Einstein gravity from rational curves in twistor space

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[Cf. also work by Andrew Hodges 1204.1930, Bo Feng & Song He 1207.3220 and 1207.4064, & Mat Bullimore 1207.3940. ]
Twistors strings $\leadsto$ remarkable progress for Yang-Mills amplitudes. What about gravity?

- Twistor-strings $\supset$ Conformal gravity [Berkovits, Witten 2004].
- Twistor-action for conformal gravity [M. 2005].

But Einstein gravity $\subset$ conformal gravity. $\leadsto$ two strategies:

1. Try to compute Einstein gravity answer from Berkovits-Witten string ($N = 4$ SUSY) [Adamo M. 2012].

2. Guess full $N = 8$ formula, at least on momentum space generalizing Hodges’ new MHV formula [Cachazo Skinner 2012, + CSM].

In this talk I review and compare the two approaches and further developments.
The S-matrix at tree-level

Definition
For field theory with field $g$ and action $S[g]$ the S-matrix is a sequence of multilinear functionals $C(g_1, \ldots, g_n)$ of $n$ solutions $g_i$ to linearized fields equs. (gives amplitudes for Scattering).

- Construct tree-level S-matrix as action $S[g]$ of solution $g$ to field equs from asymptotic data $g|\mathcal{I} = \sum_{i=1}^{n} \epsilon_i g_i|\mathcal{I}$:

$$C(g_1, \ldots, g_n) = \text{coeff. of } \prod_i \epsilon_i \text{ in } S_{CG}[g],$$

- On 4-dim space-time manifold $M$ with metric $g$:
  - Conformal gravity action $S_{CG}[g] := \frac{1}{\kappa^2} \int_M \text{Weyl}^2$.
  - Einstein gravity action $S_{EG}[g] := \frac{1}{\kappa^2} \int_M (\text{Scal} + \Lambda) d\text{vol}$.
- Notation $C$ as above and Einstein (tree) S-matrix

$$\mathcal{M}(1, \ldots, n) = \text{Coeff of } \prod_i \epsilon_i \text{ in } S_{EG}[g].$$

- Often insert Fourier modes $g_j \sim e^{iP_j \cdot x}$, momentum $P_j$, $P_j^2 = 0$, so $C, M$ become functions of $P_i$. 
Einstein field eqs \( R_{ab} = \Lambda g_{ab} \) imply conformal gravity ones
\( B_{ab} := \Box R_{ab} + \ldots = 0 \) so Einstein \( \subset \) Conformal gravity.

**Proposition (modified Maldacena after Anderson)**

*The conformal-gravity tree-level S-matrix evaluated on Einstein gravity wave functions with \( \Lambda > 0 \) gives \( \Lambda \times \) Einstein S-matrix.*

**Proof:** (Idea) If \( g \) is Einstein \( \Box R_{ab} = \Lambda g_{ab} \) developed from Einstein data, then:

\[
S_{CG}[g] = \int \text{Weyl}^2 = \text{Euler class} + \int \Lambda^2 d\text{vol}
\]

whereas

\[
S_{EG} = \int \Lambda d\text{vol}
\]

So also perturbatively \( S_{CG} = \Lambda S_{EG} \quad \Rightarrow \quad C = \Lambda M \).

Care is needed to get boundary terms right. \( \Box \)
\( N = 4 \) Super Twistor space and Minkowski space

**Spacetime** \( \mathbb{M} = (\mathbb{R}^{4|8}, g) \) coords \((x^{AA'}, \theta^{aA})\) \( A=0,1,A'=0',1', a=1,\ldots,4 \).

**Twistor space** is \( \mathbb{PT} = \mathbb{CP}^{3|4} \), homogeneous coords:
\[
Z = Z_i = (\lambda_A, \mu^{A'}, \chi^a) \in \mathbb{T} := \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^{0|4}, \quad Z \sim \zeta Z, \zeta \in \mathbb{C}^*.
\]
\( \mathbb{T} \) = fund. repn of superconformal group \( SU(2, 2|4) \).

A point \( x \in \mathbb{M} \leftrightarrow \) a line \( X = \mathbb{CP}^1 \subset \mathbb{PT} \) via incidence relations
\[
\mu^{A'} = -i x^{AA'} \lambda_A, \quad \chi^a = \theta^{aA} \lambda_A.
\]

Two points \( x, x' \) are null separated iff \( X \) and \( X' \) intersect.
Twistor-strings for conformal gravity
Berkovits-Witten 2004

Fields: \( Z = Z(\sigma, \bar{\sigma}) \), \( Y = Y_\sigma(\sigma, \bar{\sigma})d\sigma \), \( \sigma \) coord on worldsheet \( \Sigma \):

\[
Z : \Sigma \rightarrow \mathbb{PT}, \quad Y \in \Omega^{1,0}(\Sigma) \otimes T^*\mathbb{PT},
\]

action

\[
S[Z, Y, a] = \int_\Sigma Y_I \bar{\partial} Z^I, \quad \bar{\partial} Z = \frac{\partial Z}{\partial \bar{\sigma}} d\bar{\sigma}
\]

and field equations \( \bar{\partial} Z = 0 = \bar{\partial} Y \).

Data: linear conformal gravity = self-dual \( \oplus \) anti-self-dual

\[
F := (f, g) \in H^1(\mathbb{PT}', T \oplus T^*\mathbb{PT}),
\]

which perturbs action by Vertex operators

\[
V_F := V_f + V_g := \int_\Sigma f(Z)_I Y_I + g(Z)_I dZ^I.
\]

\( V_f \leftrightarrow \) deformations of \( \mathbb{C} \)-structure on \( \mathbb{PT}' \sim \bar{\partial} Z = f(Z) \),
\( V_g \) gives ‘B-field’ (noncommutative str?).
Fields: $Z = Z(\sigma, \bar{\sigma})$, $Y = Y_{\sigma}(\sigma, \bar{\sigma})d\sigma$, $\sigma$ coord on worldsheet $\Sigma$:  

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$V_f \leftrightarrow$ deformations of $\mathbb{C}$-structure on $\mathbb{P}T' \sim \bar{\partial}Z = f(Z)$, $V_g$ gives ‘B-field’ (noncommutative str?).
Path integral reduces to integral over space $\mathcal{M}^g_{d,n}$ of holomorphic maps $Z : \Sigma \to \mathbb{PT}$:

$$C(1, \ldots, n) = \sum_{g,d=0}^{\infty} \int_{\mathcal{M}^g_{d,n}} d\mu_d \langle V_{F_1}(Z(\sigma_1)) \cdots V_{F_n}(Z(\sigma_n)) \rangle_d,$$

$g =$ genus of $\Sigma$, $d =$ degree of map, $n =$ # marked points.

- For tree-amplitudes, take $g = 0$, so $\Sigma \cong \mathbb{CP}^1$.
- Coordinatize $\Sigma$ with homogeneous coords $\sigma = (\sigma_0, \sigma_1)$.
- So $\mathcal{M}_{d,n} =$ maps $Z : \mathbb{CP}^1 \to \mathbb{PT}$, degree-$d$ (weight $d$ in $\sigma$)

$$Z(\sigma) = \sum_{r=0}^{d} U_r \sigma_0^r \sigma_1^{d-r}, \quad d\mu_d = \frac{\prod_r d^4 U_r}{\text{Vol } GL(2)}.$$

- Correlator computed from green’s function

$$\langle Y_I(\sigma) Z^J_j(\sigma') \rangle_d = \frac{\delta^J_I}{(\sigma \sigma')} \frac{(\xi \sigma')^{d+1}}{(\xi \sigma)^{d+1}},$$

where $(\sigma \sigma') = \sigma_0 \sigma'_1 - \sigma_1 \sigma'_0$ and $\xi \in \Sigma$ is gauge choice.
Berkovits-Witten conjecture for conformal gravity

Path integral reduces to integral over space $\mathcal{M}_{d,n}^g$ of holomorphic maps $Z : \Sigma \to \mathbb{P}T$:

$$\mathcal{C}(1, \ldots, n) = \sum_{g,d=0}^{\infty} \int_{\mathcal{M}_{d,n}^g} d\mu_d \langle V_{F_1}(Z(\sigma_1)) \cdots V_{F_n}(Z(\sigma_n)) \rangle_d ,$$

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- Correlator computed from green's function

$$\langle Y_l(\sigma) Z_j^J(\sigma') \rangle_d = \frac{\delta^J_l}{(\sigma \sigma') (\xi \sigma)^{d+1}} (\xi \sigma')^{d+1}$$

where $(\sigma \sigma') = \sigma_0 \sigma_1' - \sigma_1 \sigma_0'$ and $\xi \in \Sigma$ is gauge choice.
Reduction from conformal to Einstein gravity

Linear Einstein SD $\oplus$ ASD fields are given by by

$$H := (h, \tilde{h}) \in H^1(\mathbb{P}T', O(2) \oplus O(-2))$$

- Introduce ‘infinity twistors’ $l_{\alpha\beta}$, $l^{\alpha\beta}$ (fermionic part $= 0$)

$$l_{\alpha\beta} = \begin{pmatrix} \varepsilon^{AB} & 0 \\ 0 & \Lambda \varepsilon_{A'B'} \end{pmatrix}, \quad l^{\alpha\beta} = \begin{pmatrix} \Lambda \varepsilon_{AB} & 0 \\ 0 & \varepsilon^{A'B'} \end{pmatrix}.$$  

$l$ has rank two when $\Lambda = 0$ ($\Lambda =$ cosmological const.) &

$$l_{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} l_{\gamma\delta}, \quad l^{\alpha\beta} l_{\beta\gamma} = \Lambda \delta_{\alpha}^{\gamma},$$

- Gives Poisson structure $\{ , \}$ and contact structure $\tau$,

$$\{ h_1, h_2 \} := l^{IJ} \partial_I h_1 \partial_J h_2, \quad \tau = l_{IJ} Z^I dZ^J,$$

Einstein $\subset$ conformal gravity: $(f^I, g_J) = (l^{IK} \partial_K h, Z^K l_{KJ} \tilde{h})$ so Einstein vertex operators:

$$V_h := \int_{\Sigma} Y \cdot \partial h := \int_{\Sigma} Y_I l^{IJ} \partial_J h, \quad V_{\tilde{h}} = \int_{\Sigma} \tilde{h} \wedge \tau.$$
Einstein amplitudes from twistor-strings

If Berkovits-Witten twistor-string correctly gives conformal gravity amplitudes, then $C = \Lambda M$ gives for Einstein, $k$ SD fields:

$$\Lambda M^k_n = \int d\mu d \langle V_{\tilde{h}_1} \cdots V_{\tilde{h}_k} V_{h_{k+1}} \cdots V_{h_n} \rangle_d$$

$$= \int d\mu d \langle \tilde{h}_1 \tau_1 \cdots \tilde{h}_k \tau_k Y_{k+1} \cdot \partial h_{k+1} \cdots Y_n \cdot \partial h_n \rangle_d$$

- Supersymmetry requires degree of map $d = k - 1$.
- RHS is polynomial degree $n$ in $\Lambda$.
- RHS $= 0$ when $\Lambda = 0$.
- $O(\Lambda)$ part gives Einstein at $\Lambda = 0$.

To check, use momentum eigenstates, momenta $P_j = \lambda_j (\tilde{\lambda}_j, \eta_j)$

$$\tilde{h}_j = \int_C s_j \, ds_j \, \bar{\delta}^2 (s_j \lambda - \lambda_j) \, e^{is_j[\mu \tilde{\lambda}_j]} , \quad h_j = \int_C \frac{ds_j}{s_j^3} \, \bar{\delta}^2 (s_j - \lambda - \lambda_j) \, e^{is_j[\mu \tilde{\lambda}_j]} ,$$

compute correlator $\langle \rangle$, integrate & compare to known results.
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$$= \int d\mu d \left\langle \tilde{h}_{1\tau_1} \cdots \tilde{h}_{k\tau_k} Y_{k+1} \cdot \partial h_{k+1} \cdots Y_n \cdot \partial h_n \right\rangle_d$$

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$$\tilde{h}_j = \int_{\mathcal{C}} ds_j \, d\bar{s}_j \, \bar{\delta}^2(s_j \lambda - \lambda_j) \, e^{is_j[[\mu \tilde{\lambda}_j]]}, \quad h_j = \int_{\mathcal{C}} \frac{d\bar{s}_j}{s_j^3} \, \bar{\delta}^2(s_j \lambda - \lambda_j) \, e^{is_j[[\mu \tilde{\lambda}_j]]},$$

compute correlator $\langle \rangle$, integrate & compare to known results.
Lemma

Each $\langle Y_i \cdot \partial h_i \tau_j \rangle$ contraction leads to a power of $\Lambda$ in the answer.

Proof: Each such contraction leads to $I^{\alpha \beta} I_{\beta \gamma} = \Lambda \delta^\alpha_\gamma$. □

Degree = 0: maps are just points $\leadsto$ easy integration over $\mathbb{P}T$, gives $\Lambda \times$ standard $k = 1$ 3 point amplitude $+$ $\Lambda^2 \times$ new term.

Degree = 1: Maps are now lines $\leftrightarrow$ points in $\mathcal{M}^{4|8}$

$$\mu^A = -ix^{AA'} \lambda_A, \quad \chi^a = \theta^{aA} \lambda_A, \quad \lambda_A = \sigma_A.$$  

fixes ‘vol $GL(2)$’, $\mathcal{M}_{1,n} = \mathcal{M}^{4|8} \times (\mathbb{C}P_1)^n$ and $d\mu_1 = d^4x$.

Three point $k = 2$:

- requires one $Y$-contraction $\langle \tilde{h_1} \tau_1 \tilde{h_2} \tau_2 Y_3 \cdot \partial h_3 \rangle$.
- Correct answer $O(\Lambda)$ and new term $O(\Lambda^2)$ are obtained

$$C(1, 2, 3) = \Lambda M(1, 2, 3) = \Lambda \frac{\langle 1 2 \rangle^2}{\langle 1 3 \rangle^2 \langle 2 3 \rangle^2} (1 + \Lambda \Box_p) \delta^{4|8} \left( \sum_i P_i \right)$$

where $\langle 1 2 \rangle = \lambda_1 A \lambda_2^A$ also set $[1 2] = \tilde{\lambda}_1 A \tilde{\lambda}_2^A$. 
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Each \( \langle Y_i \cdot \partial h_i \tau_j \rangle \) contraction leads to a power of \( \Lambda \) in the answer.

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• Correct answer \( O(\Lambda) \) and new term \( O(\Lambda^2) \) are obtained

\[
\mathcal{C}(1, 2, 3) = \Lambda \mathcal{M}(1, 2, 3) = \Lambda \frac{\langle 12 \rangle^2}{\langle 13 \rangle^2 \langle 23 \rangle^2} (1 + \Lambda \Box p) \delta^{4|8} \left( \sum_i P_i \right)
\]

where \( \langle 12 \rangle = \lambda_1 A \lambda_2^A \) also set \( [12] = \tilde{\lambda}_1 A \tilde{\lambda}_2^{A'} \).
\textbf{Lemma}

Each $\langle Y_i \cdot \partial h_i \tau_j \rangle$ contraction leads to a power of $\Lambda$ in the answer.

\textbf{Proof:} Each such contraction leads to $I^{\alpha \beta} I_{\beta \gamma} = \Lambda \delta^\alpha_\gamma$. \hfill \Box

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$$\mu^A = -i x^{AA'} \lambda_A, \quad \lambda^a = \theta^{aa} \lambda_A, \quad \lambda_A = \sigma_A.$$  

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\Lambda-dependence and low-lying examples
\(\Lambda\)-dependence and low-lying examples

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Each \(\langle Y_i \cdot \partial h_i \tau_j \rangle\) contraction leads to a power of \(\Lambda\) in the answer.

Proof: Each such contraction leads to \(I^{\alpha\beta} I_{\beta\gamma} = \Lambda \delta^\alpha_\gamma\). \(\square\)

Degree = 0: maps are just points \(\sim\) easy integration over \(\mathbb{P}T\), gives \(\Lambda \times\) standard \(k = 1\) 3 point amplitude + \(\Lambda^2 \times\) new term.

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where \(\langle 12 \rangle = \lambda_1^A \lambda_2^A\) also set \([12] = \tilde{\lambda}_1^A \tilde{\lambda}_2^{A'}\).
Does \( \Lambda = 0 \) case give zero? Case: \( d = 1, \ k = 2 \).

\[
\int \mathcal{M}_{1,n} \left< \tilde{h}_1 \tau_1 \tilde{h}_2 \tau_2 Y_3 \cdot \partial h_3 \cdots Y_n \cdot \partial h_n \right>_1 = \Lambda \mathcal{M}_n^0 = 0
\]

- \( \left< Y \cdot \partial h_i \ h_j \right> = \{ h_i, h_j \} = [i j] h_i h_j \) at \( \Lambda = 0 \) (similarly for \( \tilde{h}_j \)) so each \( Y_i \) contraction with \( h_j, \tilde{h}_j \) \( \leadsto \) the factor
  \[
  \tilde{\phi}_j := \frac{[ij] \langle \xi \ j \rangle^2}{\langle ij \rangle \langle \xi i \rangle^2}, \quad i \neq j
  \]

- \( \left< Y_i \cdot \partial h_i \ \tau_j \right> = O(\Lambda) = 0 \), so sum of \( Y_i \) contractions \( \leadsto \) factor
  \[
  -\phi^i := \sum_{j \neq i} \frac{[ij] \langle \xi \ j \rangle^2}{\langle ij \rangle \langle \xi i \rangle^2}, \quad \text{defines } \tilde{\phi}^i_j \text{ for } i = j
  \]

- \( \xi \)-independent by momentum conservation \( \sum_i \lambda_i \tilde{\lambda}_i = 0 \).
- This gives \( \prod_{i=3}^n (-\phi^i) \), but this is generically non-zero!

**Resolution:** only allow Feynman diagrams for the correlator that are connected trees.
Does $\Lambda = 0$ case give zero? Case: $d = 1$, $k = 2$.

$$\int \frac{d\mu_1}{\mathcal{M}_{1,n}} \left\langle \tilde{h}_1 \tau_1 \tilde{h}_2 \tau_2 Y_3 \cdot \partial h_3 \cdots Y_n \cdot \partial h_n \right\rangle_1 = \Lambda \mathcal{M}_n^0 = 0$$

- $\langle Y \cdot \partial h_i h_j \rangle = \{ h_i, h_j \} = [ij] h_i h_j$ at $\Lambda = 0$ (similarly for $\tilde{h}_j$) so each $Y_i$ contraction with $h_j$, $\tilde{h}_j \sim$ the factor
  $$\tilde{\phi}^i_j := \frac{[ij] \langle \xi \cdot j \rangle^2}{\langle ij \rangle \langle \xi i \rangle^2}, \quad i \neq j$$

- $\langle Y_i \cdot \partial h_i \tau_j \rangle = O(\Lambda) = 0$, so sum of $Y_i$ contractions $\sim$ factor
  $$-\phi^i_i := \sum_{j \neq i} \frac{[ij] \langle \xi \cdot j \rangle^2}{\langle ij \rangle \langle \xi i \rangle^2}, \quad \text{defines } \tilde{\phi}^i_i \text{ for } i = j$$

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**Resolution:** only allow Feynman diagrams for the correlator that are connected trees.
Connected trees and the weighted matrix-tree theorem

Proposition

If only connected trees are allowed for the contractions, the conformal gravity amplitude vanishes at $\Lambda = 0$.

Matrix tree theorem: $\sum$ Feynman tree graphs $= \det L_{n-1\, n}$

where:

- draw master graph $G$ of all possible $\langle Y_i \, h_j \rangle$, $\langle Y_i \, \tilde{h}_j \rangle$ contractions orienting line from $i$ to $j$.
  1. $n - 2$ white vertices for $Y_i \cdot \partial h_i$,
  2. $2$ black vertices for $\tilde{h}_i \tau_i$.
  3. white vertices have $n - 1$ outgoing edges, $n - 3$ incoming.
  4. black vertices $n - 2$ incoming.

- The weighted Laplacian matrix $L$ of $G$ has $(i,j)$-entries

$$L = \begin{cases} 
\tilde{\phi}_i^j & i \neq j, n - 1, n \\
0 & i = n - 1, n \\
\tilde{\phi}_i^j := - \sum_{j \neq i} \tilde{\phi}_j^i & i = j
\end{cases}$$

- Let $L_{n-1\, n} := \{L - \text{ rows & columns } n - 1 \text{ and } n\}$. 
Hodges obtained a remarkable new formula for the $n$-particle gravitational MHV amplitude as the determinant of an $n - 3 \times n - 3$ minor of an $n \times n$ matrix $\tilde{\Phi}$:

**Definition**

*The Hodges matrix* $\tilde{\Phi} := D^{-1} \tilde{\phi} D$ where $D = \text{diag}\{\langle \xi \rangle^2\}$

- $\tilde{\Phi}$ has co-rank three

\[
\sum_j \tilde{\Phi}^i_j \lambda_{jA} \lambda_{jB} = 0.
\]

following from momentum conservation.

- $\mathcal{L}_{n-1 \ n} = \tilde{\phi}_{n-1 \ n}$ (with $n - 1$ and $n$th row & column removed) as they only differ in the $n - 1$ and $n$th row and column.

- So $\det \mathcal{L}_{n-1 \ n} = \det \tilde{\phi}_{n-1 \ n} = 0$ as $\tilde{\Phi}$ and $\tilde{\phi}$ have 3-d kernel.

$C = 0$ at $\Lambda = 0$ now follows. $\Box$
MHV at order $\Lambda$, the Hodges formula

Expect gravitational MHV amplitude to be given by

$$
\mathcal{M}_n^0 = \frac{1}{\Lambda} \int d\mu_1 \left< \tilde{h}_1 \tau_1 \tilde{h}_2 \tau_2 Y_3 \cdot \partial h_3 \cdots Y_n \cdot \partial h_n \right>_{1\mid \Lambda=0}
$$

- At $O(\Lambda)$ one $Y_i$ must contract with $\tau_1$ or $\tau_2$.
- The other $n - 3$ contractions must connect remaining white vertices with one outgoing edge connecting to $i$, $n - 1$ or $n$.
- Matrix tree theorem gives sum of contributions as factor
  $$
  \det \mathcal{L}_{i\,n-1\,n} = \det \tilde{\Phi}_{i\,n-1\,n}
  $$
  multiplied by 3pt amplitude for $\mathcal{M}(i, n - 1, n)$.
- This is a version of Hodges’ MHV formula.

$$
\mathcal{M}(1, \ldots, n) = \mathcal{M}(i, n - 1, n) \det \tilde{\Phi}_{i\,n-1\,n}
$$

Note: $\tilde{\Phi}$ and Hodges formula have straightforward permutation symmetry and polynomial complexity.
\[ M_n^k = \frac{1}{\Lambda} \int_{M_{k-1,n}} d\mu_{k-1} \left< \tilde{h}_1 r_1 \cdots \tilde{h}_k r_k Y_{k+1} \cdots Y_n \cdot \partial h_{k+1} \cdots \partial h_n \right>_{k-1} \bigg|_{\Lambda=0} \]

- \( Y_i \) contractions now give generalized Hodges matrices
  \[ \phi'_i = \begin{cases} \frac{[i,j]}{(i,j)} \frac{(\xi j)^k}{(\xi i)^k} & i \neq j \\ -\sum_{l \neq i} \phi'_l & i = j \end{cases} \]  
  \( (i,j) = \sigma_i A_j^A \)

- This has co-rank \( k + 1 \) because relations
  \[ \sum_j \phi'_i \sigma_j A_1 \cdots \sigma_j A_k \frac{(\xi i)^k}{(\xi j)^k} = 0 \]
  follow from \( \sum_j \tilde{\lambda}_j \sigma_j A_1 \cdots \sigma_j A_{k-1} = 0 \).

- Matrix-tree thm gives sum of tree contractions as determinant \((n - k \) minor of \( \tilde{\phi} \)) = 0 as co-rank = \( k + 1 \).

- At \( O(\Lambda) \) with one \( \left< Y_i, r_j \right> \) contraction, Matrix-tree theorem yields answer as determinant of \( n - k - 1 \) minor of \( \tilde{\phi} \).
The Cachazo-Skinner formula for $\mathbb{PT} = \mathbb{CP}^{3|8}$

**Theorem (Cachazo, M, Skinner)**

*The tree-level S-matrix for $N = 8$ supergravity is given by*

$$
\mathcal{M}_n^k(1, \ldots, n) = \int_{\mathcal{M}_{k-1,n}} \mathcal{M}^{k-1, n} \, \text{d}\mu_{k-1} \, \text{det}'(\tilde{\Phi}^k) \text{det}'(\Phi^k) \prod_{i=1}^n \text{D}\sigma_i \, h_i(\mathcal{Z}(\sigma_i)) ,
$$

where $\tilde{\Phi}$ is conjugate to $\tilde{\phi}$ as above and

$$
\Phi^j_i = \frac{\langle \lambda(\sigma_i) \lambda(\sigma_j) \rangle}{(i \, j)} \quad i \neq j
$$

etc., has rank $k - 1$ and $\text{det}'$ is det of a minor of maximal rank divided by Vandermonde factors in $(i \, j)$.

**Proof:** Use recursion: shift momenta with complex parameter and show that residues at poles give factorized amplitudes. □

Gives full nonlinear (but perturbative) structure of Einstein equations!
Conclusions:

For Berkovits-Witten twistor-string have good evidence:

- confirmed that twistor-string gives zero at $\Lambda = 0$ as required by Maldacena argument.
- obtained Hodges formula for $k = 2$ (MHV).
- obtained part of $\tilde{\Phi}^k$ in Cachazo-Skinner formula.

But more work required for full understanding.

$N = 8$ Cachazo-Skinner formula $\leadsto$ many new avenues:

- Is there an $N = 8$ SUGRA twistor-string as well?
- What is geometric interpretation?
- Quantization?
Happy Birthday Mike!