Twistor theory for generalized complex manifolds

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1. Generalized complex geometry

The inner product and bracket on $C^\infty(T \oplus T^*)$:

$$\langle X+\xi, Y+\eta\rangle := \frac{1}{2}(\xi(Y)+\eta(X))$$

$$[X+\xi, Y+\eta] := [X,Y] + \mathcal{L}_X\eta - i_Y d\xi$$

The *Courant bracket* satisfies

$$[e_1,[e_2,e_3]] = [[e_1,e_2],e_3] + [e_2,[e_1,e_3]]$$

$$[e_1,e_1] = \pi^* d\langle e_1,e_1\rangle$$

where $e_i \in C^\infty(T \oplus T^*)$ and $\pi : T \oplus T^* \to T$ is projection.

Defn (Hitchin 2002): A *generalized complex structure* is an orthogonal endomorphism

$$\mathcal{J} : T \oplus T^* \to T \oplus T^*$$

such that $\mathcal{J}^2 = -1$ and the $+i$-eigenspace

$$L \subset (T \oplus T^*) \otimes \mathbb{C}$$

is closed under the bracket, $[L, L] \subset L$.

Eg 1.

$$\mathcal{J}_I := \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}$$

where I is a complex structure.

Eg 2.

$$\mathcal{J}_\omega := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

where ω is a symplectic structure, thought of as a map $\omega : T \to T^*$.

A generalized complex structure $\mathcal{J}$ is determined by a *pure spinor* $\Phi \in C^\infty(\wedge^\bullet T^* \otimes \mathbb{C})$.

The $+i$-eigenspace of $\mathcal{J}$,

$$L \subset (T \oplus T^*) \otimes \mathbb{C},$$

consists of (X, ξ) which annihilate Φ.

Eg 1. For $\mathcal{J}_I$ the pure spinor is $dz_1 \wedge \ldots \wedge dz_n$ and

$$L = T^{0,1} \oplus \Omega^{1,0}.$$

Eg 2. For $\mathcal{J}_\omega$ the pure spinor is $e^{i\omega}$ and

$$L = \{X - i\omega(X, -)\}.$$

A generalized complex manifold of dimension $2n$ is stratified by *type*,

$$\frac{1}{2}\text{dim} T^* \cap \mathcal{J} T^* \qquad \left(\begin{array}{ll} = n & \text{for } \mathcal{J}_I \\ = 0 & \text{for } \mathcal{J}_\omega \end{array} \right).$$

Generalized Darboux Thm (Gualtieri 2004): On an open set of constant type $= k$, a generalized complex manifold is equivalent to the product of an open set in $\mathbb{C}^k$ and an open set in $(\mathbb{R}^{2n-2k}, \omega_0)$.

Let S be a K3 surface with $\mathbb{C}$-structure I and holomorphic symplectic form $\sigma = \omega_J + i\omega_K$. Then

$$\mathcal{J}_\theta := \cos\theta \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix} + \sin\theta \begin{pmatrix} 0 & -\omega_J^{-1} \\ \omega_J & 0 \end{pmatrix}$$

interpolates between $\mathcal{J}_I$ and $\mathcal{J}_{\omega_J}$. Note that

$$\mathcal{J}_\theta = e^{-B} \mathcal{J}_{(\csc\theta)\omega_J} e^B$$

where $B = -(\cot\theta)\omega_K$ and

$$e^B := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} : T \oplus T^* \to T \oplus T^*$$

is known as a B*-field transform*.

Let M be a hyperkähler manifold with $\mathbb{C}$-structures I, J, K, and Kähler structures ω_I, ω_J, ω_K.

They give six generalized complex structures:

$$\mathcal{J}_I, \quad \mathcal{J}_J, \quad \mathcal{J}_K, \quad \mathcal{J}_{\omega_I}, \quad \mathcal{J}_{\omega_J}, \quad \mathcal{J}_{\omega_K}$$

Our goal is to assemble them all into one family.

2. Twistor spaces for HK manifolds

The set of $\mathbb{C}$-structures compatible with the hyperkähler metric is

$$\{aI + bJ + cK | a^2 + b^2 + c^2 = 1\}.$$

This is actually a holomorphic family,

$$S^2 \cong \mathbb{P}^1,$$

as there is a family of holomorphic two-forms

$$\sigma_\eta := (\omega_J + i\omega_K) + 2\eta\omega_I - \eta^2(\omega_J - i\omega_K)$$

depending holomorphically on $\eta \in \mathbb{P}^1$.

For each $\eta \in \mathbb{P}^1$, σ_η determines a $\mathbb{C}$-structure I_η on M. The $-i$-eigenspace of I_η,

$$T_\eta^{0,1} \subset T \otimes \mathbb{C},$$

consists of vectors whose interior product with $\sigma_\eta^{\wedge n}$ is zero.

These $\mathbb{C}$-structures I_η can be combined into a single $\mathbb{C}$-structure on the smooth manifold $M \times \mathbb{P}^1$.

Defn: The resulting $\mathbb{C}$-manifold $Z \to \mathbb{P}^1$ of dimension $2n+1$ is called the *twistor space* of M.

3. Generalized twistor spaces

We want the set of generalized complex structures $\mathcal{J}$ compatible with the *generalized metric*

$$G(X+\xi, Y+\eta) = \frac{1}{2}\Big(g(X,Y) + g^{-1}(\xi,\eta)\Big)\,,$$

i.e., such that $G(\mathcal{J}e_1, \mathcal{J}e_1) = G(e_1, e_2)$.

We will describe three approaches.

i) For each θ

$$\mathcal{J}_\theta := (\cos\theta)\mathcal{J}_I + (\sin\theta)\mathcal{J}_{\omega_J}$$

is compatible with G. We can replace ω_J by

$$(\cos\phi)\omega_J + (\sin\phi)\omega_K.$$

Then θ and ϕ are spherical coordinates on $S^2 \cong \mathbb{P}^1$.

We can also replace I by I_η where $\eta \in \mathbb{P}^1$.

This yields a $\mathbb{P}^1 \times \mathbb{P}^1$-family of generalized complex structures $\mathcal{J}_{\alpha,\beta}$.

The diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ parametrizes complex structures $\mathcal{J}_{I_\eta}$.

The "antipodal graph"

$$\{(\eta, -\bar{\eta}^{-1}) | \eta \in \mathbb{P}^1\} \subset \mathbb{P}^1 \times \mathbb{P}^1$$

parametrizes symplectic structures $\mathcal{J}_{\omega_\eta}$.

The generalized complex structures $\mathcal{J}_{\alpha,\beta}$ are determined by a family of pure spinors

$$\Phi_{\alpha,\beta} := i^n(\alpha-\beta)^n \exp\left(\frac{\sigma+(\alpha+\beta)\omega_I-\alpha\beta\bar{\sigma}}{i(\alpha-\beta)}\right)$$

which depend holomorphically on $(\alpha,\beta)\in\mathbb{P}^1\times\mathbb{P}^1$. When $n=1$

$$\Phi_{\alpha,\beta} := \sigma+(\alpha+\beta)\omega_I+i(\alpha-\beta)\left(1-\frac{\sigma\bar{\sigma}}{4}\right)-\alpha\beta\bar{\sigma}.$$

Recall that the $+i$-eigenspace,

$$L_{\alpha,\beta}\subset(T\oplus T^*)\otimes\mathbb{C},$$

of $\mathcal{J}_{\alpha,\beta}$ consists of (X,ξ) which annihilate $\Phi_{\alpha,\beta}$.

ii) Given a hyperkähler metric on a K3 surface, there is a corresponding positive 3-plane

$$V := \langle \omega_I, \omega_J, \omega_K \rangle \subset \mathsf{H}^2(M, \mathbb{R}).$$

The locus of holomorphic two-forms $[\sigma_\eta]$ of the $\mathbb{P}^1$-twistor family is the null conic in $\mathbb{P}(V \otimes \mathbb{C}) \cong \mathbb{P}^2$.

There is also a positive 4-plane

$$W := \left\langle \omega_I, \omega_J, \omega_K, 1 - \frac{\omega^2}{2} \right\rangle \subset \mathsf{H}^{\mathsf{even}}(M, \mathbb{R}).$$

The locus of pure spinors $[\Phi_{\alpha,\beta}]$ of the $\mathbb{P}^1 \times \mathbb{P}^1$-generalized twistor family is the null quadric surface in $\mathbb{P}(W \otimes \mathbb{C}) \cong \mathbb{P}^3$.

iii) Let (g, I_+, I_-) be a bi-Hermitian structure on M. Then

$$\mathcal{J} = \frac{1}{2}\begin{pmatrix} I_+ + I_- & -(\omega_+^{-1} - \omega_-^{-1}) \\ \omega_+ - \omega_- & -(I_+^* + I_-^*) \end{pmatrix}$$

$$\mathcal{J}' = \frac{1}{2}\begin{pmatrix} I_+ - I_- & -(\omega_+^{-1} + \omega_-^{-1}) \\ \omega_+ + \omega_- & -(I_+^* - I_-^*) \end{pmatrix}$$

give a generalized Kähler structure, i.e., a pair of commuting generalized complex structures.

If $I_+ = I_-$ then $(\mathcal{J}, \mathcal{J}') = (\mathcal{J}_{I_+}, \mathcal{J}_{\omega_+})$ comes from a genuine Kähler structure on M.

On a hyperkähler manifold, we can choose $I_+ = I_\alpha$ and $I_- = I_\beta$ from a $\mathbb{P}^1$-twistor family. For each $(\alpha, \beta) \in \mathbb{P}^1 \times \mathbb{P}^1$, (g, I_α, I_β) defines a generalized Kähler structure $(\mathcal{J}_{\alpha,\beta}, \mathcal{J}'_{\alpha,\beta})$.

Along the diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$

$$(\mathcal{J}_{\alpha,\alpha}, \mathcal{J}'_{\alpha,\alpha}) = (\mathcal{J}_{I_\alpha}, \mathcal{J}_{\omega_\alpha})$$

comes from a Kähler structure on M.

Unfortunately only $\mathcal{J}_{\alpha,\beta}$ depends holomorphically on $(\alpha, \beta) \in \mathbb{P}^1 \times \mathbb{P}^1$. ($\mathcal{J}'_{\alpha,\beta}$ depends on α and $\bar{\beta}$.)

Thm (Glover & S$-$): The generalized complex structures $\mathcal{J}_{\alpha,\beta}$ can be combined into a single generalized complex structure on the smooth manifold $X = M \times \mathbb{P}^1 \times \mathbb{P}^1$.

We call X the *generalized twistor space*.

Proof: The pure spinor on X is given by

$$\Psi = \Phi_{\alpha,\beta} \wedge d\alpha \wedge d\beta.$$

Integrability follows from $d\Psi = 0$.

Properties:

- there is a *generalized reduction* $X \to \mathbb{P}^1 \times \mathbb{P}^1$,

- the antipodal maps on the $\mathbb{P}^1 \cong S^2$s give a real structure on X,

- for each $m \in M$, $\{m\} \times \mathbb{P}^1 \times \mathbb{P}^1 \subset X$ yields a real twistor section with generalized normal bundle

$$\mathcal{O}(1,0)^{\oplus 2n} \oplus \mathcal{O}(0,1)^{\oplus 2n}.$$

4. Generalized twistor correspondence

A connection ∇ is Yang-Mills if its curvature Θ satisfies

$$\Lambda(\Theta) = \text{const.Id.}$$

The *hyperkähler twistor correspondence* gives an equivalence between (non-Hermitian) Yang-Mills connections on bundles over M and holomorphic bundles on Z.

The bundle and connection (E, ∇) on M can be pulled-back by the smooth projection $\pi : Z \to M$ to give $(\pi^* E, \pi^* \nabla^{0,1})$ on Z.

Defn: A *generalized holomorphic bundle* on X is a complex vector bundle E equipped with a flat L-connection D, where $L \subset (T \oplus T^*) \otimes \mathbb{C}$ is the $+i$-eigenspace, a Lie algebroid.

Eg. If $\mathcal{J} = \mathcal{J}_I$ is of complex type, $L^* = \Omega^{0,1} \oplus T^{1,0}$ and the L-connection decomposes into

$$D = \bar{\partial}_E + \Upsilon.$$

Then $(E, \bar{\partial}_E)$ is a holomorphic bundle and

$$\Upsilon \in \mathrm{H}^0(T^{1,0} \otimes \mathrm{End}E)$$

is a *co-Higgs field*.

Qu: Are generalized holomorphic bundles on the generalized twistor space X equivalent to some kind of bundles with connections on M?

A generalized holomorphic bundle on X should look like a co-Higgs bundle on $Z \subset X$ (over $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$) and a flat unitary bundle on $X \backslash Z$.

5. Quaternionic manifolds

$$\begin{array}{ccc}
\text{hypercomplex geom} & \subset & \text{quaternionic geom} \\
\mathsf{GL}(n,\mathbb{H})\text{-structure} & & \mathsf{GL}(n,\mathbb{H})\mathsf{GL}(1,\mathbb{H})\text{-structure} \\
\cup & & \cup \\
\text{hyperkähler geom} & \subset & \text{quaternion-Kähler geom} \\
\text{holonomy } = \mathsf{Sp}(n) & & \text{holonomy } = \mathsf{Sp}(n)\mathsf{Sp}(1)
\end{array}$$

There is a $\mathbb{P}^1$-bundle $Z \to M$ over a quaternionic manifold M whose local sections give local almost complex structures on M.

These combine to give a complex structure on Z. We call Z the *twistor space* of M.

The fibre product $Z \times_M Z$ is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over M whose local sections give pairs of local almost complex structures (I_α, I_β) on M.

On a QK manifold, (g, I_α, I_β) defines a local almost Kähler structure $(\mathcal{J}_{\alpha,\beta}, \mathcal{J}'_{\alpha,\beta})$ on M.

Qu: Do the $\mathcal{J}_{\alpha,\beta}$ combine to give a generalized complex structure on $X = Z \times_M Z$?

Eg. For $\mathbb{P}^3 \to S^4 \cong \mathbb{HP}^1$ we have $X = \mathbb{P}^3 \times_{S^4} \mathbb{P}^3$.


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