Principal Series Representations for some Infinite Dimensional Lie Groups

The Interaction of Geometry and Representation Theory: Exploring New Frontiers

and

Michael Eastwood's 60th Birthday

Schrödinger Institute, Vienna, September 10-14

Joseph A. Wolf

University of California at Berkeley

Classical Plancherel Formula

- G: semisimple Lie group, e.g. $SL(n;\mathbb{R})$, $SU(p,q)$, ...
- $Car(G)$: conj. classes $[H]$ of Cartan subgroups H of G
- $\chi \in \widehat{H}$: unitary character of $H \in [H] \in Car(G)$
- π_χ associated unitary representation of G, constructed using a "cuspidal" parabolic subgroup of G based on H
- Θ_{π_χ}: distribution character of π_χ
- Plancherel formula: if $f \in \mathcal{C}(G)$ Harish-Chandra Schwartz space then

$$f(x) = \sum_{Car(G)} \int_{\widehat{H}} \Theta_{\pi_\chi}(r_x f)\, d\mu_{[H]}(\chi)$$

Easiest: Principal Series

- θ: Cartan involution of G and $K = G^\theta$ maximal compact
- $d\theta : \quad \mathfrak{g} = \mathfrak{k} + \mathfrak{s}$, $\mathfrak{a} \subset \mathfrak{s}$ maximal abelian subspace of $\mathfrak{s}$
- $A = \exp(\mathfrak{a})$ and $M = Z_K(A)$ centralizer of A in K
- $\mathfrak{n}$: sum of positive $\mathfrak{a}$–weight spaces on $\mathfrak{g}$ and $N = \exp(\mathfrak{n})$
- $G = KAN$ Iwasawa and $P = MAN$ minimal parabolic
- $T \subset M$ and $H = (T \times A) \subset G$ are Cartan subgroups
- Given $\eta_\nu \in \widehat{M}$ and $\sigma \in \mathfrak{a}^*$ define $\chi_{\nu,\sigma} = \eta_\nu \otimes e^{i\sigma+\rho} \otimes 1$ (representation of $P = MAN$)
- Then $\pi_{\nu,\sigma} = \mathrm{Ind}_P^G(\chi_{\nu,\sigma})$ is a unitary representation of G
- DEFINITION These $\pi_{\nu,\sigma}$ form the **principal series** for G

Complex Classical Algebras

- We start with the three classical simple locally finite countable–dimensional Lie algebras $\mathfrak{g}_{\mathbb{C}} = \varinjlim \mathfrak{g}_{n,\mathbb{C}}$
- later $\mathfrak{g}$ will denote a real form of $\mathfrak{g}_{\mathbb{C}}$
- The Lie algebras $\mathfrak{g}_{\mathbb{C}}$ are the classical direct limits,
  $\mathfrak{sl}(\infty, \mathbb{C}) = \varinjlim \mathfrak{sl}(n; \mathbb{C})$,
  $\mathfrak{so}(\infty, \mathbb{C}) = \varinjlim \mathfrak{so}(2n; \mathbb{C}) = \varinjlim \mathfrak{so}(2n+1; \mathbb{C})$, and
  $\mathfrak{sp}(\infty, \mathbb{C}) = \varinjlim \mathfrak{sp}(n; \mathbb{C})$,
- Here the direct systems are given by the inclusions of the form $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$.
- We often consider the locally reductive algebra $\mathfrak{gl}(\infty; \mathbb{C}) = \varinjlim \mathfrak{gl}(n; \mathbb{C})$ along with $\mathfrak{sl}(\infty; \mathbb{C})$.

Real Classical Algebras (1)

- The real forms of the classical simple locally finite countable–dimensional complex Lie algebras $\mathfrak{g}_{\mathbb{C}}$ are
- If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(\infty; \mathbb{C})$, then $\mathfrak{g}$ is one of
  - $\mathfrak{sl}(\infty; \mathbb{R}) = \varinjlim \mathfrak{sl}(n; \mathbb{R})$, the real special linear Lie algebra;
  - $\mathfrak{sl}(\infty; \mathbb{H}) = \varinjlim \mathfrak{sl}(n; \mathbb{H})$, the quaternionic special linear Lie algebra, given by $\mathfrak{sl}(n; \mathbb{H}) := \mathfrak{gl}(n; \mathbb{H}) \cap \mathfrak{sl}(2n; \mathbb{C})$;
  - $\mathfrak{su}(p, \infty) = \varinjlim \mathfrak{su}(p, n)$, the complex special unitary Lie algebra of real rank p; or
  - $\mathfrak{su}(\infty, \infty) = \varinjlim \mathfrak{su}(p, q)$, complex special unitary algebra of infinite real rank.

Real Classical Algebras (2)

- If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(\infty; \mathbb{C})$, then $\mathfrak{g}$ is one of
  - $\mathfrak{so}(p, \infty) = \varinjlim \mathfrak{so}(p, n)$, the real orthogonal Lie algebra of finite real rank p;
  - $\mathfrak{so}(\infty, \infty) = \varinjlim \mathfrak{so}(p, q)$, the real orthogonal Lie algebra of infinite real rank; or
  - $\mathfrak{so}^*(2\infty) = \varinjlim \mathfrak{so}^*(2n)$
- If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}(\infty; \mathbb{C})$, then $\mathfrak{g}$ is one of
  - $\mathfrak{sp}(\infty; \mathbb{R}) = \varinjlim \mathfrak{sp}(n; \mathbb{R})$, the real symplectic Lie algebra;
  - $\mathfrak{sp}(p, \infty) = \varinjlim \mathfrak{sp}(p, n)$, the quaternionic unitary Lie algebra of real rank p; or
  - $\mathfrak{sp}(\infty, \infty) = \varinjlim \mathfrak{sp}(p, q)$, quaternionic unitary Lie algebra of infinite real rank.

Real Classical Algebras (3)

- If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(\infty; \mathbb{C})$, then $\mathfrak{g}$ is one of
  - $\mathfrak{gl}(\infty; \mathbb{R}) = \varinjlim \mathfrak{gl}(n; \mathbb{R})$, the real general linear Lie algebra,
  - $\mathfrak{gl}(\infty; \mathbb{H}) = \varinjlim \mathfrak{gl}(n; \mathbb{H})$, the quaternionic general linear Lie algebra;
  - $\mathfrak{u}(p, \infty) = \varinjlim \mathfrak{u}(p, n)$, the complex unitary Lie algebra of finite real rank p; or
  - $\mathfrak{u}(\infty, \infty) = \varinjlim \mathfrak{u}(p, q)$, the complex unitary Lie algebra of infinite real rank.

Some Linear Algebra

- Let $\mathfrak{g}_{\mathbb{C}}$ be one of $\mathfrak{gl}(\infty, \mathbb{C})$, $\mathfrak{sl}(\infty, \mathbb{C})$, $\mathfrak{so}(\infty, \mathbb{C})$, and $\mathfrak{sp}(\infty, \mathbb{C})$.
- For our purposes they should be described as follows
- $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$ are nondegenerately paired countable dimensional complex vector spaces
- $\mathfrak{gl}(\infty, \mathbb{C}) = \mathfrak{gl}(V_{\mathbb{C}}, W_{\mathbb{C}}) := V_{\mathbb{C}} \otimes W_{\mathbb{C}}$ consists of all finite linear combinations of the $v \otimes w : x \mapsto \langle w, x \rangle v$
- $\mathfrak{sl}(\infty, \mathbb{C}) = \mathfrak{sl}(V_{\mathbb{C}}, W_{\mathbb{C}})$ is the traceless part of $\mathfrak{gl}(\infty, \mathbb{C})$
- Then $\mathfrak{so}(\infty, \mathbb{C}) = \Lambda\mathfrak{gl}(V_{\mathbb{C}}, V_{\mathbb{C}})$ is the image of $\Lambda : v \otimes w \mapsto v \otimes w - w \otimes v$
- $\mathfrak{sp}(V_{\mathbb{C}}, V_{\mathbb{C}}) = S\mathfrak{gl}(V_{\mathbb{C}}, V_{\mathbb{C}})$ is the image of $S : v \otimes w \mapsto v \otimes w + w \otimes v$

Some Definitions

- A *Borel subalgebra* of $\mathfrak{g}_{\mathbb{C}}$ is a maximal locally solvable subalgebra
- A *parabolic subalgebra* of $\mathfrak{g}_{\mathbb{C}}$ is a subalgebra that contains a Borel
- A *(semiclosed) generalized flag* $\mathcal{F} = \{F_i\}_{i \in I}$ is an increasing family of subspaces, $F_i \subset F_j$ for $i \leqq j$, where
  - every $F \in \mathcal{F}$ belongs to an *immediate predecessor–successor pair* (IPS) $\{F', F''\}$ and
  - if $F \in \mathcal{F}$ with $F \neq F^{\perp\perp}$ then $\{F, F^{\perp\perp}\}$ is an IPS pair
- Generalized flags $\mathcal{F}$ in V and $\mathcal{G}$ in W form a *taut couple* when
  - if $F \in \mathcal{F}$ then $F^{\perp}$ is invariant by the $\mathfrak{gl}$–stabilizer of $\mathcal{G}$ and
  - if $G \in \mathcal{G}$ then $G^{\perp}$ is invariant by the $\mathfrak{gl}$–stabilizer of $\mathcal{F}$

Complex Parabolic Subalgebras

- In the $\mathfrak{so}$ and $\mathfrak{sp}$ cases one can take $V_\mathbb{C} = W_\mathbb{C}$ and $\mathcal{F} = \mathcal{G}$, and the subspaces should be isotropic or co-isotropic.
- then we speak of a generalized flag $\mathcal{F}$ in $V_\mathbb{C}$ as *self–taut*.
- THEOREM The self–normalizing parabolics in $\mathfrak{sl}(V_\mathbb{C}, W_\mathbb{C})$ and $\mathfrak{gl}(V_\mathbb{C}, W_\mathbb{C})$ are the normalizers of taut couples of generalized flags in $V_\mathbb{C}$ and $W_\mathbb{C}$. The self–normalizing parabolics in $\mathfrak{so}(V_\mathbb{C})$ and $\mathfrak{sp}(V_\mathbb{C})$ are the normalizers of self–taut generalized flags in $V_\mathbb{C}$.
- THEOREM The parabolics $\mathfrak{p}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$ are obtained from self normalizing parabolics $\widetilde{\mathfrak{p}_\mathbb{C}}$ by imposing linear comb-inations of trace conditions on $\mathfrak{gl}(\infty;\mathbb{C})$-quotients of $\widetilde{\mathfrak{p}_\mathbb{C}}$.
- CAVEAT: $\mathfrak{sl}(\infty;\mathbb{C})$ contains a Borel subalgebra of $\mathfrak{gl}(\infty;\mathbb{C})$, so $\mathfrak{sl}(\infty;\mathbb{C})$ is parabolic in $\mathfrak{gl}(\infty;\mathbb{C})$. See next slide.

Two examples

- Here are two examples showing that complex parabolics can be very different from the finite dimensional case
- Enumerate a basis of $\mathbb{C}^\infty$ by $(\mathbb{Z}^+)^n$ (or even $(\mathbb{Z}^+)^\infty$) in lexicographic order. The corresponding flag has subspaces with no immediate predecessor, and constructions involve limit ordinals.
- Enumerate a bases of $V_\mathbb{C} = \mathbb{C}^\infty$ and $W_\mathbb{C} = \mathbb{C}^\infty$ by rational numbers with pairing
$$\langle v_q, w_r \rangle = 1 \text{ if } q > r, \quad = 0 \text{ if } q \leqq r$$
Then Span$\{v_q \otimes w_r \mid q \leqq r\}$ is a Borel in $\mathfrak{gl}(\infty; \mathbb{C})$ contained in $\mathfrak{sl}(\infty; \mathbb{C})$. This shows that $\mathfrak{sl}(\infty; \mathbb{C})$ is parabolic in $\mathfrak{gl}(\infty; \mathbb{C})$.

Real Parabolic Subalgebras

- $\mathfrak{sl}$ and $\mathfrak{gl}$ cases: $\mathfrak{g}$ has inequivalent defining real representations V and W
- $\mathfrak{so}$ and $\mathfrak{sp}$ cases: $\mathfrak{g}$ has one defining representation V
- $\mathbb{D}$: algebra of all $\mathfrak{g}$-endomorphisms of V (or those of trace 0): then $\mathfrak{g}$ specified by a zero or nondegenerate $\mathbb{D}$–bilinear or $\mathbb{D}$–sesquilinear form ω on V.
- A subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is *parabolic* if its complexification $\mathfrak{p}_{\mathbb{C}}$ is parabolic in $\mathfrak{g}_{\mathbb{C}}$.
- Then $\mathfrak{p}$ is defined by infinite trace conditions on the $\mathfrak{g}$ stabilizer of a
  - $\mathfrak{sl}$ and $\mathfrak{gl}$ cases: taut couple of $\mathbb{D}$–generalized flags $\mathcal{F}$ in V and $\mathcal{G}$ in W
  - $\mathfrak{so}$ and $\mathfrak{sp}$ cases: self–taut $\mathbb{D}$–generalized flag $\mathcal{F}$ in V

Levi Components (1)

- Let $\mathfrak{p}$ be a locally finite Lie algebra, $\mathfrak{r}$ its locally solvable radical. A subalgebra $\mathfrak{l} \subset \mathfrak{p}$ is a *Levi component* if $[\mathfrak{p}, \mathfrak{p}] = (\mathfrak{r} \cap [\mathfrak{p}, \mathfrak{p}]) \inplus \mathfrak{l}$ semidirect sum.
- Every finitary Lie algebra has a Levi component
- Levi components are maximal locally semisimple subalgebras, but the converse fails
- If $X \subset V$ and $Y \subset W$ are nondegenerately paired, isotropic in the $\mathfrak{so}$ and $\mathfrak{sp}$ cases, then $\mathfrak{gl}(X, Y) \subset \mathfrak{gl}(V, W)$, $\mathfrak{sl}(X, Y) \subset \mathfrak{sl}(V, W)$, $\Lambda\mathfrak{gl}(X, Y) \subset \Lambda\mathfrak{gl}(V, V)$ and $S\mathfrak{gl}(X, Y) \subset S\mathfrak{gl}(V, V)$ are called *standard*.
- $\mathfrak{l}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ is Levi in a parabolic $\mathfrak{p}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ if and only if it is the direct sum of standard special linear subalgebras and at most one subalgebra $\Lambda\mathfrak{gl}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ in the orthogonal case, $S\mathfrak{gl}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ in the symplectic case

Levi Components (2)

- $X = \bigoplus X_i$ and $Y \bigoplus Y_i$, sums of the corresponding subspaces of V and W for $\mathfrak{l}_i$. Then
  - X and Y are nondegenerately paired,
  - $V = X \oplus Y^\perp$ and $W = Y \oplus X^\perp$ and
  - $X^\perp$ and $Y^\perp$ are nondegenerately paired
- When $\mathfrak{g}$ is defined by a bilinear or hermitian form f, identifying V and W,
  - these become $V = (X \oplus Y) \oplus (X \oplus Y)^\perp$
  - f is nondegenerate on $(X \oplus Y)^\perp$.
  - Let X' and Y' be paired maximal isotropic subspaces of $(X \oplus Y)^\perp$ and $Z' := (X' \oplus Y')^\perp \cap (X \oplus Y)^\perp$. Then $V = (X \oplus Y) \oplus (X' \oplus Y') \oplus Z'$.

Minimal Levi Components

- If $\mathfrak{l}_1 \subsetneqq \mathfrak{l}_2$ one constructs $\mathfrak{p}_1 \subsetneqq \mathfrak{p}_2$
- From now on, $\mathfrak{l}$ is a Levi component of a minimal real parabolic $\mathfrak{p} \subset \mathfrak{g}$. Then $\mathfrak{l} = \bigoplus_{i \in I} \mathfrak{l}_i$ where each $\mathfrak{l}_i$ is
  - $\mathfrak{su}(p)$, $\mathfrak{so}(p)$ or $\mathfrak{sp}(p)$ for a compact group; or
  - $\mathfrak{su}(\infty)$, $\mathfrak{so}(\infty)$ or $\mathfrak{sp}(\infty)$ for a lim–compact group.
- $\mathfrak{a}$: max $\mathbb{R}$–split toral subalg $\bigoplus \mathfrak{gl}(X'_j, Y'_j)$ annihilating $(X \oplus Y \oplus Z')$ where $\{x'_j\}$ basis of X', $\{y'_j\}$ dual basis of Y'
- $\mathfrak{m} = \widetilde{\mathfrak{l}} + \mathfrak{t}'$ where
  - $\widetilde{\mathfrak{l}_i} = \mathfrak{u}(*)$ if $\mathfrak{l}_i = \mathfrak{su}(*)$, else $\widetilde{\mathfrak{l}_i} = \mathfrak{l}_i$
  - $\mathfrak{t}'$: max imag toral in $\mathrm{Cent}_{\mathfrak{g}}((X \oplus Y) \oplus (X' \oplus Y'))$
- $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ and $P = MAN$ where $M = P \cap K$, $A = \exp(\mathfrak{a})$, and $N = \exp(\mathfrak{n})$

Closed Flags

- Semiclosed generalized flag $\mathcal{F}$ is *closed* if $F''_\alpha = (F''_\alpha)^{\perp\perp}$ for all IPS pairs (F'_α, F''_α) in $\mathcal{F}$. A parabolic defined by a closed generalized flag is *flag-closed*
- If $P = MAN$ is a flag-closed minimal parabolic and K is a maximal lim-compact subgroup of G then $\mathfrak{p} = \mathfrak{n}^\perp$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, and $G = KP$, i.e. K is transitive on G/P.

Amenability

- A topological group J (not necess. locally compact) is *amenable* if there is a *right-invariant mean* $\mu : LUC_b(J) \to \mathbb{C}$ where
  - $RUC_b(J)$: right uniformly cont. bounded functions on J
  - μ is linear, $\mu(\mathbf{1}) = 1$, and $f \geqq 0 \Rightarrow \mu(f) \geqq 0$
- minimal parabolic subgroups and maximal lim-compact subgroups of G are amenable
- If $P = MAN$ is a minimal parabolic subgroup then
  $$\mathcal{M} = \mathcal{M}(G, P)\text{: right } P\text{-invariant means on } G$$
  is non-empty. Similarly $\mathcal{M}(K, M) \neq \emptyset$.

Induction

- fix $P = MAN$: flag-closed parabolic subgroup of G
- fix τ: unitary representation of P that annihilates $\mathfrak{n}$
- $\mathbb{E}_\tau \to G/P$: associated homog. hermitian vector bundle
- $RUC_b(G/P; \mathbb{E}_\tau)$ bounded, right uniformly cont. sections
- $\mu \in \mathcal{M}$ gives seminorm $\nu_\mu(f) = \mu(||f||)$ on $RUC_b(G/P; \mathbb{E}_\tau)$
- $J(G/P; \mathbb{E}_\tau) = \{f \in RUC_b(G/P; \mathbb{E}_\tau) \mid every \nu_\mu(f) = 0\}$
- $\mathrm{Ind}\,_P^G(\tau)$: representation of G on the completion of
  $RUC_b(G/P; \mathbb{E}_\tau)/J(G/P; \mathbb{E}_\tau)$ relative to $\{\nu_\mu \mid \mu \in \mathcal{M}\}$
- $\mathrm{Ind}\,(G/P; \mathbb{E}_\tau)|_K = \mathrm{Ind}\,(K/M; \mathbb{E}_{\tau|_M})$
- Open questions: When is $\mathrm{Ind}\,(G/P; \mathbb{E}_\tau)$ factorial? unitary? Fréchet? If τ if a finite factor rep of P does $\mathrm{Ind}\,(G/P; \mathbb{E}_\tau)$ have a character? A K–character?

Tensor Representations of $U(\infty)$

- The easiest "appropriate class" of representations of M is the one met for compact factors
- $U(p)$, $Spin(p)$ or $Sp(p)$: classical, highest weight.
- In the case of $U(p)$ look at action of the symmetric group $S(p)$ on $\otimes^n(\mathbb{C}^p)$, action of $U(p)$ on tensors picked out by an irreducible summand of that action of $S(p)$.
- Kirillov and others: an analog of this for $U(\infty)$
- However this is a small class of the continuous unitary representations of $U(\infty)$. Many such don't even extend to the class of unitary operators of the form $1+$ (compact), so one can consider more general factor representations.

Type II_1 Representations of $U(\infty)$

- π: continuous unitary finite factor representation of $U(\infty)$
- character $\chi_\pi(x) = \mathrm{trace}\ \pi(x)$ (normalized trace)
- Voiculescu: parameter space is all bilateral sequences $\{c_n\}_{-\infty<n<\infty}$ such that
  (i) $\det((c_{m_i+j-i})_{1\leqq i,j\leqq N} \geqq 0$ for $m_i \in \mathbb{Z}$ and $N \geqq 0$ and
  (ii) $\sum c_n = 1$
- then the character corresponding to $\{c_n\}$ and π is
  $$\chi_\pi(x) = \prod_i p(z_i)$$
  where $\{z_i\}$ eigenvalues of x and $p(z) = \sum c_n z^n$
- Here π extends to the group of all unitary operators on the Hilbert completion of $\mathbb{C}^\infty$, such that x with $x - 1$ of trace class

Other Factor Representations of $U(\infty)$

- $\mathcal{H} = \varinjlim \mathcal{H}_n$ Hilbert space, τ bounded operator, $0 \leqq \tau \leqq 1$
- $\psi_\tau : U(\infty) \to \mathbb{C}$, $\psi_\tau(x) = \det((1-\tau) + \tau x)$, is a continuous function of positive type on $U(\infty)$
- the associated cyclic representation π_τ is
- irreducible $\Leftrightarrow \tau$ is a projection,
- type I $\Leftrightarrow \tau(1-\tau)$ is trace class,
- if $\tau(1-\tau)$ not trace class then π_τ is factorial of type
  - $II_1 \Leftrightarrow \tau - p1 \in HS$ for some $0 < p < 1$,
  - $II_\infty \Leftrightarrow (i) \tau(1-\tau)(\tau - p1)^2 \notin HS$ for some $0 < p < 1$ and (ii) ess spec(τ) meets $\{0, 1\}$,
  - $III \Leftrightarrow \tau(1-\tau)(\tau - p1)^2 \notin HS$ for all $0 < p < 1$
- (results of Stratila and Voiculescu)

Happy $2^2 \cdot 3 \cdot 5$ MikE !!