Einstein gravity from rational curves in twistor space

Lionel Mason

The Mathematical Institute, Oxford
lmason@maths.ox.ac.uk

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Work with Tim Adamo, Freddy Cachazo & David Skinner. Adamo & M arxiv:1203.1026& arxiv:1207.3602, Cachazo & Skinner arxiv:1207.0741 Cachazo, M., Skinner arxiv:1207.4712.

[Cf. also work by Andrew Hodges 1204.1930, Bo Feng & Song He 1207.3220 and 1207.4064, & Mat Bullimore 1207.3940.]

Twistor-strings and gravity

Twistors strings $\rightsquigarrow$ remarkable progress for Yang-Mills amplitudes.
What about gravity?

- Twistor-strings $\supset$ Conformal gravity [Berkovits, Witten 2004].
- Twistor-action for conformal gravity [M. 2005].

But Einstein gravity $\subset$ conformal gravity.
$\rightsquigarrow$ two strategies:

1. Try to compute Einstein gravity answer from Berkovits-Witten string ($N = 4$ SUSY) [Adamo M. 2012].
2. Guess full $N = 8$ formula, at least on momentum space generalizing Hodges' new MHV formula [Cachazo Skinner 2012, + CSM].

In this talk I review and compare the two approaches and further developments.

Definition

For field theory with field g and action $S[g]$ the S-matrix is a sequence of multilinear functionals $\mathcal{C}(g_1,\ldots,g_n)$ of n solutions g_i to linearized fields equs. (gives amplitudes for Scattering).

- Construct tree-level S-matrix as action $S[g]$ of solution g to field equs from asymptotic data $g|_{\mathscr{I}} = \sum_{i=1}^n \epsilon_i g_i|_{\mathscr{I}}$:

$$\mathcal{C}(g_1,\ldots,g_n) = \text{coeff. of } \prod_i \epsilon_i \text{ in } S_{CG}[g]\,,$$

- On 4-dim space-time manifold M with metric g:
  - Conformal gravity action $S_{CG}[g] := \frac{1}{\kappa^2}\int_M \text{Weyl}^2$.
  - Einstein gravity action $S_{EG}[g] := \frac{1}{\kappa^2}\int_M (\text{Scal} + \Lambda)\,\mathrm{d}\,\text{vol}$.
- Notation $\mathcal{C}$ as above and Einstein (tree) S-matrix

$$\mathcal{M}(1,\ldots,n) = \text{ Coeff of } \prod_i \epsilon_i \text{ in } S_{EG}[g]\,.$$

- Often insert Fourier modes $g_j \sim \mathrm{e}^{iP_j\cdot x}$, momentum P_j, $P_j^2 = 0$, so $\mathcal{C}$, $\mathcal{M}$ become functions of P_i.

From conformal gravity to Einstein gravity

Einstein field eqs $R_{ab} = \Lambda g_{ab}$ imply conformal gravity ones $B_{ab} := \Box R_{ab} + \ldots = 0$ so Einstein $\subset$ Conformal gravity.

Proposition (modified Maldacena after Anderson)

The conformal-gravity tree-level S-matrix evaluated on Einstein gravity wave functions with $\Lambda > 0$ gives $\Lambda\times$ Einstein S-matrix.

Proof: (Idea) If g is Einstein $R_{ab} = \Lambda g_{ab}$ developed from Einstein data, then:

$$S_{CG}[g] = \int \mathrm{Weyl}^2 = \mathrm{Euler\ class} + \int \Lambda^2 \mathrm{dvol}$$

whereas

$$S_{EG} = \int \Lambda \mathrm{dvol}$$

So also perturbatively $S_{CG} = \Lambda S_{EG} \quad \Rightarrow \quad \mathcal{C} = \Lambda \mathcal{M}$.
Care is needed to get boundary terms right. $\Box$

$N = 4$ Super Twistor space and Minkowski space

Spacetime $\mathbb{M} = (\mathbb{R}^{4|8}, g)$ coords $(x^{AA'}, \theta^{aA})$ $A=0,1, A'=0',1', a=1,..,4$.

Twistor space is $\mathbb{PT} = \mathbb{CP}^{3|4}$, homogeneous coords:

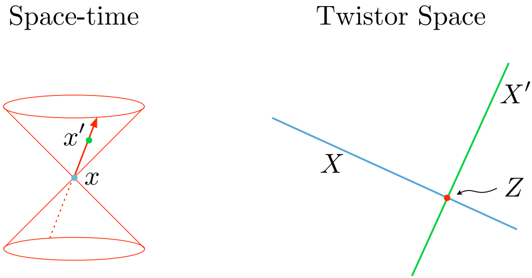
$$Z = Z_I = (\lambda_A, \mu^{A'}, \chi^a) \in \mathbb{T} := \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^{0|4}, \qquad Z \sim \zeta Z, \zeta \in \mathbb{C}^* .$$

$\mathbb{T}$ = fund. repn of superconformal group $SU(2,2|4)$.

A point $x \in \mathbb{M} \leftrightarrow$ a line $X = \mathbb{CP}^1 \subset \mathbb{PT}$ via incidence relations

$$\mu^{A'} = -ix^{AA'}\lambda_A , \qquad \chi^a = \theta^{aA}\lambda_A .$$

Two points x, x' are null separated iff X and X' intersect.



Twistor-strings for conformal gravity

Berkovits-Witten 2004

Fields: $Z = Z(\sigma, \bar\sigma)$, $Y = Y_\sigma(\sigma, \bar\sigma)\mathrm{d}\sigma$, σ coord on worldsheet Σ:

$$Z : \Sigma \to \mathbb{PT}\,, \quad Y \in \Omega^{1,0}(\Sigma) \otimes T^*\mathbb{PT}\,,$$

action

$$S[Z, Y, a] = \int_\Sigma Y_I \bar\partial Z^I\,, \qquad \bar\partial Z = \frac{\partial Z}{\partial \bar\sigma}\mathrm{d}\bar\sigma$$

and field equations $\bar\partial Z = 0 = \bar\partial Y$.

Data: linear conformal gravity = self-dual $\oplus$ anti-self-dual

$$F := (f, g) \in H^1(\mathbb{PT}', T \oplus T^*\mathbb{PT}),$$

which perturbs action by Vertex operators

$$V_F := V_f + V_g := \int_\Sigma f(Z)^I Y_I + g(Z)_I \mathrm{d}Z^I\,.$$

$V_f \leftrightarrow$ deformations of $\mathbb{C}$-structure on $\mathbb{PT}' \rightsquigarrow \bar\partial Z = f(Z)$,
V_g gives 'B-field' (noncommutative str?).

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Berkovits-Witten conjecture for conformal gravity

Path integral reduces to integral over space $\mathscr{M}_{d,n}^{g}$ of holomorphic maps $Z : \Sigma \to \mathbb{PT}$:

$$\mathcal{C}(1,\ldots,n) = \sum_{g,d=0}^{\infty} \int_{\mathscr{M}_{d,n}^{g}} \mathrm{d}\mu_d \left\langle V_{F_1}(Z(\sigma_1)) \ldots V_{F_n}(Z(\sigma_n)) \right\rangle_d \ ,$$

$g =$ genus of Σ, $d =$ degree of map, $n = \#$ marked points.

- For tree-amplitudes, take $g = 0$, so $\Sigma \cong \mathbb{CP}^1$.
- Coordinatize Σ with homogeneous coords $\sigma = (\sigma_0, \sigma_1)$.
- So $\mathscr{M}_{d,n} =$ maps $Z : \mathbb{CP}^1 \to \mathbb{PT}$, degree-$d$ (weight d in σ)

$$Z(\sigma) = \sum_{r=0}^{d} U_r \sigma_0^r \sigma_1^{d-r} \,, \qquad \mathrm{d}\mu_d = \frac{\prod_r \mathrm{d}^{4|4} U_r}{\mathrm{Vol}\ GL(2)} \,.$$

- Correlator computed from green's function

$$\langle Y_I(\sigma)\, Z_j^J(\sigma') \rangle_d = \frac{\delta_I^J}{(\sigma\,\sigma')} \frac{(\xi\,\sigma')^{d+1}}{(\xi\,\sigma)^{d+1}}$$

where $(\sigma\,\sigma') = \sigma_0\sigma_1' - \sigma_1\sigma_0'$ and $\xi \in \Sigma$ is gauge choice.

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Reduction from conformal to Einstein gravity

Linear Einstein SD $\oplus$ ASD fields are given by by

$$H := (h, \tilde{h}) \in H^1(\mathbb{PT}', \mathcal{O}(2) \oplus \mathcal{O}(-2))$$

- Introduce 'infinity twistors' $I_{\alpha\beta}$, $I^{\alpha\beta}$ (fermionic part $= 0$)

$$I_{\alpha\beta} = \begin{pmatrix} \varepsilon^{AB} & 0 \\ 0 & \Lambda\varepsilon_{A'B'} \end{pmatrix}, \qquad I^{\alpha\beta} = \begin{pmatrix} \Lambda\varepsilon_{AB} & 0 \\ 0 & \varepsilon^{A'B'} \end{pmatrix}.$$

  I has rank two when $\Lambda = 0$ ($\Lambda =$ cosmological const.) &

$$I^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} I_{\gamma\delta}, \quad I^{\alpha\beta} I_{\beta\gamma} = \Lambda \delta^{\alpha}_{\gamma},$$

- Gives Poisson structure $\{,\}$ and contact structure τ,

$$\{h_1, h_2\} := I^{IJ}\partial_I h_1 \partial_J h_2, \qquad \tau = I_{IJ} Z^I \mathrm{d} Z^J,$$

Einstein $\subset$ conformal gravity: $(f^I, g_J) = (I^{IK}\partial_K h, Z^K I_{KJ}\tilde{h})$ so Einstein vertex operators:

$$V_h := \int_\Sigma Y \cdot \partial h := \int_\Sigma Y_I I^{IJ} \partial_J h, \quad V_{\tilde{h}} = \int_\Sigma \tilde{h} \wedge \tau.$$

Einstein amplitudes from twistor-strings

If Berkovits-Witten twistor-string correctly gives conformal gravity amplitudes, then $\mathcal{C} = \Lambda\mathcal{M}$ gives for Einstein, k SD fields:

$$\begin{aligned}\Lambda\,\mathcal{M}_n^k &= \int_{\mathscr{M}_{d,n}} \mathrm{d}\mu_d \left\langle V_{\tilde{h}_1} \cdots V_{\tilde{h}_k} V_{h_{k+1}} \cdots V_{h_n} \right\rangle_d \\ &= \int_{\mathscr{M}_{d,n}} \mathrm{d}\mu_d \left\langle \tilde{h}_1\tau_1 \cdots \tilde{h}_k\tau_k Y_{k+1}\cdot\partial h_{k+1} \cdots Y_n \cdot \partial h_n \right\rangle_d\end{aligned}$$

- Supersymmetry requires degree of map $d = k - 1$.
- RHS is polynomial degree n in Λ.
- RHS $= 0$ when $\Lambda = 0$.
- $O(\Lambda)$ part gives Einstein at $\Lambda = 0$.

To check, use momentum eigenstates, momenta $P_j = \lambda_j(\tilde{\lambda}_j, \eta_j)$

$$\tilde{h}_j = \int_{\mathbb{C}} s_j \, \mathrm{d}s_j \, \bar{\delta}^2(s_j\lambda - \lambda_j)\, \mathrm{e}^{is_j[[\mu\,\tilde{\lambda}_j]]}\,, \quad h_j = \int_{\mathbb{C}} \frac{\mathrm{d}s_j}{s_j^3} \, \bar{\delta}^2(s_j - \lambda - \lambda_j)\, \mathrm{e}^{is_j[[\mu\,\tilde{\lambda}_j]]}\,,$$

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Lemma

Each $\langle Y_i \cdot \partial h_i \, \tau_j \rangle$ contraction leads to a power of Λ in the answer.

Proof: Each such contraction leads to $I^{\alpha\beta} I_{\beta\gamma} = \Lambda \delta^\alpha_\gamma$.□

Degree $= 0$: maps are just points $\rightsquigarrow$ easy integration over $\mathbb{PT}$, gives $\Lambda\times$ standard $k = 1$ 3 point amplitude $+\Lambda^2\times$ new term.

Degree $= 1$: Maps are now lines $\leftrightarrow$ points in $\mathbb{M}^{4|8}$

$$\mu^{A'} = -ix^{AA'}\lambda_A \,, \qquad \chi^a = \theta^{aA}\lambda_A \,, \qquad \lambda_A = \sigma_A \,.$$

fixes 'vol $GL(2)$', $\mathscr{M}_{1,n} = \mathbb{M}^{4|8} \times (\mathbb{CP}_1)^n$ and $\mathrm{d}\mu_1 = \mathrm{d}^4 x$.

Three point $k = 2$:

- requires one Y-contraction $\langle \tilde{h}_1\tau_1 \tilde{h}_2\tau_2 Y_3 \cdot \partial h_3 \rangle$.
- Correct answer $O(\Lambda)$ and new term $O(\Lambda^2)$ are obtained

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Does $\Lambda = 0$ case give zero? Case: $d = 1$, $k = 2$.

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- $\langle Y \cdot \partial h_i \, h_j \rangle = \{h_i, h_j\} = [i\, j]\, h_i h_j$ at $\Lambda = 0$ (similarly for $\tilde{h}_j$) so each Y_i contraction with h_j, $\tilde{h}_j \rightsquigarrow$ the factor

$$\tilde{\phi}^i_j := \frac{[ij]\langle \xi\, j\rangle^2}{\langle ij\rangle \langle \xi i\rangle^2}, \qquad i \neq j$$

- $\langle Y_i \cdot \partial h_i \, \tau_j \rangle = O(\Lambda) = 0$, so sum of Y_i contractions $\rightsquigarrow$ factor

$$-\tilde{\phi}^i_i := \sum_{j \neq i} \frac{[ij]\langle \xi\, j\rangle^2}{\langle ij\rangle \langle \xi i\rangle^2}, \qquad \text{defines } \tilde{\phi}^i_j \text{ for } i = j$$

- ξ-independent by momentum conservation $\sum_i \lambda_i \tilde{\lambda}_i = 0$.
- This gives $\prod_{i=3}^n (-\tilde{\phi}^i_i)$, but this is generically non-zero!

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Connected trees and the weighted matrix-tree theorem

Proposition

If only connected trees are allowed for the contractions, the conformal gravity amplitude vanishes at $\Lambda = 0$.

Matrix tree theorem: $\sum$ Feynman tree graphs $= \det \mathcal{L}_{n-1\,n}$ where:

- draw master graph G of all possible $\langle Y_i\, h_j\rangle$, $\langle Y_i\, \tilde{h}_j\rangle$ contractions orienting line from i to j.
  1. $n-2$ white vertices for $Y_i \cdot \partial h_i$,
  2. 2 black vertices for $\tilde{h}_i \tau_i$.
  3. white vertices have $n-1$ outgoing edges, $n-3$ incoming.
  4. black vertices $n-2$ incoming.
- The weighted Laplacian matrix $\mathcal{L}$ of G has (i,j)-entries

$$\mathcal{L} = \begin{cases} \tilde{\phi}^i_j & i \neq j, n-1, n \\ 0 & i = n-1, n \\ \tilde{\phi}^i_i := -\sum_{j\neq i} \tilde{\phi}^i_j & i = j \end{cases}$$

- Let $\mathcal{L}_{n-1\,n} := \{\mathcal{L}-$ rows & columns $n-1$ and $n\}$.

Hodges obtained a remarkable new formula for the n-particley gravitational MHV amplitude as the determinant of an $n-3 \times n-3$ minor of an $n \times n$ matrix $\tilde{\Phi}$:

Definition

The Hodges matrix $\tilde{\Phi} := D^{-1}\tilde{\phi}D$ *where* $D = \mathrm{diag}\{\langle\xi\, i\rangle^2\}$

- $\tilde{\Phi}$ has co-rank three

$$\sum_j \tilde{\Phi}^i_j \lambda_{jA}\lambda_{jB} = 0\,.$$

  following from momentum conservation.
- $\mathcal{L}_{n-1\,n} = \tilde{\phi}_{n-1\,n}$ (with $n-1$ and nth row & column removed) as they only differ in the $n-1$ and nth row and column.
- So $\det \mathcal{L}_{n-1\,n} = \det \tilde{\phi}_{n-1\,n} = 0$ as $\tilde{\Phi}$ and $\tilde{\phi}$ have 3-d kernel.

$\mathcal{C} = 0$ at $\Lambda = 0$ now follows.□

MHV at order Λ, the Hodges formula

Expect gravitational MHV amplitude to be given by

$$\mathcal{M}_n^0 = \frac{1}{\Lambda}\int_{\mathscr{M}_{1,n}} \mathrm{d}\mu_1 \left\langle \tilde{h}_1 \tau_1 \tilde{h}_2 \tau_2 Y_3 \cdot \partial h_3 \cdots Y_n \cdot \partial h_n \right\rangle_1 \Big|_{\Lambda=0}$$

- At $O(\Lambda)$ one Y_i must contract with τ_1 or τ_2.
- The other $n-3$ contractions must connect remaining white vertices with one outgoing edge connecting to i, $n-1$ or n.
- Matrix tree theorem gives sum of contributions as factor

$$\det \mathcal{L}_{i\,n-1\,n} = \det \tilde{\Phi}_{i\,n-1\,n}$$

  multiplied by 3pt amplitude for $\mathcal{M}(i, n-1, n)$.
- This is a version of Hodges' MHV formula.

$$\mathcal{M}(1,\ldots,n) = \mathcal{M}(i,n-1,n) \det \tilde{\Phi}_{i\,n-1\,n}$$

Note: $\tilde{\Phi}$ and Hodges formula have straightforward permutation symmetry and polynomial complexity.

$$\mathcal{M}_n^k = \frac{1}{\Lambda}\int_{\mathscr{M}_{k-1,n}} \mathrm{d}\mu_{k-1} \left\langle \tilde{h}_1\tau_1 \cdots \tilde{h}_k\tau_k Y_{k+1}\cdot\partial h_{k+1}\cdots Y_n\cdot\partial h_n \right\rangle_{k-1}\Bigg|_{\Lambda=0}$$

- Y_i contractions now give generalized Hodges matrices

$$\tilde{\phi}_i^j = \begin{cases} \frac{[i\,j]}{(i\,j)}\frac{(\xi\,j)^k}{(\xi\,i)^k} & i \neq j \\ -\sum_{l\neq i}\tilde{\phi}_i^l & i = j \end{cases} \qquad (i\,j) = \sigma_{i\underline{A}}\sigma_j^{\underline{A}}$$

- This has co-rank $k+1$ because relations

$$\textstyle\sum_j \tilde{\phi}_i^j \sigma_{jA_1}\ldots\sigma_{jA_k}\frac{(\xi\,i)^k}{(\xi\,j)^k} = 0$$

  follow from $\sum_j \tilde{\lambda}_j\sigma_{jA_1}\ldots\sigma_{jA_{k-1}} = 0$.
- Matrix-tree thm gives sum of tree contractions as determinant ($n-k$ minor of $\tilde{\phi}$) $= 0$ as co-rank $= k+1$.
- At $O(\Lambda)$ with one $\langle Y_i\,\tau_j\rangle$ contraction, Matrix-tree theorem yields answer as determinant of $n-k-1$ minor of $\tilde{\phi}$.

The Cachazo-Skinner formula for $\mathbb{PT} = \mathbb{CP}^{3|8}$

Theorem (Cachazo, M, Skinner)

The tree-level S-matrix for $N = 8$ supergravity is given by

$$\mathcal{M}_n^k(1, \ldots, n) = \int_{\mathscr{M}_{k-1,n}} \mathrm{d}\mu_{k-1} \, \mathrm{det}'(\tilde{\Phi}^k)\mathrm{det}'(\Phi^k) \prod_{i=1}^{n} \mathrm{D}\sigma_i \; h_i\left(Z(\sigma_i)\right),$$

where $\tilde{\Phi}$ is conjugate to $\tilde{\phi}$ as above and

$$\Phi_i^j = \frac{\langle \lambda(\sigma_i)\, \lambda(\sigma_j)\rangle}{(i\, j)} \qquad\qquad i \neq j$$

etc., has rank $k-1$ and det$'$ *is* det *of a minor of maximal rank divided by Vandermonde factors in $(i\,j)$.*

Proof: Use recursion: shift momenta with complex parameter and show that residues at poles give factorized amplitudes. □

Gives full nonlinear (but perturbative) structure of Einstein equations!

Conclusions:

For Berkovits-Witten twistor-string have good evidence:

- confirmed that twistor-string gives zero at $\Lambda = 0$ as required by Maldacena argument.
- obtained Hodges formula for $k = 2$ (MHV).
- obtained part of $\tilde{\Phi}^k$ in Cachazo-Skinner formula.

But more work required for full understanding.

$N = 8$ Cachazo-Skinner formula $\rightsquigarrow$ many new avenues:

- Is there an $N = 8$ SUGRA twistor-string as well?
- What is geometric interpretation?
- Quantization?

Happy Birthday MikE!