On the BGG complexes in singular infinitesimal character

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Regular orbits, BGG sequence

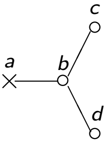
- G/P a homogeneous model for a parabolic geometry, M its curved analogues ($\mathcal{G} \mapsto M$, $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$)
- an invariant differential operator D on M acts between bundles associated to irreducible representations $\mathbb{V}_\lambda$, $\mathbb{V}_{\lambda'}$ of P weights of representations for the source and the target should be on the same orbit of the affine action of the Weyl group
- regular orbits are classified by weights $\lambda = \sum_i \lambda_i \omega_i$, where ω_i, $i = 1, \ldots, k$ are fundamental weights
  $\rho = (1, \ldots, 1)$ λ_i ... non-negative integers (finite-dimensional representations of $\mathfrak{g}$)
  a big classes of invariant differentail operators are constructed for orbits in a regular infinitesimal character (i.e. the orbit of $\lambda + \rho$ contains an element *inside* the dominant Weyl chamber)
  - in particular, operators in the BGG sequences
  the shape of the orbit is described by the parabolic Hasse

singular orbits

- singular orbits are orbits containing λ such that $\lambda + \rho$ is on the boundary of the dominant Weyl chamber, i.e. λ_i are integers bigger or equal to -1 number of coefficients equal to $-1 =$ a degree of degeneracy .. size of the isotropic subgroup
- many interesting operators correspond to orbits in singular character
  the problem is to develop a systematic construction of invariant operators in for orbits in a singular character

Contents

Conformal case

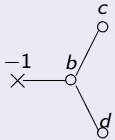
- 

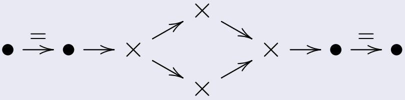
- $$a, b, c, d \geq 0, \in \mathbb{Z}, \text{ conformal geometry in dim 6}$$

  $$\text{the bottom case } a = b = c = d = 0, \text{de Rham}$$

$$\begin{tikzcd}
 & & & \bullet \arrow[dr, "d+1"] & & & \\
\bullet \arrow[r, "a+1"] & \bullet \arrow[r, "b+1"] & \bullet \arrow[ur, "c+1"] \arrow[dr, "d+1"'] & & \bullet \arrow[r, "b+1"] & \bullet \arrow[r, "a+1"] & \bullet \\
 & & & \bullet \arrow[ur, "c+1"'] & & &
\end{tikzcd}$$

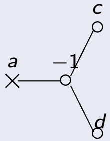
Singular orbits, I

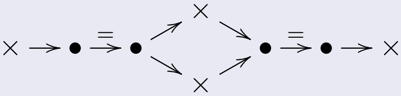




b=c=d=0

Δ^2

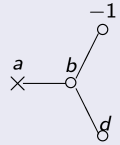


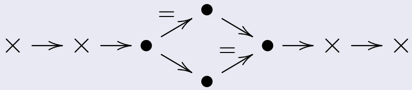


a=c=d=0

Δ

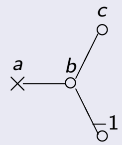
Singular orbits, II

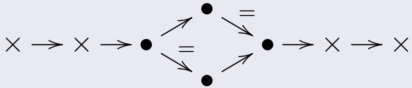




a=b=d=0

D_+



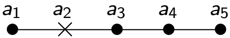


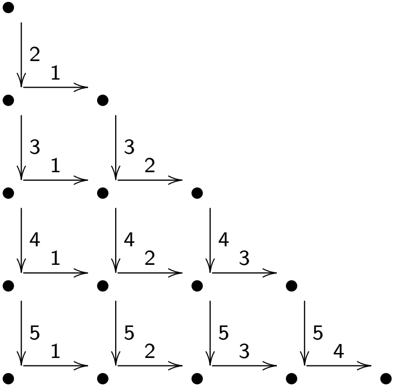
a=b=c=0

D_-

Grassmannian case

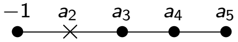
Hasse diagram for





Singular case, I

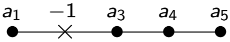
Hasse diagram for $a_1 = -1$, others non-negative



$$
\begin{array}{cccccc}
x & & & & & \\
 & & & & & \\
\bullet & = & \bullet & & & \\
 & \downarrow & & & & \\
\bullet & = & \bullet & x & & \\
 & \downarrow & & & & \\
\bullet & = & \bullet & x & x & \\
 & \downarrow & & & & \\
\bullet & = & \bullet & x & x & x
\end{array}
$$

Singular case, II

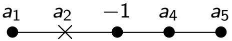
Hasse diagram for $a_2 = -1$, others non-negative



$$
\begin{array}{cccccc}
\bullet & & & & & \\
= & & & & & \\
\bullet & x & & & & \\
 & & \downarrow & & & \\
x & \bullet & = & \bullet & & \\
 & & \downarrow & & & \\
x & \bullet & = & \bullet & x & \\
 & & \downarrow & & & \\
x & \bullet & = & \bullet & x & x
\end{array}
$$

Singular case, III

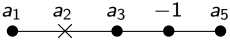
Hasse diagram for $a_3 = -1$, others non-negative



$$
\begin{array}{ccccccc}
x & & & & & & \\
\bullet & & \bullet & & & & \\
= & \rightarrow & = & & & & \\
\bullet & & \bullet & x & & & \\
x & & x & \bullet & = & \bullet & \\
 & & & & \downarrow & & \\
x & & x & \bullet & = & \bullet & x
\end{array}
$$

Singular case, IV

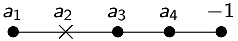
Hasse diagram for $a_4 = -1$, others non-negative



$$
\begin{array}{cccccccc}
x & & & & & & & \\
x & & x & & & & & \\
\bullet & & \bullet & & \bullet & & & \\
= & \to & = & \to & = & & & \\
\bullet & & \bullet & & \bullet & x & & \\
x & & x & = & x & \bullet & = & \bullet
\end{array}
$$

Singular case, V

Hasse diagram for $a_5 = -1$, others non-negative



$$
\begin{array}{ccccccccc}
x & & & & & & & & \\
x & & x & & & & & & \\
x & & x & & x & & & & \\
\bullet & & \bullet & & \bullet & & \bullet & & \\
= & \to & = & \to & = & \to & = & & \\
\bullet & & \bullet & & \bullet & & \bullet & & x
\end{array}
$$

$|1|$-graded case

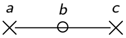
B. Boe, D. Hunziker: Kostant modules in block of category $\mathcal{O}_S$, Communications in Algebra, 37 (2009), 323-356.

- $(\mathfrak{g}, \mathfrak{h})$ Cartan subalgebra of a simple Lie algebra
- Δ ... the set of simple roots, W .. the Weyl group
- $S \subset \Delta$, (non-crossed roots) are $1-1$ with parabolic P_S parabolic Hasse $W^P \subset W$
- typ of a singularity - $J \subset \Delta$

Theorem

The structure of singular orbits for a couple $(\mathfrak{g}, \mathfrak{p})$ can be described by a structure of the regular orbits for a suitable another couple $(\mathfrak{g}', \mathfrak{p}')$. There is an explicit description for all cases of this correspondence. Singular orbits are realized as a Lie algebra cohomology of suitable simple modules in the category $\mathcal{O}_S$.

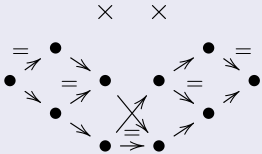
Lagrangian conctact case



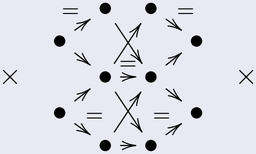
$$a, b, c \geq 0, \in \mathbb{Z},$$

$$\begin{tikzcd}[column sep=small]
 & & \text{-a-b-3,a,b+c+1} \arrow[r] \arrow[ddr] & \text{-a-b-c-4,a,b} \arrow[dr] & & \\
 & \text{-a-2,a+b+1,c} \arrow[ur] \arrow[dr] & & & \text{-a-b-c-4,a+b+1,-b-2} \arrow[dr] & \\
\text{a,b,c} \arrow[ur] \arrow[dr] & & \text{-a-2,a+b+c+2,-c-2} \arrow[uur] \arrow[r] \arrow[ddr] & \text{-a-b-3,a+b+c+2,-b-c-2} \arrow[ur] \arrow[dr] & & \text{-b-c-3,b,-a-b-3} \\
 & \text{a,b+c+1,-c-2} \arrow[ur] \arrow[dr] & & & \text{-b-2,b+c+1,-a-b-c-4} \arrow[ur] & \\
 & & \text{a+b+1,c,-b-c-3} \arrow[uur] \arrow[r] & \text{b,c,-a-b-c-4} \arrow[ur] & &
\end{tikzcd}$$

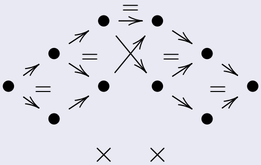
$a = -1,\ b, c \geq 0, \in \mathbb{Z}$



$b = -1,\ a, c \geq 0, \in \mathbb{Z}$



$c = -1,\ a, b \geq 0, \in \mathbb{Z}$



Contents

Conformal case

Theorem (M. Eastwood, J. Slovák)

Let $(M, [g])$ be a manifold with a given conformal structure. Then for each singular (two-point) orbit, there exists an invariant differential operator with the corresponding source and target.

A Grassmannian case

- R.Baston: Quaternionic complexes, Jour. Geom. Phys., 8, 1992, 29-52
- $|1|$-graded case, a first order G_0–structure, i.e. a reduction of the frame bundle of M to the structure group G_0.
  Here $G = PSL(n+1,\mathbb{H})$, so $\mathfrak{g}$ is a real form of $\mathfrak{sl}(2n+2,\mathbb{C})$. Then P turns out to be the stabilizer of a quaternionic line in $\mathbb{H}^{n+1}$ and $G_0 \cong S(GL(1,\mathbb{H})GL(n,\mathbb{H})) \subset GL(4n,\mathbb{R})$. The resulting geometry is called an *almost quaternionic structure* on a manifold M of dimension $4n$ (see e.g. Salamon). It is given by a rank 3subbundle $Q \subset L(TM, TM)$ which can be locally spanned by I, J, and IJ for two anti commuting almost complex structures I and J on M. Going to complexification and to two fold covering (locally), this is equivalent to a tensor product decomposition of $TM \otimes \mathbb{C} \simeq S^A \otimes S^{A'}$ with S^A being of rank 2 and $S^{A'}$ being of rank n.

The harmonic curvature decomposes into two parts. The torsion part can be determined as a specific component of the torsion of an arbitrary linear connection on TM which is compatible with the G_0–structure. This component is independent of the choice of the connection. It is exactly the obstruction to torsion freeness in the sense of first order structures, and its vanishing is also equivalent to torsion freeness of the corresponding regular normal parabolic geometry. Torsion free geometries of that type are called *quaternionic* structures.

Theorem

A (suitably convex, complex) quaternionic manifold M has a twistor space Z related to M by a double fibration.
In each singular orbit, the Penrose transform computes cohomologies of suitable vector bundles by a resolution of a length $2n$, which exhaust all spaces in the singular orbit. Some operators in resolutions are non-standard invariant operators.

Lagrangian contact case

parabolic subalgebra $\times\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!\times$
Split real form with $G := SL(n+2, \mathbb{R})$ leads to Lagrangean (or Legendrean) contact structures. Such a structure on a manifold M of dimension $2n+1$ is given by a codimension one subbundle $H \subset TM$ which defines a contact structure, and a fixed decomposition of $H = E \oplus F$ such that the Lie bracket of two sections of E (or two sections of F) is a section of H.
There are three components in the harmonic curvature. The $(0,2)$– and $(2,0)$–parts are torsions which are the obstructions to integrability of the subbundles $E, F \subset TM$. Vanishing of these two components is equivalent to torsion freeness of the corresponding parabolic geometry.

Theorem (A.Čap, VS)

Let $\mathfrak{g} = sl(4, \mathbb{R})$ *and a nested pair of parabolic subalgebras* $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ *is given by* $\times\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\times$ $\quad \times\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ$ *respectively.*
For any $\mathfrak{p}$ *module* V *with the highest weight* $\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\circ}\!\!-\!\!\!-\!\!\overset{c}{\circ}$ *with* $a, b, c \in \mathbb{Z}, b, c \geq 0,$ *there is the corresponding relative BGG sequence of invariant operators*

$$\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\circ}\!\!-\!\!\!-\!\!\overset{c}{\times} \quad \to \quad \overset{a,}{\times}\!\!-\!\!\!-\!\!\overset{b+c+1,}{\circ}\!\!-\!\!\!-\!\!\overset{-c-2}{\times} \quad \to \quad \overset{a+b+1,}{\times}\!\!-\!\!\!-\!\!\overset{c,}{\circ}\!\!-\!\!\!-\!\!\overset{-b-c-3}{\times}.$$

These sequences include all anti-diagonal parts of all singular orbits.
The similar statement for $\overset{a}{\circ}\!\!-\!\!\!-\!\!\overset{b}{\circ}\!\!-\!\!\!-\!\!\overset{c}{\times}$ *with* $a, b, c \in \mathbb{Z}, a, b \geq 0,$ *covers all diagonal parts of all singular orbits.*

Contents

Problem

Complex function theory

$$f : \mathbb{R}^2 \to \mathbb{C}$$

$$\bar{\partial} f = 0;$$

Clifford analysis

$\mathbb{S}$ - spinor module for $Spin(n)$

$$f = f(x) : \mathbb{R}^n \to \mathbb{S}$$

$$Df = \sum_1^n e_i \partial_{x_i} f = 0$$

Several complex variables

$$f = f(z_1, \ldots, z_n) : (\mathbb{R}^2)^k \to \mathbb{C}$$

$$\bar{\partial}_{z_1} f = \ldots = \bar{\partial}_{z_k} f = 0$$

Several Clifford variables

$\underline{x} = \sum_i e_i x_i \in End(S)$

$$f = f(\underline{x}_1, \ldots, \underline{x}_k) : (\mathbb{R}^n)^k \to \mathbb{S}$$

$$D_1 f = \ldots = D_n f = 0$$

Massless fields in dimension 4

Spinors in dimension 4 are elements in S^A, resp. $S^{A'}$. The (complexified) space $\mathbb{C}^4$ can be identified with $S_A \otimes S_{A'}$.
A higher spin field (spin s) has values in $\odot^s S^{A'}$. The massless field equation is

$$\nabla_{AA'} \varphi^{A' \dots E'} = 0.$$

These equations can also be considered as a generalization of the Cauchy-Riemann equations from dimension 2 to dimension 4.

Resolution for $D = (D_1, \dots, D_k)$

- Algebraic methods - Hilbert syzygy theorem, Gröbner basis, computational algebraic analysis -
  A. Damiano (Italy), D. Struppa (USA), I. Sabadini, F Colombo (Italy)
  see the web page www.tlc185.com/coala
- Clifford analysis methods
- Both methods are summarized in
  F. Colombo, I. Sabadini, F. Sommen, D. Struppa: Analysis of Dirac Systems and Computational Algebra, Birkhäuser,2004;
- Representation theory methods (parabolic geometry)
  - Verma module homomorphisms in singular character (P. Franek, Praha)
  - Penrose transform (L. Krump, VS, T. Salač, Praha)

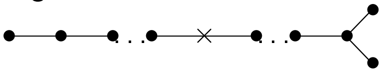
Dimension 4

The Baston quaternionic complexes in dimension 4.
A description of the first operators in the singular resolutions:
A flat case (the big cell $\mathfrak{g}_-$ in G/P) maps on $\mathbb{H}^n$ with values in $\mathfrak{g}_0$-modules
$\mathbb{CH}^n : x_a \simeq x_{AA'}$, $A = 1, 2, \ldots, 2n$, $A' = 1, 2; a = 1, \ldots, n$.

- a Laplace type equation $\nabla_{A'[A}\nabla^{A'}_{B]}\varphi = 0$, $a_2 = -1$, others equal to zero
- the Fueter (Dirac) equation in n variables $\nabla_{AA'}\varphi^{A'} = 0$, $a_3 = -1$, other trivial
- the Maxwell equation in n variables $\nabla_{AA'}\varphi^{A'B'} = 0$, $a_4 = -1$, other trivial
- the massless field equation $(s = 3)$ in n variables $\nabla_{AA'}\varphi^{A'B'C'} = 0$, $a_5 = -1$, other trivial

Higher dimensions

A choice of a parabolic geometry: $n = 2m$
S an irreducible spinor representation for the Clifford algebra $\mathcal{C}_n$
$f : x \in \mathbb{R}^{nk} \mapsto S$, $x = (x_1, \dots, x_k)$; $x_i \in \mathbb{R}^n$
$Df = (D_1 f, \dots, D_k f)$; $D_i f = \sum_j e_j \partial/\partial x_{ij} f$
we want $\mathfrak{g}$ with $\mathfrak{g}_1 \simeq \mathbb{R}^{nk}$, $\mathfrak{g}_0^{ss} \simeq \mathfrak{sl}(k) \times \mathfrak{so}(n)$
we get $\mathfrak{g} = \mathfrak{so}(k, k+n)$, but $\mathfrak{g}$ is $|2|$-graded with the Dynkin diagram



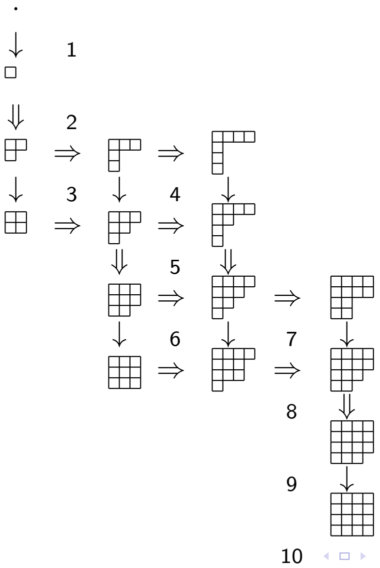
The operator $Df = (D_1 f, \dots, D_k f)$ act from spinors to k copies of the spinors. There is just one G-invariant first order operator inducing it, and it fixes the whole weight of the Levi factor of G.

Facts: The orbit is in a singular character with a double degeneration.
It is possible to compute the form of the orbit.

Young diagram is diagram with k rows, each row having $\lambda_i, i = 1, \dots, k$ left justified boxes.
Example:

$$\lambda = (4, 3, 2, 2, 0, \dots, 0),$$

Such symbol denotes below the irreducible $\mathfrak{g}_0^{ss}$-module $V_\lambda \otimes S$, where S is an irreducible Spin(n)-module. The orbit in the case $k = 4$ is displayed below.


1
2
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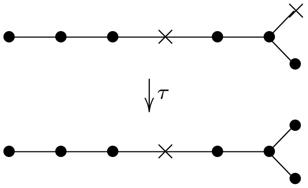
the fibration of flag manifolds, the direct image of a suitable relative BGG complex upstairs
e.g., 4 variables in dimension 8 (a stable case)



The direct image of a suitable relative BGG sequence upstairs induces a locally exact complex below.

The BGG sequence for the singular orbit

Theorem (L. Krump, T. Salač)

There is an exact sequence of invariant differential operators on the big cell $\mathfrak{g}_-$ starting with the Dirac operator in k variable.

Note that the first operator is acting on spinor valued functions on the whole $\mathfrak{g}_-$ but it contains only derivatives in direction $\mathfrak{g}_{-1}$. Solutions are usually called monogenic functions.

The generalized Dolbeault resolution

Theorem (T. Salač)

The sequence of invariant differential operators on G/P starting with the k-Dirac operator remains on $\mathfrak{g}_{-1}$ a resolution of the sheaf of monogenic sections, if we take the real analytic sections which do not depend on $\mathfrak{g}_{-2}$-variables.

The theorem proves an old conjecture for the analogue of the Dolbeault resolution for k Dirac operators for a general stable case $k \leq m$.