Vladimir S. Matveev (Jena)

Degrees of mobility of projective and h-projective structures:

The projective part is joint with A. Fedorova.
The h-projective part is joint with S. Rosemann.

Vienna 11.9.2012; The Interaction of Geometry and Representation Theory

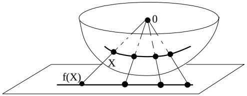
Main definition and some history

Def. Let g be a Riemannian or pseudo-Riemannian metric on M. A metric $\bar{g}$ on the same manifold M is geodesically equivalent to g, if every g-geodesic is a (possibly, reparameterized) $\bar{g}$-geodesic.

The notion is natural and the study of geodesically equivalent metrics was one of the favorite topics in the XIXth/beginning of XXth century: Beltrami, Dini, Levi-Civita, Painlevé, Eisenhart, Weyl, Veblen, Thomas obtained nontrivial results.

The subject has a revival now: many new methods appear and many classical problems were solved.

Examples of Lagrange 1789 and of Dini 1869



Radial projection $f : S^2 \to \mathbb{R}^2$ takes geodesics of the sphere to geodesics of the plane, because geodesics on the sphere/plane are intersection of planes containing 0 with the sphere/plane.

Example (Dini 1869) The metric $(X(x) - Y(y))(dx^2 + dy^2)$ is geodesically equivalent to $\left(\frac{1}{Y(y)} - \frac{1}{X(x)}\right)\left(\frac{dx^2}{X(x)} + \frac{dy^2}{Y(y)}\right)$, (if they have sense).

The equation for geodesic equivalence and the degree of mobility

Theorem (Sinjukov 1962). Let g be a metric. The metrics $\bar g$ that are geodesically equivalent to g are in one-to-one correspondence with the solutions of the following system of PDE on the (0,2)-tensorfield $a = a_{ij}$ and (0,1)-tensorfield λ_i such that $det(a) \neq 0$ at all points:

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}. \qquad (*)$$

The one-to-one correspondence is given by

$$\bar g \longrightarrow \left(a = \left(\frac{\det(\bar g)}{\det(g)}\right)^{\frac{1}{n+1}} g \bar g^{-1} g, \lambda = \tfrac{1}{2} d\,\mathrm{trace}_g(a) \right).$$

Historical remark. In dim 2, the Sinjukov-equations were known to Liouville 1889. There exist two improvements of this system of equations (Bolsinov-M$\sim$ 2002 and Eastwood-M$\sim$ 2007) that made the system more convenient for certain goals.

Def. The dimension of the space of solutions of this system is called the degree of mobility and is denoted by $D(g)$.

Main question of my talk and the previously known answers

Question we will partially answer: What values can the degree of mobility accept?

For every dimension n, we would like to have a list of all possible degrees of mobility of all possible metrics. We work locally, or on simply-connected manifolds.

Answer in dimension 2 (Koenigs 1886). In dimension 2, the possible degrees of mobility are $\{1, 2, 3, 4, 6\}$.
Maximal degree of mobility (Weyl and Eisenhart, 1920th). It is $\frac{(n+1)(n+2)}{2}$ and is accepted on the metrics of constant curvature.
Submaximal degree of mobility (Sinjukov-Mikes 1962). It is $\frac{(n-2)(n-1)}{2} + 1$.
Answer in dimension 3 (Kiosak-M$\sim$ 2011): $D(g) \in \{1, 2, 10\}$.

Main result

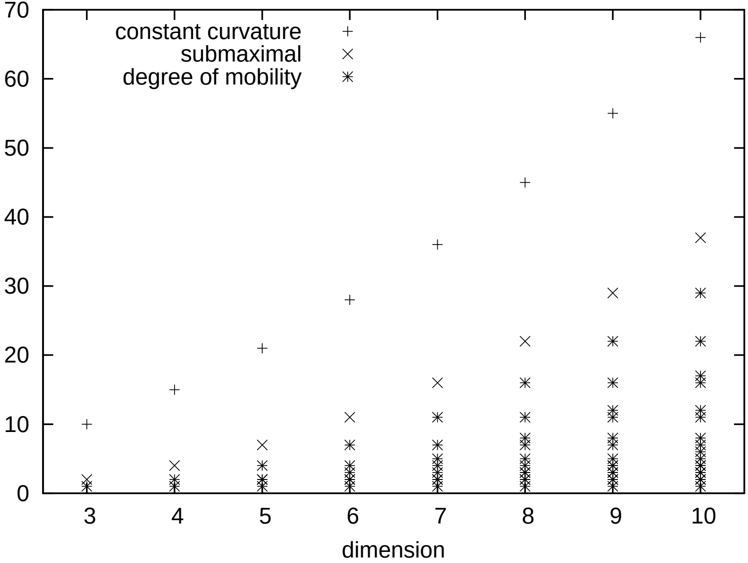
Thm. Let g be a metric of Riemannian or Lorentz signature on a simply-connected manifold of dimension ≥ 3. Assume there exists at least one metric that is geodesically equivalent to g, but is not affinely equivalent to g. Then, the degree of mobility can accept (precisely) one of the following numbers only

- 2;
- $\frac{k(k+1)}{2} + \ell$, where $k \in \{0, ..., n-2\}$ and $\ell \in \{1, ..., \left[\frac{n-k+1}{3}\right]\}$.
- $\frac{(n+2)(n+1)}{2}$ (constant curvature case).

Remark. The submaximal degree of mobility $\frac{(n-2)(n-1)}{2} + 1$ corresponds to $k = n-2$ and $\ell = 1$.

Historical remark. In the Riemannian case the answer is due to Shandra 2000 and Kiosak-Matveev-Mikes-Shandra 2010: the principal idea is due to Shandra but the paper S2000 contains a mistake that was corrected in KMMS2010; so the new result is actually the Lorentz case.

We did not expect that the answer in the Lorentz and Riemannian signatures coincide. As we know (in particular from recent results of Dennis The - Boris Kruglikov) it is rather typical that the dimensions of the space of solutions of geometric equations on a pseudo-Riemannian manifold depend on the signature of the metric.



constant curvature
submaximal
degree of mobility
0
10
20
30
40
50
60
70
3
4
5
6
7
8
9
10
dimension

The schema of my explanation of the result.

Naive approach that probably would not work. Given an explicit metric, it is a purely algorithmic "**Maple**ized" task to find the dimension of the space of solution – one needs to prolong the system and calculate the rank of the prolongations. If the metric is not fixed, apriori there could be many combinatorial possibilities for the behavior of the prolonged system and in most problems of such type only the maximal and submaximal dimensions could be found.

- I will first explain the "Shandra" idea in the Riemannian case
- I will give an example of a Lorentz signature such that the Riemannian proof does not work, show the difficulties in the transition Riemannian $\longrightarrow$ Lorentz, and sketch how we could overcome them.
- And then I come to applications and to the other topic of the talk – *h*-projectively equivalent metrics

Principle observation

Thm (Kiosak-Matveev 2011; Riemannian case is in Solodovnikov 1956). Let g be a metric on an $n \geq 3$-dimensional connected manifold with $D(g) \geq 3$. Then, there exists a constant B such that for any solution (a, λ) of the equations $a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}$ there exists a function μ such that in addition the following two equations are fulfilled:

$$\begin{aligned} \lambda_{i,j} &= \mu g_{ij} - B a_{ij} \\ \mu_{,i} &= 2B\lambda_i. \end{aligned}$$

Remark. Theorem is nontrivial. We went until 5th prolongation to prove it.

Remark. The constant B depends on the metric (and is the same for all solutions (a, λ)).

Geodesically equivalent metrics as parallel sections.

We assume that $n = dim(M) \geq 3$ and $D(g) \geq 3$. Then, metrics $\bar{g}$ that are geodesically equivalent to g are in one-to-one correspondence to the solutions (a, λ, μ) of the system of equations

$$\begin{array}{rcll} a_{ij,k} & = & \lambda_i g_{jk} + \lambda_j g_{ik} & (*) \\ \lambda_{i,j} & = & \mu g_{ij} - B a_{ij} & \\ \mu_{,i} & = & 2B\lambda_i, & \end{array}$$

i.e., are parallel sections on the following connection on $S_2 M \otimes T^*M \otimes \mathbb{R}$:

$$D_k \begin{pmatrix} a_{ij} \\ \lambda_i \\ \mu \end{pmatrix} = \begin{pmatrix} a_{ij,k} - \lambda_i g_{jk} - \lambda_j g_{ik} \\ \lambda_{i,k} - \mu g_{ik} + B a_{ik} \\ \mu_{,k} - 2B\lambda_k \end{pmatrix}.$$

Remark. It is NOT the prolongation connection of the equation $(*)$. Its parallel sections are essentially the same as that of the prolongation connection of the equation $(*)$, but the connections are different. The system of equations is in a certain sense projectively invariant and the investigation of its projective properties is a joint project with R. Gover.

Let us study this connection in details.

If the constant $B \neq 0$, one can think $B = 1$; otherwise replace g by $\frac{1}{B} g$. Let us first assume $B = 1$; then the connection has the following form

$$D_k \begin{pmatrix} a_{ij} \\ \lambda_i \\ \mu \end{pmatrix} = \begin{pmatrix} a_{ij,k} - \lambda_i g_{jk} - \lambda_j g_{ik} \\ \lambda_{i,k} - \mu g_{ik} + a_{ik} \\ \mu_{,k} - 2\lambda_k \end{pmatrix}.$$

Principal Observation. The parallel sections of this connection are in one-to-one correspondence with the parallel symmetric $(0,2)$-tensors on the cone $(\hat{M}, \hat{g}) = (\mathbb{R}_{>0} \times M, dt^2 + t^2 g)$.

The one-to-one correspondence is given by

$$(a, \lambda, \mu) \mapsto \begin{pmatrix} \mu & -t \cdot \lambda_1 & \dots & -t \cdot \lambda_n \\ -t \cdot \lambda_1 & t^2 \cdot a_{11} & \dots & t^2 \cdot a_{1n} \\ \vdots & \vdots & & \vdots \\ -t \cdot \lambda_n & t^2 \cdot a_{n1} & \dots & t^2 \cdot a_{nn} \end{pmatrix}.$$

Proof is an easy exercise – write down the Levi-Civita connection of the cone and see that it coincides with the above one.

Main message of the previous slide

Assume $D(g) \geq 3$ and $B = B(g) \neq 0$; we assume $B = 1$. Then,

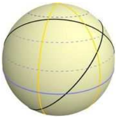
$$\boxed{\text{The metrics geodesically equivalent to } g} \quad \overset{1:1}{\longleftrightarrow} \quad \boxed{\text{Nondegenerate parallel symmetric } (0,2)\text{-tensors on the cone manifold } (\hat{M}, \hat{g}) = (\mathbb{R}_{>0} \times M, dt^2 + t^2 g).}$$

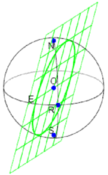
Illustration of the transition (geodesically equivalent metrics) $\longrightarrow$ (parallel symmetric (0,2)-tensors).

We take the standard sphere $S^n \subset \mathbb{R}^{n+1}$. It has constant curvature and therefore the maximal degree of mobility $\frac{(n+1)(n+2)}{2}$.

Let us explain that in this case metrics geodesically equivalent to g_{sphere} are in $1:1$ correspondence with parallel symmetric $(0,2)$-tensors on $(\mathbb{R}^{n+1}, g_{euclidean}) = (\mathbb{R}_{>0} \times S^n, dr^2 + r^2 g_{sphere})$.

Fact. Geodesics of the sphere are the great circles, that are the intersections of the 2-planes containing the center of the sphere with the sphere.





Beltrami (1865) observed:

For every $A \in SL(n+1) \xrightarrow{\text{we construct}} a : S^n \to S^n, a(x) := \frac{A(x)}{|A(x)|}$

- a is a diffeomorphism
- a takes great circles (geodesics) to great circles (geodesics)

Thus, $SL(n+1)$ produces geodesically equivalent metrics; and two elements of $SL(n+1)$ generate the same geodesically equivalent metric if $A' = AO$ for some orthogonal O. We known that with the help of the equivalence $A \longrightarrow AO$ one can make any matrix A symmetric, so the set of the metrics geodesically equivalent to g are in 1:1 correspondence with symmetric $(n+1) \times (n+1)-$matrices.

From the other side, the value of B for the metric of constant curvature is $B = +1$, and the cone $(\mathbb{R}_{>0} \times S^n, \hat{g})$ is $(\mathbb{R}^{n+1}, g_{euclidean})$. The symmetric covariantly constant tensors on $(\mathbb{R}^{n+1}, g_{euclidean})$ are also in 1:1 correspondence with symmetric $(n+1) \times (n+1)-$matrices.

It is easy to calculate the dimension of the space of the parallel symmetric $(0,2)$-tensors on the Riemannian cone.

By de Rham decomposition theorem, any Riemannian metric is (locally) decomposable into the direct product:

$$(\hat{M}, \hat{g}) = \underbrace{(M_0, g_0)}_{\text{flat}} + \underbrace{(M_1, g_1)}_{\text{irreduc.}} + ... + \underbrace{(M_\ell, g_\ell)}_{\text{irreduc.}}.$$

Every covariantly constant (0,2)-tensor on M is the direct sum of the covariantly constant tensors living on the pieces. The flat piece (of dimension k) gives a $\frac{k(k+1)}{2}$-dimensional space of parallel symmetric (0,2)-tensors, and each of the others (M_i, g_i) gives a one-dimensional space $C \cdot g_i$, so all together we obtain $\frac{k(k+1)}{2} + \ell$.

How big could be ℓ? Since $(\hat{M}, \hat{g})$ is the cone, all irreducible pieces (M_i, g_i), $i > 0$, are also cones and are therefore at least 3-dimensional, since the only two-dimensional cone manifold has the metric $dt^2 + t^2 dx^2$ and is flat. Thus, ℓ is at most $\left[\frac{n+1-k}{3}\right]$. **The Theorem is proved under the additional assumptions we assumed on the way to make the life simpler.**

What additional assumptions we assumed in the explanation?

We assumed that the initial g is Riemannian and $B > 0$.

- If $B < 0$, then the normalization $g \mapsto \frac{1}{B} g$ makes the Riemannian metric to be negatively defined, and we end up with the cone of the signature $(\underbrace{-, ..., -,}_{n} +)$. This is not a big problem though: the covariantly constant symmetric tensors of the Lorentzian manifolds are well-understood:
  Thm (Eisenhart): Every covariantly constant symmetric $(0, 2)$-tensor on a indecomposable Lorentzian manifold has the form $\mathrm{const} \cdot g_{ij} + v_i v_j$ for a covariantly constant (0,1)-tensor v.
  Using this result, one can essentially repeat the argumentation from the Riemannian case
- Suppose $B = 0$. Then, the following theorem (probably Gorbatyi 1982; the paper is impossible to find) works:
  Thm. If g is Riemannian (of dim ≥ 3), has $D(g) \geq 3$ and $B = 0$, then for any open subset with compact closure one can find a geodesically equivalent metric with $B > 0$.

What difficulties arise in the Lorentz case?

- The case $B > 0$ corresponds to the case $B < 0$ in the Riemannian case and no difficulties appear.
- The case $B < 0$ is more complicated. We need to describe the covariantly constant $(0,2)$-tensors on cone manifolds of signature $(\underbrace{-,...,-}_{n-1},+,+)$:

  Thm. Every covariantly constant symmetric $(0,2)$-tensor on a indecomposable manifold (M^{n+1}, g) of signature $(n-1,2)$ has the form (for $a, b, c, d \in \mathbb{R}$
  $a \cdot v_i v_j + b \cdot (v_i u_j + v_j u_i) + c \cdot u_i u_j + d \cdot g_{ij}$
  for covariantly constant (0,1)-tensors u and v.

  With the help of this theorem, the Riemannian arguments still work and the answer survives.

What if $B = 0$? Bad Example:

Thm (Gorbatyi?). If g is Riemannian of dim ≥ 3, has $D(g) \geq 3$ and $B = 0$, then for any open subset with compact closure one can find a geodesically equivalent metric with $B > 0$.

Thm (nontrivial).

- In the Lorentz signature, there exist examples of metrics with $D(g) \geq 3$ such that $B = 0$ and such that all metrics geodesically equivalent to g still have $B = 0$.
- The smallest dimension when the examples are possible is 6 (in the Einstein case: 10); which implies (by careful counting) that the list of the degrees of mobility remain unchanged.

$$g = \begin{bmatrix} 0 & x_1 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha(x_3, x_4)(a - x_2)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha(x_3, x_4)(a - x_2)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta(x_5, x_6)(b - x_2)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta(x_5, x_6)(b - x_2)^2 \end{bmatrix}$$

General philosophic observations

If you prove a result which is $\underbrace{\textcolor{red}{\text{nontrivial}}}_{\textbf{you learned a lot}}$, then in the neighborhood of these results there are at least two more.

The dimensions of the space of projective vector fields

Def. A projective vector field is a vector field whose flow sends geodesics to geodesics. We denote by *proj* the vector space of projective vector fields and by *iso* the space of Killing vector fields.

Thm. *Let $(M^{n\geq 3}, g)$ be a connected simply connected manifold of Riemannian or Lorentzian signature. Then, the number*

$$dim(proj(g)) - dim(iso(g))$$

is one from the following list:

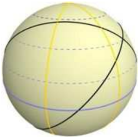
- 0 *or* 1*;*
- $\frac{k(k+1)}{2} + \ell - 1$, *where* $k \in \{0, ..., n-2\}$ *and* $\ell \in \{1, ..., \left[\frac{n-k+1}{3}\right]\}$.
- $\frac{(n+2)(n+1)}{2} - 1$ *(constant curvature case).*

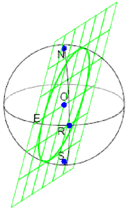
$D(g)$ has values from the following list:

- 2;
- $\frac{k(k+1)}{2} + \ell$, where $k \in \{0, ..., n-2\}$ and $\ell \in \{1, ..., \left[\frac{n-k+1}{3}\right]\}$.
- $\frac{(n+2)(n+1)}{2}$ (constant curvature case).

Example and idea of the proof.

Consider again the standard sphere: we have seen that projective transformations are essentially elements of $SL(n+1)$, and the isometries are the elements of $SO(n+1)$.





From the other side, we know that $SL(n+1)$ is the group of volume-preserving linear (=centro-affine) transformations of $\mathbb{R}^{n+1}$, and the Killing vector fields are orthogonal transformations.

We see in this "round" example that we have a natural surjective mapping

$$\Phi : \text{ centro-affine trns of } \mathbb{R}^{n+1} \longrightarrow \text{ projective trns of } \mathbb{S}^n$$

with 1-dimensional kernel and sending isometries to isometries.

The phenomena survives for all metrics.

Thm. Let g be a metric on a connected $M^{n\geq 3}$ with $D(g)\geq 3$ and $B\neq 0$. Assume there exists a metric which is geodesically equivalent to g and is not affine equivalent to g. Then, there exists a (canonical) surjective linear mapping

$$\Phi : aff(\hat{g}) \to proj(g)$$

with one-dimensional kernel that sends isometries to isometries.

Example. The group of projective transformations of the standard $S^n \subset \mathbb{R}^{n+1}$ is (essentially) $SL(n+1)$; the isometry group of S^n is $SO(n+1)$.
The cone over S^n is $\mathbb{R}^{n+1}\setminus\{0\}$ and its group of affine transformations is $GL(n+1)=\mathbb{R}\times SL(n+1)$; the group of the isometries of $\mathbb{R}^{n+1}\setminus\{0\}$ is again $SO(n+1)$.

On the cone $(\hat{M}, \hat{g})$ we can easily count all affine transformations (modulo isometries):

$$(\hat{M}, \hat{g}) = \underbrace{(M_0, g_0)}_{\text{flat } k\text{-dim.}} + \underbrace{(M_1, g_1)}_{\text{irreduc.}} + ... + \underbrace{(M_\ell, g_\ell)}_{\text{irreduc.}}.$$

The piece (M_0, g_0) gives $\frac{k(k+1)}{2}$ and each pieces (M_i, g_i) give one (the cone manifolds have a homothety vector field).
Thus, if $D(g) \geq 3$, then $dim(proj(g)) - dim(iso(g)) = D(g) - 1$.

What if the initial metric is Einstein?

Remark. The condition that the metric is Einstein is a projectively invariant property (Mikes 1982; Kiosak-Matveev 2009).

Theorem. Let g be a metric of Riemannian or Lorentz signature on a simply-connected manifold of dimension ≥ 3. Assume there exists at least one metric that is geodesically equivalent to g, but is not affinely equivalent to g. Then, the degree of mobility can accept one of the following numbers only

- $\frac{k(k+1)}{2} + \ell$, where $k \in \{0, ..., n-4\}$ and $\ell \in \{1, ..., \left[\frac{n-k+1}{5}\right]\}$.
- $\frac{(n+2)(n+1)}{2}$ (constant curvature case).

Compare the lists for Einstein and possibly non-Einstein metric

Possibly non-Einstein metric

- 2;
- $\frac{k(k+1)}{2} + \ell$, where $k \in \{0, ..., n-2\}$ and $\ell \in \{1, ..., \left[\frac{n-k+1}{3}\right]\}$.
- $\frac{(n+2)(n+1)}{2}$ (constant curvature case).

Einstein metric

- $\frac{k(k+1)}{2} + \ell$, where $k \in \{0, ..., n-4\}$ and $\ell \in \{1, ..., \left[\frac{n-k+1}{5}\right]\}$.
- $\frac{(n+2)(n+1)}{2}$ (constant curvature case).

Corollary (proved before by Kiosak-Matveev 2009). 4-dimensional Einstein manifolds of nonconstant curvature are geodesically rigid: Unparameterized geodesics determine the affine connection uniquely.

This corollary answers the question asked by Weyl in 1924, Petrov 1966, Ehlers-Pirani-Schild 1972.

This corollary is closely related to a possible joint project(publication) with Andrzej Trautman initiated by Nurowski.

The "extended" system in the Einstein case.

Possibly non-Einstein metric with $D(g) \geq 3$: geodesically equivalent metrics are essentially parallel sections of the following connection on $S_2M \otimes T^*M \otimes \mathbb{R}$:

$$D_k \begin{pmatrix} a_{ij} \\ \lambda_i \\ \mu \end{pmatrix} = \begin{pmatrix} a_{ij,k} - \lambda_i g_{jk} - \lambda_j g_{ik} \\ \lambda_{i,k} - \mu g_{ik} + B a_{ik} \\ \mu_{,k} - 2B\lambda_k \end{pmatrix}.$$

Einstein metrics: geodesically equivalent metrics are essentially parallel sections of the following connection on $S_2M \otimes T^*M \otimes \mathbb{R}$:

$$D_k \begin{pmatrix} a_{ij} \\ \lambda_i \\ \mu \end{pmatrix} = \begin{pmatrix} a_{ij,k} - \lambda_i g_{jk} - \lambda_j g_{ik} \\ \lambda_{i,k} - \mu g_{ik} + \frac{R}{n(n-1)} a_{ik} \\ \mu_{,k} - 2\frac{R}{n(n-1)}\lambda_k \end{pmatrix},$$

where R is the scalar curvature

In the Einstein case, the connection D is essentially the tractor "Thomas-Eastwood" connection on the projective tractor bundle
The Einstein metrics in the projective class are normal solutions from the yesterday talk of R. Gover.
A nontrivial observation: If we have an Einstein metric in the projective class, all solutions of the "metrisability equation" are normal.

Explanation of differences in answers for Einstein and possibly non-Einstein metrics

Possibly non-Einstein metric

- 2;
- $\frac{k(k+1)}{2} + \ell$, where $k \in \{0, ..., n-2\}$ and $\ell \in \{1, ..., \left[\frac{n-k+1}{3}\right]\}$.
- $\frac{(n+2)(n+1)}{2}$ (constant curvature case).

Einstein metric

- $\frac{k(k+1)}{2} + \ell$, where $k \in \{0, ..., n-4\}$ and $\ell \in \{1, ..., \left[\frac{n-k+1}{5}\right]\}$.
- $\frac{(n+2)(n+1)}{2}$ (constant curvature case).

Explanation in the Riemannian case: recall that we have the decomposition

$$(\hat{M}, \hat{g}) = \underbrace{(M_0, g_0)}_{\text{flat of dim. } k} + \underbrace{(M_1, g_1)}_{\text{irreduc.}} + ... + \underbrace{(M_\ell, g_\ell)}_{\text{irreduc.}}.$$

Now, if M is Einstein then all (M_i, g_i) should be Ricci-flat. Since they are cone manifolds, each block (M_i, g_i) has the dimension ≥ 5.

What if M is compact?

Thm. Let g be a pseudo-Riemannian metric (arbitrary signature!) of nonconstant curvature on a closed connected M. Then, if $D(g) \geq 3$, every metric geodesically equivalent to g is affinely equivalent to g.

Idea of the proof. The case $B = 0$ can be eliminated by a trick I will not explain. Now, the case $B \neq 0$ corresponds to the cones over a closed manifold and cones over closed manifolds do not have nontrivial parallel symmetric $(0, 2)$-tensors:

- **Riemannian case:** Gallot 1978/Tanno 1978 (wrong proof by Obata 1965).
- **Arbitrary signature case under the assumption that the manifold is complete:** Alekseevsky-Cortes-Galaev-Leistner 2009,
- **No assumptions.** Matveev-Mounoud 2011.

The h-projective analog of the story (joint with S. Rosemann)

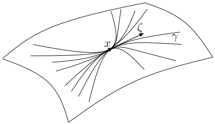
Assume now that M^{2n} carries a complex structure J.

Def. Let $\nabla = \left(\Gamma^i_{jk}\right)$ be a symmetric affine connection on (M^{2n}, J) compatible with J (i.e., $\nabla J = 0.$) An h-planar curve $c : I \to M$, $c : t \mapsto x(t)$ on (M, g) is given as solution of

$$\frac{d^2x^a}{dt^2} + \Gamma^a_{bc}\frac{dx^b}{dt}\frac{dx^c}{dt} \quad = \alpha(t)\frac{dx^a}{dt} + \beta(t)\frac{dx^k}{dt}J^a_k$$

$$(= (\alpha(t) + i \cdot \beta(t)) \cdot \frac{dx}{dt}.)$$

- $\exists$ infinitely many h-planar curves γ with $\gamma(0) = x$ and $\dot\gamma(0) = \zeta$ for each $x \in M$ and $\zeta \in T_xM$.



- reparameterized geodesics satisfy $\nabla_{\dot\gamma}\dot\gamma = \alpha\dot\gamma$.

Def. Two metrics g and $\bar g$ (compatible with the same J) are h-projectively equivalent, if every h-planar curve for ∇^g is an h-planar curve for $\nabla^{\bar g}$.

Some history of *h*-projective equivalence

- Introduced by T. Otsuki, Y. Tashiro in 1954.
- Between 1960-1980 actively studied in Japanese (Obata, Yano) and Soviet (Odessa and Kazan) differential geometry schools.
- Reinvented recently independently under other names:
  - *Apostolov, Calderbank, Gauduchon, (Toennesen-Friedman) (JDG 2004 and many other publications)*:
    Local and global classification of pairs of h-projectively equivalent Kähler Riemannian metrics.
  - *Kiyohara (Notes of AMS 1997) and Topalov-Kiyohara (PAMS 2011)*:
    Non-degenerate pairs of h-projectively equivalent metrics have Liouville-integrable geodesic flow and applications of this.
- Few classical problems were solved recently, in particular the **Yano-Obata conjecture.**: *Fedorova-Kiosak-Matveev-Rosemann (PLMS 2012) and Matveev-Rosemann (JDG to appear)*.

Equation for the h-projective equivalence (Mikes - Domashev 1978)

Metrics $\bar{g}$ that are h-projectively equivalent to g are in $1:1$ correspondence with the nondegenerate solutions (a_{ij}, λ_i) of

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} + \lambda_a J^a{}_i J_{jk} + \lambda_b J^b{}_j J_{ik} \qquad (**)$$

Here (a, λ) and g are related by the formula

$$a = \left(\frac{\det(\bar{g})}{\det(g)}\right)^{\frac{1}{2(n+1)}} g\bar{g}^{-1}g, \lambda = \tfrac{1}{4}d\mathrm{trace}_g(a)).$$

We see that the equation is similar to the one in the projective case. In particular it is linear and of finite type.

Def. The dimension of the space of solutions of this system is called the degree of mobility of (g, J).

Remark. There also exist a h-projectively invariant version of the equations $(**)$ (Matveev-Rosemann 2011/Calderbank 2011). Investigation of the existence of solutions of this system in dimension 4 is a joint project with Th. Mettler.

Thm. Let g be a Riemannian metric on a simply-connected manifold of dimension $2n \geq 4$. Assume there exists at least one metric that is h-projectively equivalent to g, but is not affinely equivalent to g. Then, the degree of mobility can accept one of the following numbers only

- $k^2 + \ell$, where $k \in \{0, ..., n-1\}$ and $\ell \in \{1, ..., \left[\frac{n-k+1}{2}\right]\}$.
- $(n+1)^2$ (constant curvature case).

For example:

- $n = 2$:
  $D(g, J) = 1, 2, 9$
- $n = 3$:
  $D(g, J) = 1, 2, 5, 16$
- $n = 4$:
  $D(g, J) = 1, 2, 3, 5, 10, 25$
- $n = 5$:
  $D(g, J) = 1, 2, 3, 5, 6, 10, 17, 36$

Principle observation

Theorem (Fedorova-Kiosak-Matveev-Rosemann (PLMS 2012)).
Let g be a metric on $2n \geq 4$-dimensional connected manifold with $D(g) \geq 3$. Then, there exists a constant B such that for any solution (a, λ) of the equations
$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} + \lambda_a J^a{}_i J_{jk} + \lambda_b J^b{}_j J_{ik} \quad (**)$
there exists a function μ such that the following two equations are fulfilled:

$$\begin{aligned} \lambda_{i,j} &= \mu g_{ij} - B a_{ij} \\ \mu_{,i} &= -2B\lambda_i. \end{aligned}$$

This is again a connection!!!
Question. What geometric object lies behind this connection?

Calabi Kähler cone

Recall that the cone over (M, g) is the manifold

$$(\hat{M}, \hat{g}) = (\mathbb{R}_{>0} \times M, dt^2 + t^2 g).$$

One can of course do this construction for Kähler manifolds as well; in this case we do not obtain nice properties of $\hat{g}$ though.
There exists though a Kähler analog of the cone construction (called "conification " in folklore)

$$(M^{2n}, g, J) \longmapsto (\hat{M}^{2n+2} = \mathbb{R}^2 \times M, \hat{g}, \hat{J})$$

that produces a $(2n+2)$-dimensional Kähler manifold.

The connection in the previous slide (assuming $B \neq 0$) is essentially the Levi-Civita connection on the Calabi cone and parallel sections of it are essentially the parallel symmetric $(0, 2)$-tensors.
Thus, the Shandra idea still works.

Conclusion

The topic is hot – join!!!

Thanks a lot