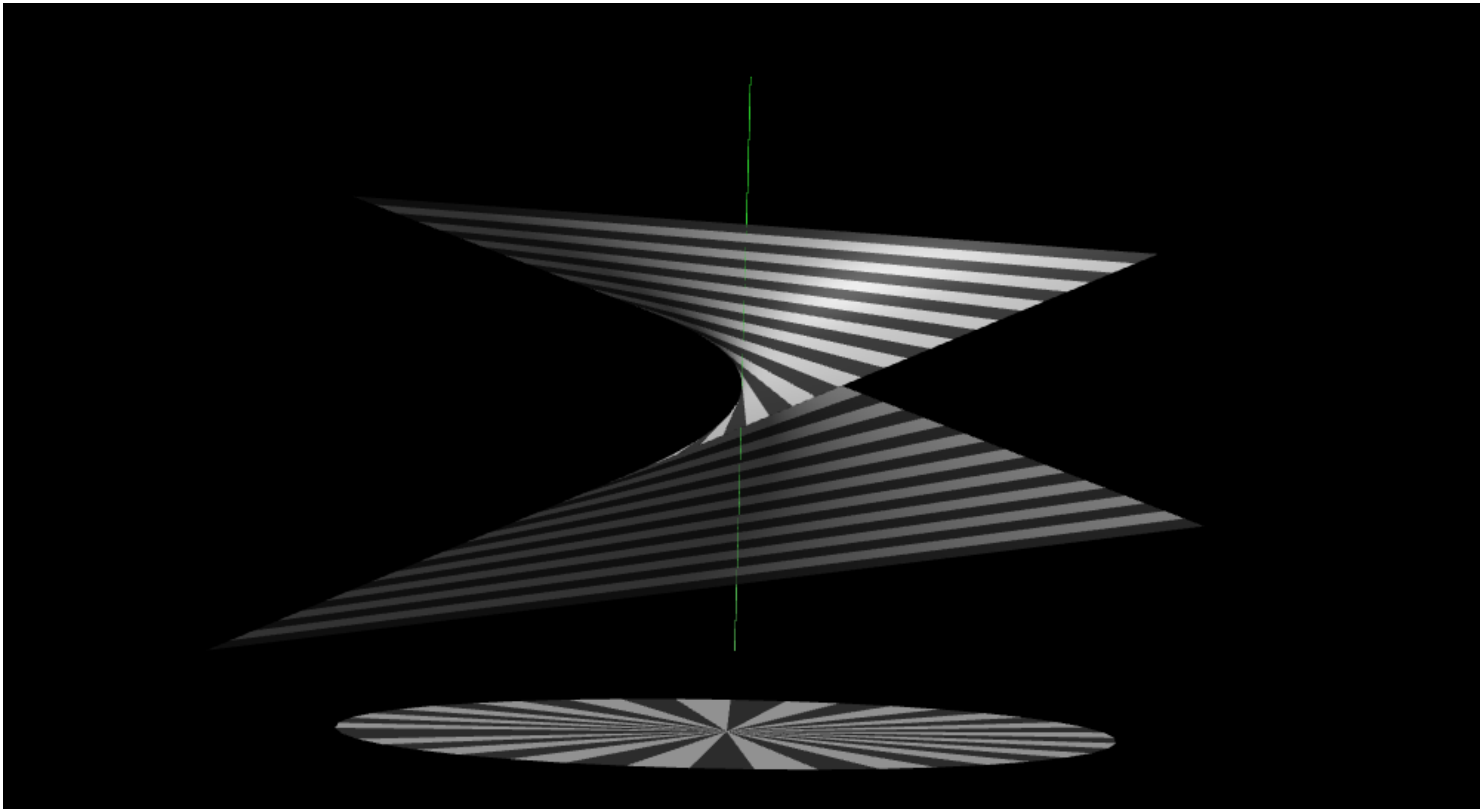

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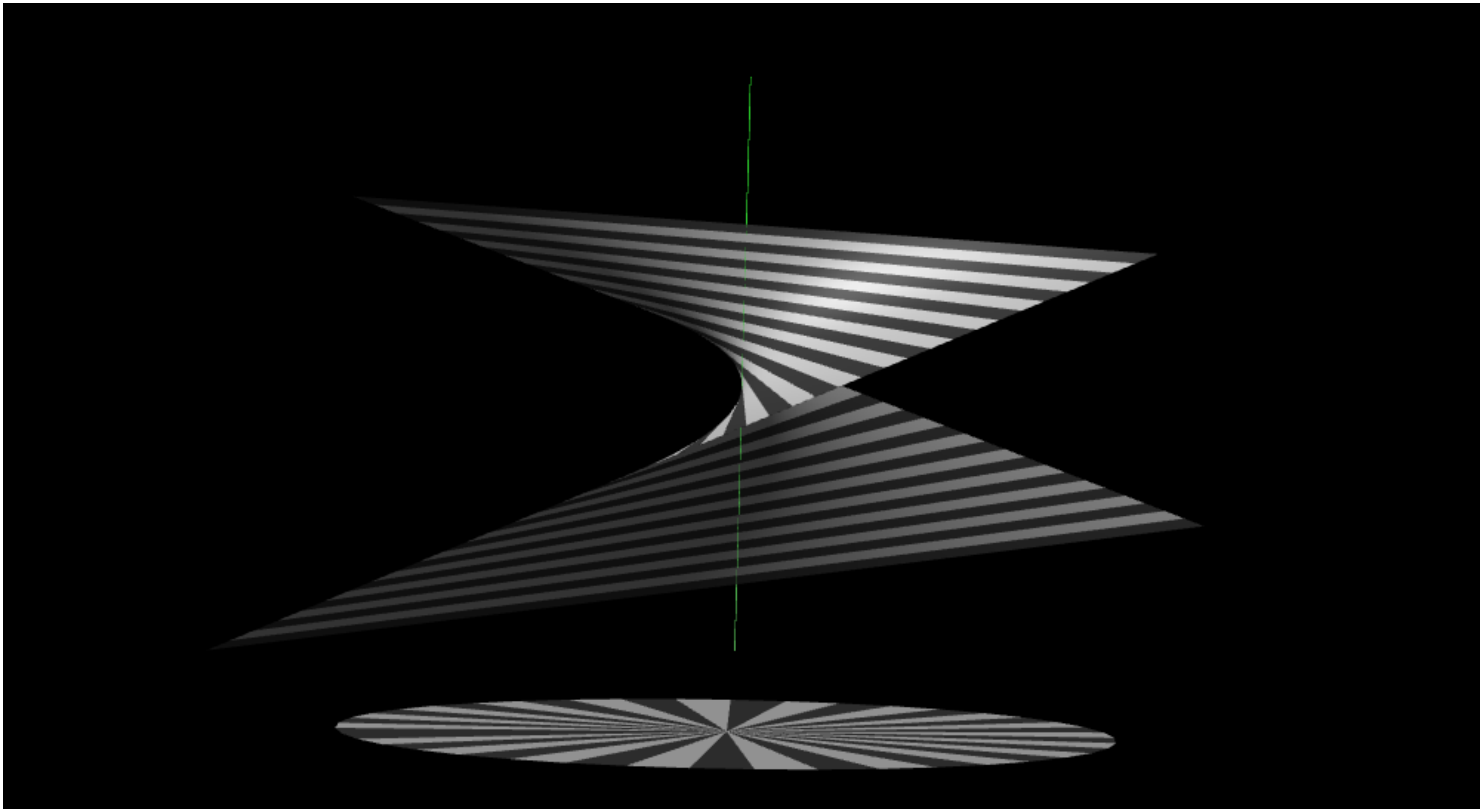

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One says that X is minimal model of M.

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Stategy: replace neighborhood of each orbifold point with ALE Ricci-flat manifold.

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Felix Klein's Table of Singularities (1884)

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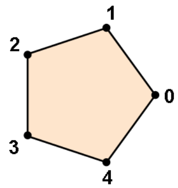

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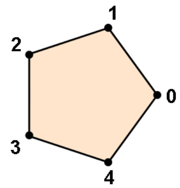

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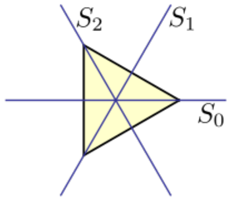

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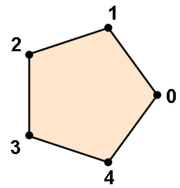

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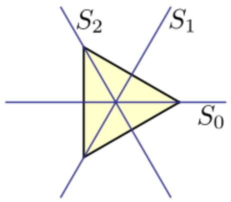

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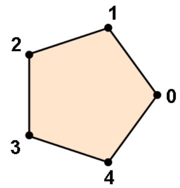

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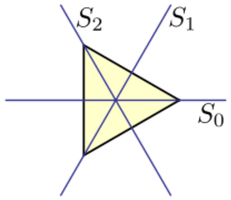

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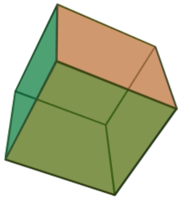

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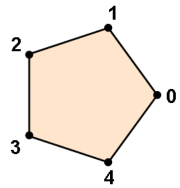

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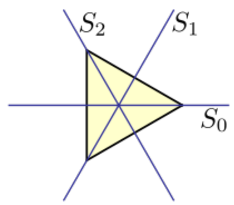

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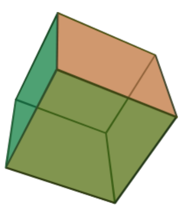

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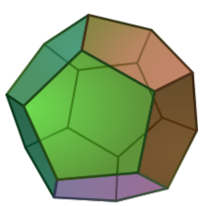

$I^* \quad \longleftrightarrow \quad w^2 + x^3 + y^5 = 0$

Prototypical Klein singularity:

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Two ways to get rid of a singularity:

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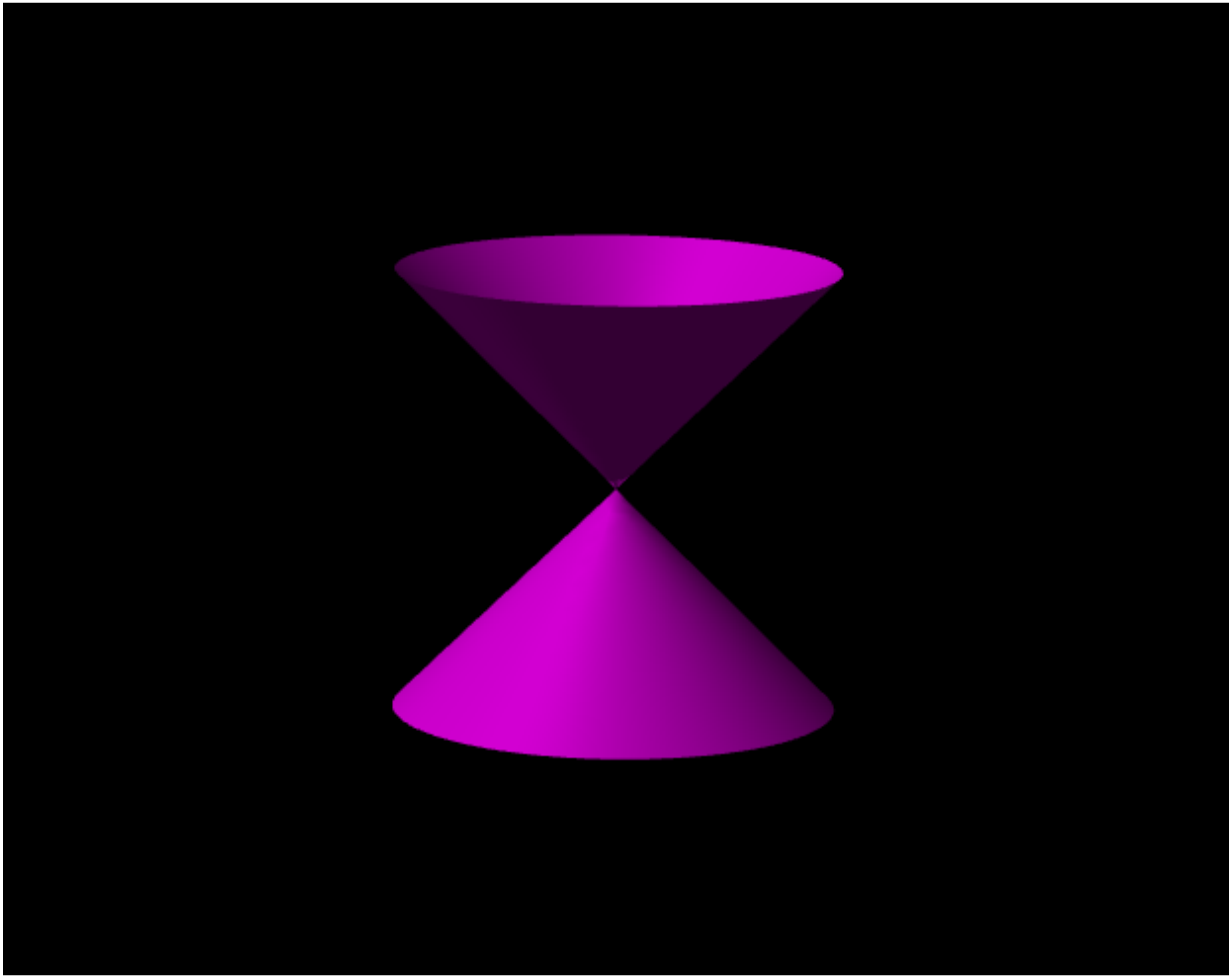

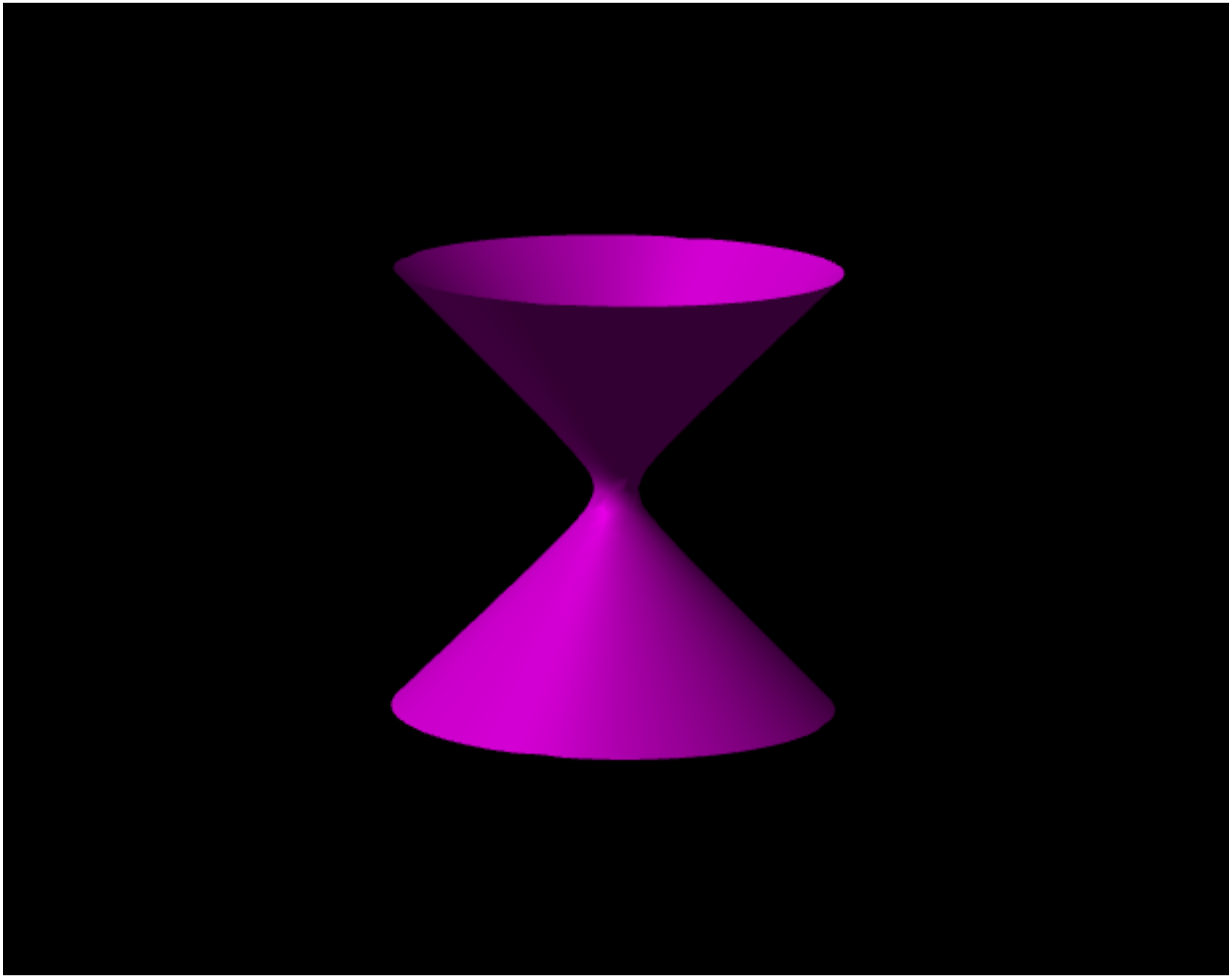

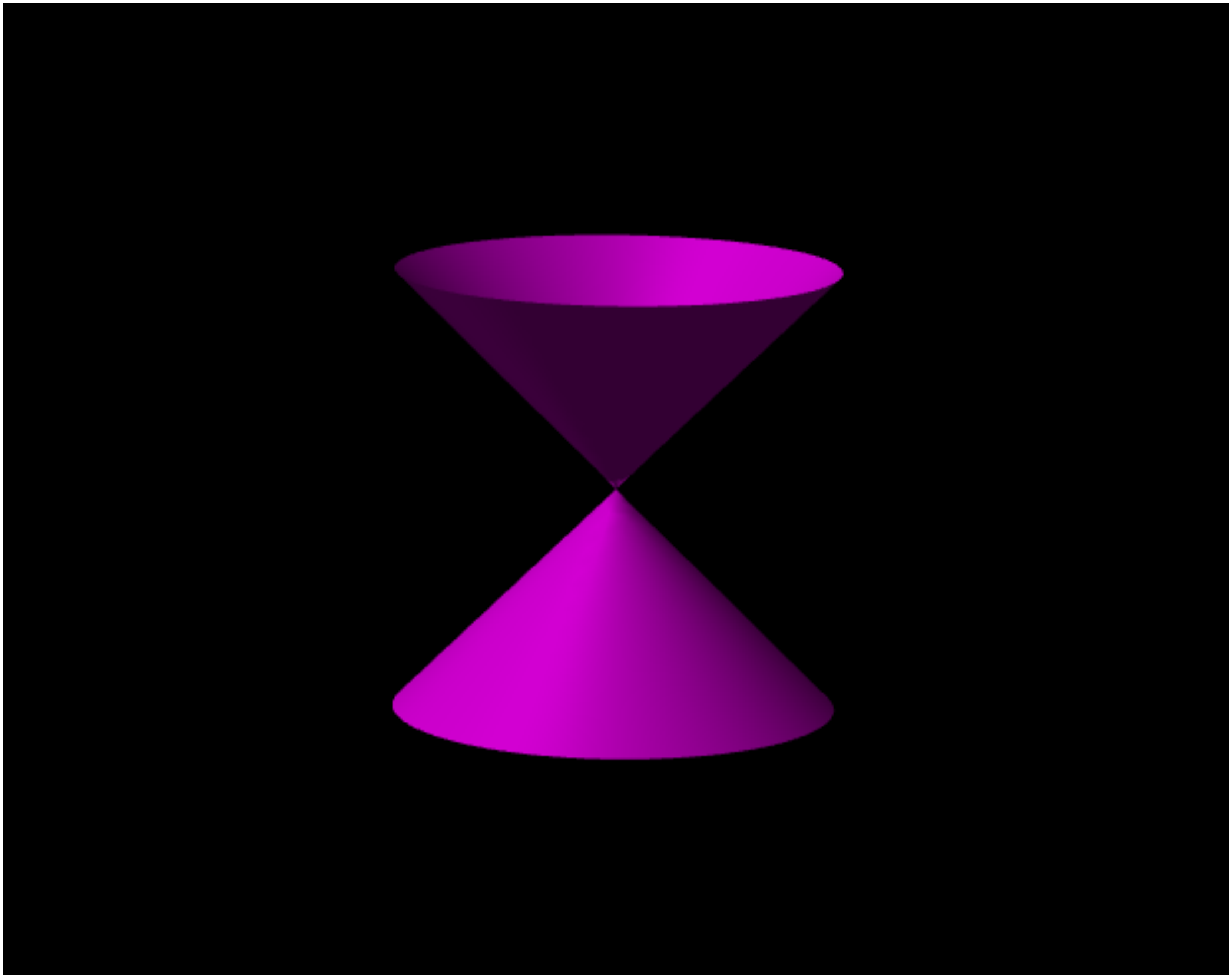

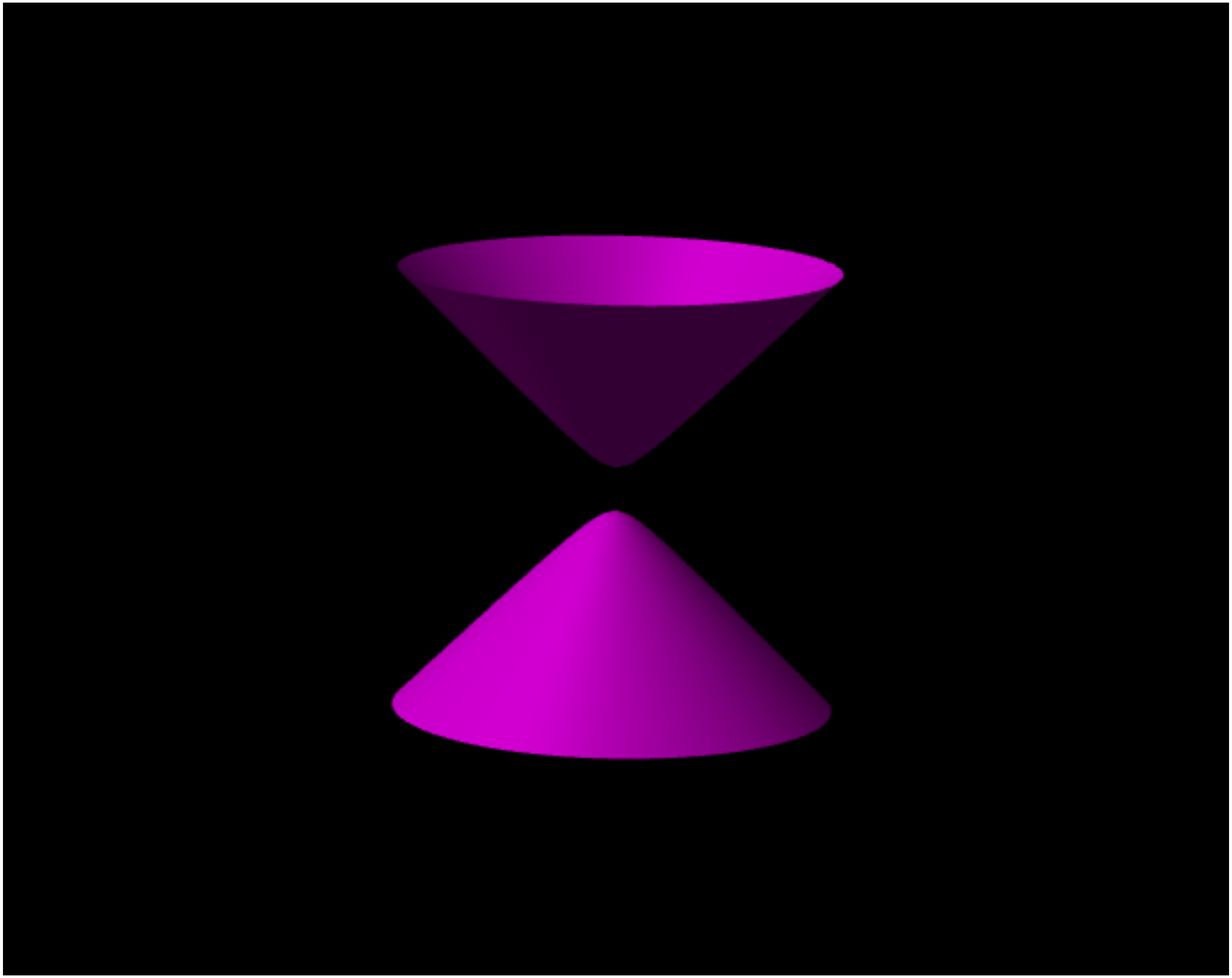

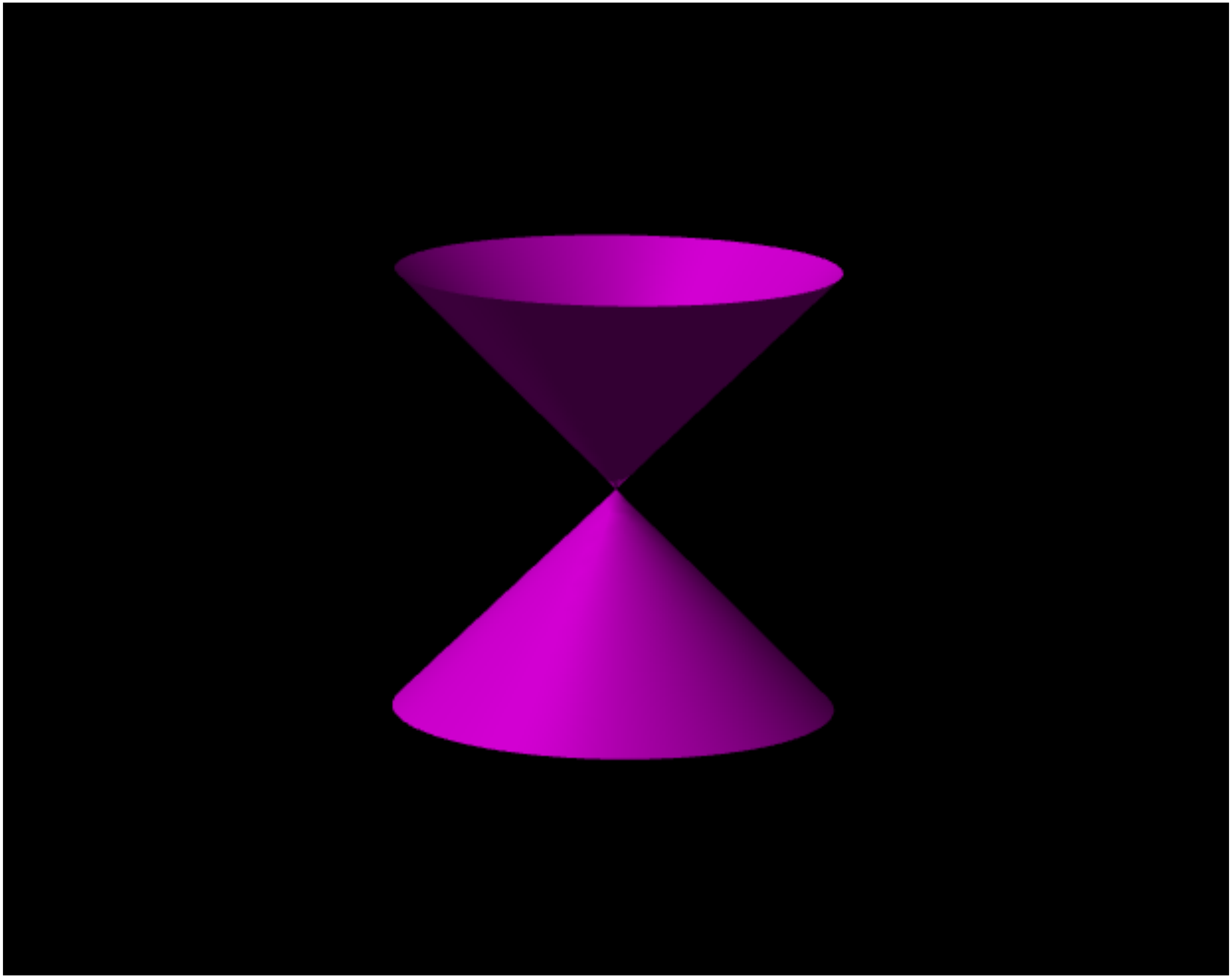

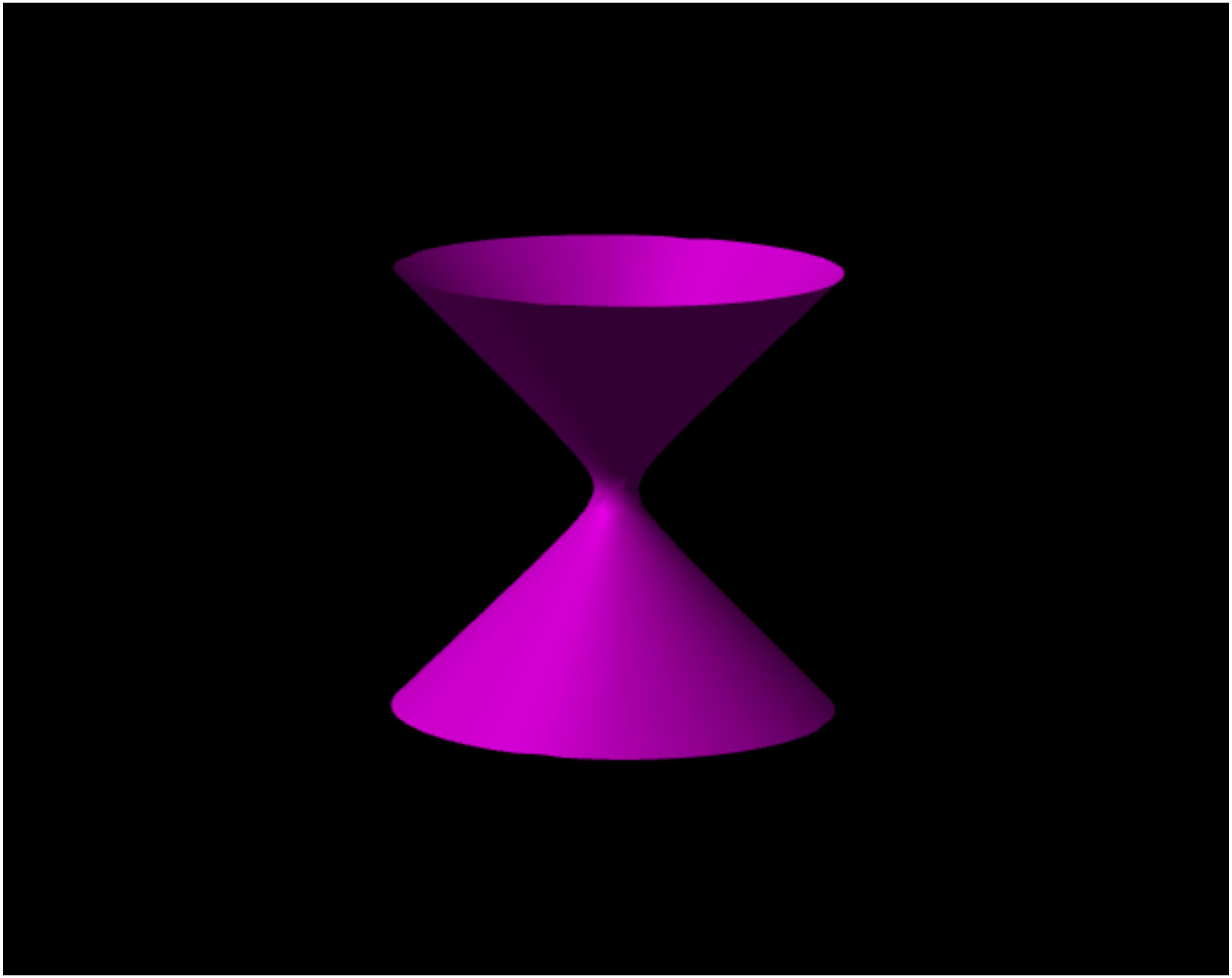

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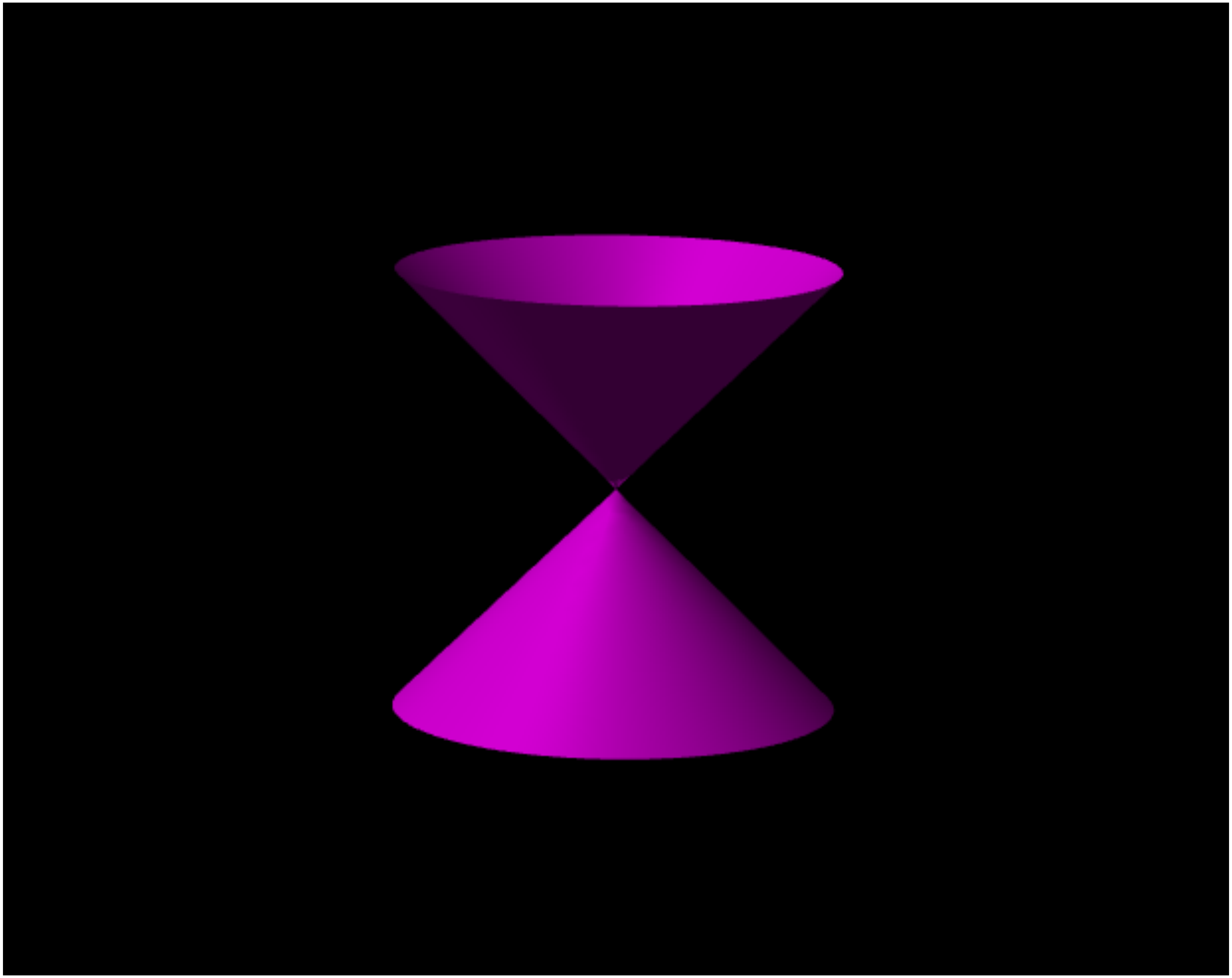

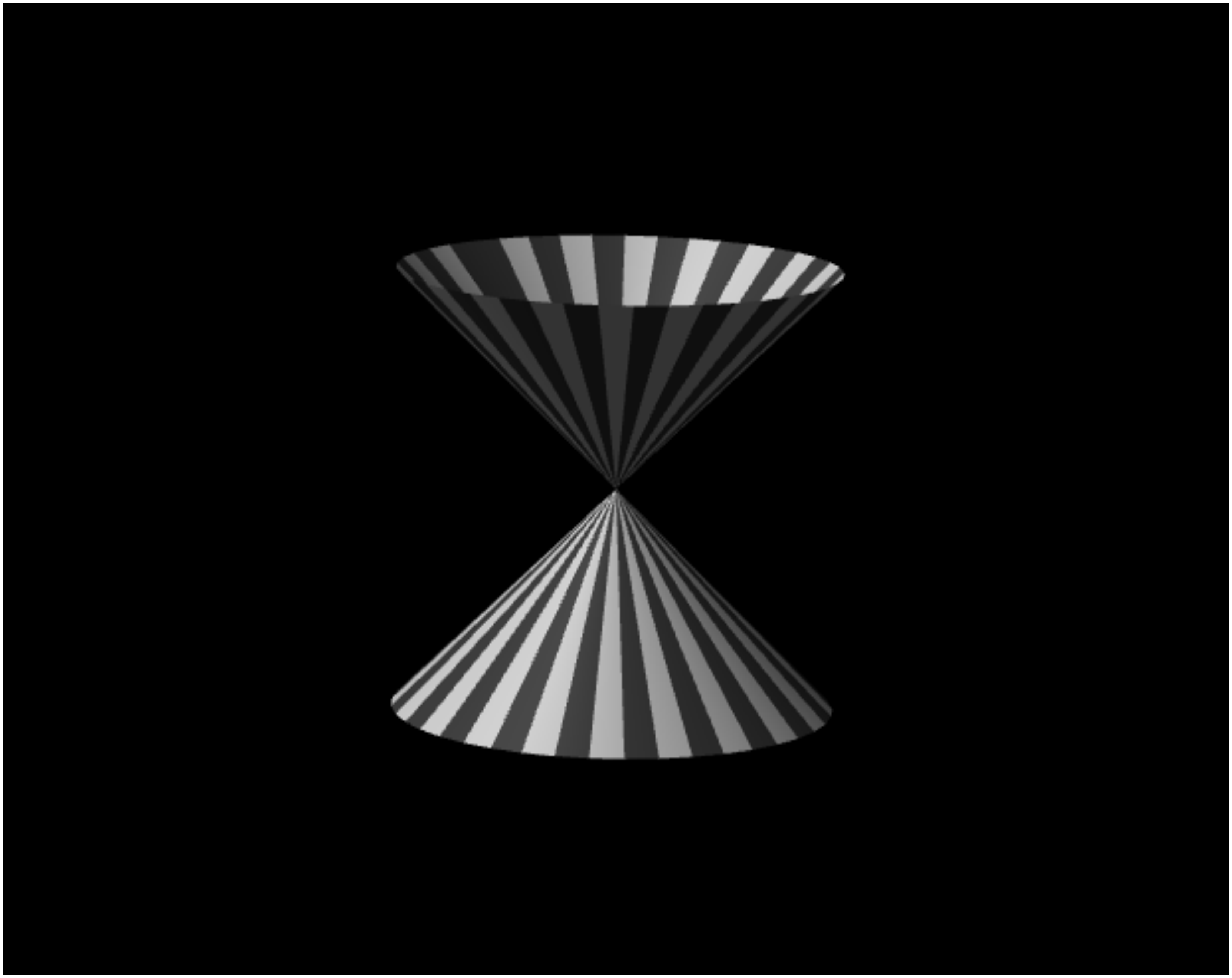

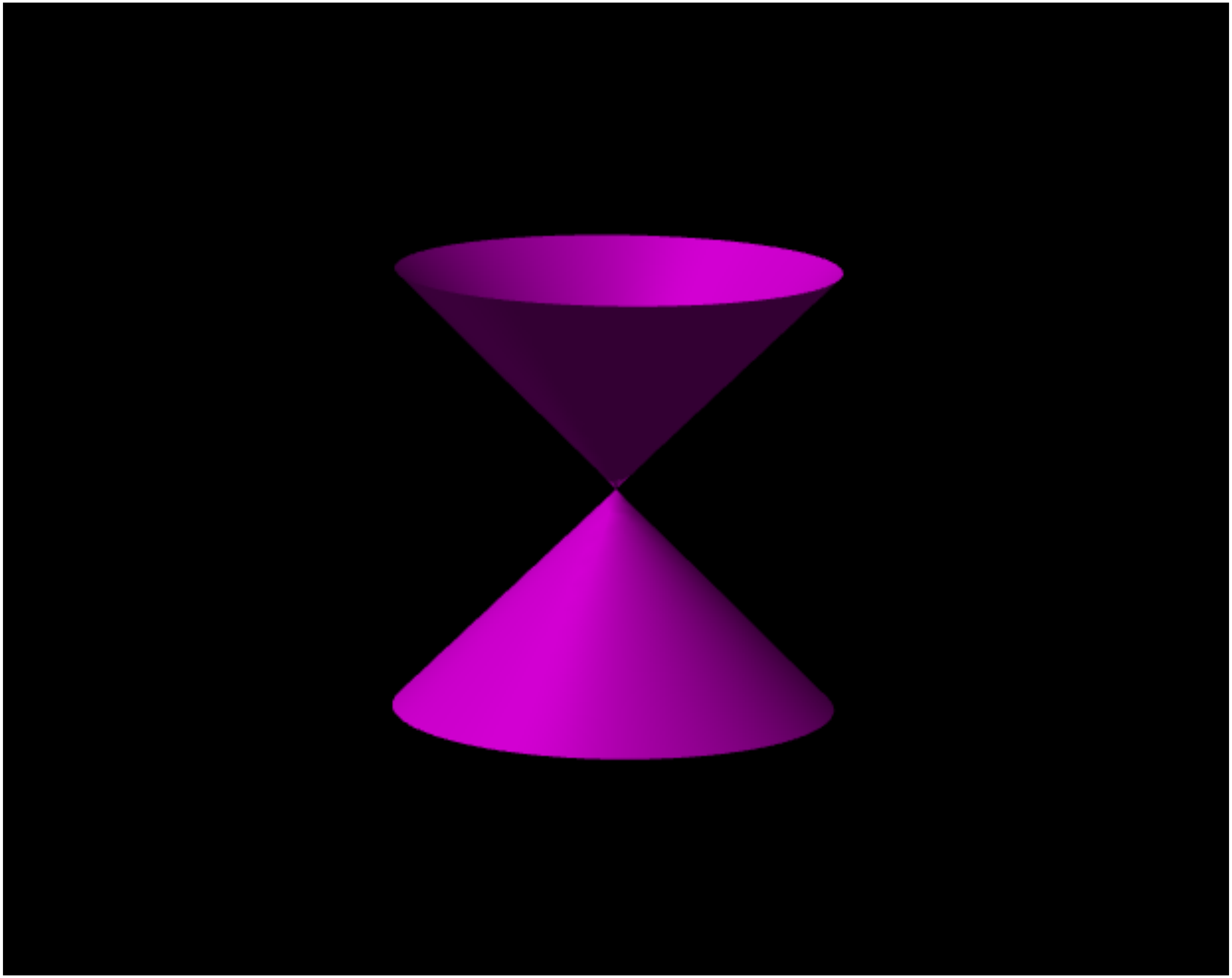

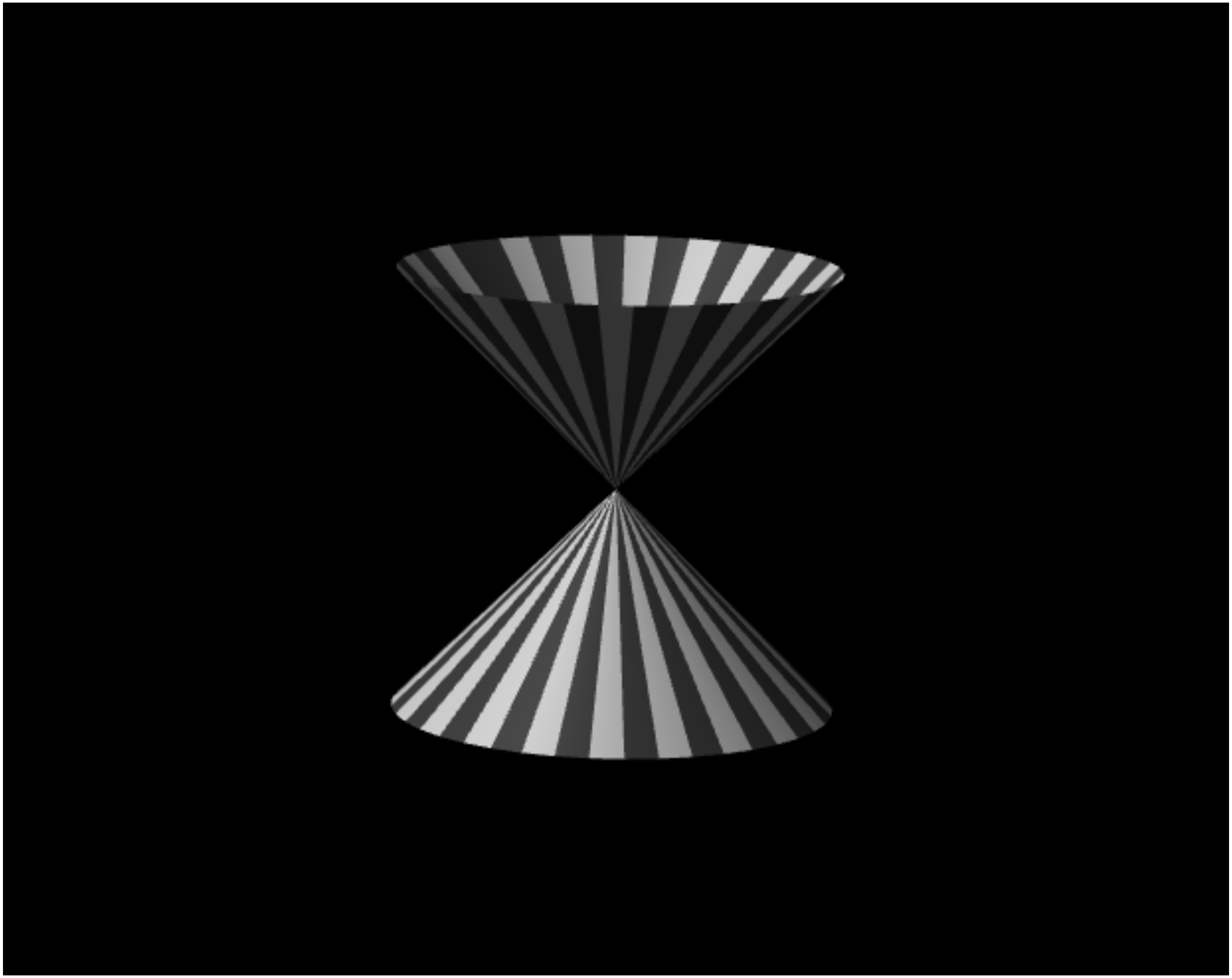

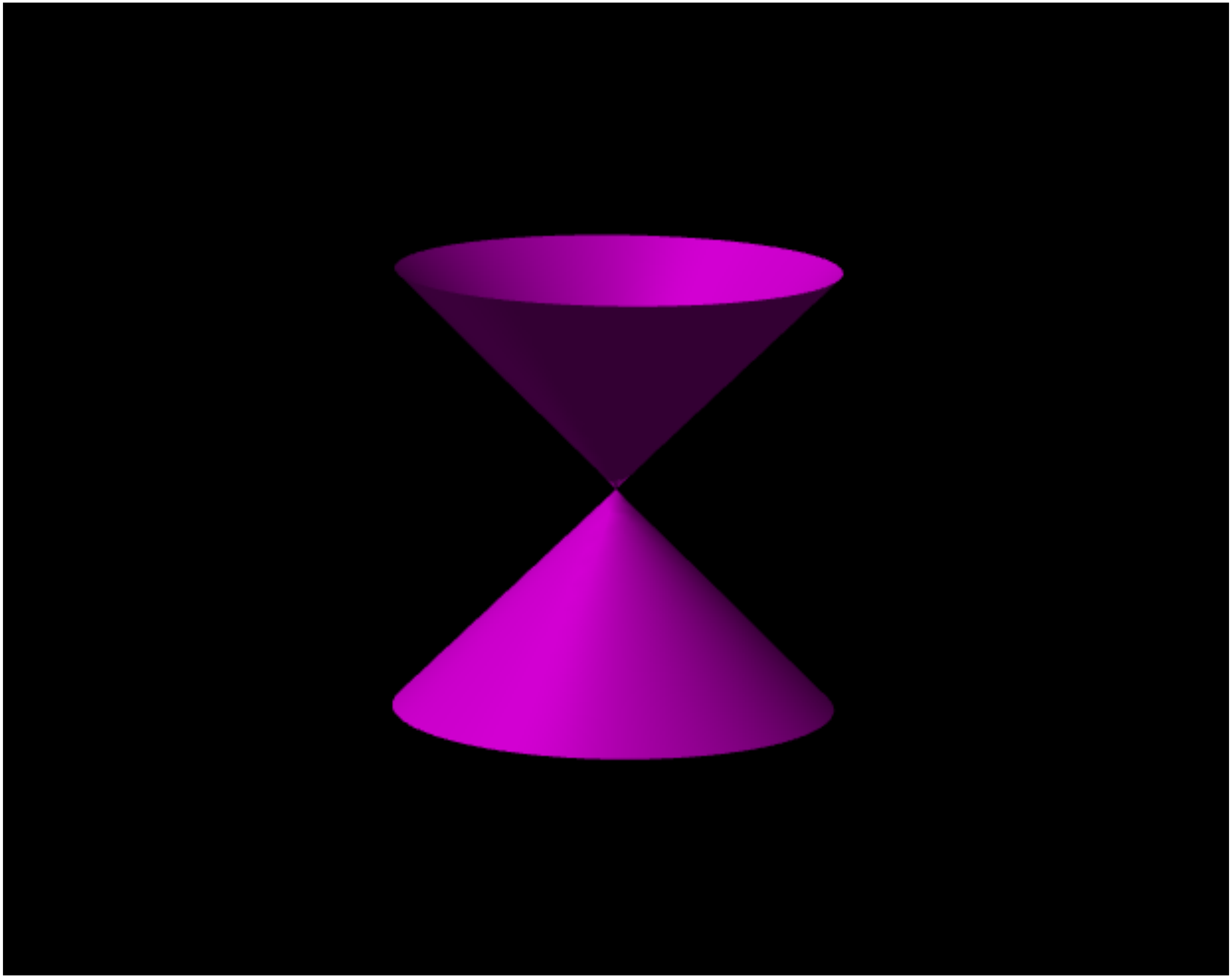

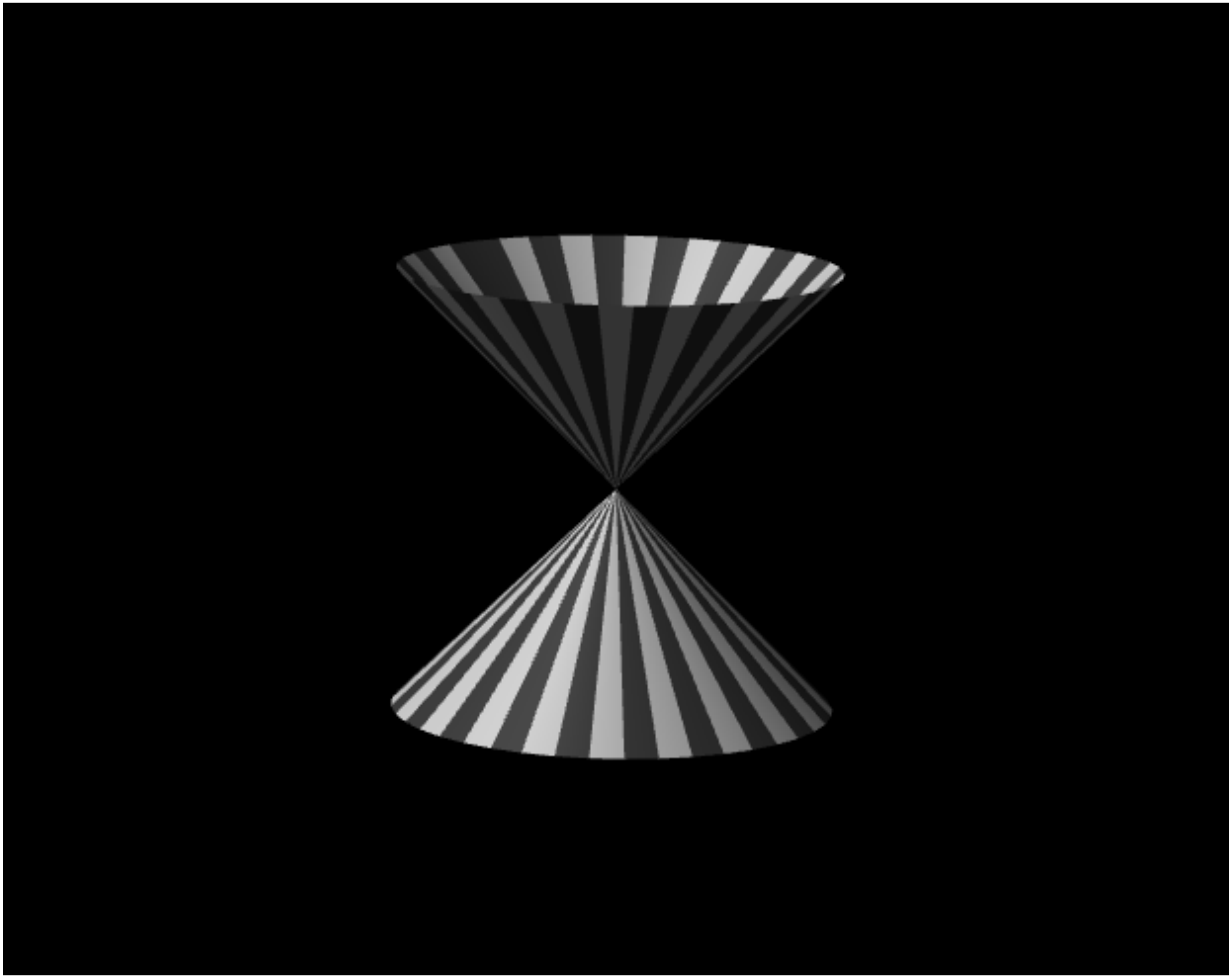

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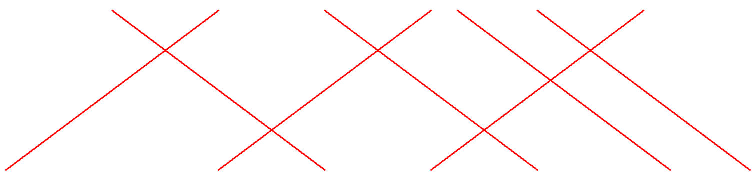

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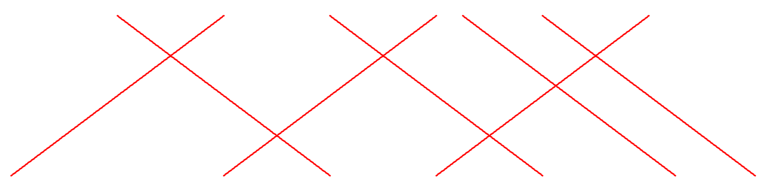

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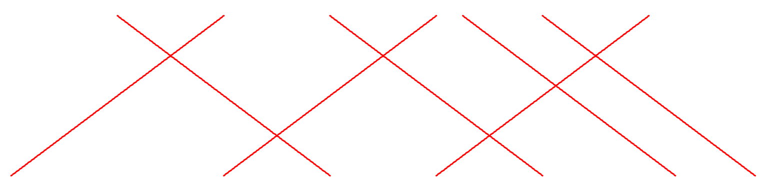

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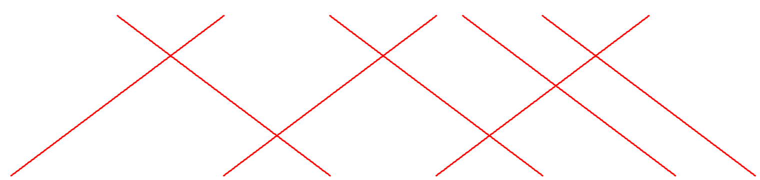

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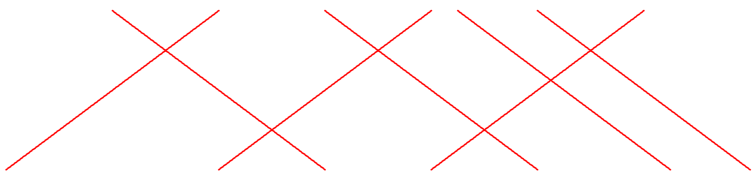

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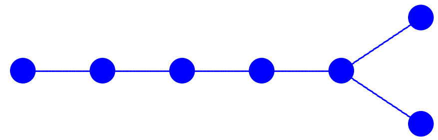

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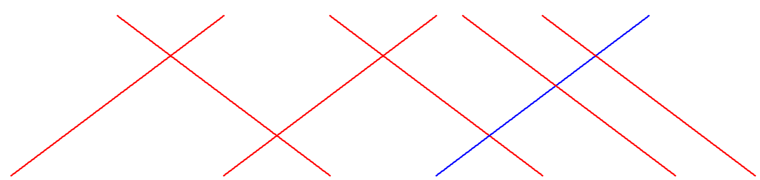

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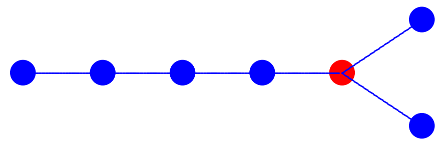

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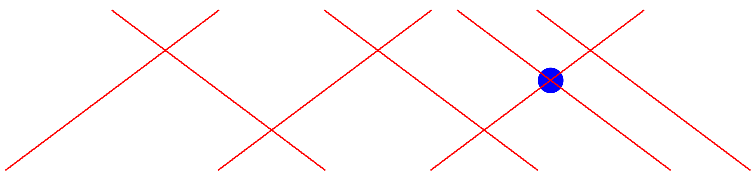

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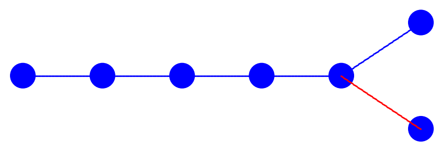

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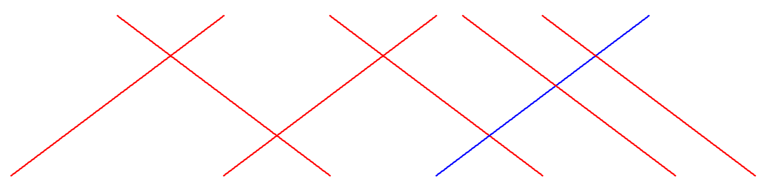

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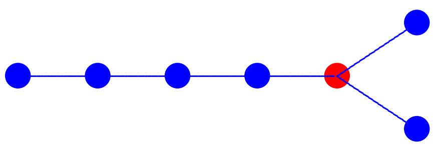

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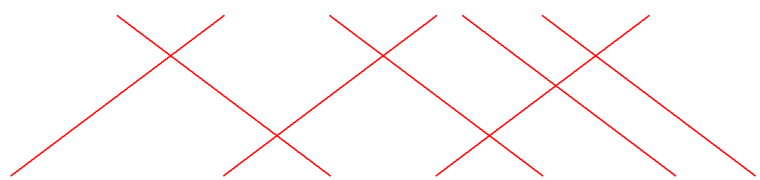

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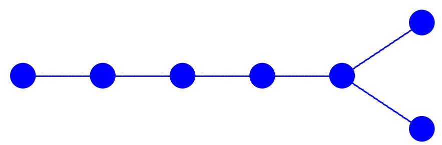

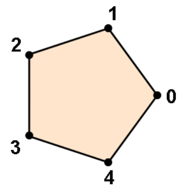

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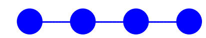

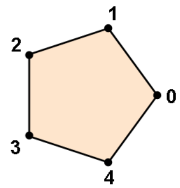

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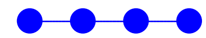

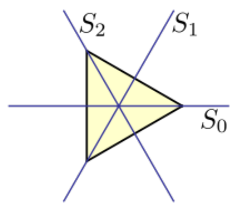

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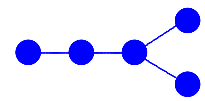

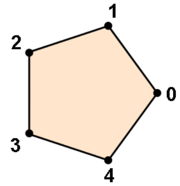

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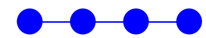

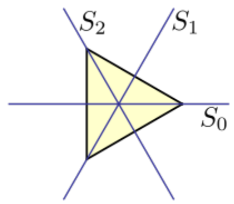

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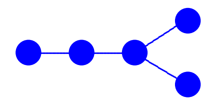

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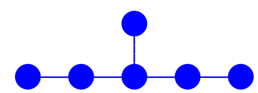

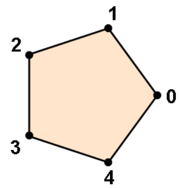

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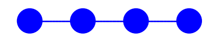

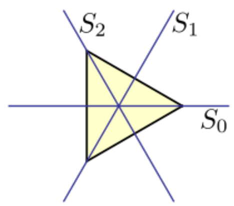

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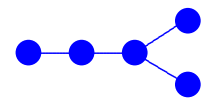

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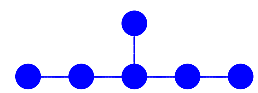

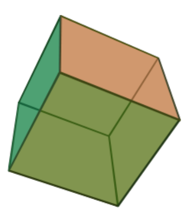

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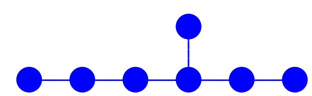

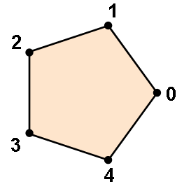

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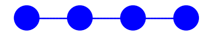

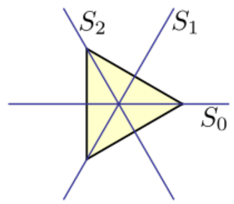

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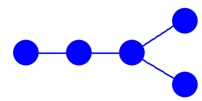

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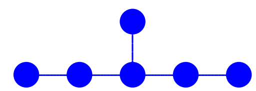

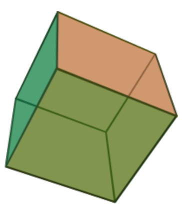

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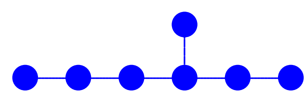

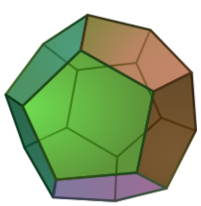

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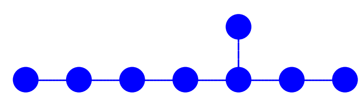

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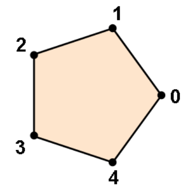
Reproduces Dynkin diagram of crepant resolution!



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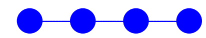

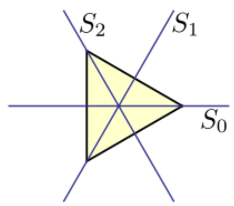

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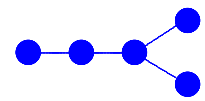

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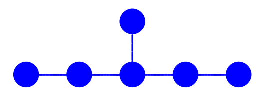

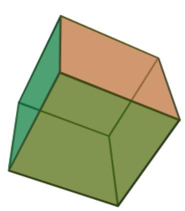

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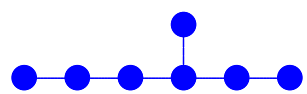

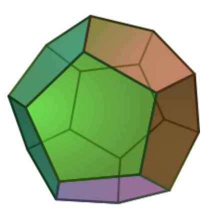

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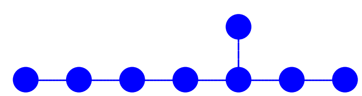

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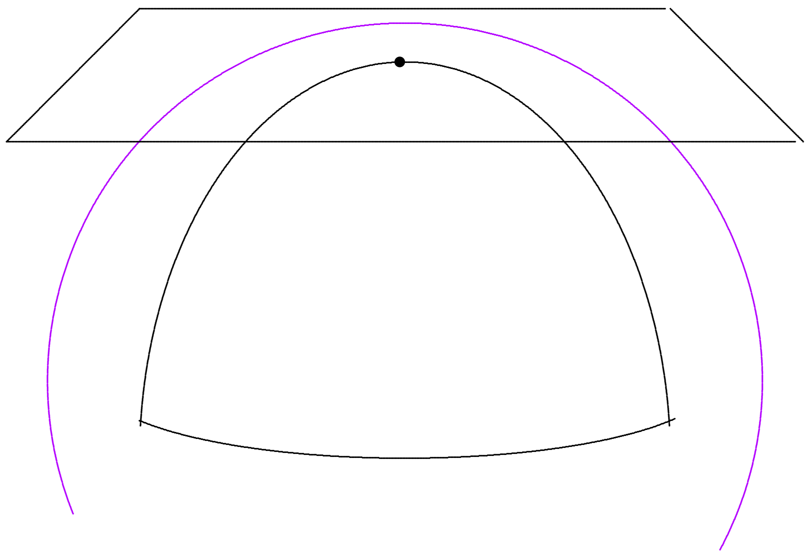

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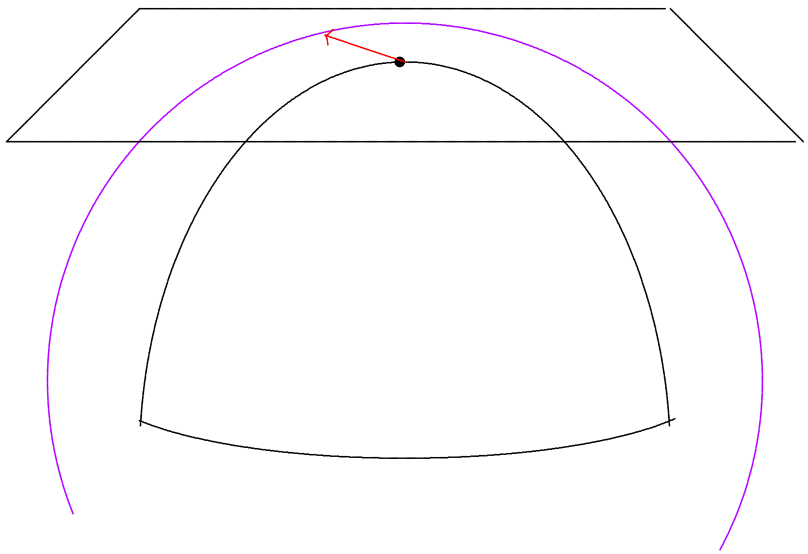

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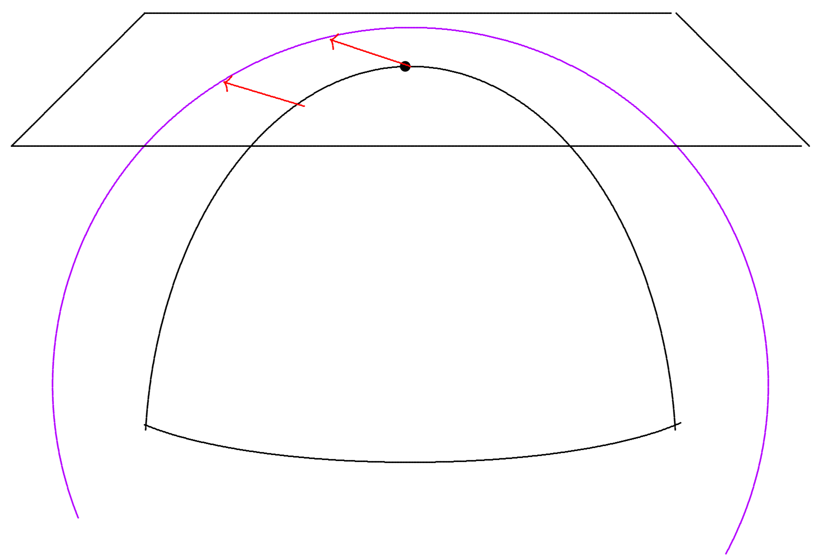

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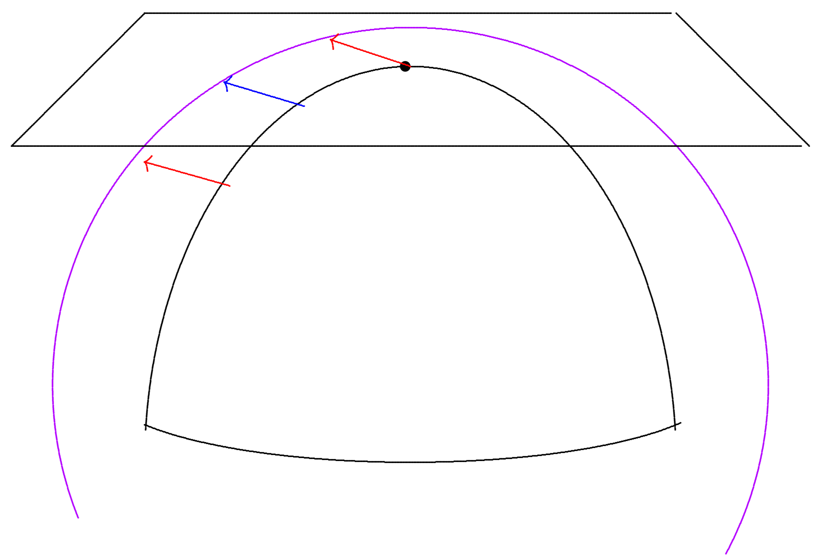

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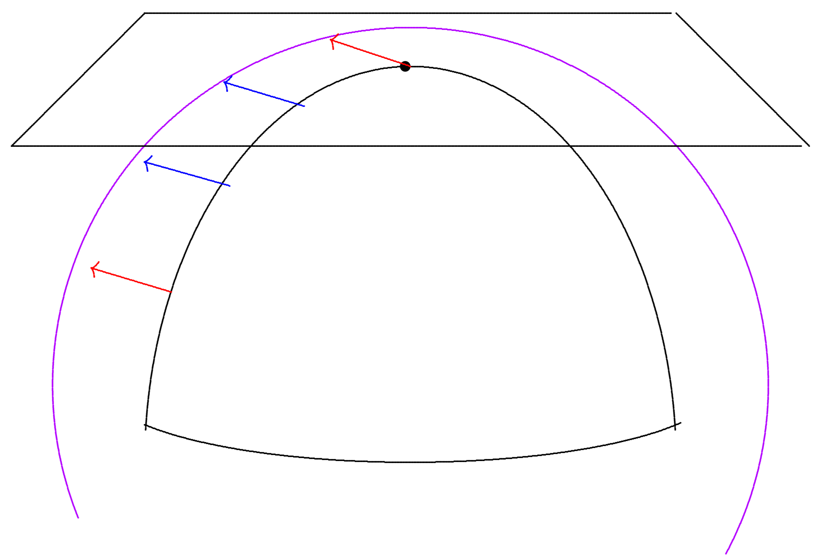

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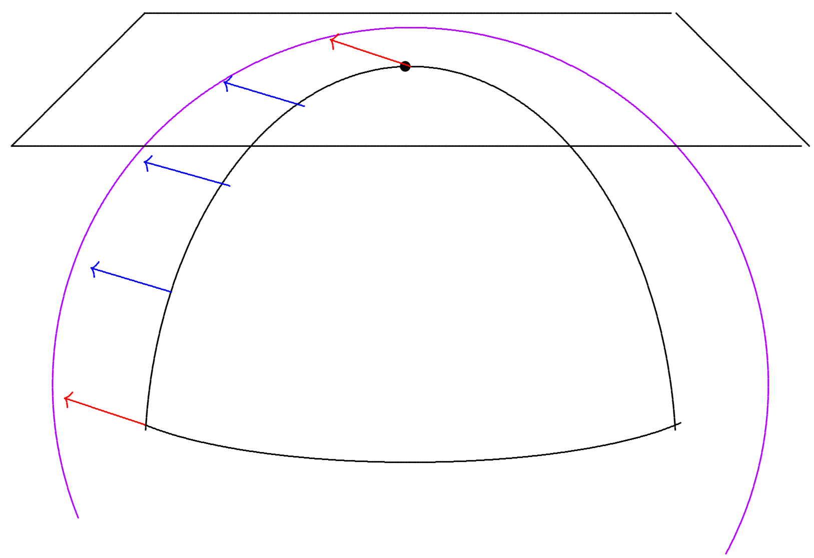

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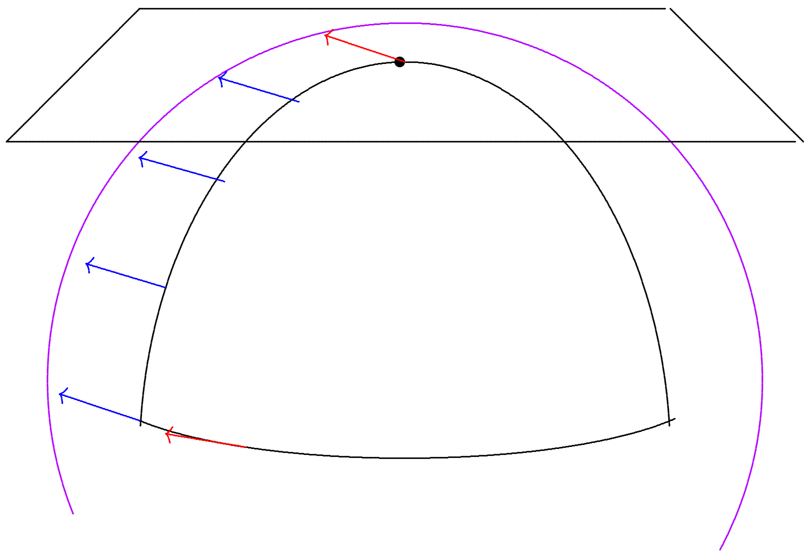

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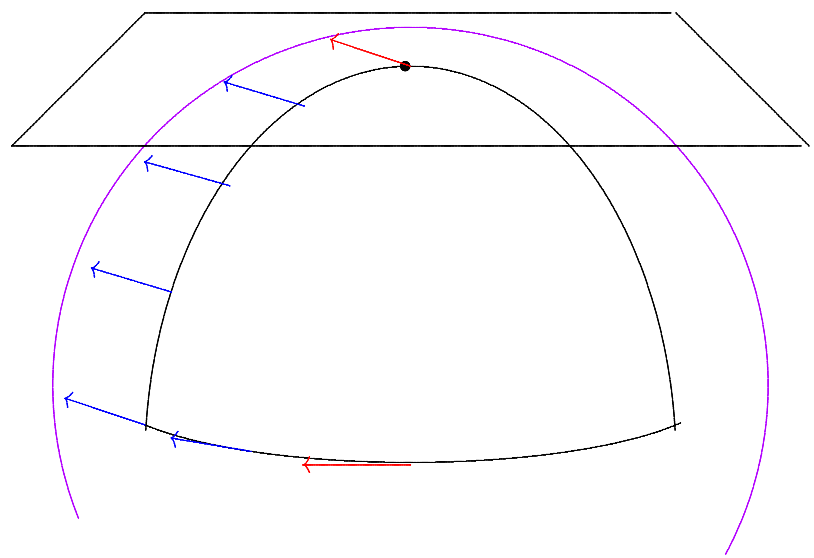

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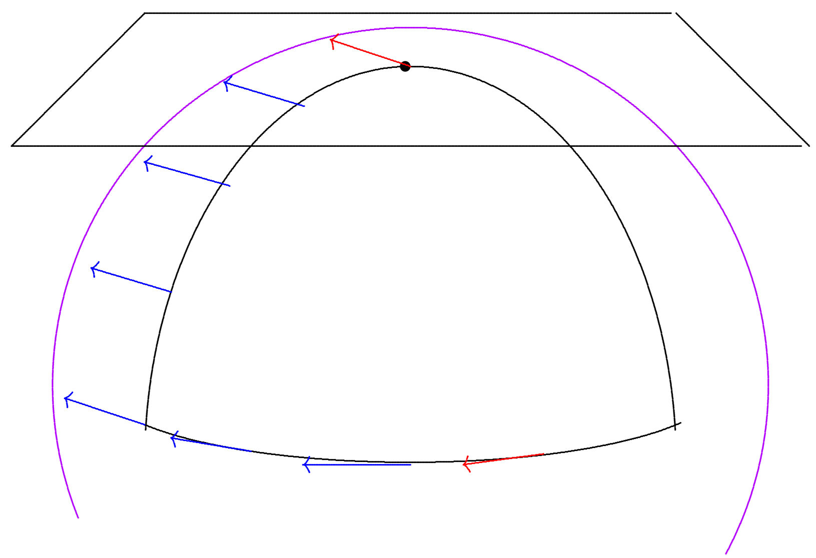

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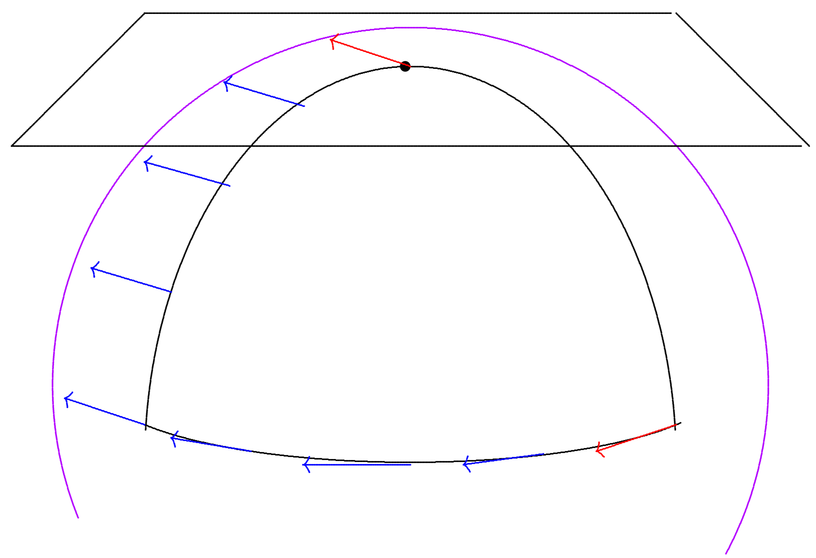

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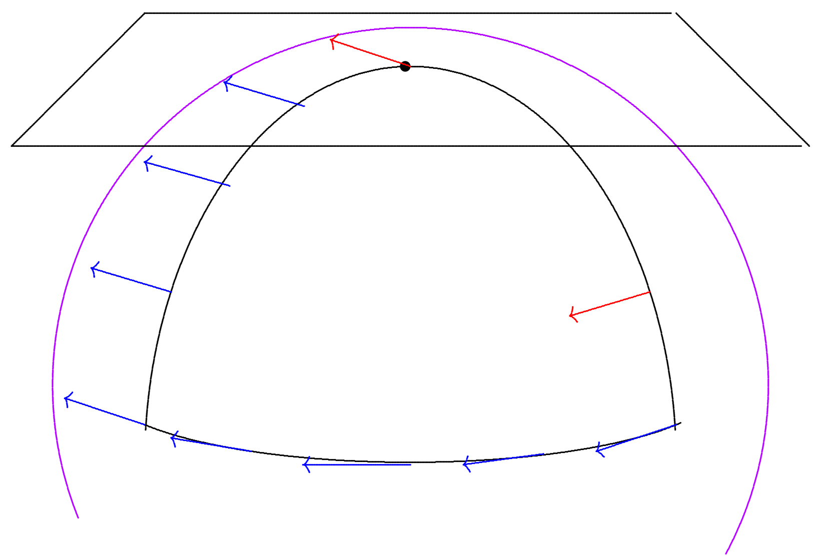

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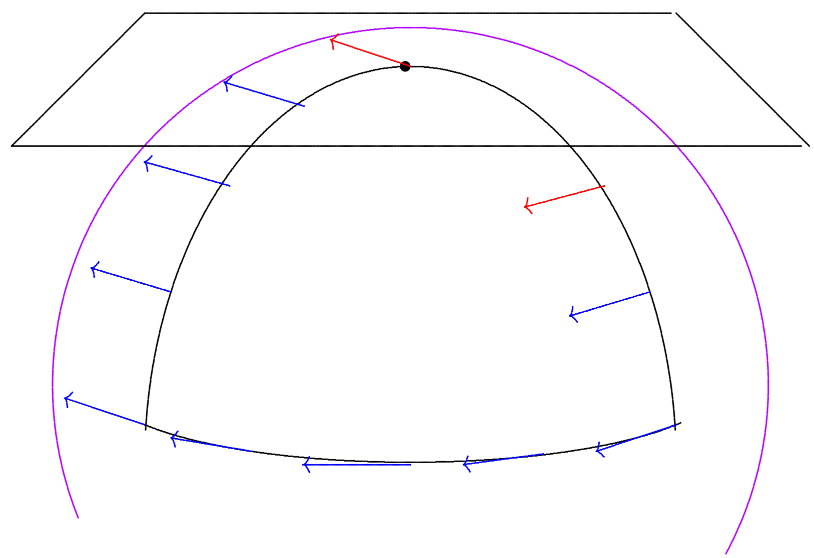

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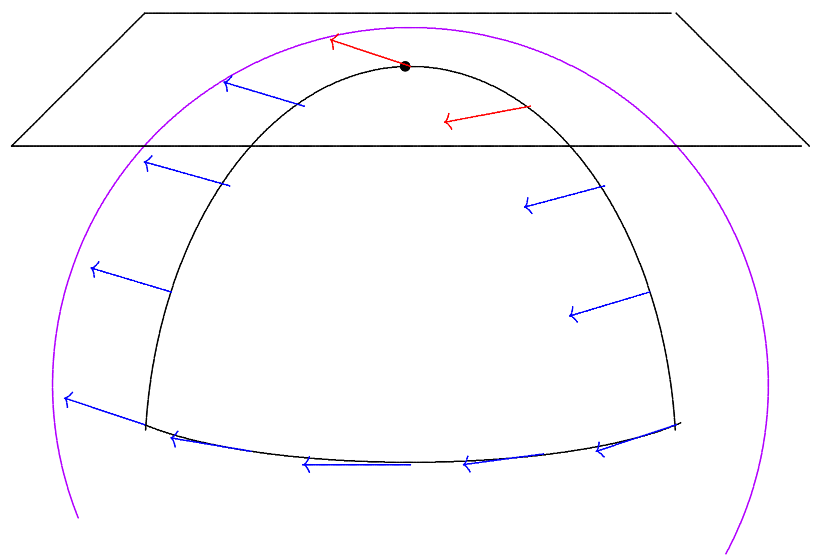

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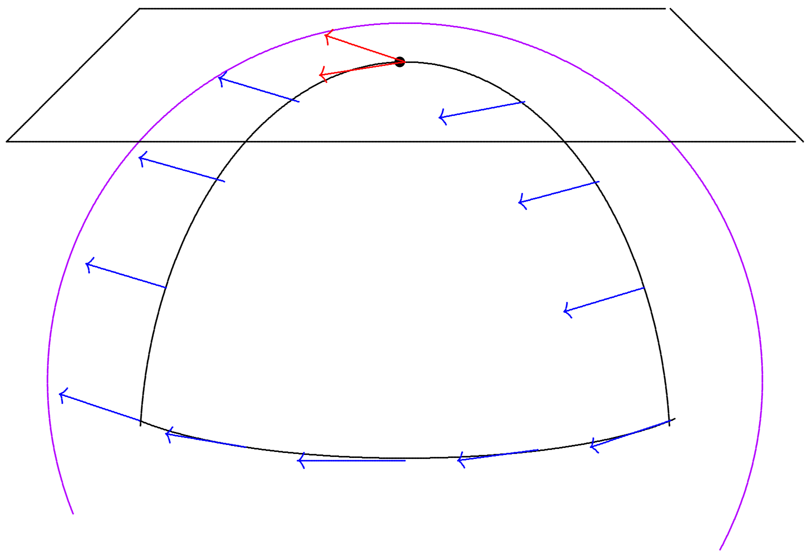

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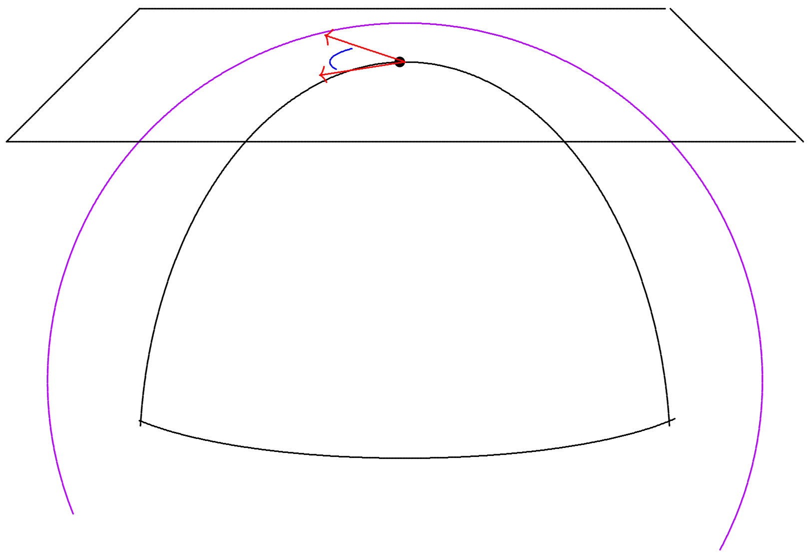
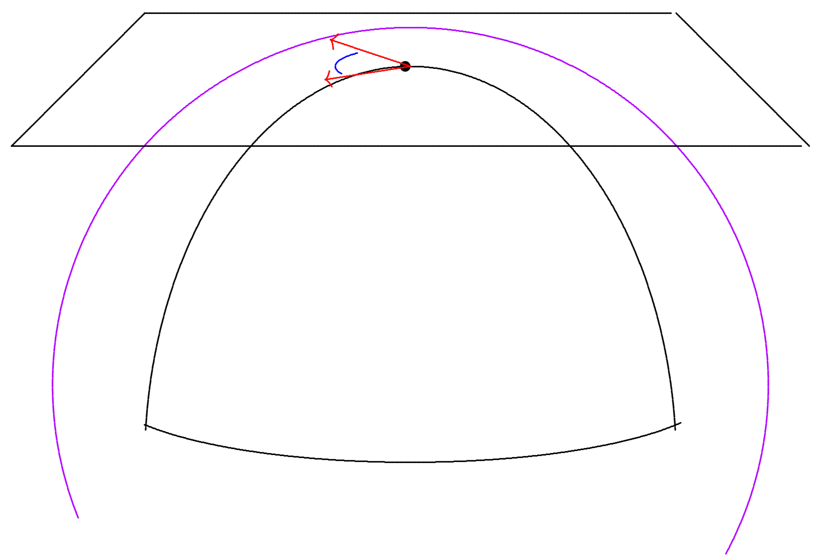


(M^n, g): holonomy $\subset O(n)$



Kähler metrics:

(M^{2m}, g):

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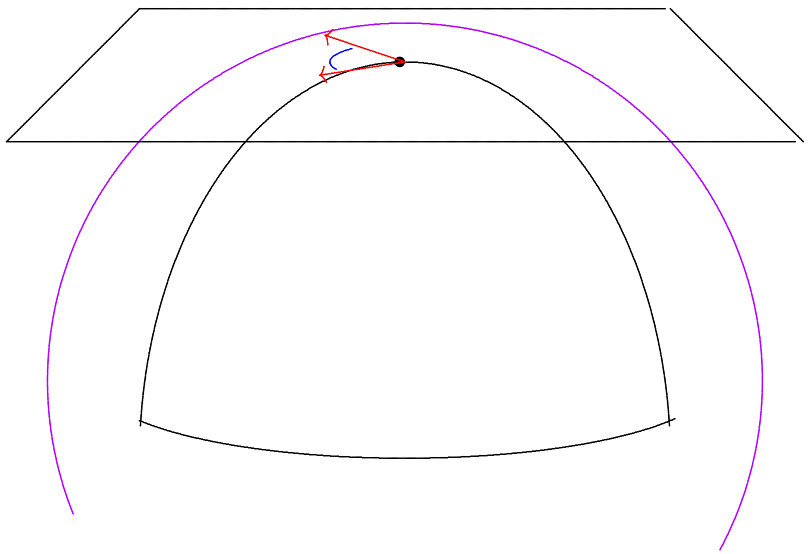
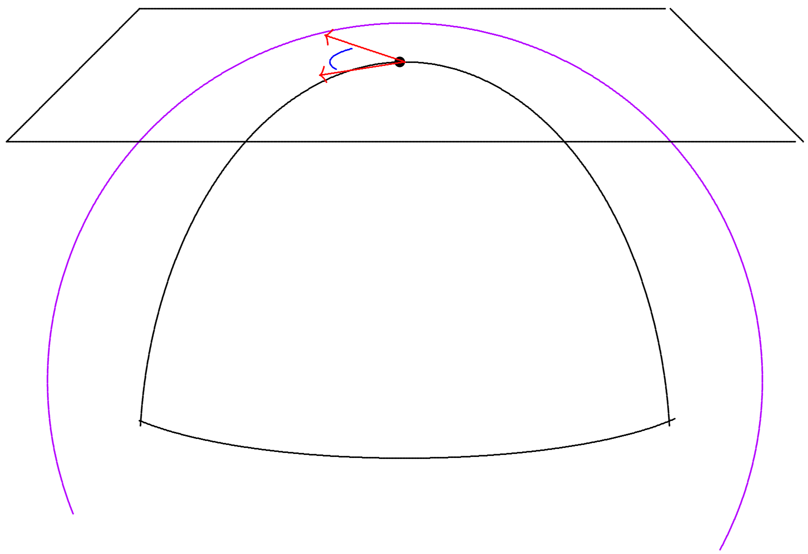


Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$



Kähler metrics:

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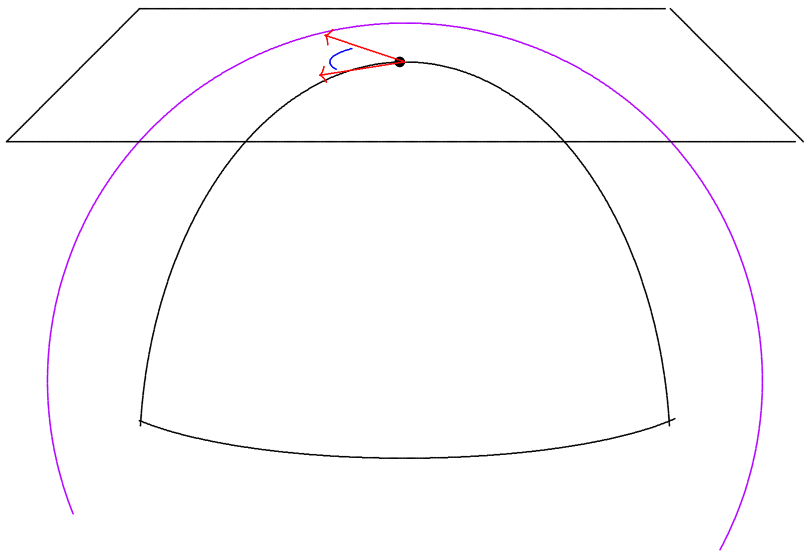
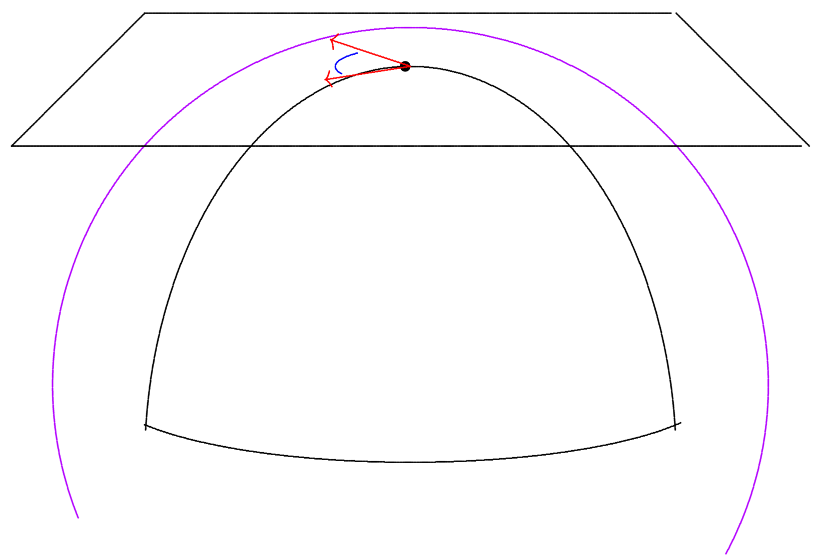


Hyper-Kähler metrics:

$(M^{4\ell}, g)$ Hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(\ell)$



Hyper-Kähler metrics:

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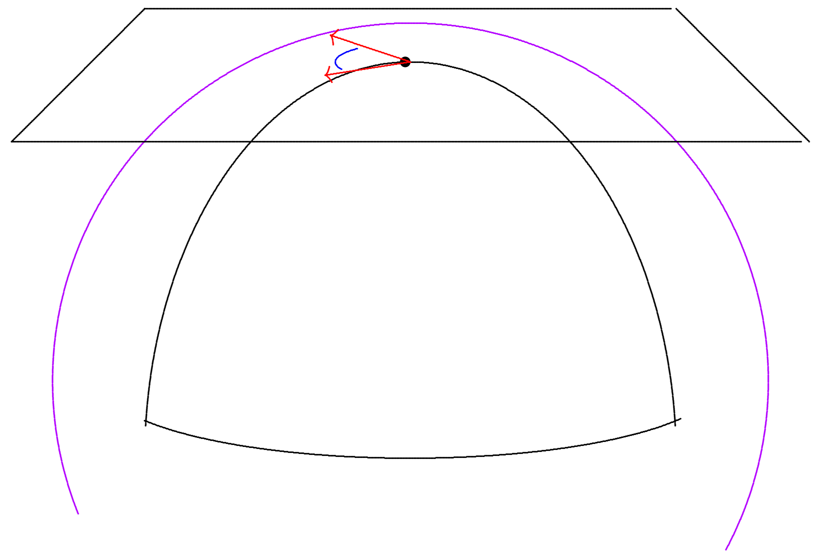

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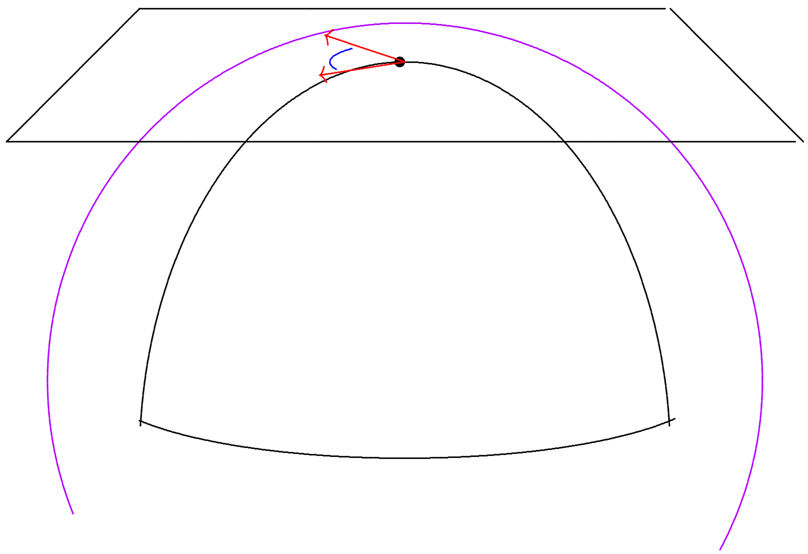

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Construction depends on ζ: $3k$ parameters.

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