CR–tractors and the Fefferman Space

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ABSTRACT. We develop the natural tractor calculi associated to conformal and CR structures as a fundamental tool for the study of Fefferman's construction of a canonical conformal class on the total space of a circle bundle over a non-degenerate CR manifold of hypersurface type. In particular, we construct and treat the basic objects that relate the natural bundles and natural operators on the two spaces. This is illustrated with several applications: We prove that a number of conformally invariant overdetermined systems, including Killing form equations and the equations for twistor spinors, admit non-trivial solutions on any Fefferman space. We show that, on a Fefferman space, the space of infinitesimal conformal isometries naturally decomposes into a direct sum of subspaces, which admit an interpretation as solutions of certain CR invariant PDE's. Finally we explicitly analyse the relation between tractor calculus on a CR manifold and the complexified conformal tractor calculus on its Fefferman space, thus obtaining a powerful computational tool for working with the Fefferman construction.

1. INTRODUCTION

Given a non-degenerate CR manifold M of hypersurface type, the Fefferman space $\tilde{M}$, associated to M, is the total space of a circle bundle over M, on which the given CR structure induces a natural (indefinite) conformal structure. Fefferman's original construction in [16] is applied to smooth boundaries of strictly pseudoconvex domains and it was a vision of Fefferman to exploit this relationship to study the invariant theory and geometry of these objects in terms of conformal geometry. Despite the subsequent and increasing interest in these, and related issues of CR geometry, this remarkable construction has not been fully explored. The explanation surely lies in the complicated relationship between natural bundles and operators on the two structures. The main aim of this work is to present a treatment of the Fefferman space that provides both a conceptual and calculationally practical solution to this problem at the level of the basic foundational issues. We illustrate the power of the approach by recovering very quickly a string of applications that we discuss below.

The Fefferman construction was generalised to abstract CR manifolds in [5] using the canonical Chern-Moser Cartan connection. A main application of this and [16] was that chains in M can be interpreted as the projections of null geodesics in $\tilde{M}$. This crucially depends on the fact that the canonical Cartan connection of the conformal structure on $\tilde{M}$ is closely related to the CR Cartan connection on M. In both [5] and [16] the necessary verification of this relationship was evidently very complicated, with many details of the calculation sketched or omitted.

A different approach to the Fefferman construction was developed by J. Lee in [30]. For a choice of a pseudo-Hermitian structure (i.e., a contact form) on M, Lee used the associated Webster-Tanaka connection, to directly define a metric on the Fefferman space $\tilde{M}$. Then he showed that changing the contact form only leads to a conformal rescaling of that metric. A feature of this approach is that it directly leads to explicit formulae. On the other hand, the relation between the Cartan connections is not directly visible in this picture. Using Lee's approach, it was realised that Fefferman spaces provide particularly interesting examples of conformal structures. For example, it was shown in [34] for CR-dimension one, and in [2] in general, that (slightly modified) Fefferman spaces always admit nontrivial twistor spinors.

Conformal and CR geometry have been very active areas of research recently. Questions related to invariant operators with symbol a power of the Laplacian, respectively the sub-Laplacian, as well as to Branson's Q-curvature have received particular interest. These studies are driven not only by differential geometry but also through the role of these objects in geometric analysis. It is not surprising then that there is renewed interest in Fefferman's construction as a bridge between CR geometry and conformal geometry. In [17], via the Fefferman metric, a CR Q-curvature was defined and studied, while in [24] the Fefferman metric was used as one of the two main construction techniques for the CR invariant powers of the sub-Laplacian. In these treatments the use of the Fefferman structure is relatively straightforward since the central objects involved are density bundles and operators between them. To develop analogous results involving, for example, tensor or spinor bundles requires a significantly deeper understanding of the Fefferman space and its precise geometric relationship to the underlying CR structure. The problem is that the relation between such bundles on a Fefferman space and irreducible bundles on the underlying CR manifold is very complicated in general.

We solve this problem by introducing tractor bundles and tractor connections as a new tool in the study of the Fefferman construction; tensor and spinor bundles are captured as subquotients in these. The tractor bundles are natural vector bundles associated to a CR strucure or a conformal structure (and more generally to a so-called parabolic geometry) which are endowed with canonical invariant linear connections. They are equivalent to the Cartan princpal bundle and the canonical Cartan connection, see [9], but are easier to handle. We show that the relation between conformal tractor bundles on a Fefferman space $\tilde{M}$ and CR tractor bundles on the underlying CR manifold M is rather simple. For example, CR standard tractors on M may be identified with conformal standard tractors on $\tilde{M}$ which are parallel for the canonical tractor connection in the direction of the fibers of $\tilde{M} \to M$. This is simpler than the relation between the canonical Cartan geometries, which involves extension of the structure group and equivariant extension of the Cartan connection. Irreducbile bundles then naturally occur as subquotients of tractor bundles, so we obtain a vehicle for effectively carrying the conceputal and calculational details of the relationship.

To have tractors at our disposal, in addition to the CR structure on M, we have to choose a certain root of the canonical bundle, compare with [24]. This is no restriction locally for arbitrary structures, and is no restriction globally in the embedded case. This slightly richer structure leads to an immediate payoff: we automatically get a canonical spin structure on the Fefferman space, and we exactly recover Fefferman's original construction (which is an $(n + 2)$-fold covering of the one in [5]) for embedded structures. Moreover, using Tanaka's version of the canonical Cartan connection, the construction automatically extends to the class of partially integrable almost CR structures, which is much larger than the integrable ones. While a canonical conformal structure is obtained without the integrability assumption, the close relation between the canonical Cartan connections surprisingly is true only in the integrable case. We prove this in the language of tractor connections in Theorem 2.3.

Having this at hand, we apply the powerful tools available for parabolic geometries to study the relation between a CR manifold and its Fefferman space as well as the conformal geometry of Fefferman spaces in Section 3. We obtain a short and conceptual proof of the existence of non-trivial twistor spinors, new results on the existence and construction of odd degree conformal Killing forms, as well as a natural decomposition of conformal Killing vector fields (i.e., infinitesimal conformal isometries). A crucial ingredient in all this is that on the conformal standard tractor bundle of a Fefferman space one obtains a parallel, orthogonal complex structure. In a companion article [11] to this one we show that Fefferman spaces are characterised (up to local conformal isometry) by the existence of such a complex structure, which can be viewed as a restriction on the so-called conformal holonomy group. This also leads to a new proof and extension of Sparling's characterisation of Fefferman spaces from [27].

In Section 4, the abstract developments from preceding sections are converted into an explicit calculus. We first show that the tractor calculus developed in [24] recovers precisely the complexified normal CR standard tractor bundle. This result should be of independent interest. Next, we study the consequences of the existence of a parallel, orthogonal complex structure in terms of conformal tractor calculus. In the case of a Fefferman space $\tilde{M} \to M$, we obtain explicit relations between CR tractor calculus on M and conformal tractor calculus on $\tilde{M}$. Using this, we completely describe how a contact form/pseudo-Hermitian structure on the CR manifold may be related to an equivalent structure on the Fefferman space. We illustrate the utility of this by converting the developments of Section 3 into explicit formulae. Moreover, we explicitly compute the metric in the conformal class associated to a choice of CR scale, thus tying in with Lee's approach. We also obtain explicit relations between Webster-Tanaka connections on M and Levi-Civita connections on $\tilde{M}$. As an application, we discuss a tractor interpretation of Einstein-type structures (cf. [33]) in CR geometry.

2. The Fefferman Space

2.1. CR manifolds. An *almost CR structure* of hypersurface type on a smooth manifold M of dimension $2n+1$ is a rank n complex subbundle H of the tangent bundle TM. We denote by $J : H \to H$ the complex structure on the subbundle. The quotient $Q := TM/H$ is a real line bundle on M. Let $q : TM \to Q$ be the obvious surjection. For two sections ξ, $\eta \in \Gamma(H)$ the expression $q([\xi, \eta])$ is bilinear over smooth functions, and so we obtain a skew symmetric bundle map $\mathcal{L} : H \times H \to Q$ given by $\mathcal{L}(\xi(x), \eta(x)) = q([\xi, \eta](x))$. The almost CR structure is called *non-degenerate* if $\mathcal{L}(\xi, \eta) = 0$ for all η implies $\xi = 0$. This is equivalent to the fact that H, viewed as a real subbundle of TM, defines a contact structure on M.

Looking at the complexified tangent bundle $T_{\mathbb{C}}M$, the complex structure on H is equivalent to a splitting of the subbundle $H_{\mathbb{C}}$ into the direct sum of the holomorphic part $H^{1,0}$ and the antiholomorphic part $H^{0,1} = \overline{H^{1,0}}$. The almost CR structure is called *integrable* or a *CR structure* if the subbundle $H^{1,0} \subset T_{\mathbb{C}}M$ is involutive, i.e., the space of its sections is closed under the Lie bracket. A weakening of this condition, called *partial integrability*, is obtained by requiring that the bracket of two sections of $H^{1,0}$ is a section of $H_{\mathbb{C}}$. This is equivalent to $\mathcal{L}$ being of type $(1,1)$, that is $\mathcal{L}(J\xi, J\eta) = \mathcal{L}(\xi, \eta)$ for all ξ, $\eta \in H$. Throughout the paper, we will only deal with non-degenerate partially integrable CR structures, and unless explicitly specified, we will assume integrability.

Let $Q_{\mathbb{C}}$ be the complexification of Q and let $q_{\mathbb{C}} : T_{\mathbb{C}}M \to Q_{\mathbb{C}}$ be the complex linear extension of q. The *Levi form* $\mathcal{L}_{\mathbb{C}}$ of an almost CR structure is the $Q_{\mathbb{C}}$-valued Hermitian form on $H^{1,0}$ induced by $(\xi, \eta) \mapsto 2iq_{\mathbb{C}}([\xi, \bar{\eta}])$. Assuming partial integrability, $\mathcal{L}$ can be naturally identified with the imaginary part of $\mathcal{L}_{\mathbb{C}}$, and so under this assumption, non-degeneracy of the almost CR structure also can be characterised by non-degeneracy of the Levi form.

Choosing a local trivialisation of Q and using the induced trivialisation of $Q_{\mathbb{C}}$, $\mathcal{L}_{\mathbb{C}}$ gives rise to a Hermitian form. If (p, q) is the signature of this form, then one also says that M is non-degenerate of signature (p, q). If $p \neq q$, then such local trivialisations of Q necessarily fit together to give a global trivialisation. In the case of symmetric signature (p, p) we assume that a global trivialisation of Q exists.

A global trivialisation of Q is equivalent to a ray subbundle of the line bundle of contact forms for $H \subset TM$, so we obtain a notion of positivity for contact forms.

Generic real hypersurfaces in complex manifolds give the prototypal examples of CR structures, and form an important class for many applications. Let $\mathcal{M}$ be a complex manifold of complex dimension $n + 1$ and let $M \subset \mathcal{M}$ be a smooth real hypersurface. For each point $x \in M$ the tangent space T_xM is a subspace of the complex vector space $T_x\mathcal{M}$ of real codimension one. This implies that the maximal complex subspace H_x of T_xM must be of complex dimension n. Of course, these subspaces fit together to define a smooth subbundle $H \subset TM$, which by construction is equipped with a complex structure. Since the bundle $H^{1,0} \subset T_{\mathbb{C}}M$ can be viewed as the intersection of the involutive subbundles $T_{\mathbb{C}}M$ and $T^{1,0}\mathcal{M}$ of $T_{\mathbb{C}}\mathcal{M}|_M$, we see that we always obtain a CR structure in this way. This structure is non-degenerate if H defines a contact structure on M, which is satisfied generically. Examples of this type are usually referred to as *embedded CR manifolds*, in particular for $\mathcal{M} = \mathbb{C}^{n+1}$.

2.2. The Fefferman space for the homogeneous model. For the case of the homogeneous model, the Fefferman construction can be easily described and this motivates the general construction. Fix a complex vector space $\mathbb{V}$ of dimension $n + 2$, endowed with a Hermitian inner product $\langle\ ,\ \rangle$ of signature $(p+1, q+1)$, where $p + q = n$. Let $C \subset \mathbb{V}$ be the cone of nonzero null vectors, and let M be the image of C in the complex projectivisation $\mathcal{P}\mathbb{V} \cong \mathbb{C}P^{n+1}$. Hence M is the space of those complex lines in $\mathbb{V}$ which are null with respect to $\langle\ ,\ \rangle$. Note that M is a smooth real hypersurface in $\mathcal{P}\mathbb{V}$. The resulting CR structure on M is easily described explicitly: Given a null line $\ell \subset \mathbb{V}$, the CR subspace $H_\ell M \subset T_\ell M$ is the image of the complex orthogonal complement $\ell^\perp$ of ℓ under the tangent map of the obvious projection $C \to M$. Hence we obtain an isomorphism $\ell^\perp/\ell \to H_\ell M$. It is easy to see that under this isomorphism the Levi form, at ℓ, corresponds (up to a nonzero multiple) to the Hermitian form on $\ell^\perp/\ell$ induced by $\langle\ ,\ \rangle$ in the obvious way. In particular, M is non-degenerate of signature (p, q).

Let G be the special unitary group of $(\mathbb{V}, \langle\ ,\ \rangle)$. The standard linear action of G on $\mathbb{V}$ restricts to an action on C and then descends to a smooth left action of G on M. Since the CR structure on M is completely determined by the Hermitian form $\langle\ ,\ \rangle$ it is evident the G acts on M by CR automorphisms. Elementary linear algebra shows that G acts transitively on C and thus also on M. Fixing a null line $\ell \subset V$ and denoting by $P \subset G$ the stabiliser of ℓ, we obtain an identification of M with the homogeneous space G/P.

Note that the action of G on M is not effective. The kernel of the action is the centre $Z(G)$ which consists of those multiples of the identity which lie in G. Hence the possible factors are the $(n + 2)$nd roots of unity, and $Z(G) \cong \mathbb{Z}_{n+2}$. It is a classical result that the action on M induces an isomorphism between $G/Z(G)$ and the group of CR automorphisms of M. So $M = (G/Z(G))/(P/Z(G))$ as a homogeneous space for its group of CR automorphisms.

To keep track of the full group G we incorporate the principal $\mathbb{C}^*$-bundle $C \to M$ as a part of the structure, since the lift of the G-action to this bundle separates the points in the centre. We shall work with this richer structure.

The Fefferman space $\tilde{M}$ arises naturally from the underlying real picture. Let $\mathcal{P}_{\mathbb{R}}\mathbb{V}$ be the real projectivisation of $\mathbb{V}$ and let $\tilde{M}$ be the image of C in this real projective space. That is, $\tilde{M}$ is the space of all real lines in $\mathbb{V}$ which are null for the inner product $\langle\ ,\ \rangle_{\mathbb{R}}$, the real part of $\langle\ ,\ \rangle$. The space $\tilde{M}$ is a smooth hypersurface in $\mathbb{R}P^{2n+3}$, and we have an obvious projection $C \to \tilde{M}$, which is a principal bundle with fibre group $\mathbb{R}^*$.

Any real null line generates a complex null line containing it. This gives rise to a smooth projection $\pi : \tilde{M} \to M$, which is a fiber bundle over M, with fibre the space $\mathbb{R}P^1 \cong S^1$ of real lines in $\mathbb{C}$.

Fixing an element $v \in C$ and denoting by $\tilde{\ell}$ the real line $\mathbb{R}v$, the tangent map in v of the projection $C \to \tilde{M}$ identifies $T_{\tilde{\ell}}\tilde{M}$ with the quotient of the real orthocomplement of $\tilde{\ell}$ by $\tilde{\ell}$. The inner product $\langle\ ,\ \rangle_{\mathbb{R}}$ thus induces an inner product on $T_{\ell}\tilde{M}$ and changing the point v leads to a positive rescaling of this product. Hence we obtain a well-defined conformal structure on $\tilde{M}$.

Let $\tilde{G}$ be the connected component of the identity of the orthogonal group of $(\mathbb{V}, \langle\ ,\ \rangle_{\mathbb{R}})$, and let $\tilde{P} \subset \tilde{G}$ be the stabiliser of a real null line. Then as above we obtain a transitive action of $\tilde{G}$ on $\tilde{M}$ which leads to an identification $\tilde{M} \cong \tilde{G}/\tilde{P}$. By construction, $\tilde{G}$ acts by conformal isometries on $\tilde{M}$. It is a classical result, this action identifies $\tilde{G}/Z(\tilde{G})$ with the group of conformal isometries of $\tilde{M}$.

We have noted above that the subgroup $G \subset \tilde{G}$ acts transitively on C, so it also acts transitively on $\tilde{M}$. Taking a real null line and the complex null line generated by it as the base points of $\tilde{M}$ and M, we see that $G \cap \tilde{P} \subset P$, and $G \cap \tilde{P}$ is the stabiliser of a real null line, so $\tilde{M} \cong G/(G \cap \tilde{P})$.

The inclusion $G \hookrightarrow \tilde{G}$ may be viewed as inducing a reduction of structure group of the bundle $\tilde{G} \to \tilde{M}$ from $\tilde{P}$ to $G \cap \tilde{P} \subset P$. This reduction is determined by the complex structure on $\mathbb{V}$, which is orthogonal for the inner product $\langle\ ,\ \rangle_{\mathbb{R}}$. Equivalently, we can view this complex structure as a splitting $\mathbb{V} \otimes \mathbb{C} = \mathbb{V}^{1,0} \oplus \mathbb{V}^{0,1}$, of the complexification of $\mathbb{V}$, into a holomorphic and an anti-holomorphic part. The inner product $\langle , \rangle_{\mathbb{R}}$ induces a non-degenerate complex bilinear form on $\mathbb{V} \otimes \mathbb{C}$. Since the complex structure is orthogonal for $\langle\ ,\ \rangle_{\mathbb{R}}$, the subspaces $\mathbb{V}^{1,0}$ and $\mathbb{V}^{0,1}$ are isotropic for this complex inner product, while at the same time the induced pairing between the two spaces exactly comes from viewing $\langle\ ,\ \rangle$ as a complex bilinear map $\mathbb{V} \times \bar{\mathbb{V}} \to \mathbb{C}$.

2.3. The canonical Cartan connection. For an arbitrary non-degenerate CR manifold (M, H), the canonical Cartan connection gives a description which is appropriate for generalising the above construction for the homogeneous space $(G/Z(G))/(P/Z(G))$. Various constructions of the canonical Cartan bundle and Cartan connection can be found in [12, 15, 28, 36, 40]. The outcome may be described as follows: First, one builds a principal fibre bundle $p : \underline{\mathcal{G}} \to M$ with structure group $P/Z(G)$; this may be obtained as an extension of an adapted frame bundle of $H \to M$ or as a subbundle of the (co)frame bundle of the total space of the line bundle $Q \to M$. The principal bundle can be endowed with a *Cartan connection* $\underline{\omega} \in \Omega^1(\underline{\mathcal{G}}, \mathfrak{g})$, where $\mathfrak{g}$ denotes the Lie algebra of G. This generalises the trivialisation of the tangent bundle of $G/Z(G)$ by left translations.

Explicitly, we require that $\underline{\omega}$ defines a trivialisation of $T\underline{\mathcal{G}}$, which is $P/Z(G)$-equivariant and reproduces the generators of fundamental vector fields. If one requires the curvature of $\underline{\omega}$ to satisfy a normalisation condition, which will be discussed in detail below, then the pair $(\underline{\mathcal{G}}, \underline{\omega})$ is uniquely determined up to isomorphism.

For later purposes, it will be very important to extend the structure group from $P/Z(G)$ to P. In the case of the homogeneous model, the way to expose the centre is via its action on the restriction of the tautological bundle to the hyperquadric. To generalise this to arbitrary CR manifolds, one proceeds as follows: The natural choice of a complex line bundle on a CR manifold is provided by the canonical bundle $\mathcal{K}$. By definition, $\mathcal{K}$ is the $(n+1)$st complex exterior power of the annihilator of $H^{0,1}$ in the complexified cotangent bundle. In the case of the homogeneous model one shows that $\mathcal{K}$ is associated to the $\mathbb{C}^*$-bundle $C \to G/P$ with respect to the representation $z \mapsto z^{-n-2}$, so the null cone may be naturally viewed as the dual of a $(n+2)$nd root of the canonical bundle.

When dealing with a general CR manifold M, we will always assume that we have chosen a complex line bundle $\mathcal{E}(1,0) \to M$ together with a duality between $\mathcal{E}(1,0)^{\otimes(n+2)}$ and the canonical bundle $\mathcal{K}$. In general, such a choice may not exist globally but locally it poses no problem. Moreover, for CR manifolds embedded in $\mathbb{C}^{n+1}$ the canonical bundle is trivial, so the required identification exists globally in this setting. For $w, w' \in \mathbb{R}$ such that $w' - w \in \mathbb{Z}$, the map $\lambda \mapsto |\lambda|^{2w}\bar{\lambda}^{(w'-w)}$ is a well defined one-dimensional representation of $\mathbb{C}^*$. Hence we can define a complex line bundle $\mathcal{E}(w, w')$ over M by forming the associated bundle to the frame bundle of $\mathcal{E}(1,0)$ with respect to this representation. By construction we get $\mathcal{E}(w', w) = \overline{\mathcal{E}(w, w')}$, $\mathcal{E}(-w, -w') = \mathcal{E}(w, w')^*$ and $\mathcal{E}(k, 0) = \mathcal{E}(1,0)^{\otimes k}$ for $k \in \mathbb{N}$. Finally, by definition $\mathcal{K} \cong \mathcal{E}(0, -n-2)$ in this notation.

There is a natural inclusion of the real line bundle $Q := TM/H$ into the density bundle $\mathcal{E}(1,1)$ which is defined as follows. For a local nonzero section α of $\mathcal{E}(1,0)$ one can, by definition, view $\alpha^{-(n+2)}$ as a section of the canonical bundle $\mathcal{K}$. Then by [30, Lemma 3.2], there is a unique positive contact form θ, with respect to which $\alpha^{-(n+2)}$ is length normalised. Mapping $\xi \in TM$ to $\theta(\xi)\alpha\bar{\alpha}$ then descends to an inclusion, which is CR invariant.

Recall that the Cartan connection on $\underline{\mathcal{G}} \to M$ identifies TM with a bundle associated to the Cartan bundle $\underline{\mathcal{G}} \to M$, see e.g. [9, 2.8]. Thus, also the dual $\mathcal{K}^*$ of $\mathcal{K}$ is associated to $\underline{\mathcal{G}}$, so its frame bundle is a quotient of $\underline{\mathcal{G}}$. By the definition of $\mathcal{E}(1,0)$, its frame bundle is an $(n+2)$-fold covering of the frame bundle of $\mathcal{K}^*$. Pulling back the covering to $\underline{\mathcal{G}}$, we obtain an $(n+2)$-fold covering $\mathcal{G}$ of

$\underline{\mathcal{G}}$. The principal action of $P/Z(G)$ on $\underline{\mathcal{G}}$ lifts to a right action of P on $\mathcal{G}$. This can be used to make the bundle $\mathcal{G} \to M$ into a principal P-bundle in such a way that $\mathcal{E}(1,0) = \mathcal{G} \times_P (\mathbb{V}^1)^*$, where $\mathbb{V}^1 \subset \mathbb{V}$ is the null line stabilised by P. A normal Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ on $\mathcal{G}$ is obtained by pulling back $\underline{\omega}$. As with the structure $(\underline{\mathcal{G}}, \underline{\omega})$, the pair $(\mathcal{G}, \omega)$ is determined by $(M, H, \mathcal{E}(1,0))$ up to isomorphism. We will show in Section 4.3 below that the Cartan structure $(\mathcal{G}, \omega)$ may be recovered from the tractor bundle introduced in [24].

2.4. The Fefferman space. Suppose we have given a partially integrable almost CR manifold (M, H) with a fixed choice $\mathcal{E}(1,0)$ of an $(n+2)$nd root of the anticanonical bundle as in Section 2.3. Let $\mathcal{E}(-1,0)$ be the dual bundle to $\mathcal{E}(1,0)$ and define the *Fefferman space* $\tilde{M}$ of M to be space of real lines in $\mathcal{E}(-1,0)$. We state this more precisely as follows. Let $\mathcal{F}$ be the bundle obtained by removing the zero section in $\mathcal{E}(-1,0)$. Since $\mathcal{E}(-1,0)$ is a complex line bundle, we get a free right action of $\mathbb{C}^*$ on $\mathcal{F}$ which is transitive on each fibre. Restricting this action to the subgroup $\mathbb{R}^*$, we can define $\tilde{M}$ as the quotient $\mathcal{F}/\mathbb{R}^*$. Hence $\tilde{M} \to M$ is a principal fibre bundle with structure group $\mathbb{C}^*/\mathbb{R}^* \cong U(1)$.

Theorem 2.1. *Let $\tilde{M}$ be the Fefferman space of $(M, H, \mathcal{E}(1,0))$. Then the CR Cartan bundle $\mathcal{G} \to M$ can be naturally viewed as a principal bundle over $\tilde{M}$ with structure group $G \cap \tilde{P}$. The normal CR Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ also gives a Cartan connection on $\mathcal{G} \to \tilde{M}$.*

Denoting by (p,q) the signature of (M,H), the Fefferman space $\tilde{M}$ naturally carries a conformal spin structure of signature $(2p+1,\ 2q+1)$.

Proof. We have noted in Section 2.3 that we may identify $\mathcal{E}(-1,0)$ with $\mathcal{G} \times_P \mathbb{V}^1$. By construction, we can therefore view $\tilde{M}$ as the associated bundle $\mathcal{G} \times_P \mathcal{P}_{\mathbb{R}}\mathbb{V}^1$ with fibre the space of real lines in $\mathbb{V}^1$. Since G acts transitively on the cone of nonzero null vectors, P acts transitively on the space of real lines in $\mathbb{V}^1$. By definition (and as observed in Section 2.2 above) the stabiliser of one of these lines is $G \cap \tilde{P}$, whence $\mathcal{P}_{\mathbb{R}}\mathbb{V}^1 \cong P/(G \cap \tilde{P})$. Now $\mathcal{G} \times_P (P/(G \cap \tilde{P}))$ can be naturally identified with the orbit space $\mathcal{G}/(G \cap \tilde{P})$. Hence we can view $\mathcal{G}$ as a principal bundle over $\tilde{M}$ with structure group $G \cap \tilde{P}$.

Let $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ be the normal CR Cartan connection. By definition, ω provides a trivialisation of $T\mathcal{G}$, which is equivariant for the action of the structure group P and reproduces the generators (in $\mathfrak{p}$) of fundamental vector fields. But P-equivariancy of course implies equivariancy for the actions of the subgroup $G \cap \tilde{P} \subset P$, and the fundamental vector fields on $\mathcal{G} \to \tilde{M}$ are exactly those fundamental vector fields on $\mathcal{G} \to M$ whose generators lie in $\mathfrak{g} \cap \tilde{\mathfrak{p}}$. Hence ω also defines a Cartan connection on $\mathcal{G} \to \tilde{M}$. From Section 2.2 we know that the inclusion $G \hookrightarrow \tilde{G}$ induces a diffeomorphism $G/(G \cap \tilde{P}) \to \tilde{G}/\tilde{P}$. Denoting by $\tilde{\mathfrak{g}}$ the Lie algebra of $\tilde{G}$, we obtain an inclusion $\mathfrak{g} \to \tilde{\mathfrak{g}}$ which induces a linear isomorphism $\mathfrak{g}/(\mathfrak{g} \cap \tilde{\mathfrak{p}}) \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. By construction, this isomorphism is equivariant for the actions of $G \cap \tilde{P}$ on both sides coming from the adjoint action. The Cartan connection

ω on $\mathcal{G} \to M$ gives rise to an identification of the tangent bundle $T\tilde{M}$ with the associated bundle $\mathcal{G} \times_{G\cap\tilde{P}} \mathfrak{g}/(\mathfrak{g} \cap \tilde{\mathfrak{p}})$. Since $\tilde{G} \cong SO(2p+2,\ 2q+2)$ and $\tilde{P}$ is the stabiliser of a null line in the standard representation, it is well-known that the natural action of $\tilde{G}$ on $\tilde{G}/\tilde{P}$ respects an oriented conformal structure of signature $(2p+1,\ 2q+1)$. This gives rise to an orientation and a conformal class of inner products of the same signature on $T_{e\tilde{P}}(\tilde{G}/\tilde{P}) = \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$, which are invariant under the natural action of $\tilde{P}$. Via the above isomorphism, we obtain corresponding structures on $\mathfrak{g}/(\mathfrak{g} \cap \tilde{\mathfrak{p}})$ which are invariant under $G \cap \tilde{P}$. Passing to the associated bundle, we obtain an oriented conformal structure on $\tilde{M}$. Thus it remains to construct a natural spin structure. We only discuss this for $p, q > 0$, if $p = 0$ or $q = 0$, the argument is similar.

The group $\tilde{G} \cong SO_0(2p+2,\ 2q+2)$ is homotopy equivalent to its maximal compact subgroup $(O(2p+2) \times O(2q+2)) \cap SO_0(2p+2,\ 2q+2)$. Hence the fundamental group of $\tilde{G}$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the spin group $\mathrm{Spin}(2p+2,\ 2q+2)$ is the two-fold covering corresponding to the diagonal subgroup. On the other hand, the group $G \cong SU(p+1,\ q+1)$ is homotopy equivalent to its maximal compact subgroup, which is isomorphic to $S(U(p+1) \times U(q+1))$. Each of the unitary groups has fundamental group $\mathbb{Z}$ and hence $\pi_1(G) = \{(k, -k) : k \in \mathbb{Z}\} \subset \mathbb{Z} \times \mathbb{Z}$. Then the homomorphism $\pi_1(G) \to \pi_1(\tilde{G})$ induced by the inclusion $G \hookrightarrow \tilde{G}$ is reduction modulo two in both components. Hence its image lies in the diagonal subgroup which means that there is a lift to an inclusion $G \hookrightarrow \mathrm{Spin}(2p+2, 2q+2)$.

Restricting this lift to $G \cap \tilde{P}$ we obtain a homomorphism to the stabiliser $\tilde{P}^{Sp} \subset \mathrm{Spin}(2p+2,\ 2q+2)$ of the chosen real null line in $\mathbb{V}$. It is well known that the two-fold covering map from this stabiliser onto $\tilde{P}$ projects onto the covering $\mathrm{Spin}(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}) \to SO_0(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})$. Hence denoting by K the kernel of the composition $G \cap \tilde{P} \to \tilde{P}^{Sp} \to \mathrm{Spin}(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})$, we conclude that $\mathcal{G}/K \to \tilde{M}$ defines a two fold covering of a subbundle of the conformal frame bundle of $\tilde{M}$. By construction this is compatible with the projection $\mathrm{Spin}(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}) \to SO_0(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})$, and hence defines a conformal spin structure on $\tilde{M}$. $\square$

Remarks 2.2. (1) Although the theorem holds without assuming integrability, we will soon restrict to the case of CR structures, for which there is a nice relationship between the normal Cartan connections associated to the two structures. In the integrable case, the theorem is a minor variation of known results. The main difference between our construction and the ones in [5], [30], [29] is the use of the additional bundle $\mathcal{E}(1, 0)$ instead of the anticanonical bundle. Thus we obtain an $n+2$-fold covering of the spaces constructed in those articles. An immediate advantage of this is the existence of a canonical spin structure as proved above.

(2) Another advantage of our construction is that it exactly recovers Fefferman's original construction from [16] for boundaries of strictly pseudoconvex domains. This can be proved directly by showing, similarly to the conformal

case treated in [10], that the CR standard tractor bundle together with its filtration, Hermitian metric and connection, can be constructed from the ambient tangent bundle, the ambient (pseudo-Kähler) metric, and its Levi-Civita connection. Since this needs several non-trivial verifications, it will be taken up elsewhere.

(3) We will give an explicit formula for the conformal structure on $\tilde{M}$ in Proposition 4.7 below.

2.5. Standard tractors. Having extended the structure group of the CR Cartan bundle to P, we can work with the standard representation $\mathbb{V}$ of G (which does not make sense for $G/Z(G)$). This leads to standard tractors which will provide a simple relationship between the CR structure on M and the conformal structure on $\tilde{M}$, and at the same time keep track of the associated Cartan connections. See [9] for the general relation between tractor bundles and Cartan geometries and [10] for more details on the conformal case.

Restricting the standard representation of G to the subgroup P, we obtain the associated bundle $\mathcal{T} := \mathcal{G} \times_P \mathbb{V}$. This is called the *standard tractor bundle* of $(M, H, \mathcal{E}(1,0))$. By construction, it is a rank $n+2$ complex vector bundle over M, which comes equipped with a Hermitian inner product h of signature $(p+1,\ q+1)$ that is induced by $\langle\ ,\ \rangle$. The P-invariant subspace $\mathbb{V}^1 \subset \mathbb{V}$ gives rise to a natural subbundle $\mathcal{T}^1 \subset \mathcal{T}$, which is a complex line bundle isomorphic to $\mathcal{E}(-1,0)$. Moreover, the fibres of $\mathcal{T}^1$ are all null with respect to h. Since $P \subset SU(\mathbb{V})$, a choice of a nonzero element $\tau \in \Lambda^{n+2}\mathbb{V}$ induces a trivialisation of the highest complex exterior power $\Lambda^{n+2}\mathcal{T}$.

Of course, we can restrict the standard representation further to the subgroup $G \cap \tilde{P} \subset P$ and obtain an associated vector bundle $\tilde{\mathcal{T}} := \mathcal{G} \times_{G\cap\tilde{P}} \mathbb{V} \to \tilde{M}$. The Hermitian inner product on $\mathbb{V}$ is G-invariant, so it gives rise to a Hermitian bundle metric on $\tilde{\mathcal{T}}$ of signature $(p+1,\ q+1)$. Taking the real part of this defines a real bundle metric $\tilde{h}$ of signature $(2p+2,\ 2q+2)$ on $\tilde{\mathcal{T}}$. The real line $\mathbb{V}^1_{\mathbb{R}} \subset \mathbb{V}$, stabilised by $G \cap \tilde{P}$, gives rise to a real line subbundle $\tilde{\mathcal{T}}^1 \subset \tilde{\mathcal{T}}$ and each of these lines is null with respect to $\tilde{h}$. Thus, defining $\tilde{\mathcal{T}}^0$ to be the real orthogonal complement of $\tilde{\mathcal{T}}^1$, we obtain a filtration $\tilde{\mathcal{T}} = \tilde{\mathcal{T}}^{-1} \supset \tilde{\mathcal{T}}^0 \supset \tilde{\mathcal{T}}^1$ by smooth subbundles. The real volume form $\tau \wedge \bar{\tau}$ on $\mathbb{V}$ induces a trivialisation of the highest real exterior power $\Lambda^{2n+4}\tilde{\mathcal{T}}$.

Theorem 2.3. *The Cartan connection ω on $\mathcal{G}$ induces a tractor connection $\nabla^{\tilde{\mathcal{T}}}$ on the bundle $\tilde{\mathcal{T}} \to \tilde{M}$, and $(\tilde{\mathcal{T}}, \tilde{\mathcal{T}}^1, \tilde{h}, \nabla^{\tilde{\mathcal{T}}})$ is a standard tractor bundle for the natural conformal structure on $\tilde{M}$. The tractor connection $\nabla^{\tilde{\mathcal{T}}}$ is normal if and only if the almost CR structure (M, H) is integrable.*

Proof. As a representation of $\tilde{P}$, the $(2n+2)$nd tensor power of $\mathbb{V}^1_{\mathbb{R}}$ is dual to the highest exterior power $\Lambda^{2n+2}(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})$. This immediately implies that $\tilde{\mathcal{T}}^1$ is a density bundle on $\tilde{M}$, which (in the conventions of [10] but using a tilde to indicate density bundles on $\tilde{M}$) is $\tilde{\mathcal{E}}[-1]$. In the same way we conclude that

$\tilde{\mathcal{T}}/\tilde{\mathcal{T}}^0 \cong \tilde{\mathcal{E}}[1]$ and $\tilde{\mathcal{T}}^0/\tilde{\mathcal{T}}^1 \cong T\tilde{M} \otimes \tilde{\mathcal{E}}[-1]$. Finally the relation between the $\tilde{P}$-invariant conformal class of inner products on $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ and the standard representation shows that the conformal class on $\tilde{M}$ constructed in Theorem 2.1 comes from the metric on $T\tilde{M} \otimes \tilde{\mathcal{E}}[-1] \cong \tilde{\mathcal{T}}^0/\tilde{\mathcal{T}}^1$ induced by $\tilde{h}$.

Since the representation $\mathbb{V}$ of $G \cap \tilde{P}$ is the restriction of a representation of G, we may invoke the mechanism of [9, Theorem 2.7] to get a tractor connection on the associated bundle $\tilde{\mathcal{T}}$ from the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. For a smooth section $s \in \Gamma(\tilde{\mathcal{T}})$, consider the corresponding $(G \cap \tilde{P})$–equivariant smooth function $f : \mathcal{G} \to \mathbb{V}$. For a smooth vector field ξ on $\tilde{M}$, choose a lift $\bar{\xi} \in \mathfrak{X}(\mathcal{G})$, and define the function corresponding to $\nabla^{\tilde{\mathcal{T}}}_\xi s$ by $u \mapsto (\bar{\xi} \cdot f)(u) + \omega(\bar{\xi}(u))(f(u))$, where in the second summand we use the action of $\mathfrak{g}$ on $\mathbb{V}$. As in the proof of [9, Theorem 2.7] one concludes that this defines a linear connection on $\tilde{\mathcal{T}}$. By construction, this connection is compatible with the bundle metric $\tilde{h}$ and the volume form on $\tilde{\mathcal{T}}$. Sections of $\tilde{\mathcal{T}}^1$ correspond to functions with values in $\mathbb{V}^1_{\mathbb{R}}$. If f is such a function, then so is $\bar{\xi} \cdot f$ for any vector field $\bar{\xi}$. If s is the corresponding section and $s(x) \neq 0$, we see that $\nabla^{\tilde{\mathcal{T}}}_\xi s(x) \in \tilde{\mathcal{T}}^1_x$ if and only if $\omega(\bar{\xi}) \in \mathfrak{g} \cap \tilde{\mathfrak{p}}$, the stabiliser in $\mathfrak{g}$ of $\mathbb{V}^1_{\mathbb{R}}$. But this means that $\bar{\xi}$ projects to zero on $\tilde{M}$, so $\xi(x) = 0$. Hence, we have verified that $\nabla^{\tilde{\mathcal{T}}}$ is non-degenerate in the sense of [9, 2.5], compare also with [10, 2.2], and thus it is a tractor connection. Hence we see that $(\tilde{\mathcal{T}}, \tilde{\mathcal{T}}^1, \tilde{h}, \nabla^{\tilde{\mathcal{T}}})$ is a standard tractor bundle for the natural conformal class on $\tilde{M}$ in the sense of [10, 2.2].

To discuss normality, we need to compare the curvatures of $\nabla^{\mathcal{T}}$ and $\nabla^{\tilde{\mathcal{T}}}$. This is best done in the picture of the curvature function. Initially, the curvature of ω is defined as the $\mathfrak{g}$-valued two form on $\mathcal{G}$ given by $(\xi, \eta) \mapsto d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$. This is easily seen to be horizontal and P-equivariant, so it can be interpreted as a two form κ on M with values in $\mathcal{G} \times_P \mathfrak{g} = \mathfrak{su}(\mathcal{T})$. The Cartan connection ω gives rise to an isomorphism $TM \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$. Therefore, κ may be viewed as a section of the associated bundle $\mathcal{G} \times_P (\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g})$. The curvature function is the P-equivariant function $\mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ corresponding to this section. We also write κ for the curvature function.

Likewise, $T\tilde{M} \cong \mathcal{G} \times_{G \cap \tilde{P}} (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})$, so the corresponding curvature function has values in $\Lambda^2(\mathfrak{g}/(\mathfrak{g} \cap \tilde{\mathfrak{p}}))^* \otimes \tilde{\mathfrak{g}}$. To talk about conformal normality, we just have to use the isomorphism $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \cong \mathfrak{g}/(\mathfrak{g} \cap \tilde{\mathfrak{p}})$ obtained in the proof of Theorem 2.1 to interpret the curvature function $\tilde{\kappa}$ as having values in $\Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$.

By [9, Proposition 2.9], the curvature of the tractor connection induced by a Cartan connection is induced by the curvature of the Cartan connection. Hence both κ and $\tilde{\kappa}$ are induced by the curvature of ω, which implies that for $u \in \mathcal{G}$, the map $\tilde{\kappa}(u) : \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}) \to \tilde{\mathfrak{g}}$ is the composition

$$\Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}) \xrightarrow{\cong} \Lambda^2(\mathfrak{g}/(\mathfrak{g} \cap \tilde{\mathfrak{p}})) \to \Lambda^2(\mathfrak{g}/\mathfrak{p}) \xrightarrow{\kappa(u)} \mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}. \tag{2.1}$$

Now it is well known that a normal conformal Cartan connection must be torsion free, which means that if $\nabla^{\tilde{\mathcal{T}}}$ is normal, then $\tilde{\kappa}(u)$ must have values in $\tilde{\mathfrak{p}} \subset \tilde{\mathfrak{g}}$. This is only possible if $\kappa(u)$ has values in $\mathfrak{g} \cap \tilde{\mathfrak{p}} \subset \mathfrak{p} \subset \mathfrak{g}$. This means that the Cartan connection ω has to be torsion free, which implies that (M, H) is integrable, see [12, 4.16].

To prove the other implication, first note that there is an abelian subalgebra $\tilde{\mathfrak{p}}_+ \subset \tilde{\mathfrak{p}}$, which consists of all maps that annihilate $\mathbb{V}^1_{\mathbb{R}}$ and map the real orthocomplement $\mathbb{V}^0_{\mathbb{R}}$ of this line to $\mathbb{V}^1_{\mathbb{R}}$. It is easy to see that $\tilde{\mathfrak{p}}_+$ is the annihilator of $\tilde{\mathfrak{p}}$ with respect to the trace form. Thus by the non-degeneracy of the trace form it gives rise to an isomorphism $\tilde{\mathfrak{p}}_+ \cong (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^*$. Likewise, there is a subalgebra $\mathfrak{p}_+ \subset \mathfrak{p}$, which consists of all maps that annihilate the complex line $\mathbb{V}^1$ and map its complex orthocomplement $\mathbb{V}^0$ to $\mathbb{V}^1$. The (real) trace form on $\mathfrak{g}$ induces an isomorphism $\mathfrak{p}_+ \cong (\mathfrak{g}/\mathfrak{p})^*$.

Now the conformal normalisation condition can be stated as follows. Take dual bases $\{\tilde{X}_j\}$ of $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ and $\{\tilde{Z}_j\}$ of $\tilde{\mathfrak{p}}_+$. Then for each $u \in \mathcal{G}$ and each $X \in \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ the expression

$$\sum_{j=0}^{2n+1} [\tilde{Z}_j, \tilde{\kappa}(u)(\tilde{X}, \tilde{X}_j)] \tag{2.2}$$

has to vanish. To obtain appropriate bases, we choose elements $X_0, \dots, X_{2n} \in \mathfrak{g} \subset \tilde{\mathfrak{g}}$ which project onto a basis of $\mathfrak{g}/\mathfrak{p}$, and put $\tilde{X}_j := X_j + \tilde{\mathfrak{p}}$. Next, take $\tilde{X}_{2n+1} := i\,\mathrm{id} + \tilde{\mathfrak{p}}$. Under the isomorphism $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \to \mathfrak{g}/(\mathfrak{g} \cap \tilde{\mathfrak{p}})$ the element $\tilde{X}_{2n+1}$ corresponds to a nonzero element of $\mathfrak{p}/(\mathfrak{g} \cap \tilde{\mathfrak{p}})$, so $\{\tilde{X}_j : j = 0, \dots, 2n+1\}$ is a basis of $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. Let $\{\tilde{Z}_j\}$ be the dual basis of $\tilde{\mathfrak{p}}_+$. As a linear map on $\mathbb{V}$ we can decompose each of the $\tilde{Z}_j$ uniquely as $Z_j + \hat{Z}_j$, where Z_j is complex linear and $\hat{Z}_j$ is conjugate linear.

First observe that by (2.1), $\tilde{\kappa}(u)$ factors through $\Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}) \to \Lambda^2(\mathfrak{g}/\mathfrak{p})$, which, since $X_{2n+1} \in \mathfrak{p}$, implies that the term with $j = 2n+1$ does not contribute to (2.2). For the same reason, it suffices to take in (2.2) $\tilde{X} = X + \tilde{\mathfrak{p}}$ for $X \in \mathfrak{g}$. Further, we know that $\kappa(u)$ has values in $\mathfrak{g}$, and in particular is complex linear. Therefore,

$$[\tilde{Z}_j, \tilde{\kappa}(u)(\tilde{X}, \tilde{X}_j)] = [Z_j, \tilde{\kappa}(u)(\tilde{X}, \tilde{X}_j)] + [\hat{Z}_j, \tilde{\kappa}(u)(\tilde{X}, \tilde{X}_j)]$$

is the decomposition into complex linear and conjugate linear parts. Using (2.1) we conclude that the complex linear part of (2.2) is given by

$$\sum_{j=0}^{2n} [Z_j, \kappa(u)(X+\mathfrak{p},\ X_j+\mathfrak{p})]. \tag{2.3}$$

Now since each X_j is complex linear, the real trace of $X_j \circ \hat{Z}_k$ vanishes for all k, hence we obtain the real traces $\mathrm{tr}(X_j \circ Z_k) = \delta_{jk}$. For $j = 2n+1$, we have

$X_j = i\,\mathrm{id}$, which shows that for $k \le 2n$ the map Z_k has vanishing complex trace, so it lies in $\mathfrak{su}(\mathbb{V})$. Take a nonzero element $v \in \mathbb{V}^1_{\mathbb{R}}$. Then iv lies in the real orthocomplement of $\mathbb{V}^1_{\mathbb{R}}$, so by the definition of $\tilde{\mathfrak{p}}_+$, $\tilde{Z}_k(iv) = av$ for some $a \in \mathbb{R}$. But then $X_{2n+1} \circ \tilde{Z}_k$ maps iv to aiv, and looking at an appropriate basis, that extends $\{v, iv\}$, one sees that $\mathrm{tr}(X_{2n+1} \circ \tilde{Z}_k) = 2a$, so $a = 0$ for $k \le 2n$. Hence $\tilde{Z}_k$ vanishes on the complex line $\mathbb{V}^1$, so the same is true for Z_k. Since $(\mathbb{V}^1)^\perp$ is a complex subspace of $\mathbb{V}$ which is contained in the real orthocomplement of $\mathbb{V}^1_{\mathbb{R}}$, it is mapped to $\mathbb{V}^1_{\mathbb{R}}$ by $\tilde{Z}_k$ and hence to $\mathbb{V}^1$ by Z_k for $k \le 2n$. Hence we have verified that $\{Z_0, \dots, Z_{2n}\} \subset \mathfrak{p}_+$, and this is a basis, which is dual to the basis $\{X_j + \mathfrak{p} : j \le 2n\}$ of $\mathfrak{g}/\mathfrak{p}$.

But if κ is the curvature function of a torsion free normal parabolic geometry of type (G, P), then (2.3) always vanishes. This is shown in the proof of Theorem 3.8 of [14]. Alternatively, it follows by translating the last part of the proof of Theorem 4.1 below (which is independent of the current considerations) into a statement on the curvature function. The same arguments show that $\kappa(u)$ has values in $\mathfrak{g} \cap \tilde{\mathfrak{p}} \subset \mathfrak{p}$. This shows that each summand in (2.2) lies in $[\tilde{\mathfrak{p}}_+, \tilde{\mathfrak{p}}] = \tilde{\mathfrak{p}}_+$, and we have just seen that the complex linear part of the whole sum vanishes. Hence the linear map $\varphi : \mathbb{V} \to \mathbb{V}$ defined by (2.2) is conjugate linear and contained in $\tilde{\mathfrak{p}}_+$, and we claim that this already implies $\varphi = 0$.

The complex subspace $(\mathbb{V}^1)^\perp$ of $\mathbb{V}$ is contained in the real orthocomplement of $\mathbb{V}^1_{\mathbb{R}}$. Since $\varphi \in \tilde{\mathfrak{p}}_+$, this implies that $\varphi((\mathbb{V}^1)^\perp) \subset \mathbb{V}^1_{\mathbb{R}}$. But by conjugate linearity, $\varphi((\mathbb{V}^1)^\perp)$ is a complex subspace of $\mathbb{V}$, so φ vanishes on $(\mathbb{V}^1)^\perp$. Hence fixing a nonzero element $v \in \mathbb{V}^1_{\mathbb{R}}$, there must be an element $w \in \mathbb{V}$ such that $\varphi(x) = \langle v, x\rangle w$. If x is chosen in such a way that $\langle v, x\rangle = i$, then $\varphi(x) \in \mathbb{V}^1_{\mathbb{R}}$, so $w = iav$ for some $a \in \mathbb{R}$. But now one immediately verifies that $x \mapsto \langle v, x\rangle iav$ is symmetric for $\langle\ ,\ \rangle_{\mathbb{R}}$, so $\varphi \in \mathfrak{so}(\mathbb{V})$ only if $\varphi = 0$. □

Remarks 2.4. (1) The standard tractor bundle $(\tilde{\mathcal{T}}, \tilde{\mathcal{T}}^1, \tilde{h}, \nabla^{\tilde{\mathcal{T}}})$ is equivalent to a principal bundle $\tilde{\mathcal{G}} \to \tilde{M}$ with structure group $\tilde{P}$ endowed with a Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$. The bundle $\tilde{\mathcal{G}}$ can be constructed as an adapted frame bundle for $\tilde{\mathcal{T}}$, see [10, 2.2], which implies that $\tilde{\mathcal{G}} = \mathcal{G} \times_{G \cap \tilde{P}} \tilde{P}$. Since this implies $\tilde{\mathcal{T}} = \tilde{\mathcal{G}} \times_{\tilde{P}} \mathbb{V}$, we obtain a Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ by [9, Theorem 2.7]. By construction, $\tilde{\omega}$ restricts to ω on $T\mathcal{G} \subset T\tilde{\mathcal{G}}|_{\mathcal{G}}$, which uniquely determines $\tilde{\omega}$ by the defining properties of Cartan connections.

(2) Without the assumption of integrability, the tractor connection $\nabla^{\tilde{\mathcal{T}}}$ differs from the conformal normal standard tractor connection. While we will restrict to the integrable case for the rest of this paper, there is scope to use our results in the non-integrable case. The route to this should be to compute the difference to the normal tractor connection explicitly, say in terms of the Nijenhuis tensor of (M, H), and then translate the results below (most of which do hold for the induced tractor connection in the non-integrable case) to results for the normal tractor connection.

3. Conformal Geometry of Fefferman Spaces

3.1. The canonical complex structure on standard tractors. Let $(M, H, \mathcal{E}(1,0))$ be a CR manifold with Fefferman space $\tilde{M}$. Then we have defined the conformal standard tractor bundle $\tilde{\mathcal{T}}$ on $\tilde{M}$ as $\mathcal{G} \times_{G\cap\tilde{P}} \mathbb{V}$. Consider multiplication by i as a linear map from $\mathbb{V}$ to itself. Since $\langle\ ,\ \rangle$ is Hermitian, this map lies in $\mathfrak{so}(\mathbb{V}, \langle\ ,\ \rangle_{\mathbb{R}}) = \tilde{\mathfrak{g}}$. Moreover, the action of any element of $G \cap \tilde{P}$ commutes with this map since G consists of complex linear maps. Consequently, the corresponding constant map $\mathcal{G} \to \tilde{\mathfrak{g}}$ is $(G \cap \tilde{P})$-equivariant and hence gives rise to a section $\mathbb{J}$ of the associated bundle $\mathcal{G} \times_{G\cap\tilde{P}} \tilde{\mathfrak{g}} \cong \tilde{\mathcal{G}} \times_{\tilde{P}} \tilde{\mathfrak{g}}$, the *adjoint tractor bundle* $\tilde{\mathcal{A}}$ of $\tilde{M}$. Observe that by construction $\tilde{\mathcal{A}} = \mathfrak{so}(\tilde{\mathcal{T}})$, so the standard tractor connection induces a linear connection $\nabla^{\tilde{\mathcal{A}}}$ on $\tilde{\mathcal{A}}$, called the *adjoint tractor connection*. Since the tangent bundle $T\tilde{M}$ is the associated bundle $\tilde{\mathcal{G}} \times_{\tilde{P}} (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})$, there is a natural projection $\tilde{\Pi} : \tilde{\mathcal{A}} \to T\tilde{M}$.

Theorem 3.1. *Let M be a CR manifold with Fefferman space $\tilde{M}$, and let $\mathbb{J} \in \Gamma(\tilde{\mathcal{A}})$ be the section constructed above. Then we have:*

(1) *$\mathbb{J}$ makes $\tilde{\mathcal{T}}$ into a complex vector bundle, it is orthogonal for $\tilde{h}$, and $\nabla^{\tilde{\mathcal{A}}}\mathbb{J} = 0$.*
(2) *The vector field $\mathbf{k} := \tilde{\Pi}(\mathbb{J}) \in \mathfrak{X}(\tilde{M})$ is nowhere vanishing and generates the vertical bundle of $\tilde{M} \to M$. For the conformal Cartan curvature $\tilde{\kappa} \in \Omega^2(M, \tilde{\mathcal{A}})$ we have $i_{\mathbf{k}}\tilde{\kappa} = 0$ and $\mathbf{k}$ is a conformal Killing field.*

Proof. (1) Since $\mathbb{J}$ corresponds to multiplication by i in $\mathbb{V}$, it clearly satisfies $\mathbb{J}^2 = -\mathrm{id}$. Since $\tilde{h}$ corresponds to a Hermitian form on $\mathbb{V}$, $\mathbb{J}$ is orthogonal (or equivalently skew symmetric). By the definition of $\mathbb{J}$, if $s \in \Gamma(\tilde{\mathcal{T}})$ corresponds to $f : \mathcal{G} \to \mathbb{V}$, then $\mathbb{J}s$ corresponds to if. Now for any tangent vector ξ on $\mathcal{G}$, we have $\xi \cdot (if) = i(\xi \cdot f)$ and $\omega(\xi)$ is complex linear. By definition of $\nabla^{\tilde{\mathcal{T}}}$ this implies that $\nabla^{\tilde{\mathcal{T}}}\mathbb{J}s = \mathbb{J}\nabla^{\tilde{\mathcal{T}}}s$ for any section s. Since $\nabla^{\tilde{\mathcal{A}}}$ is induced by $\nabla^{\tilde{\mathcal{T}}}$, this shows that $\nabla^{\tilde{\mathcal{A}}}\mathbb{J} = 0$.

(2) Fix an element $F \in \mathfrak{p}$ which acts by multiplication by i on $\mathbb{V}^1$. Then $i\,\mathrm{id} - F$ acts trivially on $\mathbb{V}^1$ and thus lies in $\tilde{\mathfrak{p}}$. Consequently, the isomorphism $\mathfrak{g}/(\mathfrak{g} \cap \tilde{\mathfrak{p}}) \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ induced by the inclusion $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ maps $F + (\mathfrak{g} \cap \tilde{\mathfrak{p}})$ to $i\,\mathrm{id} + \tilde{\mathfrak{p}}$. Since $F \in \mathfrak{p}$ but $F \notin (\mathfrak{g} \cap \tilde{\mathfrak{p}})$, this implies that $i\,\mathrm{id} + \tilde{\mathfrak{p}}$ is a nonzero element in the kernel of the projection $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \to \mathfrak{g}/\mathfrak{p}$, which represents $T\pi : T\tilde{M} \to TM$. Hence the vector field $\mathbf{k}$ is nowhere vanishing and therefore generates the vertical bundle of $\tilde{M} \to M$. We have already observed in the proof of Theorem 2.3 that $\tilde{\kappa}$ comes from the tractor curvature on $\mathcal{T}$. Since $\mathbf{k}$ lies in the vertical subbundle of $\tilde{M} \to M$, this implies $i_{\mathbf{k}}\tilde{\kappa} = 0$. Hence $\mathbb{J}$ satisfies

$$\nabla^{\tilde{\mathcal{A}}}\mathbb{J} + i_{\tilde{\Pi}(\mathbb{J})}\kappa = 0 \tag{3.1}$$

and this is equivalent to $\tilde{\Pi}(\mathbb{J})$ being a conformal Killing field, compare with [23, Proposition 2.2], [7, Proposition 3.2]. □

This result has some immediate consequences. Using the tractor metric, we can identify the bundle $\tilde{\mathcal{A}} = \mathfrak{so}(\tilde{\mathcal{T}})$ with the real second exterior power $\Lambda^2\tilde{\mathcal{T}}$. Since $\mathbb{J}$ is a complex structure then, obviously, as a section of $\Lambda^2\tilde{\mathcal{T}}$ it is non-degenerate, i.e., the $(n+2)$-fold wedge product of $\mathbb{J}$ with itself is a nowhere vanishing section of the real line bundle $\Lambda^{2n+4}\tilde{\mathcal{T}}$. Hence for each $1 \le k \le n+2$ we obtain a nonzero section $\mathbb{J} \wedge \cdots \wedge \mathbb{J}$ (k factors) of the bundle $\Lambda^{2k}\tilde{\mathcal{T}}$. Since the normal tractor connections on the exterior powers of $\tilde{\mathcal{T}}$ are induced by $\nabla^{\tilde{\mathcal{T}}}$, all these sections are parallel.

The filtration $\tilde{\mathcal{T}}^1 \subset \tilde{\mathcal{T}}^0 \subset \tilde{\mathcal{T}}$ from Section 2.5 induces a filtration of the exterior powers of $\tilde{\mathcal{T}}$ (see e.g. [4]). Generalising the projection $\tilde{\mathcal{T}} \to \tilde{\mathcal{T}}/\tilde{\mathcal{T}}^0 \cong \mathcal{E}[1]$, there is a natural projection $\Lambda^j\tilde{\mathcal{T}} \to \Lambda^{j-1}T^*\tilde{M} \otimes \mathcal{E}[j]$. Hence the parallel section $\mathbb{J} \wedge \cdots \wedge \mathbb{J}$ gives rise to a weighted $(2k-1)$-form on $\tilde{M}$. There is a conformally invariant first order differential operator defined on sections of $\Lambda^{j-1}T^*\tilde{M} \otimes \mathcal{E}[j]$, which is called the conformal Killing operator, since for $j = 1$ its solutions are conformal Killing fields, see [38] and references therein.

The conformal Killing operator factors through the composition of the induced connection on $\Lambda^j\tilde{\mathcal{T}}$ with an invariant differential operator which splits the projection $\Lambda^j\tilde{\mathcal{T}} \to \Lambda^{j-1}T^*\tilde{M} \otimes \mathcal{E}[j]$. Moreover, any parallel section of $\Lambda^j\tilde{\mathcal{T}}$ is obtained by applying the splitting operator to its projection. In particular, parallel sections correspond to special solutions of the conformal Killing equation, which are called normal conformal Killing forms in [32]. All these facts are an extremely special case of the machinery of BGG sequences, whose general version has been developed in [13] and [6].

Corollary 3.2. *Let $\tilde{M}$ be a Fefferman space. For $1 \le k \le 2n+1$, let A_k be the space of normal conformal Killing k-forms on $\tilde{M}$. Then $A_k \neq \{0\}$ for odd k and there is a natural map $A_k \to A_{k+2}$, which is injective for $k < n+1$ and surjective for $k > n+1$.*

Proof. From above we know that A_k is isomorphic to the space of parallel sections of $\Lambda^{k+1}\tilde{\mathcal{T}}$. Hence $\mathbb{J} \wedge \cdots \wedge \mathbb{J}$ (k factors) projects to a nonzero element of A_{2k-1}. Taking the wedge product with $\mathbb{J}$, maps parallel sections of $\Lambda^j\tilde{\mathcal{T}}$ to parallel sections of $\Lambda^{j+2}\tilde{\mathcal{T}}$, and hence induces a map $A_j \to A_{j+2}$. The injectivity and surjectivity properties are purely algebraic consequences of the non-degeneracy of $\mathbb{J}$. □

An explicit formula for the odd degree conformal Killing forms which exist on any Fefferman space is given in Corollary 4.3 below.

3.2. Relating tractor bundles. We next discuss the relation between sections of natural vector bundles on M and on $\tilde{M}$. Natural vector bundles on $(M, H, \mathcal{E}(1,0))$ are in bijective correspondence with representations of P via forming associated bundles to the Cartan bundle. Similarly, natural vector bundles on a conformal manifold are determined by representations of $\tilde{P}$. Given a representation of $\tilde{P}$ on a vector space W, then, in the special case of a Fefferman space, we have $\tilde{\mathcal{G}} \times_{\tilde{P}} W \cong \mathcal{G} \times_{G \cap \tilde{P}} W$, so it is only the restriction of the representation to $G \cap \tilde{P}$ that matters.

Assume that we have given representations of P and of $\tilde{P}$ on a vector space W, which are compatible in the sense that their restrictions to $G \cap \tilde{P}$ coincide. Then sections of $\mathcal{G} \times_P W \to M$ are in bijective correspondence with P-equivariant functions $\mathcal{G} \to W$. On the other hand, sections of $\tilde{\mathcal{G}} \times_{\tilde{P}} W \cong \mathcal{G} \times_{G \cap \tilde{P}} W$ are in bijective correspondence with $(G \cap \tilde{P})$-equivariant functions $\mathcal{G} \to W$. Hence $\Gamma(\mathcal{G} \times_P W \to M)$ may be identified with a subspace of $\Gamma(\tilde{\mathcal{G}} \times_{\tilde{P}} W \to \tilde{M})$.

There is a simple source for compatible representations: Since $\tilde{G}$ contains both P and $\tilde{P}$ as subgroups, we may use the restrictions of representations of $\tilde{G}$ to the two subgroups. Given such a representation on a vector space $\mathbb{W}$, the associated vector bundle $\tilde{\mathcal{W}} = \tilde{\mathcal{G}} \times_{\tilde{P}} \mathbb{W} \to \tilde{M}$ is the (conformal) $\mathbb{W}$-tractor bundle. Similarly, since $\mathbb{W}$ is also a representation of G by restriction, the downstairs bundle $\mathcal{W} = \mathcal{G} \times_P \mathbb{W} \to M$ is a (CR) tractor bundle on M.

In this way, we obtain a simple relation between conformal tractor bundles on $\tilde{M}$ and CR tractor bundles on $\tilde{M}$. This is the central tool developed in this article. Of course historically, and for many applications, irreducible bundles are the geometric objects studied. However, in contrast to the situation with tractors, the relation between irreducible bundles on M and on $\tilde{M}$ is typically complicated. In many cases of interest, such relations can be deduced from the tractor picture.

The relation between the two types of tractor bundles can be made explicit using the canonical normal tractor connections. These are linear connections induced by the canonical Cartan connections $\tilde{\omega}$ and ω, which exist on each tractor bundle, see [9]. We have noted in Remark 2.4 that $\tilde{\omega}$ is determined by the fact that its restriction to $T\mathcal{G}$ coincides with ω. As in the case of the standard representation discussed in Section 2.5, this implies that, viewing $\tilde{\mathcal{W}}$ as $\mathcal{G} \times_{G \cap \tilde{P}} \mathbb{W}$, the connection $\nabla^{\tilde{\mathcal{W}}}$ is obtained from ω via the mechanism of [9, 2.7].

Proposition 3.3. *Let $(M, H, \mathcal{E}(1,0))$ be a CR manifold with Fefferman space $\tilde{M}$, $\mathbb{W}$ a representation of $\tilde{G}$, and $\tilde{\mathcal{W}} \to \tilde{M}$ and $\mathcal{W} \to M$ the corresponding tractor bundles. Let $\mathbf{k} = \tilde{\Pi}(\mathbb{J}) \in \mathfrak{X}(\tilde{M})$ be the vector field constructed in 3.1. Then we have:*

(1) *A section $\varphi \in \Gamma(\tilde{\mathcal{W}})$ lies in $\Gamma(\mathcal{W})$ if and only if $\nabla^{\tilde{\mathcal{W}}}_{\mathbf{k}} \varphi = 0$.*
(2) *The restriction of $\nabla^{\tilde{\mathcal{W}}}$ to $\Gamma(\mathcal{W}) \subset \Gamma(\tilde{\mathcal{W}})$ descends to a linear connection on the bundle $\mathcal{W} \to M$, which coincides with the normal tractor connection $\nabla^{\mathcal{W}}$.*

Proof. (1) Since $\tilde{\mathcal{W}} \cong \mathcal{G} \times_{G \cap \tilde{P}} \mathbb{W}$, sections of $\tilde{\mathcal{W}}$ are in bijective correspondence with $(G \cap \tilde{P})$-equivariant functions $\mathcal{G} \to \mathbb{W}$. A section φ lies in the subspace $\Gamma(\mathcal{W})$ if and only if the corresponding function f is actually P-equivariant. Since $P/(G \cap \tilde{P})$ is connected, this equivariancy can be checked infinitesimally. It is equivalent to the fact that for each $u \in \mathcal{G}$ and one (or equivalently any) element

$A \in \mathfrak{p} \setminus (\mathfrak{g} \cap \tilde{\mathfrak{p}})$ we have $\zeta_A(u) \cdot f = -A(f(u))$. In the left hand side of this expression the fundamental vector field differentiates f, while on the right hand side A acts by the infinitesimal representation on $f(u) \in \mathbb{W}$. Since $A \in \mathfrak{p}$ we have $\zeta_A(u) = \omega_u^{-1}(A)$. Choosing for A the element F from the proof of Theorem 3.1 we see from the definition of the tractor connection that $\omega_u^{-1}(F) \cdot f + F(f(u))$ is exactly the value at u of the function representing $\nabla_{\mathbf{k}}\varphi$.

(2) Let $\xi \in \mathfrak{X}(M)$ be a vector field and let $\tilde{\xi}$ be a lift of ξ to $\tilde{M}$. For $\varphi \in \Gamma(\mathcal{W}) \subset \Gamma(\tilde{\mathcal{W}})$ we have $\nabla^{\tilde{\mathcal{W}}}_{\mathbf{k}}\varphi = 0$, so since $\mathbf{k}$ spans the vertical bundle of $\tilde{M} \to M$, we see that $\nabla^{\tilde{\mathcal{W}}}_{\tilde{\xi}}\varphi$ depends only on ξ and not on the choice of the lift. From the fact that $\tilde{\xi}$ is projectable, one immediately concludes that $[\mathbf{k}, \tilde{\xi}]$ lies in the vertical subbundle of $\tilde{M} \to M$, so $\nabla^{\tilde{\mathcal{W}}}_{[\mathbf{k},\tilde{\xi}]}\varphi = 0$. The curvature $R^{\tilde{\mathcal{W}}}$ of $\nabla^{\tilde{\mathcal{W}}}$ is induced by the Cartan curvature of $\tilde{\omega}$. Hence $R^{\tilde{\mathcal{W}}}(\mathbf{k}, \tilde{\xi})(\varphi) = 0$ by part (2) of Theorem 3.1, and expanding the definition of the curvature we conclude that $\nabla^{\tilde{\mathcal{W}}}_{\mathbf{k}}\nabla^{\tilde{\mathcal{W}}}_{\tilde{\xi}}\varphi = 0$. Therefore, $\nabla^{\tilde{\mathcal{W}}}_{\tilde{\xi}}\varphi$ is an element of $\Gamma(\mathcal{W})$ which we denote by $\nabla_\xi\varphi$. It is straightforward to verify that this defines a linear connection on $\mathcal{W} \to M$ which by construction coincides with the tractor connection induced by ω. □

In particular, we can apply this result to the standard representation $\mathbb{V}$ to characterise sections of $\mathcal{T} \to M$ among sections of $\tilde{\mathcal{T}} \to \tilde{M}$. Note that, by construction for $s, t \in \Gamma(\mathcal{T}) \subset \Gamma(\tilde{\mathcal{T}})$ the function $\tilde{h}(s,t)$ is constant along the fibres of $\pi : \tilde{M} \to M$ and descends to the real part of $h(s,t)$.

3.3. Twistor spinors on Fefferman spaces. We next describe a surprising application of Proposition 3.3. It is known that a certain variant of Fefferman spaces always admits nontrivial twistor spinors. This has been shown for CR dimension one in [34] and in general in [2] by direct and involved computations. Here we obtain a conceptual proof, without any computations, via a simple analog of the proof of the existence of parallel spinors on Calabi-Yau manifolds.

Suppose that $\mathbb{W}$ is a representation of $\tilde{G}$ as in Section 3.2 above. Even if $\mathbb{W}$ is irreducible as a representation of $\tilde{G}$, it may well happen that as a representation of G, $\mathbb{W}$ decomposes into several irreducible components. If $\mathbb{W} = \mathbb{W}_1 \oplus \cdots \oplus \mathbb{W}_\ell$ as a representation of G, then this decomposition is also valid as a representation of $G \cap \tilde{P}$. Hence for a Fefferman space, the corresponding conformal tractor bundle $\tilde{\mathcal{W}}$ admits a decomposition into a direct sum of bundles. Each of the summands corresponds to a CR tractor bundle $\mathcal{W}_i \to M$. For each i, we can view $\Gamma(\mathcal{W}_i)$ as a subspace of $\Gamma(\tilde{\mathcal{W}})$. Since the tractor connection on $\tilde{\mathcal{W}}$ can be obtained from the Cartan connection ω on $\mathcal{G}$, this decomposition is compatible with the tractor connection, and the restriction of $\nabla^{\tilde{\mathcal{W}}}$ to the subspace $\Gamma(\mathcal{W}_i)$ descends to the normal CR tractor connection on $\mathcal{W}_i \to M$.

Corollary 3.4. *Let $(M, H, \mathcal{E}(1,0))$ be a CR manifold with Fefferman space $\tilde{M}$. Then, for the canonical spin structure on $\tilde{M}$, there is a two parameter family of nonzero twistor spinors.*

Proof. From the proof of Theorem 2.1 we may take $\tilde{G}$ to be the spin group. Denoting by $\mathbb{S}$ the basic spin representation, the resulting conformal tractor bundle $\tilde{\mathcal{S}} \to \tilde{M}$ is known as the bundle of local twistors, which was introduced by Penrose (see [37] and references therein) in dimension four. The corresponding tractor connection is the local twistor transport and it is well-known that parallel sections of this bundle are in bijective correspondence with twistor spinors [3, 18]. Hence it suffices to show that the bundle $\tilde{\mathcal{S}} \to \tilde{M}$ admits a two-parameter family of parallel sections.

But the restriction of the basic spin representation to $SU(p+1, q+1) \subset \mathrm{Spin}(2p+2, 2q+2)$ splits into a direct sum of irreducibles among which there are two copies of the trivial representation (see e.g. [41]). Hence we obtain two copies of $C^\infty(M, \mathbb{C})$ in $\Gamma(\tilde{\mathcal{S}})$, on which the spin tractor connection descends to the exterior derivative. Hence constant functions in these two copies give rise to a two parameter family of parallel sections of $\tilde{\mathcal{S}}$. □

3.4. Relating adjoint tractors. An important example of a conformal tractor bundle, which is in general indecomposable but which admits a direct sum splitting on Fefferman spaces, is the adjoint tractor bundle $\tilde{\mathcal{A}}$. To apply the ideas of Section 3.3 we have to understand the restriction of the adjoint representation of $\tilde{G} = SO_0(2p+2, 2q+2)$ to the subgroup $G = SU(p+1, q+1)$. Given a real linear map $f : \mathbb{V} \to \mathbb{V}$ which is skew symmetric with respect to $\langle\ ,\ \rangle_{\mathbb{R}}$, we can uniquely split f into a complex linear part f_1 and a conjugate linear part. Then f_1 is skew Hermitian with respect to $\langle\ ,\ \rangle$ so it can be written as the sum of an element of $\mathfrak{su}(\mathbb{V})$ and a real multiple of $i\,\mathrm{id}$. On the other hand, mapping φ to $\langle\ , \varphi(\ )\rangle$ defines a linear isomorphism between the space of those conjugate linear endomorphisms of $\mathbb{V}$ which are also skew symmetric with respect to $\langle\ ,\ \rangle_{\mathbb{R}}$, and the space of complex bilinear skew symmetric maps $\mathbb{V} \times \mathbb{V} \to \mathbb{C}$. Hence we see that $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R} \oplus \Lambda^2_{\mathbb{C}}\mathbb{V}^*$ and this decomposition is invariant under the action of G. Since all the summands are irreducible for G, this is the complete decomposition.

From this elementary representation theory, it follows that the adjoint tractor bundle $\tilde{\mathcal{A}}$ splits into three pieces that are preserved by $\nabla^{\tilde{\mathcal{A}}}$. It is easy to make this splitting explicit. Let $\{\ ,\ \}$ be the algebraic bracket on $\tilde{\mathcal{A}}$ induced by the commutator of endomorphisms of $\tilde{\mathcal{T}}$, and let B be the real trace form, i.e., the non-degenerate bilinear form mapping two endomorphisms to the real trace of their composition. By construction, we then have $B(\mathbb{J}, \mathbb{J}) = -(2n+4)$. Given a section $s \in \Gamma(\tilde{\mathcal{A}})$ we can write the conjugate linear part of s as $\frac{1}{4}\{\mathbb{J}, \{s, \mathbb{J}\}\}$. Finally, the trace part of s only sits in the complex linear part and is given by $(-1/(2n+4))B(s, \mathbb{J})\mathbb{J}$. Hence we conclude that the full decomposition of

$s \in \Gamma(\tilde{\mathcal{A}})$ is given as

$$(3.2)\quad s = \left(s - \frac{1}{4}\{\mathbb{J}, \{s, \mathbb{J}\}\} + \frac{1}{2n+4} B(s, \mathbb{J})\mathbb{J}\right) + \frac{-1}{2n+4} B(s, \mathbb{J})\mathbb{J} + \frac{1}{4}\{\mathbb{J}, \{s, \mathbb{J}\}\}.$$

Since $\nabla^{\tilde{\mathcal{A}}}$ satisfies a Leibniz rule with respect to $\{\,,\,\}$ and $\mathbb{J}$ and B are parallel, we see that, as expected, this decomposition is preserved by $\nabla^{\tilde{\mathcal{A}}}$. In particular, we see that for the CR adjoint tractor bundle $\mathcal{A} \to M$ we have

$$\Gamma(\mathcal{A}) = \left\{s \in \Gamma(\tilde{\mathcal{A}}) : \{s, \mathbb{J}\} = 0,\ B(s, \mathbb{J}) = 0,\ \nabla^{\tilde{\mathcal{T}}}_{\mathbf{k}} s = 0\right\}$$

and the restriction of $\nabla^{\tilde{\mathcal{A}}}$ to this subspace descends to the normal adjoint tractor connection on $\mathcal{A}$.

We can apply this to obtain a complete description of infinitesimal conformal isometries of a Fefferman space. As noted in the proof of Theorem 3.1, infinitesimal conformal isometries of $\tilde{M}$ are in bijective correspondence with smooth sections $s \in \Gamma(\tilde{\mathcal{A}})$ such that $\nabla^{\tilde{\mathcal{A}}} s + i_{\tilde{\Pi}(s)}\tilde{\kappa} = 0$. First we need the following result on $\tilde{M}$.

Lemma 3.5. *Suppose that $s \in \Gamma(\tilde{\mathcal{A}})$ satisfies $\nabla^{\tilde{\mathcal{A}}} s + i_{\tilde{\Pi}(s)}\tilde{\kappa} = 0$. Then $\{s, \mathbb{J}\}$ is a parallel section of $\tilde{\mathcal{A}}$, and the function $B(s, \mathbb{J})$ is constant.*

Proof. For $\xi \in \mathfrak{X}(M)$ we obtain using the Leibniz rule and that $\mathbb{J}$ is parallel

$$\nabla^{\tilde{\mathcal{A}}}_{\xi}\{s, \mathbb{J}\} = \{\nabla^{\tilde{\mathcal{A}}}_{\xi} s, \mathbb{J}\} = -\left\{\tilde{\kappa}(\Pi(s), \xi), \mathbb{J}\right\},$$

which vanishes since $\tilde{\kappa}$ has complex linear values.

Similarly, naturality of B implies that

$$\xi \cdot B(s, \mathbb{J}) = B(\nabla^{\tilde{\mathcal{A}}}_{\xi} s, \mathbb{J}) = -B\left(\tilde{\kappa}(\Pi(s), \xi), \mathbb{J}\right),$$

which vanishes since $\tilde{\kappa}$ has values in $\mathfrak{su}(\tilde{\mathcal{T}})$. □

Theorem 3.6. *Let $(M, H, \mathcal{E}(1,0))$ be a CR manifold with Fefferman space $\tilde{M}$. Then the space of conformal Killing fields on $\tilde{M}$ naturally splits into a direct sum $A_1 \oplus \mathbb{R}\mathbf{k} \oplus A_2$, where A_1 is isomorphic to the space of infinitesimal CR automorphisms of M. The space A_1 can be identified with the space of solutions of a second order CR invariant linear differential operator defined on the bundle $Q = TM/H$. The space A_2 can be identified with a subspace of the joint kernel of two CR invariant linear first order differential operators defined on $H \otimes \mathcal{E}(-1,1)$.*

Proof. For a conformal Killing field consider the corresponding section $s \in \Gamma(\tilde{\mathcal{A}})$ and its decomposition $s = s_1 + a\mathbb{J} + s_2$ according to (3.2). By Lemma

3.5, the function a is constant and $\nabla^{\tilde{\mathcal{A}}}s_2 = 0$. By [23, Proposition 2.2] or [7, Corollary 3.5], this implies $i_{\tilde{\Pi}(s_2)}\tilde{\kappa} = 0$ and therefore $i_{\tilde{\Pi}(s)}\tilde{\kappa} = i_{\tilde{\Pi}(s_1)}\tilde{\kappa}$. Hence s_1 and s_2 independently satisfy the infinitesimal automorphism equation, and $\tilde{\Pi}(s) = \tilde{\Pi}(s_1) + a\mathbf{k} + \tilde{\Pi}(s_2)$ is the decomposition claimed in the theorem.

Since $\mathbf{k}$ hooks trivially into $\tilde{\kappa}$ we see that $\nabla^{\tilde{\mathcal{A}}}_{\mathbf{k}}s_1 = 0$, so $s_1 \in \Gamma(\mathcal{A})$. Since the connection $\nabla^{\tilde{\mathcal{A}}}$ and the curvature $\tilde{\kappa}$ descend to their CR counterparts, s_1 satisfies $\nabla^{\mathcal{A}}s_1 + i_{\Pi(s_1)}\kappa = 0$, where $\Pi : \mathcal{A} \to TM$ is the natural projection and κ is the CR tractor curvature. This equation is equivalent to s_1 giving rise to an infinitesimal CR automorphism, see [7, Proposition 3.2]. The interpretation of infinitesimal CR automorphisms as solutions of an invariant operator follows using the BGG machinery, see [7, Theorem 3.4].

Since s_2 is conjugate linear it may be viewed as lying in $\Gamma(\Lambda^2_{\mathbb{C}}\mathcal{T}^*)$ and it is parallel for the normal tractor connection on that bundle. In analogy with the exterior powers of the conformal standard tractor bundle in Section 3.1, there is a natural projection on $\Lambda^2_{\mathbb{C}}\mathcal{T}^*$ to an irreducible quotient bundle, which here is $H \otimes \mathcal{E}(-1,1)$. The machinery of BGG sequences can be used to construct invariant differential operators between irreducible bundles from tractor connections. The first operators in a BGG sequence are defined on the irreducible quotient of the tractor bundle in question. The other bundles occurring in the sequence are irreducible subquotients of the bundles of forms with values in the initial tractor bundle. The isomorphism type of these subquotients can be computed using representation theory. If we start from the tractor bundle $\Lambda^2_{\mathbb{C}}\mathcal{T}^*$, then these computations show that one obtains two invariant operators defined on $H \otimes \mathcal{E}(-1,1)$, one having values in $S^2H \otimes \mathcal{E}(-2,0)$ and the other having values in the tensor product of tracefree endomorphisms of H with $\mathcal{E}(-1,1)$. Again by representation theory, these two invariant operators must be of first order.

A particular consequence of the construction of BGG sequences is that projecting a parallel section of a tractor bundle to the irreducible quotient, one always obtains a section which lies in the kernel of the first operators in the BGG sequence. Hence s_2 projects onto a section of $H \otimes \mathcal{E}(-1,1)$ which is annihilated by the two invariant operators discussed above. □

Remarks 3.7. (1) The decomposition of conformal Killing fields is described explicitly in Section 4.11 below.

(2) There is a general classification of first order invariant differential operators on arbitrary parabolic geometries, see [39]. In particular, this implies that the two first order operators occurring in the theorem are both given by taking one Webster-Tanaka derivative and then projecting to the given irreducible component.

3.5. Relating densities and weighted tractors. Another source of representations of P and $\tilde{P}$ which are compatible in the sense of Section 3.2 is provided by density bundles. Let $\rho : P \to \mathbb{C}^*$ be the representation defined by the action of P on $\mathbb{V}^1$. Then we can form the representation $g \mapsto \rho(g)^{-w}\overline{\rho(g)}^{-w'}$, and the corresponding associated bundle is the density bundle $\mathcal{E}(w, w') \to M$. Restricting this representation to $G \cap \tilde{P}$ we obtain multiplication by $\lambda^{-w-w'}$ (for $\lambda \in \mathbb{R}^*$), so the corresponding associated bundle is the complexified density bundle $\tilde{\mathcal{E}}_{\mathbb{C}}[w + w'] := \tilde{\mathcal{E}}[w + w'] \otimes \mathbb{C}$.

Recall that for any natural vector bundle on a conformal manifold, one has the *fundamental derivative* or fundamental D-operator, see [9, Section 3]. We will denote this operator by $\tilde{\mathbb{D}}$. In particular, we get an operator, on $\tilde{M}$, $\Gamma(\tilde{\mathcal{A}}) \times \Gamma(\tilde{\mathcal{E}}_{\mathbb{C}}[w]) \to \Gamma(\tilde{\mathcal{E}}_{\mathbb{C}}[w])$ written as $(s, \alpha) \mapsto \tilde{\mathbb{D}}_s \alpha$.

Proposition 3.8. *A complex density $\alpha \in \Gamma(\tilde{\mathcal{E}}_{\mathbb{C}}[w + w'] \to \tilde{M})$ lies in the subspace $\Gamma(\mathcal{E}(w, w') \to M)$ if and only if $\tilde{\mathbb{D}}_{\mathbb{J}} \alpha = (w - w') i \alpha$.*

Proof. Consider the $(G \cap \tilde{P})$-equivariant function $f : \mathcal{G} \to \mathbb{C}$ representing α. As in the proof of Proposition 3.3 we conclude that $\alpha \in \mathcal{E}(w, w')$ if and only if $\omega_u^{-1}(F) \cdot f = -(w' - w) i f(u)$, where $F \in \mathfrak{p}$ is the element from the proof of Theorem 3.1. Of course, the right-hand side represents the function $(w - w') i \alpha$, so it suffices to verify that the left-hand side represents $\tilde{\mathbb{D}}_{\mathbb{J}} \alpha$.

To see this, let us view $\mathcal{G}$ as a subset of $\tilde{\mathcal{G}}$ and extend f equivariantly to a function $\tilde{f} : \tilde{\mathcal{G}} \to \mathbb{C}$. By definition of $\tilde{\mathbb{D}}$, the density $\tilde{\mathbb{D}}_{\mathbb{J}} \alpha$ is represented by the function $(\tilde{\omega}^{-1} \circ \varphi) \cdot \tilde{f}$, where $\varphi : \tilde{\mathcal{G}} \to \tilde{\mathfrak{g}}$ is the function representing $\mathbb{J}$. For $u \in \mathcal{G}$ we have $\varphi(u) = i\,\mathrm{id}$ and hence this function evaluates in u to $\tilde{\omega}_u^{-1}(i\,\mathrm{id}) \cdot \tilde{f}$. As we have noted in the proof of Theorem 3.1, we have $F - i\,\mathrm{id} \in \tilde{\mathfrak{p}}$. Since this element annihilates the real line $\mathbb{V}^1_{\mathbb{R}}$, it acts trivially on powers of this, viz. real one dimensional representations. Since $\tilde{f}$ is $\tilde{P}$-equivariant, we conclude that $\tilde{\omega}^{-1}(F - i\,\mathrm{id}) \cdot \tilde{f} = 0$ and hence $\tilde{\omega}^{-1}(i\,\mathrm{id}) \cdot \tilde{f} = \tilde{\omega}^{-1}(F) \cdot \tilde{f}$. Since $\tilde{\omega}$ restricts to ω on $T\mathcal{G} \subset T\tilde{\mathcal{G}}|_{\mathcal{G}}$ we conclude that $\tilde{\omega}_u^{-1}(F) = \omega_u^{-1}(F)$ for $u \in \mathcal{G}$, and since this is tangent to $\mathcal{G}$ we obtain $\omega_u^{-1}(F) \cdot \tilde{f} = \omega_u^{-1}(F) \cdot f$. □

Using this result we can now easily extend the characterisation of Proposition 3.3 to weighted tractor bundles, i.e., tensor products of tractor bundles with density bundles. Given a representation $\mathbb{W}$ of $\tilde{G}$, we consider the tractor bundles $\mathcal{W} \to M$ and $\tilde{\mathcal{W}} \to \tilde{M}$ as in Section 3.2. We define $\mathcal{W}(w, w') := \mathcal{W} \otimes \mathcal{E}(w, w')$ and $\tilde{\mathcal{W}}_{\mathbb{C}}[w + w'] := \tilde{\mathcal{W}} \otimes \mathcal{E}_{\mathbb{C}}[w + w']$. If $\mathbb{W}$ is a complex representation, then the tensor product can be taken over $\mathbb{C}$, otherwise it is understood as a real tensor product. The basic operator on such bundles is the *double-D-operator* [9, 20, 21], which is obtained by coupling the fundamental derivative to the tractor connection. This is well defined, since both the fundamental derivative and the tractor connection satisfy a Leibniz rule. In particular, the conformal double-D defines an operator $\tilde{\mathbb{D}}^{\nabla} : \Gamma(\tilde{\mathcal{A}}) \otimes \Gamma(\tilde{\mathcal{W}}_{\mathbb{C}}[w + w']) \to \Gamma(\tilde{\mathcal{W}}_{\mathbb{C}}[w + w'])$. Explicitly, for $s \in \Gamma(\tilde{\mathcal{A}})$, $\alpha \in \Gamma(\mathcal{E}_{\mathbb{C}}[w + w'])$ and $\varphi \in \Gamma(\tilde{\mathcal{W}})$ we have

$$\tilde{\mathbb{D}}_s^{\nabla}(\alpha \otimes \varphi) = (\tilde{\mathbb{D}}_s \alpha) \otimes \varphi + \alpha \otimes \nabla^{\tilde{\mathcal{W}}}_{\Pi(s)} \varphi.$$

The above with Proposition 3.3 gives the following result.

Corollary 3.9. *A section $\varphi \in \Gamma(\tilde{\mathcal{W}}_{\mathbb{C}}[w+w'])$ lies in the subspace $\Gamma(\mathcal{W}(w,w'))$ if and only if $\tilde{\mathbb{D}}^{\nabla}_{\mathbb{J}}\varphi = (w-w')i\varphi$.*

3.6. Relating fundamental derivatives and double-D-operators. The fundamental derivative is defined on any natural bundle on a manifold endowed with an arbitrary parabolic geometry. The CR version of these operators can be coupled to CR tractor connections to obtain a CR version of double-D-operators. Both types of operators play a central role in the invariant theory and invariant operator theory of conformal, CR geometry [20, 21] (and other parabolic geometries [9]). Our next task is to relate the CR versions of these operators to the conformal variants on the Fefferman space. The basis for this is a simple identity in the conformal setting.

Lemma 3.10. *Let $\tilde{\mathcal{W}} \to \tilde{M}$ be any conformally natural bundle on a Fefferman space, and let $\mathbb{J} \in \Gamma(\tilde{\mathcal{A}})$ be the canonical complex structure. Then for arbitrary sections $s \in \Gamma(\tilde{\mathcal{A}})$ and $\varphi \in \Gamma(\tilde{\mathcal{W}})$ we get*

$$\tilde{\mathbb{D}}_{\mathbb{J}}\tilde{\mathbb{D}}_s\varphi = \tilde{\mathbb{D}}_s\tilde{\mathbb{D}}_{\mathbb{J}}\varphi + \tilde{\mathbb{D}}_{(\nabla^{\tilde{\mathcal{A}}}_{\mathbf{k}}s-\{\mathbb{J},s\})}\varphi.$$

Proof. By Section 3.7 of [9], the difference $\tilde{\mathbb{D}}_{\mathbb{J}}\tilde{\mathbb{D}}_s\varphi - \tilde{\mathbb{D}}_s\tilde{\mathbb{D}}_{\mathbb{J}}\varphi$ is given by $\tilde{\mathbb{D}}_{[\mathbb{J},s]}\varphi$, where $[\mathbb{J},s]$ denotes the Lie bracket of the two adjoint tractor fields. A formula for this Lie bracket is given in [9, Proposition 3.6], and together with Proposition 3.2 of that reference this gives

$$[\mathbb{J},s] = \tilde{\mathbb{D}}_{\mathbb{J}}s - \nabla^{\tilde{\mathcal{A}}}_{\Pi(s)}\mathbb{J} - \tilde{\kappa}(\Pi(\mathbb{J}),\Pi(s)) = \tilde{\mathbb{D}}_{\mathbb{J}}s,$$

where we have used that $\mathbb{J}$ is parallel and $\Pi(\mathbb{J}) = \mathbf{k}$ hooks trivially into the Cartan curvature. Applying again Proposition 3.3 of [9] we get $\tilde{\mathbb{D}}_{\mathbb{J}}s = \nabla^{\tilde{\mathcal{A}}}_{\mathbf{k}}s - \{\mathbb{J},s\}$, which implies the result. □

From this, we immediately get the relation between double-D-operators.

Proposition 3.11. *Consider a representation $\mathbb{W}$ of $\tilde{G}$ and the corresponding weighted tractor bundles on M and $\tilde{M}$ as in Section 3.5. Then for any $s \in \Gamma(\mathcal{A}) \subset \Gamma(\tilde{\mathcal{A}})$ the double-D-operator $\tilde{\mathbb{D}}^{\nabla}_s$ acting on $\Gamma(\tilde{\mathcal{W}}_{\mathbb{C}}[w+w'])$ preserves the subspace $\Gamma(\mathcal{W}(w,w'))$ and by restriction gives the CR double-D-operator $\mathbb{D}^{\nabla}_s$ on that subspace.*

Proof. The description of $\Gamma(\mathcal{A}) \subset \Gamma(\tilde{\mathcal{A}})$ in Section 3.4 shows that $\tilde{\nabla}^{\tilde{\mathcal{A}}}_{\mathbf{k}}s = \{\mathbb{J},s\} = 0$. By Lemma 3.10, the fundamental derivatives with respect to $\mathbb{J}$ and to s commute. By the proof of that lemma, this also implies that $[\mathbb{J},s] = 0$ and using Proposition 3.6 of [9] this gives $[\mathbf{k},\Pi(s)] = 0$. Since $\mathbf{k}$ hooks trivially into $\tilde{\kappa}$, and the curvature of any tractor connection is given by the action of $\tilde{\kappa}$, we conclude that also $\nabla^{\tilde{\mathcal{W}}}_{\mathbf{k}}$ and $\nabla^{\tilde{\mathcal{W}}}_{\Pi(s)}$ commute. Hence $\tilde{\mathbb{D}}^{\nabla}_s$ commutes with $\tilde{\mathbb{D}}^{\nabla}_{\mathbb{J}}$, and this with Corollary 3.9 shows that the subspace $\Gamma(\tilde{\mathcal{W}}_{\mathbb{C}}[w+w'])$ is preserved by $\tilde{\mathbb{D}}^{\nabla}_s$.

In Section 3.2 we have already seen that conformal tractor connections restrict to their CR counterparts. Hence to see that $\tilde{\mathbb{D}}^{\nabla}_s$ restricts to its CR counterpart it suffices to show that $\tilde{\mathbb{D}}_s$ restricts to $\mathbb{D}_s$ on $\Gamma(\mathcal{E}(w,w')) \subset \Gamma(\tilde{\mathcal{E}}_{\mathbb{C}}[w+w'])$. The fact that $s \in \Gamma(\mathcal{A}) \subset \Gamma(\tilde{\mathcal{A}})$ implies that the corresponding equivariant function $f : \tilde{\mathcal{G}} \to \tilde{\mathfrak{g}}$ has the property that $f(\mathcal{G}) \subset \mathfrak{g} \subset \tilde{\mathfrak{g}}$. Consequently, the vector field $\xi \in \mathfrak{X}(\tilde{\mathcal{G}})$ characterised by $\tilde{\omega}(\xi(u)) = f(u)$ for all $u \in \tilde{\mathcal{G}}$ has the property that its restriction to $\mathcal{G}$ is tangent to $\mathcal{G}$. Since $\tilde{\omega}$ restricts to ω on vectors tangent to $\mathcal{G}$, we see that $\xi|_{\mathcal{G}} \in \mathfrak{X}(\mathcal{G})$ is the vector field associated to $s \in \Gamma(\mathcal{A})$ via the CR Cartan connection. Now the result follows immediately from the definition of the fundamental derivative. □

3.7. Complexified adjoint tractors. A deeper understanding and richer theory of fundamental derivatives and double-D-operators is revealed, in this context, by passing to the complexification $\tilde{\mathcal{A}}_{\mathbb{C}}$ of the adjoint tractor bundle. Recall from Section 3.4 that $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R} \oplus \Lambda^2_{\mathbb{C}}\mathbb{V}^*$ as a $\mathfrak{g}$-module, with the first two summands corresponding to complex linear maps and the last summand corresponding to conjugate linear maps. Since the last summand is already complex, its complexification splits into a holomorphic and an anti-holomorphic part. The holomorphic part is isomorphic to $\Lambda^2_{\mathbb{C}}\mathbb{V}^*$ and since $\mathbb{V}^* \cong \bar{\mathbb{V}}$ via the Hermitian form, the anti-holomorphic part is isomorphic to $\Lambda^2_{\mathbb{C}}\mathbb{V}$. Moreover, the isomorphism between conjugate linear maps φ in $\tilde{\mathfrak{g}}$ and skew symmetric complex bilinear maps on $\mathbb{V}$ is given by $\varphi \mapsto \langle\,,\varphi(\,)\rangle$, which implies that the complex structure on $\Lambda^2_{\mathbb{C}}\mathbb{V}^*$ corresponds to mapping φ to $v \mapsto -i\varphi(v)$.

Passing to associated bundles, we see that the subspace of elements of $\tilde{\mathcal{A}}_{\mathbb{C}}$ which anti-commute with $\mathbb{J}$ is isomorphic to $\Lambda^2_{\mathbb{C}}\tilde{\mathcal{T}}^* \oplus \Lambda^2_{\mathbb{C}}\tilde{\mathcal{T}}$. The two summands are characterised by $is = -\mathbb{J} \circ s$ respectively $is = \mathbb{J} \circ s$. Note that $is = \mp\mathbb{J} \circ s$ implies that $\{\mathbb{J}, s\} = \mp 2is$.

Since we deal with unweighted tractor bundles here, the double-D-operator $\tilde{\mathbb{D}}^{\nabla}_{\mathbb{J}}$ is simply given by the tractor connection $\tilde{\nabla}_{\mathbf{k}}$. Now by Corollary 3.9, a section $s \in \Gamma(\Lambda^2_{\mathbb{C}}\tilde{\mathcal{T}}^*)$ lies in the subspace $\Gamma(\Lambda^2\mathcal{T}^*(-1,1)) \subset \Gamma(\Lambda^2_{\mathbb{C}}\tilde{\mathcal{T}}^*)$ if and only if

$$\tilde{\nabla}_{\mathbf{k}} s = \tilde{\mathbb{D}}^{\nabla}_{\mathbb{J}} s = -2is = \{\mathbb{J}, s\}.$$

In the same way, $s \in \Gamma(\Lambda^2_{\mathbb{C}}\tilde{\mathcal{T}})$ lies in the subspace $\Gamma(\Lambda^2\mathcal{T}(1,-1))$ if and only if $\tilde{\nabla}_{\mathbf{k}} s = \{\mathbb{J}, s\}$.

Now consider a complex weighted tractor bundle $\tilde{\mathcal{W}}_{\mathbb{C}}[w+w']$ as in Section 3.6. For a section φ of this bundle, we can view $\tilde{\mathbb{D}}^{\nabla}\varphi$ as a section of

$$L(\tilde{\mathcal{A}},\ \tilde{\mathcal{W}}_{\mathbb{C}}[w+w']) \cong L_{\mathbb{C}}(\tilde{\mathcal{A}}_{\mathbb{C}},\ \tilde{\mathcal{W}}_{\mathbb{C}}[w+w']).$$

Hence we can form $\tilde{\mathbb{D}}_s^\nabla \varphi$ for all sections $s \in \Gamma(\tilde{\mathcal{A}}_{\mathbb{C}})$. In close analogy to the proof of Proposition 3.11 we then conclude that if $s \in \Gamma(\tilde{\mathcal{A}}_{\mathbb{C}})$ sits in either of the subspaces $\Gamma(\Lambda^2\mathcal{T}^*(-1,1))$ or $\Gamma(\Lambda^2\mathcal{T}(1,-1))$, then the operator $\tilde{\mathbb{D}}_s^\nabla$ preserves the subspaces $\Gamma(\mathcal{W}(w,w')) \subset \Gamma(\tilde{\mathcal{W}}_{\mathbb{C}}[w+w'])$ for all w, w'.

Recall that $\tilde{\mathfrak{g}} \cong \tilde{\mathfrak{g}}^*$, via the Killing form, and the decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R} \oplus \Lambda^2\mathbb{V}^*$ is orthogonal with respect to the Killing form. These results extend to the complexification. In the complexification of the last factor of the decomposition, the holomorphic and anti-holomorphic parts are both isotropic with respect to the Killing form, which induces a duality between the two parts. Using this duality (in the language of associated bundles), we can interpret $\tilde{\mathbb{D}}^\nabla$ as an operator mapping sections of $\tilde{\mathcal{W}}_{\mathbb{C}}[w+w']$ to sections of $\tilde{\mathcal{A}}_{\mathbb{C}} \otimes \tilde{\mathcal{W}}_{\mathbb{C}}[w+w']$, and this target splits according to the splitting of $\tilde{\mathcal{A}}_{\mathbb{C}}$. From Proposition 3.11 and the above considerations we get the following result:

Theorem 3.12. *Let $\tilde{\mathcal{W}}_{\mathbb{C}}[w+w']$ be a weighted complex tractor bundle. Consider the double-D-operator as an operator*

$$\tilde{\mathbb{D}}^\nabla : \Gamma(\tilde{\mathcal{W}}_{\mathbb{C}}[w+w']) \to \Gamma(\tilde{\mathcal{A}}_{\mathbb{C}} \otimes \tilde{\mathcal{W}}_{\mathbb{C}}[w+w']).$$

(1) *Passing to the tracefree part of the complex linear part in the $\tilde{\mathcal{A}}_{\mathbb{C}}$-component, one obtains an operator which descends to the CR double-D, viewed as*

$$\mathbb{D}^\nabla : \Gamma(\mathcal{W}(w,w')) \to \Gamma(\mathcal{A} \otimes \mathcal{W}(w,w')).$$

(2) *If one forms the holomorphic part of the conjugate linear part in the $\tilde{\mathcal{A}}_{\mathbb{C}}$-component, then the result descends to an operator*

$$\Gamma(\mathcal{W}(w,w')) \to \Gamma(\Lambda^2\mathcal{T} \otimes \mathcal{W}(w-1,w'+1)).$$

(3) *If one forms the anti-holomorphic part of the conjugate linear part in the $\tilde{\mathcal{A}}_{\mathbb{C}}$-component, then the result descends to an operator*

$$\Gamma(\mathcal{W}(w,w')) \to \Gamma(\Lambda^2\mathcal{T}^* \otimes \mathcal{W}(w+1,w'-1)).$$

The theorem gives, via the Fefferman space, a geometric interpretation to the operators in parts (2) and (3) which were constructed directly (but without a conceptual interpretation) in [20]. These will be described explicitly in Section 4.12.

4. Tractor Calculus on a Fefferman Space

In this section, we will describe the (complexified) standard tractor bundle and the tractor calculus on a Fefferman space more explicitly. To do this, we show that the version of CR tractors introduced in [24] describes the normal CR standard tractor bundle, and relate it to the calculus on the Fefferman space. Hence we obtain an explicit description in terms of a chosen pseudo-hermitian structure on the underlying CR manifold.

4.1. Pseudo-hermitian structures. We review some facts about pseudohermitian structures on a CR manifold (M, H), primarily to fix the conventions, which follow [24]. We will assume that M is orientable, which implies that the annihilator $H^\perp$ of H in T^*M admits a nonvanishing global section. From the non-degeneracy of the CR structure such a section θ is a contact form on M, and it is called a *pseudohermitian structure*. We fix an orientation on $H^\perp$ and restrict consideration to θ's which are positive relative to this orientation. The *Levi form* of θ is the Hermitian form $\mathcal{L}^\theta$ on $H^{1,0} \subset TM \otimes \mathbb{C}$ defined by

$$\mathcal{L}^\theta(Z, \bar{W}) = -2id\theta(Z, \bar{W}),$$

so this exactly corresponds to $\mathcal{L}^{\mathbb{C}}$ introduced in Section 2.1 under the trivialisation given by θ.

Given a pseudohermitian structure θ, define the *Reeb field* r to be the unique vector field on M satisfying

$$\theta(r) = 1 \quad \text{and} \quad i_r d\theta = 0. \tag{4.1}$$

An *admissible coframe* is a set of complex valued forms $\{\theta^\alpha\}$, $\alpha = 1, \ldots, n$, which satisfy $\theta^\alpha(r) = 0$ and whose restrictions to $H^{1,0}$ are complex linear and form a basis for $(H^{1,0})^*$. We will use lower case Greek indices to refer to frames for $T^{1,0}$ or its dual. We may also interpret these indices abstractly, so will denote by $\mathcal{E}^\alpha$ the bundle $H^{1,0}$ (or its space of sections) and by $\mathcal{E}_\alpha$ its dual, and similarly for the conjugate bundles or for tensor products thereof. By integrability and (4.1), we have

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}$$

for a smoothly varying Hermitian matrix $h_{\alpha\bar{\beta}}$, which we may interpret as the matrix of the Levi form in the frame θ^α, or as the Levi form itself in abstract index notation. Using the inclusion $Q \hookrightarrow \mathcal{E}(1,1)$ from Section 2.3, the Levi form $\mathcal{L}_{\mathbb{C}}$ itself can be viewed as a canonical section of $\mathcal{E}_{\alpha\bar{\beta}}(1,1)$ which we also denote by $\mathbf{h}_{\alpha\bar{\beta}}$. By $\mathbf{h}^{\alpha\bar{\beta}} \in \mathcal{E}^{\alpha\bar{\beta}}(-1,-1)$ we denote its inverse. These will be used to raise and lower indices without further mention.

By ∇ we denote the *Webster-Tanaka connections* (on various bundles) associated to θ. In particular, these satisfy $\nabla\theta = 0$, $\nabla h = 0$, $\nabla\mathbf{h} = 0$, $\nabla r = 0$, and $\nabla J = 0$, so the decomposition $T_{\mathbb{C}}M = H^{1,0}M \oplus H^{0,1}M \oplus \mathbb{C}r$ is invariant under ∇.

Therefore, if we decompose a tensor field relative to this splitting (and/or its dual), we may calculate the covariant derivative componentwise. Each of the components may be regarded as a section of a tensor product of $\mathcal{E}^\alpha$ or its dual or conjugates thereof. Therefore we will often restrict consideration to the action of the connection on $\mathcal{E}^\alpha$ or $\mathcal{E}_\alpha$. We will use indices α, $\bar{\alpha}$, 0 for components with respect to the frame $\{\theta^\alpha, \theta^{\bar{\alpha}}, \theta\}$ and its dual, so that the 0-components incorporate weights. If f is a (possibly density-valued) tensor field, we will denote components of the (tensorial) iterated covariant derivatives of f in such a frame by preceding

∇'s, e.g. $\nabla_\alpha\nabla_0\cdots\nabla_{\bar\beta}f$. As usual, such indices may alternately be interpreted abstractly. So, for example, if $f_\beta\in\mathcal{E}_\beta(w,w')$, we will consider ∇f as the triple $\nabla_\alpha f_\beta\in\mathcal{E}_{\alpha\beta}(w,w')$, $\nabla_{\bar\alpha}f_\beta\in\mathcal{E}_{\bar\alpha\beta}(w,w')$, $\nabla_0 f_\beta\in\mathcal{E}_\beta(w-1,w'-1)$.

4.2. CR tractor calculus. Our next task is to show that the calculus introduced in [24] is consistent with the (complexified) CR standard tractor bundle and connection as described here. The defining feature of the bundle $\mathcal{T}_\Phi$ constructed in [24] is that any choice of a pseudohermitian structure θ on M gives rise to an identification

$$\mathcal{T}_\Phi \overset{\theta}{=} \mathcal{E}(1,0)\oplus\mathcal{E}_\alpha(1,0)\oplus\mathcal{E}(0,-1).$$

For a section $T_\Phi\in\mathcal{T}_\Phi$ one writes

$$[T_\Phi]_\theta=\begin{pmatrix}\sigma\\ \tau_\beta\\ \rho\end{pmatrix},$$

or equivalently

$$T_\Phi=\sigma Y_\Phi+\tau_\beta W_\Phi{}^\beta+\rho Z_\Phi,$$

for $\sigma\in\mathcal{E}(1,0)$, $\tau_\beta\in\mathcal{E}_\beta(1,0)$, $\rho\in\mathcal{E}(0,-1)$ and sections $Y_\Phi\in\mathcal{T}_\Phi(-1,0)$, $W_\Phi{}^\beta\in\mathcal{T}_\Phi{}^\beta(-1,0)$, and $Z_\Phi\in\mathcal{T}_\Phi(0,1)$ which depend on θ. Changing scale from θ to $\hat\theta=e^\Upsilon\theta$, the expression for $[T_\Phi]_{\hat\theta}$ is determined by

$$\hat W_\Phi{}^\alpha=W_\Phi{}^\alpha+\Upsilon^\alpha Z_\Phi,\qquad \hat Y_\Phi=Y_\Phi-\Upsilon_\beta W_\Phi{}^\beta-\frac{1}{2}(\Upsilon_\beta\Upsilon^\beta-i\Upsilon_0)Z_\Phi,$$

while Z_Φ is independent of the choice of θ and e.g. $\Upsilon_\alpha:=\nabla_\alpha\Upsilon$. In particular, this shows that Z_Φ gives rise to an isomorphism from $\mathcal{E}(0,-1)$ onto a subbundle $\mathcal{T}^1\subset\mathcal{T}_\Phi$. For two sections T_Φ and T'_Φ the quantity $\sigma\overline{\rho'}+\rho\overline{\sigma'}+\mathbf{h}^{\alpha\bar\beta}\tau_\alpha\tau'_{\bar\beta}$ is independent of the choice of θ so one obtains a well-defined hermitian metric $h^{\Phi\bar\Psi}$ on $\mathcal{T}_\Phi$. Note that the subbundle $\mathcal{T}^1$ is isotropic for $h^{\Phi\bar\Psi}$. Taking $\mathcal{T}^0$ to be the orthocomplement of $\mathcal{T}^1$, we obtain $\mathcal{T}^1\subset\mathcal{T}^0\subset\mathcal{T}_\Phi$, and for any choice of θ the elements of $\mathcal{T}^0$ are characterised by $\sigma=0$. Projecting onto the σ-component shows that $\mathcal{T}_\Phi/\mathcal{T}^0\cong\mathcal{E}(1,0)$, while projecting onto the middle component we have $\mathcal{T}^0/\mathcal{T}^1\cong\mathcal{E}_\beta(1,0)$. The filtration of $\mathcal{T}_\Phi$ can be equivalently described as a composition series which we write as

$$\mathcal{T}_\Phi=\mathcal{E}(1,0)\in\mathcal{E}_\alpha(1,0)\in\mathcal{E}(0,-1).$$

Now the bundles $\mathcal{E}(0,-1)$ and $\mathcal{E}(1,0)$ are, by definition, conjugate to $\mathcal{E}(-1,0)$ respectively $\mathcal{E}(0,1)$. Via the Levi form, the bundle $\mathcal{E}_\beta(1,0)$ is identified with the conjugate of $H^{1,0} \otimes \mathcal{E}(-1,0)$. Therefore, the conjugate bundle $\mathcal{T}_{\bar{\Phi}}$ to $\mathcal{T}_{\Phi}$, which via $h^{\Phi\bar{\Psi}}$ is identified with the dual bundle $\mathcal{T}^{\Phi}$, has a composition series

$$\mathcal{T}^{\Phi} = \mathcal{E}(0,1) \in \mathcal{E}^{\alpha}(-1,0) \in \mathcal{E}(-1,0).$$

which is exactly as for the standard tractor bundle from Section 2.5. For $\mathcal{T}_{\bar{\Phi}}$, we obtain a canonical section $Z_{\bar{\Phi}} \in \mathcal{T}_{\bar{\Phi}}(1,0)$ which maps to the line subbundle. Clearly given a choice of contact form we also have the projectors $Y_{\bar{\Phi}} \in \mathcal{T}_{\bar{\Phi}}(0,-1)$ and $W_{\alpha\bar{\Phi}} \in \mathcal{T}_{\alpha\bar{\Phi}}(1,0)$. We can use the Hermitian metric to raise and lower tractor indices. For example, we obtain $Z^{\Phi} \in \mathcal{T}^{\Phi}(1,0)$ which represents the natural inclusion $\mathcal{E}(-1,0) \hookrightarrow \mathcal{T}^{\Phi}$ as well as the natural projection $\mathcal{T}_{\Phi} \to \mathcal{E}(1,0)$.

4.3. Normality. The next step in [24] is to introduce a linear connection on $\mathcal{T}_{\Phi}$. Since we often have to use this tractor connection coupled to a Webster-Tanaka connection, the best move is to denote both by ∇. Which connection is acting is determined by the objects it acts on, so this should cause no confusion. In [24] the extension of the linear connection to the complexified tangent bundle is provided directly: in the display (3.3) of that reference the authors produce explicit formulae for $\nabla_{\alpha} T_{\Phi}$, $\nabla_{\bar{\beta}} T_{\Phi}$ and $\nabla_0 T_{\Phi}$, for a section T_{Φ} in terms of a choice of θ, and verify that this definition is independent of the choice. It is also verified there that the connection is Hermitian. On the other hand, by construction it is compatible with the complex structure on $\mathcal{T}_{\Phi}$.

The connection on $\mathcal{T}_{\Phi}$ coupled with the Tanaka-Webster connection acts on the projectors as follows.

$$\begin{aligned}
\nabla_{\beta} Y_{\Phi} &= i A_{\alpha\beta} W_{\Phi}{}^{\alpha} + T_{\beta} Z_{\Phi}, \\
\nabla_{\beta} W_{\Phi}{}^{\alpha} &= -\delta_{\beta}^{\alpha} Y_{\Phi} - P_{\beta}{}^{\alpha} Z_{\Phi}, \\
\nabla_{\beta} Z_{\Phi} &= 0,
\end{aligned}$$

and

$$\begin{aligned}
\nabla_{\bar{\beta}} Y_{\Phi} &= P_{\alpha\bar{\beta}} W_{\Phi}{}^{\alpha} - T_{\bar{\beta}} Z_{\Phi}, \\
\nabla_{\bar{\beta}} W_{\Phi}{}^{\alpha} &= i A_{\bar{\beta}}{}^{\alpha} Z_{\Phi}, \\
\nabla_{\bar{\beta}} Z_{\Phi} &= \mathbf{h}_{\alpha\bar{\beta}} W_{\Phi}{}^{\alpha},
\end{aligned}$$

and

$$\begin{aligned}
\nabla_0 Y_{\Phi} &= \frac{i}{n+2} P Y_{\Phi} + 2i T_{\alpha} W_{\Phi}{}^{\alpha} + i S Z_{\Phi}, \\
\nabla_0 W_{\Phi}{}^{\alpha} &= -i P_{\beta}{}^{\alpha} W_{\Phi}{}^{\beta} + \frac{i}{n+2} P W_{\Phi}{}^{\alpha} + 2i T^{\alpha} Z_{\Phi}, \\
\nabla_0 Z_{\Phi} &= -i Y_{\Phi} + \frac{i}{n+2} P Z_{\Phi}.
\end{aligned}$$

where the quantities on the right-hand-side that we have not defined above are torsion and curvature components of the Webster-Tanaka connection. The definitions can be found in [24].

These formulae then determine the connection of $\mathcal{T}_\Phi$ in an obvious way. In particular, taking the covariant derivative of ρZ_Φ for a locally nonvanishing section $\rho \in \mathcal{E}(0,-1)$ and factoring by $\mathcal{T}^1$, the resulting tensorial map $TM \to \mathcal{T}_\Phi/\mathcal{T}^1$ is injective. Indeed passing further to $\mathcal{T}_\Phi/\mathcal{T}^0$ exactly extracts the coefficient of the Reeb field by the formula for $\nabla_0 Z_\Phi$ while the formula for $\nabla_{\bar{\beta}} Z_\Phi$ shows that the middle component will be injective on $H^{0,1}$.

Theorem 4.1. *The bundle $\mathcal{T}_\Phi$ can be naturally identified with the dual of the normal standard tractor bundle in such a way that the filtration, the Hermitian metric and the connection ∇ are mapped to their canonical counterparts.*

Proof. Let us write $\mathcal{A} \to M$ to denote the bundle of skew Hermitian endomorphisms of $\mathcal{T}_\Phi$. The filtration of $\mathcal{T}_\Phi$ gives rise to a filtration on $\mathcal{A}$, while the commutator defines a tensorial Lie bracket. Hence $\mathcal{A}$ becomes a bundle of filtered Lie algebras modelled on $\mathfrak{su}(p+1, q+1)$ and thus an abstract adjoint tractor bundle for $\mathcal{T}$ in the sense of [9, Section 2.2]. The filtration has the form $\mathcal{A} = \mathcal{A}^{-2} \supset \mathcal{A}^{-1} \supset \cdots \supset \mathcal{A}^2$, and the component $\mathcal{A}^j$ is characterised by the facts that for $i = -1, 0, 1$ its elements map $\mathcal{T}_\Phi^i$ to $\mathcal{T}_\Phi^{i+j}$, where $\mathcal{T}_\Phi^\ell = \mathcal{T}_\Phi$ for $\ell < -1$ and $\mathcal{T}_\Phi^\ell = 0$ for $\ell > 1$.

A corresponding principal bundle, with structure group the subgroup $P \subset SU(p+1, q+1)$ from Section 2.2, can be constructed as the frame bundle for $\mathcal{T}_\Phi$ sensitive to the filtration structure. This is an adapted frame bundle in the sense of [9, Section 2.2]. The connection ∇ on $\mathcal{T}_\Phi$ from above induces a connection on $\mathcal{A}$, and one immediately verifies that the non-degeneracy property observed above implies that this is a tractor connection. In view of Section 2.12 of [9] we therefore only have to verify that the curvature Ω of ∇, which is computed in [24] satisfies the normalisation condition.

An explicit formula for the normalisation condition can be found in [12, Section 2.5]. Translated to geometric terms, this reads as

$$0 = \sum_j \{\eta_j, \Omega(\xi, \xi_j)\} + \frac{1}{2} \sum_j \Omega(\Pi(\{\eta_j, A\}), \xi_j) \tag{4.2}$$

for all vector fields ξ, where $A \in \mathcal{A}$ satisfies $\Pi(A) = \xi$, the ξ_j form a real local frame for TM and the η_j form the dual frame for T^*M. As in Section 3.4, the brackets $\{\ ,\ \}$ denote the tensorial Lie bracket on $\mathcal{A}$ induced by the Lie bracket on $\mathfrak{g}$. Moreover, one uses the natural identification $T^*M \cong \mathcal{A}^1$, see [9, Section 2.8]. Since the formula for Ω in [24] refers to a choice of θ, we may assume that $\eta_0 = \theta$ and $\xi_0 = r$ while the remaining elements form dual frames for H and H^*. This implies that $\eta_0 \in \mathcal{A}^2$ and hence $\{\eta_0, A\} \in \mathcal{A}^0 = \ker(\Pi)$ for all $A \in \mathcal{A}$. On the other hand, if $\Pi(A) = \xi \in H$, then $A \in \mathcal{A}^1$, which implies that $\Pi(\{\eta_j, A\}) = 0$

for all j. For $A \in \mathcal{A}$ such that $\Pi(A) = \xi_0 = r$, one computes directly that $\Pi(\{\eta_j, A\}) = -J\xi_j$. Then vanishing of the second sum in (4.2) follows from the fact that $\Omega_\alpha{}^\alpha{}_\Phi{}^\Psi = 0$ which is formula (3.4) of [24].

To analyse the first sum in (4.2), we use the matrix representation of Ω from [24]. The bracket $\{\,,\,\}$ may then be computed as a commutator of matrices, by representing a one-form φ by the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ \varphi_\alpha & 0 & 0 \\ -i\varphi_0 & -\varphi_{\bar\beta} & 0 \end{pmatrix}.$$

From this, and the formula for Ω in [24], we immediately conclude that the term with $j = 0$ does not contribute to the sum. Hence we are left with real dual frames of H and H^*, and it suffices to show that the expression vanishes if one sums over complex dual frames. Then the computation can be done directly, and the identities for the curvature quantities derived in [24] immediately show that all traces which show up in the result vanish. ◻

4.4. Conformal standard tractors with a parallel and orthogonal complex structure. For conventions on conformal structures and results on conformal tractor calculus, we refer to [1, 26], but we use tildes in the notation to distinguish conformal objects from CR objects. Let $\tilde M$ be a smooth manifold of dimension $2n+2$ endowed with a conformal structure $[g]$ of signature $(2p+1,\ 2q+1)$. By $\mathbf{g} \in \tilde{\mathcal{E}}_{(ab)}[-2]$ we denote the conformal metric. Denoting the standard tractor bundle by $\tilde{\mathcal{T}}^A$, we have a conformally invariant metric $\tilde h$ of signature $(2p+2,\ 2q+2)$ and a composition series

$$\tilde{\mathcal{T}}^A = \tilde{\mathcal{E}}[1] \in \tilde{\mathcal{E}}_a[1] \in \tilde{\mathcal{E}}[-1].$$

Let X^A be the canonical section of $\tilde{\mathcal{T}}^A[1]$ which represents the inclusion $\tilde{\mathcal{E}}[-1] \to \tilde{\mathcal{T}}^A$. Tractor indices can be raised and lowered using $\tilde h$. For example, we obtain a natural section $X_A \in \tilde{\mathcal{T}}_A[1]$, which represents the natural projection $\tilde{\mathcal{T}}^A \to \tilde{\mathcal{E}}[1]$. We shall raise and lower indices in this way without further mention.

A Weyl structure is a splitting of the filtration of the tractor bundle. Evidently this is equivalent to a section Y_A of $\tilde{\mathcal{T}}_A[-1]$ such that $X^AY_A = 1$ and $Y^AY_A = 0$. We are most interested in splittings that arise from a choice of scale. A scale is a section s of $\tilde{\mathcal{E}}_+[1]$, the positive ray subbundle in $\tilde{\mathcal{E}}[1]$. (This determines a metric from the conformal class viz. $g = s^{-2}\mathbf{g}$.) There is a conformal generalisation of the exterior derivative $\tilde d$, see [4]. This arises from the restriction of the exterior derivative on the total space of the conformal metric bundle to differential forms which are homogeneous for the obvious $\mathbb{R}_+$-action. Thus this operator is conformally invariant, first order and, for example, maps sections of $\tilde{\mathcal{E}}[1]$ to sections of $\tilde{\mathcal{T}}_A/\tilde{\mathcal{E}}[-1]$. Then Y_A is the unique null (weighted) tractor which maps to $s^{-1}\tilde d s$

under the canonical quotient map $\tilde{\mathcal{T}}_A[-1] \to \tilde{\mathcal{T}}_A[-1]/\tilde{\mathcal{E}}[-2]$. Henceforth Y_A will mean the section on $\tilde{M}$ arising from a scale in this way. Having made this choice, we obtain

$$\tilde{\mathcal{T}}^A \stackrel{s}{=} \tilde{\mathcal{E}}[1] \oplus \tilde{\mathcal{E}}_a[1] \oplus \tilde{\mathcal{E}}[-1].$$

and we write $\mathbb{Z}$ for the complementary projector/injector. That is, a triple (σ, μ_a, ρ) from the direct sum represents the element $\sigma Y^A + \mathbb{Z}^A{}_a \mu^a + \rho X^A \in \tilde{\mathcal{T}}^A$. Under a change of scale $s \mapsto e^{-\Upsilon} s$ these projectors transform according to

$$\hat{\mathbb{Z}}^{Ab} = \mathbb{Z}^{Ab} + \Upsilon^b X^A, \quad \hat{Y}^A = Y^A - \Upsilon_b \mathbb{Z}^{Ab} - \frac{1}{2} \Upsilon_b \Upsilon^b X^A. \tag{4.3}$$

where $\Upsilon_a = d\Upsilon$. The tractor metric $\tilde{h}$ is characterised by $(\sigma, \mu_a, \rho) \mapsto 2\sigma\rho + \mathbf{g}_{ab}\mu^a\mu^b$. On the other hand, the conformal metric $\mathbf{g}$ is recovered from the tractor metric by the expression

$$\mathbf{g}_{ab}\xi^a\eta^b = \tilde{h}_{AB}\mathbb{Z}^A{}_a\mathbb{Z}^B{}_b\xi^a\eta^b. \tag{4.4}$$

Note that although the projector $\mathbb{Z}^A{}_a$ depends on a choice of metric, from the conformal class, it follows easily from (4.3), and the inner product relations amongst the projectors, that the expression on the right-hand side is independent of this choice.

We use the same symbol $\tilde{\nabla}$ for the Levi-Civita connections determined by a choice of scale, and also for the canonical tractor connections, the distinction is again by context. Using the coupled connection, the tractor connection is then determined by

$$\tilde{\nabla}_a X^A = \mathbb{Z}^A{}_a, \quad \tilde{\nabla}_a \mathbb{Z}^A{}_b = -\tilde{P}_{ab}X^A - Y^A\mathbf{g}_{ab}, \quad \tilde{\nabla}_a Y^A = \tilde{P}_{ab}\mathbb{Z}^{Ab}, \tag{4.5}$$

where $\tilde{P}_{ab}$ is the conformal Rho-tensor (or Schouten-tensor).

Suppose that the standard tractor bundle is endowed with a complex structure $\mathbb{J}$ which is orthogonal (or equivalently skew symmetric) with respect to the tractor metric. In abstract indices we have $\mathbb{J}_A{}^B$ with $\mathbb{J}_A{}^B\mathbb{J}_B{}^C = -\delta_A^C$ and $\mathbb{J}_{AB} = -\mathbb{J}_{BA}$. Using $\mathbb{J}$ we obtain a canonical section $\mathsf{K}^A := (\mathbb{J}X)^A = X^B\mathbb{J}_B{}^A \in \tilde{\mathcal{T}}^A[1]$. Since $\mathbb{J}$ is orthogonal, we immediately obtain $\mathsf{K}^A\mathsf{K}_A = X^AX_A = 0$ as well as $\mathsf{K}^AX_A = -\mathsf{K}^AX_A$ so that K is null and orthogonal to X.

Since $\mathsf{K}^AX_A = 0$, the element $\mathbf{k}^a := \mathsf{K}^A\mathbb{Z}_A{}^a \in \tilde{\mathcal{E}}^a$ is independent of the choice of scale. (In the case of a Fefferman space this is, by construction, the conformal Killing field $\mathbf{k}$ from Theorem 3.1.) Since in any scale $\mathsf{K}^A - \mathbb{Z}^A{}_b\mathbf{k}^b = X^A\rho$, for some density ρ, it follows from (4.4), and that K^A is null, that $\mathbf{k}^a$ is null for the conformal structure.

As mentioned K is, by construction, independent of any choice of scale. On the other hand a choice of scale determines a dual object viz. $\mathsf{L}^A := Y^B\mathbb{J}_B{}^A$. Arguing in a manner similar to the above, we see that L^A is null and orthogonal to Y, while $\mathsf{L}^A\mathsf{K}_A = 1$ since $\mathbb{J}$ is orthogonal.

Definition. A conformal scale $s \in \mathcal{E}_+[1]$ on $\tilde{M}$ is called *preferred* (with respect to the complex structure $\mathbb{J}$) if and only if for the fundamental derivative $\tilde{\mathbb{D}}$ we have $\tilde{\mathbb{D}}_{\mathbb{J}} s = 0$.

Proposition 4.2. *Let $\tilde{M}$ be a smooth manifold of dimension $2n+2$ endowed with a conformal structure of signature $(2p+1, 2q+1)$ and an orthogonal parallel complex structure $\mathbb{J}$ on the standard tractor bundle $\tilde{\mathcal{T}}$. Let K^A the canonical weighted tractors constructed above and let $\mathbf{k}^a = \mathsf{K}^A \mathbb{Z}_A{}^a$ be the conformal Killing field underlying $\mathbb{J}$. Let s be a preferred scale on $\tilde{M}$, let L^A be the associated weighted tractor, put $\ell_a := \mathsf{L}_A \mathbb{Z}^A{}_a$. Then in the scale s we have:*

(1) $\tilde{\nabla}_a \mathbf{k}^a = 0$, *so $\mathbf{k}^a$ is a Killing field for the metric determined by s.*
(2) $\mathsf{K}^A = \mathbb{Z}^A{}_b \mathbf{k}^b$, $\mathsf{L}_A = \mathbb{Z}_A{}^a \ell_a$, *and hence $\mathbf{k}^a \ell_a = 1$ and $\ell^a \ell_a = 0$.*
(3) $\mathbf{k}^b \tilde{\nabla}_b \mathbf{k}^a = \ell_b \tilde{\nabla}^b \mathbf{k}^a = 0$.
(4) $\ell_a = \tilde{P}_{ab} \mathbf{k}^b$ *and* $\mathbf{k}^b \tilde{\nabla}_b \ell_a = 0$.
(5) *The almost complex structure $\mathbb{J}$ is explicitly given by*

$$\mathbb{J}_{AB} = 2Y_{[A} \mathbb{Z}_{B]}{}^b \mathbf{k}_b + \mathbb{Z}_A{}^a \mathbb{Z}_B{}^b \tilde{\nabla}_a \mathbf{k}_b + 2X_{[A} \mathbb{Z}_{B]}{}^b \tilde{P}_{bc} \mathbf{k}^c.$$

Proof. Recall the definition of the conformally invariant *tractor-D* operator, which acts on arbitrary weighted tractor fields. For a conformal tractor bundle $\tilde{\mathcal{W}}$, the operator D_A maps sections of $\tilde{\mathcal{W}}[w]$ to sections of $\tilde{\mathcal{T}}_A \otimes \tilde{\mathcal{W}}[w-1]$. In our notation, it is given by

$$D_A t := (2n + 2w) w Y_A t + (2n + 2w) \mathbb{Z}_A{}^a \tilde{\nabla}_a t - X_A (\tilde{\nabla}^a \tilde{\nabla}_a + w\tilde{P}) t, \tag{4.6}$$

where $\tilde{P} = \tilde{P}_a{}^a$.

For a vector field v^a define the tractor $V^A \in \tilde{\mathcal{T}}^A[1]$ as being given, in a scale s', by $\tilde{V}^A := \mathbb{Z}^A{}_a v^a - (1/(2n+2)) X^A \tilde{\nabla}'_a v^a$, where we write $\tilde{\nabla}'$ for covariant derivatives with respect to s'. It is easily verified that this defines a conformally invariant operation. By Lemma 2.1 of [23], the fact that $\mathbf{k}^a$ is a conformal Killing is equivalent to the corresponding tractor K^A satisfying $D_A K_B = -D_B K_A$, while from Proposition 2.2 there, the differential splitting operator relating conformal Killing fields to sections of the adjoint tractor bundle satisfying equation (3.1) is given by $v^a \mapsto (1/(2n+2)) D_{[A} \tilde{V}_{B]}$, where $[\cdots]$ indicates that we take the skew part over the enclosed indices. Since $\mathbb{J}$ is parallel and $\mathbf{k} = \Pi(\mathbb{J})$, we can recover $\mathbb{J}$ from $\mathbf{k}$ as $\mathbb{J}^A{}_B = (1/(2n+2)) D^A K_B$. But then $\mathsf{K}_B = \mathbb{J}^A{}_B X_A = K_B$. That is, in any scale s' we have $\mathsf{K}^A = \mathbb{Z}^A{}_a \mathbf{k}^a - (1/(2n+2)) X^A \tilde{\nabla}'_a \mathbf{k}^a$.

By the formula for $\tilde{\mathbb{D}}$ in [8] the equation $\tilde{\mathbb{D}}_{\mathbb{J}} s = 0$ expands to

$$n \mathbf{k}^a \tilde{\nabla}'_a s - s \tilde{\nabla}'_a \mathbf{k}^a = 0.$$

This holds for any s'. But, by construction for the metric determined by s, we have $\tilde{\nabla}_a s = 0$, and since s is nowhere vanishing, (1) follows.

Thus, in the preferred scale s, we have $\mathsf{K}^A = \mathbb{Z}^A{}_b \mathbf{k}^b$, and

$$0 = \mathsf{K}^A Y_A = (\mathbb{J}_B{}^A X^B) Y_A = (\mathbb{J}_B{}^A Y_A) X^B = -\mathsf{L}_B X^B.$$

Since we already know that $\mathsf{L}_B Y^B = 0$, we get $\mathsf{L}_A = \mathbb{Z}_A{}^a \ell_a$ and then $\mathsf{K}^A \mathsf{L}_A = 1$ and $\mathsf{L}^A \mathsf{L}_A = 0$ imply the rest of (2). The formula for $\mathbb{J}$ in (5) is then obtained by expanding $\mathbb{J}^A{}_B = (1/(2n+2)) D^A \mathsf{K}_B$.

Using (5) to expand $\mathsf{L}^A = \mathbb{J}_B{}^A Y^B$, we obtain $\mathsf{L}_A = \mathbb{Z}_A{}^b \tilde{P}_{bc} \mathbf{k}^c$ and hence the formula for ℓ_a in (4). Note that this implies $\tilde{P}_{ab} \mathbf{k}^a \mathbf{k}^b = 1$, which is familiar from Sparling's characterisation of Fefferman spaces, see [27].

Differentiating $\mathbf{k}^a \mathbf{k}_a = 0$, we get $\mathbf{k}^a \tilde{\nabla}_b \mathbf{k}_a = 0$, which implies first equation in (3) by the skew symmetry of $\tilde{\nabla}_b \mathbf{k}_a$. Next, from the definition of L, we get $\mathsf{L}^A \mathbb{J}_{AB} = -Y_B$, and expanding this using (5), the second part of (3) follows.

Since $\mathbf{k}^a$ is a Killing field, its Lie derivative annihilates $\tilde{P}_{ab}$. Of course the Lie bracket of $\mathbf{k}^a$ with itself vanishes, and so the Lie derivative by $\mathbf{k}^a$ annihilates $\ell_a = \tilde{P}_{ab} \mathbf{k}^b$. This reads as $0 = \mathbf{k}^b \tilde{\nabla}_b \ell_a - \ell^b \tilde{\nabla}_b \mathbf{k}_a$, and the second summand vanishes by (3). □

From the formula in part (5), we immediately get an explicit formula for the normal conformal Killing forms obtained from $\mathbb{J}$ in Corollary 3.2.

Corollary 4.3. *For each $j = 1, \dots, n-1$, the form defined in a preferred scale s as $\mathbf{k}_{[a}(\tilde{\nabla}_{a_1} \mathbf{k}_{b_1}) \cdots (\tilde{\nabla}_{a_j} \mathbf{k}_{b_j]})$ defines a normal conformal Killing $(2j+1)$-form on $\tilde{M}$.*

In a preferred scale the corresponding metric from the conformal class may be put in the form

$$g_{ab} = 2\mathbf{k}_a \odot \ell_b + \tilde{h}_{ab} \tag{4.7}$$

where $\tilde{h}_{ab}$ annihilates $\mathbf{k}^a$ and ℓ^a.

In the case of a Fefferman space, we will shortly describe such a metric in terms of tensors on M. Before doing that, we will study the decomposition of the tangent spaces induced by a choice of preferred scale in more detail.

4.5. Decomposition of the tangent bundle. Let us write V for the line subbundle in $T\tilde{M}$ spanned by $\mathbf{k}$. Since $\mathbf{k}$ is null, the orthocomplement $\mathbf{k}^\perp$ contains V, and defining $\tilde{H} := \mathbf{k}^\perp / V$, and $\tilde{Q} := T\tilde{M}/\mathbf{k}^\perp$, we obtain a composition series for $T\tilde{M}$, namely

$$T\tilde{M} = \tilde{Q} \in \tilde{H} \in V. \tag{4.8}$$

The developments in Section 4.4 above show that a choice of a preferred scale s leads to a splitting of the filtration $V \subset \mathbf{k}^\perp \subset T\tilde{M}$. Since $\ell_a \mathbf{k}^a = 1$, we see that ℓ^a spans a line subbundle in $T\tilde{M}$ which is complementary to $\mathbf{k}^\perp$, and that $\tilde{H}_s := \mathbf{k}^\perp \cap \ell^\perp$ is a corank one subbundle of $\mathbf{k}^\perp$ complementary to V. In particular, a choice of preferred scale induces an identification of $\tilde{H}_s = \mathbf{k}^\perp \cap \ell^\perp$ with $\tilde{H}$. Let $\mathbb{I}_b^a$ be the projector onto $\mathbf{k}^\perp \cap \ell^\perp$, i.e.,

$$\mathbb{I}_b^a = \delta_b^a - \mathbf{k}^a \ell_b - \ell^a \mathbf{k}_b. \tag{4.9}$$

By duality, the whole picture carries over to the cotangent bundle in an obvious way. In particular, we obtain a composition series

$$T^*\tilde{M} = V^* \Subset \tilde{H}^* \Subset \tilde{Q}^*. \tag{4.10}$$

The splitting determined by a preferred scale s comes from the line subbundle spanned by ℓ_a and the annihilator of ℓ^a. The corresponding decomposition of a one-form ω explicitly reads as

$$\omega_a = \omega_b \mathbf{k}^b \ell_a + \omega_b \ell^b \mathbf{k}_a + \mathbb{I}_a^b \omega_b. \tag{4.11}$$

Combining the projectors $\mathbb{I}$ and $\mathbb{Z}$ we obtain

$$\tilde{W}_A{}^a := \mathbb{Z}_A{}^b \mathbb{I}_b^a = \mathbb{Z}_A{}^a - \mathsf{K}_A \ell^b - \mathsf{L}_A \mathbf{k}^b. \tag{4.12}$$

Viewed as a map $T^*\tilde{M} \to \tilde{\mathcal{T}}_A$, this annihilates the subbundle spanned by $\mathbf{k}_a$ and ℓ_b, and is injective on $\tilde{H}_s^*$. In terms of this, the form $\tilde{h}_{ab}$ in the metric (4.7) is given by

$$\tilde{h}_{ab} = s^{-2} \tilde{W}^A{}_a \tilde{W}^B{}_b \tilde{h}_{AB}.$$

Now $\mathbb{J}$ preserves the (weighted) tractor subspace spanned pointwise by X and K and similarly (in a choice of preferred scale s) it preserves the subspace spanned pointwise by Y and L. Thus $\mathbb{J}$ determines a canonical complex structure $\tilde{J}$ on the subquotient bundle $\tilde{H} = \mathbf{k}^\perp / V$ of $T\tilde{M}$. We may equally view this as an almost complex structure on $\tilde{H}_s = \mathbf{k}^\perp \cap \ell^\perp \subset T\tilde{M}$. In this picture, it is given by $\tilde{J}_a{}^b = \mathbb{J}_A{}^B \tilde{W}^A{}_a \tilde{W}_B{}^b$. From parts (3) and (4) of Proposition 4.2 and (4.12) we see that $\mathbb{Z}_A{}^a \mathbb{Z}_B{}^b \tilde{\nabla}_a \mathbf{k}_b = \tilde{W}_A{}^a \tilde{W}_B{}^b \tilde{\nabla}_a \mathbf{k}_b$, so part (5) of Proposition 4.2 immediately gives

$$\mathbb{J}^A{}_B = X^A \mathsf{L}_B - \mathsf{K}^A Y_B + \tilde{W}^A{}_a \tilde{W}_B{}^b \tilde{\nabla}^a \mathbf{k}_b + Y^A \mathsf{K}_B - \mathsf{L}^A X_B. \tag{4.13}$$

which in turns shows that $\tilde{J}_a{}^b = \tilde{\nabla}_a \mathbf{k}^b$.

We will also need the complexified version of the decomposition of the tangent and cotangent bundles. The composition series (4.8) and (4.10) carry over to the complexified setting without changes. The main additional input is that the complexification of the subquotient $\tilde{H}$ splits into a holomorphic and an anti-holomorphic part. We will use upper Greek indices for the holomorphic, and bared upper Greek indices for the anti holomorphic part. Correspondingly, the projectors $\mathbb{I}_b^a$ give rise to $\mathbb{I}_b^\alpha$ and $\mathbb{I}_b^{\bar{\alpha}}$. Correspondingly, we get $\tilde{W}_A{}^\alpha$ and $\tilde{W}_A{}^{\bar{\alpha}}$.

4.6. The complexified standard tractor bundle. To relate the conformal calculus developed so far to CR tractor calculus, it will convenient to complexify the standard tractor bundle. This complexification splits into its holomorphic and anti-holomorphic parts via $t \mapsto (\frac{1}{2}(t - i\mathbb{J}t), \frac{1}{2}(t + i\mathbb{J}t))$. We write $\tilde{\mathcal{T}}^A \otimes \mathbb{C} = \tilde{\mathcal{T}}^\Phi \oplus \tilde{\mathcal{T}}^{\bar{\Phi}}$ and we will view $\tilde{\mathcal{T}}^A$ as a (real) subbundle in the complexification. In terms of this splitting the complex linear extension of $\mathbb{J}$ is diagonalised. Concerning the tractor metric, we first consider the unique Hermitian extension $\mathcal{H}_{AB}$ of $\tilde{h}_{AB}$ on $\tilde{\mathcal{T}}_A$. Explicitly, $\mathcal{H}_{AB} = \tilde{h}_{AB} - i\mathbb{J}_{AB}$. This then extends to a complex bilinear form on $\tilde{\mathcal{T}}^A \otimes \mathbb{C}$. Writing $\tilde{h}$ and $\mathbb{J}$ for the complex linear extensions of these tractors, in the matrix notation we have

$$\mathbb{J}_B{}^A = \begin{pmatrix} i\delta_\Psi^\Phi & 0 \\ 0 & -i\delta_{\bar{\Psi}}^{\bar{\Phi}} \end{pmatrix} \quad \text{and} \quad \tilde{h}_{AB} = \begin{pmatrix} 0 & \mathcal{H}_{\Phi\bar{\Psi}} \\ \mathcal{H}_{\bar{\Phi}\Psi} & 0 \end{pmatrix},$$

where $\mathcal{H}_{\Phi\bar{\Psi}}$ is $\frac{1}{2}\mathcal{H}_{AB}$, or more accurately

$$\frac{1}{2}\mathcal{H}_{AB} = \begin{pmatrix} 0 & \mathcal{H}_{\Phi\bar{\Psi}} \\ 0 & 0 \end{pmatrix},$$

and $\mathcal{H}_{\bar{\Phi}\Psi}$ is the conjugate object.

We write $\tilde{Z}_\Phi$ and $\tilde{Z}_{\bar{\Phi}}$ for the holomorphic and anti-holomorphic parts of the canonical section X_A of the weighted conformal standard tractor bundle $\mathcal{T}_A[1]$. That is,

$$X_A = (\tilde{Z}_\Phi, \tilde{Z}_{\bar{\Phi}}).$$

From this and $\mathsf{K}_B = \mathbb{J}^A{}_B X_A$, it follows immediately that

$$\mathsf{K}_B = (-i\tilde{Z}_\Phi, i\tilde{Z}_{\bar{\Phi}}) \iff \mathsf{K}^B = (i\tilde{Z}^\Phi, -i\tilde{Z}^{\bar{\Phi}}).$$

For a choice of preferred scale on $\tilde{M}$, we write $\frac{1}{2}\tilde{Y}^\Phi$ and $\frac{1}{2}\tilde{Y}^{\bar{\Phi}}$ for the holomorphic and anti-holomorphic parts of Y^A, i.e., $Y^A = \frac{1}{2}(\tilde{Y}^\Phi, \tilde{Y}^{\bar{\Phi}})$. (The normalisation on $\tilde{Y}^\Phi$ means that we have $\tilde{Y}^\Phi \tilde{Z}_\Phi = 1$ which simplifies calculations and is consistent with [24].) It follows that

$$\mathsf{L}^A = \frac{1}{2}(i\tilde{Y}^\Phi, -i\tilde{Y}^{\bar{\Phi}}) \iff \mathsf{L}_A = \frac{1}{2}(-i\tilde{Y}_\Phi, i\tilde{Y}_{\bar{\Phi}}).$$

Finally, we also have the complexified versions of the $\tilde{W}$-projectors. The fact that $\tilde{W}$ is complex linear implies that its complexification preserves the decomposition into holomorphic and anti-holomorphic part, so we have $\tilde{W}_A^a = (\tilde{W}_\Phi^\alpha, \tilde{W}_{\bar{\Phi}}^{\bar{\alpha}})$, and no combination of barred and unbarred indices.

4.7. The case of a Fefferman space. If $\tilde{M}$ is the Fefferman space of a CR manifold M, then there are several refinements of the above picture. First observe that there is a special class of (conformal) scales on $\tilde{M}$, namely those coming from CR scales on M. A CR scale on M simply is a choice of positive contact form θ. As observed in Section 2.3, the bundle $Q = TM/H$ naturally includes into $\mathcal{E}(1,1)$. Now θ defines a linear map $Q \to \mathbb{R}$ and by complex linear extension a section of the dual bundle $\mathcal{E}(-1,-1)$. This section can be viewed as U^{-1}, for a positive section U of $\mathcal{E}(1,1)$. Now as observed in Section 3.5, we may also view U as a section of $\mathcal{E}_{\mathbb{C}}[2]$, which is easily seen to be in $\mathcal{E}_{+}[2]$. Hence its square root can be used as a scale s on $\tilde{M}$. We will also call such a conformal scale a *CR scale*. By dint of context this should cause no confusion. By Proposition 3.8, sections of $\tilde{\mathcal{E}}(1,1)$ are characterised among sections of $\tilde{\mathcal{E}}[2]$ by $\tilde{\mathbb{D}}_{\mathbb{J}} s = 0$. So in fact CR scales are exactly preferred scales in the sense introduced in the last section and we can carry over the results that hold for these scales. In particular, in a CR scale $\mathbf{k}^a$ is a Killing field and $Y_A \mathsf{K}^A = 0 = \mathsf{L}^A X_A$. In the subsequent, calculations scales, when chosen, will be CR scales.

From the discussion of $\tilde{\mathbb{D}}$ in Section 4.4 we see that on density bundles, $\tilde{\mathbb{D}}_{\mathbb{J}}$ differs from $\tilde{\nabla}_{\mathbf{k}}$ by a multiple of $\tilde{\nabla}_a \mathbf{k}^a$, so in a CR-scale the two operators coincide. By definition this implies that in a CR scale, the double-D-operator $\tilde{\mathbb{D}}^{\nabla}_{\mathbb{J}}$ on any weighted tractor bundle coincides with $\tilde{\nabla}_{\mathbf{k}}$.

Proposition 4.4.

(1) *The holomorphic and anti-holomorphic parts $\tilde{Z}^{\Phi}$ and $\tilde{Z}^{\bar{\Phi}}$ of $X^A \in \Gamma(\tilde{\mathcal{T}}^A_{\mathbb{C}}[-1])$ lie in the subspaces $\Gamma(\mathcal{T}(1,0))$ respectively $\Gamma(\mathcal{T}^{\bar{\Phi}}(0,1))$ and coincide with the sections Z^{Φ} and $Z^{\bar{\Phi}}$ from Section 4.2.*

(2) *For a choice s of CR-scale, the sections $\tilde{Y}^{\Phi}$ and $\tilde{Y}^{\bar{\Phi}}$ from Section 4.6 lie in the subspaces $\Gamma(\mathcal{T}(0,-1))$ and $\Gamma(\mathcal{T}(-1,0))$ and coincide with the sections Y^{Φ} and $Y^{\bar{\Phi}}$ from Section 4.2.*

Proof. (1) By definition, $2\tilde{Z}^{\Phi} = X^A - i\mathsf{K}^A$. Now $\mathbf{k}^a \tilde{\nabla}_a X^A = \mathbf{k}^a \mathbb{Z}^A{}_a = \mathsf{K}^A$. Since $\mathbb{J}$ is parallel, this shows that $\tilde{Z}^{\Phi}$ is an eigenvector for $\tilde{\nabla}_{\mathbf{k}} = \tilde{\mathbb{D}}^{\nabla}_{\mathbb{J}}$ with eigenvalue i, so by Corollary 3.9 it lies in $\Gamma(\mathcal{T}(1,0)) \subset \Gamma(\tilde{\mathcal{T}} \otimes \tilde{\mathcal{E}}_{\mathbb{C}}[1])$. Viewed as an inclusion $\tilde{\mathcal{E}}_{\mathbb{C}}[-1] \to \tilde{\mathcal{T}}_{\mathbb{C}}$, $\tilde{Z}^{\Phi}$ represents the complex linear extension of X^A. From Section 2.5 we see that this represents the inclusion $\mathcal{E}(-1,0) \to \mathcal{T}$, which is given by Z^{Φ}. The other statement follows in the same way.

(2) Next consider the tractor field Y^A coming from a choice $U \in \mathcal{E}(1,1) \subset \tilde{\mathcal{E}}_{+}[2]$ of CR scale on $\tilde{M}$. Applying the fundamental derivative, we obtain $\tilde{\mathbb{D}}U \in \tilde{\mathcal{A}}^* \otimes \tilde{\mathcal{E}}[2]$. Hence $U^{-1}\tilde{\mathbb{D}}U \in \tilde{\mathcal{A}}^* \cong \tilde{\mathcal{A}}$, where we use the trace-form for the last identification. We claim that this is the *grading element* E_s associated to the square root s of U, i.e., its eigenspaces represent the splitting of $\tilde{\mathcal{T}}$ according to U. Via the scale U, the adjoint tractor bundle can be identified with $T\tilde{M} \oplus \mathfrak{co}(T\tilde{M}) \oplus T^*\tilde{M}$, and the sum of the first and last part is orthogonal to the middle part with respect

to the trace-form. Also, the middle part splits into the orthogonal direct sum of multiples of $\mathrm{id}_{T\tilde{M}}$ and $\mathfrak{so}(T\tilde{M})$. In the scale determined by U, we have $\tilde{\nabla}U = 0$ so from the formula for $\tilde{\mathbb{D}}$ in [8] (or [20]) we get $\tilde{\mathbb{D}}_t U = 0$ for $t \in T\tilde{M} \oplus \mathfrak{so}(T\tilde{M}) \oplus T^*\tilde{M}$. Hence $U^{-1}\tilde{\mathbb{D}}U \in \Gamma(\tilde{\mathcal{A}})$ is a multiple of E_s. By definition $E_s \circ E_s$ has trace two, so we can compute

$$U^{-1}\tilde{\mathbb{D}}U = \frac{1}{2}E_s U^{-1}\tilde{\mathbb{D}}_{E_s}U = \frac{1}{2}E_s U^{-1}2U = E_s,$$

where we have used that the algebraic action of E_s on $\mathcal{E}[w]$ is given by multiplication by w. By Proposition 3.11, since U lies in $\mathcal{E}(1,1) \subset \tilde{\mathcal{E}}[2]$, for a section t of $\mathcal{A}$, the section $\tilde{\mathbb{D}}_t U$ lies in $\mathcal{E}(1,1)$ and is equal to the CR fundamental derivative $\mathbb{D}_t U$. Now one can play the same game as above in the CR world to show that $U^{-1}\mathbb{D}U$, viewed as a section of $\mathcal{A}$, equals $\frac{1}{2}E_U$, where E_U is the CR grading element determined by the scale U. (The factor $\frac{1}{2}$ is caused by the fact the ± 1 eigenspaces of E_U have each real dimension 2, so applying the real trace-form to two copies of E_U, one obtains 4 rather than 2.) This means that E_U is twice the component of E_s in the decomposition of $\tilde{\mathcal{A}}$ from Section 3.4, and hence twice the complex linear part of E_s.

Now consider the tractor field Y^A determined by s. Viewed as a projection $\tilde{\mathcal{T}} \to \tilde{\mathcal{T}}^1$, this is the projection onto the eigenspace of E_s with eigenvalue 1. Explicitly, $Y^A = \frac{1}{2}E_s \circ (E_s + \mathrm{id})$. A direct computation using $E_s \circ \mathbb{J} \circ E_s = 0$ shows that $\frac{1}{2}E_U \circ (E_U + \mathrm{id})$ is twice the complex linear part of this projection. Hence decomposing Y^A into holomorphic and anti-holomorphic parts, we obtain $Y^A = \frac{1}{2}(Y^{\Phi}, Y^{\bar{\Phi}})$, for the weighted CR tractors determined by the scale U as in Section 4.2. □

4.8. Relating the tangent bundles. Our next task is to interpret the decomposition of the tangent bundle from Section 4.5 in the special case of a Fefferman space $\pi : \tilde{M} \to M$. The subbundle $V \subset T\tilde{M}$, spanned by $\mathbf{k}$, is the vertical subbundle of π.

Lemma 4.5. *Let $\pi : \tilde{M} \to M$ be a Fefferman space. Let θ be a contact form on M and consider the corresponding CR scale on $\tilde{M}$. Then we have $\mathbf{k}_a = 2\pi^*\theta$, $\tilde{\nabla}_a \mathbf{k}_b = 2\pi^* d\theta$ and the vector field $2\ell^a$ is the unique null lift of the Reeb field associated to θ.*

In particular, the subbundle $\tilde{H} \subset T\tilde{M}$ from Section 4.5 is exactly the preimage of the CR subbundle $H \subset TM$.

Proof. The definition of $\tilde{W}$ in (4.12) reads as

$$\mathbb{Z}^A{}_a = \mathsf{L}^A k_a + \mathsf{K}^A \ell_a + \tilde{W}^A{}_a.$$

Since $\tilde{\nabla}_a X^A = \mathbb{Z}^A{}_a$, we have $\mathbf{k}_a = \mathsf{K}_A \mathbb{Z}^A_a = \mathsf{K}_A \tilde{\nabla}_a X^A$. Since $X^A = (Z^{\Phi}, Z^{\bar{\Phi}})$ and $K_A = (-iZ_{\Phi}, iZ_{\bar{\Phi}})$, this is $-2iZ_{\Phi}\tilde{\nabla}_a Z^{\Phi}$. Using $Z^{\Phi}Z_{\Phi} = 0$, we see that for any non-vanishing $\sigma \in \mathcal{E}(-1,0)$ we have $Z_\Phi \tilde{\nabla}_a Z^\Phi = Z_\Phi \sigma^{-1} \tilde{\nabla}_a \sigma Z^\Phi$. Now $\sigma Z_\Phi \in \Gamma(\mathcal{T}) \subset \Gamma(\tilde{\mathcal{T}})$, so Proposition 3.3 implies that the one-form $Z_\Phi \sigma^{-1} \tilde{\nabla} \sigma Z^\Phi$ is the pullback of $Z_\Phi \sigma^{-1} \nabla \sigma Z^\Phi = Z_\Phi \nabla Z^\Phi$. From the formulae for the CR tractor connection we obtain $Z_\Phi \nabla Z^\Phi = i\theta$. Thus $k_a = 2\pi^* \theta_a$, so $\mathbf{k}^\perp$ coincides with the preimage of H.

Next by part (1) of Proposition 4.2, $\tilde{\nabla}_a \mathbf{k}_b$ is skew symmetric, so it coincides with $\frac{1}{2}$ times the exterior derivative of $\mathbf{k}_a$, which equals $\pi^* d\theta$.

In view of these two results, the equations $\ell^a \mathbf{k}_a = 1$ and $\ell^b \tilde{\nabla}_b \mathbf{k}_a$ observed in Proposition 4.2 imply that the value of $2\ell^a$ at each point of $\tilde{M}$ projects onto the Reeb field of $\tilde{\theta}$. This pins down ℓ^a uniquely up to adding $f\mathbf{k}^a$ for some smooth function f. But $(\ell^a + f\mathbf{k}^a)(\ell_a + f\mathbf{k}_a) = 2f$, which completes the proof. □

Collecting the results, we see that for a Fefferman space $\pi : \tilde{M} \to M$, the filtration $V \subset \mathbf{k}^\perp \subset T\tilde{M}$ from Section 4.5 has the form $\ker(T\pi) \subset T\pi^{-1}(H) \subset T\tilde{M}$. Note further that the resulting identification of H with $\tilde{H}/V$ is compatible with the complex structures on both bundles, since they were both induced from the complex structure on the tractor bundle.

Next given a choice of CR scale, we can explicitly identify sections of the CR subbundle $H \to M$ with sections of the corresponding subbundle $\tilde{H} = \mathbf{k}^\perp \cap \ell^\perp \subset T\tilde{M}$. Since it will be useful later, we do this in a complexified picture and in a weighted version.

Proposition 4.6. *Let $\pi : \tilde{M} \to M$ be a Fefferman space, and fix some CR scale. Let $\tilde{H} \otimes \mathbb{C} = \tilde{H}^\alpha \oplus \tilde{H}^{\bar{\alpha}}$ be the decomposition of the complexification of $\tilde{H} = \mathbf{k}^\perp \cap \ell^\perp$ into holomorphic and anti-holomorphic parts. Then for arbitrary weights w and w', sections of $H^\alpha(w, w')$ are in bijective correspondence with sections ξ of $\tilde{H}^\alpha[w + w']$ such that $\tilde{\nabla}_{\mathbf{k}} \xi = (w - w' + 1)i\xi$. Likewise, sections of $H^{\bar{\alpha}}(w, w')$ are in bijective correspondence with sections ξ of $\tilde{H}^{\bar{\alpha}}[w + w']$ such that $\tilde{\nabla}_{\mathbf{k}} \xi = (w - w' - 1)i\xi$.*

Proof. Let us first treat the real subbundles H and $\tilde{H}$. The flow lines of $\mathbf{k}^a$ are exactly the fibres of π. From this, one easily concludes that a vector field $\xi \in \mathfrak{X}(\tilde{M})$ is projectable if and only if the Lie derivative $\mathcal{L}_{\mathbf{k}} \xi$ is vertical, and thus $\mathcal{L}_{\mathbf{k}} \xi = \ell(\mathcal{L}_{\mathbf{k}} \xi)\mathbf{k}$ (where we view ℓ as a 1-form). If ξ is a section of the subbundle $\mathbf{k}^\perp \cap \ell^\perp$, then $\ell(\xi) = 0$, and hence $\ell(\mathcal{L}_{\mathbf{k}} \xi) = -(\mathcal{L}_{\mathbf{k}} \ell)(\xi)$. In Section 4.4 we have observed that $\ell_a = \tilde{P}_{ab} \mathbf{k}^b$. Also there we noted that since, in a CR scale, $\mathbf{k}$ is Killing, we have $\mathcal{L}_{\mathbf{k}} \ell = 0$ and it follows that a section ξ of $\mathbf{k}^\perp \cap \ell^\perp$ is projectable if and only if $\mathcal{L}_{\mathbf{k}} \xi = 0$. Since $\tilde{\nabla}$ is torsion free, we get $\mathcal{L}_{\mathbf{k}} \xi = \tilde{\nabla}_{\mathbf{k}} \xi - \tilde{\nabla}_\xi \mathbf{k}$. Thus, sections of H are in bijective correspondence with sections ξ of $\tilde{H}$ such that $\tilde{\nabla}_{\mathbf{k}} \xi = \tilde{\nabla}_\xi \mathbf{k}$.

Now, from Proposition 4.2, $\tilde{\nabla}_\xi \mathbf{k}$ is the complex structure on $\tilde{H}$ applied to ξ, so on sections of $\tilde{H}^\alpha$ this coincides with $i\xi$, and on sections of $\tilde{H}^{\bar{\alpha}}$ it coincides with $-i\xi$. Thus we obtain the result for $w = w' = 0$.

To conclude the proof, we observe that a CR scale is a preferred scale and hence is killed by $\tilde{\mathbb{D}}_{\mathbb{J}}$. Since powers of this scale are used to trivialise density bundles, we conclude that $\tilde{\mathbb{D}}_{\mathbb{J}} = \tilde{\nabla}_{\mathbf{k}}$ on density bundles. Using this, the general result immediately follows from Proposition 3.8. □

4.9. Computing a metric from the conformal class. A choice of CR scale determines a metric from the canonical conformal class on a Fefferman space. We are now ready to compute this explicitly. Choose a contact form θ on M, and let $U \in \mathcal{E}(1,1)$ be the corresponding CR scale. Choose a section σ of $\mathcal{E}(-1,0) \subset \tilde{\mathcal{E}}_{\mathbb{C}}[-1]$ such that $\sigma\bar{\sigma} = U^{-1}$. Locally, we define a smooth function $\gamma_\sigma : \tilde{M} \to \mathbb{R}$ by requiring that $\tilde{\sigma} = \sigma e^{i\gamma_\sigma}$ is a positive real section of $\tilde{\mathcal{E}}[-1]$. By definition, this implies that $\tilde{\sigma} = s^{-1}$, where s is the CR scale determined by U. Using that the points in a fibre of $\pi : \tilde{M} \to M$ determine how the real line $\tilde{\mathcal{T}}^1$ sits inside the complex line $\mathcal{T}^1$, one easily verifies that γ_σ defines a local coordinate for each fibre.

On the other hand, recall that the real part of the Levi form defines a non-degenerate bundle metric on the CR subbundle H. We can uniquely extend this to a (degenerate) bundle metric $L : TM \times TM \to \mathbb{R}$ by requiring that the Reeb vector field inserts trivially into L. This is called the *degenerate Levi metric*.

Proposition 4.7. *Let $\pi : \tilde{M} \to M$ be a Fefferman space, θ a contact form, $U \in \Gamma(\mathcal{E}(1,1))$ the corresponding CR scale. Choose $\sigma \in \Gamma(\mathcal{E}(1,0))$ such that $\sigma\bar{\sigma} = U$ and consider the (local) one-form γ_σ defined above.*

Then the one-form $\tau := \ell_a$

$$\tau = -\frac{i}{2}\pi^*(\sigma^{-1}\nabla\sigma - \bar{\sigma}^{-1}\nabla\bar{\sigma}) - \frac{1}{n+2}\tilde{\theta}\pi^*(P) + d\gamma_\sigma$$

depends only on θ and the metric g_θ in the conformal class corresponding to the CR scale s determined by U is given by

$$g_\theta = \pi^* L + 4\tau \odot \pi^*\theta$$

where L is the degenerate Levi metric.

Proof. We start by computing $\ell_a = \mathsf{L}_B \mathbb{Z}^B{}_a$. Since $\mathbb{Z}^B{}_a = \tilde{\nabla}_a X^B$, we have to calculate $\mathsf{L}_B \tilde{\nabla}_a X^B$. Recall that

$$\tilde{\nabla}_a X^B = (\tilde{\nabla}_a Z_\Psi, \tilde{\nabla}_a Z^{\bar{\Psi}}) \quad \text{and} \quad \mathsf{L}_B = \frac{1}{2}(-iY_\Psi, iY_{\bar{\Psi}}).$$

The section $\tilde{\sigma} = s^{-1}$ is parallel for $\tilde{\nabla}$, which implies $\sigma\tilde{\nabla}_a Z_\Psi = e^{-i\gamma_\sigma}\tilde{\nabla}_a\tilde{\sigma} Z_\Psi$. We apply the Leibniz rule to rewrite this and obtain $(\tilde{\nabla}_a \sigma Z_\Psi + i\sigma Z_\Psi \tilde{\nabla}_a\gamma_\sigma)$. So $\tilde{\nabla}_a Z_\Psi = \sigma^{-1}\tilde{\nabla}_a\sigma Z_\Psi + iZ_\Psi\tilde{\nabla}_a\gamma_\sigma$. Contracting this with iY_Ψ, we get $i\sigma^{-1}Y_\Psi\tilde{\nabla}_a\sigma Z_\Psi - \tilde{\nabla}_a\gamma_\sigma$. Now $\sigma Z_\Psi \in \Gamma(\mathcal{T}) \subset \Gamma(\tilde{\mathcal{T}})$, which implies that the one-form $i\sigma^{-1}Y_\Psi\tilde{\nabla}\sigma Z_\Psi$ is the pullback of $i\sigma^{-1}Y_\Psi\nabla\sigma Z_\Psi$. But by the Leibniz rule and the formulae for ∇Z^Φ we have

$$i\sigma^{-1}Y_\Psi\nabla\sigma Z_\Psi = i\sigma^{-1}\nabla\sigma + \frac{1}{n+2}\theta P.$$

Thus $iY_\Psi\tilde{\nabla}Z_\Psi = \pi^*(i\sigma^{-1}\nabla\sigma + (1/(n+2))\theta P) - \nabla\gamma_\sigma$ and averaging this with its conjugate brings us to

$$\tau = -\frac{i}{2}\pi^*(\sigma^{-1}\nabla\sigma - \bar{\sigma}^{-1}\nabla\bar{\sigma}) - \frac{1}{n+2}\tilde{\theta}\pi^*(P) + d\gamma_\sigma.$$

A simple computation shows that $\frac{i}{2}\pi^*(\sigma^{-1}\nabla\sigma - \bar{\sigma}^{-1}\nabla\bar{\sigma}) - d\gamma_\sigma$ depends only on θ and not on the choice of σ.

In view of the formula (4.7) for g_{ab}, it remains to discuss the quantity $\tilde{h}_{ab}$ occurring there. Since it annihilates $\mathbf{k}^a$, at a point $x \in \tilde{M}$, it descends to $T_{\pi(x)}M$. Since $\ell^a\tilde{h}_{ab} = 0$, this descended quantity annihilates the Reeb field. On the other hand, from the construction of $\tilde{h}_{ab}$ via the tractor metric it follows that the restriction to $H_{\pi(x)}$ coincides with the real part of the Levi form. But this immediately implies that $\tilde{h}_{ab}$ is the pullback of L. □

4.10. Relating preferred connections. Let $\pi : \tilde{M} \to M$ be a Fefferman space. Choosing a contact form θ (on M) we get the Webster-Tanaka connection on M and an induced CR scale on $\tilde{M}$. We next want to compare the Levi-Civita connection associated to the latter with the downstairs Webster-Tanaka connection.

Proposition 4.8. *Let $\pi : \tilde{M} \to M$ be a Fefferman space, θ a choice of contact form on M and s the corresponding CR scale on $\tilde{M}$. Let us denote by ∇ the corresponding Webster-Tanaka connections (if necessary coupled to CR tractor connections) and by $\tilde{\nabla}$ the corresponding Levi-Civita connections (if necessary coupled to conformal tractor connections). By ∇^H we denote the restriction of ∇ to directions in $H \subset TM$.*

(1) *For a complex conformal weighted tractor bundle $\tilde{\mathcal{W}}[w+w']$, and any section $f \in \Gamma(\mathcal{W}(w,w')) \subset \Gamma(\tilde{\mathcal{W}}[w+w'])$, $\mathbb{I}_a^c\tilde{\nabla}_c f$ descends to $\nabla^H f$, and $2\ell^a\tilde{\nabla}_a f$ descends to $\nabla_0 f + (i(w-w')/(n+2))Pf$.*
(2) *For $\xi^a \in \Gamma(H^\alpha(w,w')) \subset \Gamma(\tilde{H}^a[w+w'])$, $\mathbb{I}_c^a\mathbb{I}_b^d\tilde{\nabla}_d\xi^c$ descends to $\nabla^H\xi^\alpha$, and $2\mathbb{I}_c^a\ell^b\tilde{\nabla}_b\xi^c$ descends to $\nabla_0\xi^\alpha + iP^\alpha{}_\beta\xi^\beta + (i(w-w')/(n+2))P\xi^\alpha$.*
(3) *Consider the extension of the projector/injector $\tilde{W}_A{}^a$ associated to s to the complexified tractor bundle. Then the decomposition into holomorphic and antiholomorphic part has the form $\tilde{W}_A{}^a = (\tilde{W}_\Phi{}^\alpha, \tilde{W}_{\bar{\Phi}}{}^{\bar{\alpha}})$, and the two components descend to the CR-objects $W_\Phi{}^\alpha \in \Gamma(\mathcal{E}_\Phi{}^\alpha(-1,0))$ and $W_{\bar{\Phi}}{}^{\bar{\alpha}} \in \Gamma(\mathcal{E}_{\bar{\Phi}}{}^{\bar{\alpha}}(0,-1))$ from Section 4.2.*

Proof. (1) For unweighted tractor bundles, $\tilde{\nabla}$ and ∇ are just tractor connections, so the results follow from Proposition 3.3. By the Leibniz rule it therefore suffices to prove the result for densities. Assume first, that $f \in \Gamma(\mathcal{E}(-1,0))$. Working on M, we can use $Y_\Phi Z^\Phi = 1$ to compute

$$\nabla f = \nabla Y_\Phi f Z^\Phi = Y_\Phi \nabla f Z^\Phi + f Z^\Phi \nabla Y_\Phi.$$

From the formulae for the components of ∇Y_Φ in Section 4.3 we see that $Z^\Phi \nabla^H Y_\Phi = 0$ and $Z^\Phi \nabla_0 Y_\Phi = (i/(n+2))P$. On the other hand, fZ^Φ is an unweighted tractor, so we know that $\tilde{\nabla} f Z^\Phi$ descends to $\nabla f Z^\Phi$. Using the Leibniz rule once more, we get

$$Y_\Phi \mathbb{I}_b^d \tilde{\nabla}_d f Z^\Phi = \mathbb{I}_b^d \tilde{\nabla}_d f + Y_\Phi f \mathbb{I}_b^d \tilde{\nabla}_d Z^\Phi,$$

and this descends to $\nabla^H f$. By (4.5), $\mathbb{I}_b^d \tilde{\nabla}_d X^A = \mathbb{I}_b^d \mathbb{Z}^A{}_d = \tilde{W}^A{}_b$. Since Z^Φ lies in the complex subspace generated by X^A, we see that $\mathbb{I}_b^d \tilde{\nabla}_d Z^\Phi$ lies in the complex subspace generated by $\tilde{W}^A{}_b$. Since both Y_A and L_A hook trivially into $\tilde{W}^A{}_b$ (and hence into any element of that complex subspace), we conclude that $\mathbb{I}_b^d \tilde{\nabla}_d f$ descends to $\nabla^H f$. Passing to powers of f and $\bar{f}$, we see that this holds for arbitrary densities.

To deal with ∇_0, recall from Lemma 4.5 that $2\ell^a$ is a lift of the Reeb field. Since fZ^Φ is an unweighted tractor, $2Y^\Phi \ell^a \tilde{\nabla}_a f Z^\Phi$ descends to

$$Y^\Phi \nabla_0 f Z^\Phi = \nabla_0 f - \frac{i}{n+2} P f.$$

Since $\tilde{\nabla}_a X^A = \mathbb{Z}^A{}_a$, we see that $\ell^a \tilde{\nabla}_a Z^\Phi$ lies in the complex subspace generated by $\ell^a \mathbb{Z}^A{}_a = \mathsf{L}^A$, which implies that $\tilde{Y}_\Phi \ell^a \tilde{\nabla}_a Z^\Phi = 0$, so $2\ell^a \tilde{\nabla}_a f$ descends to $\nabla_0 f - (i/(n+2))Pf$. Using powers of f and its conjugate we obtain the general formula.

(3) In the proof of Proposition 4.6 we have noted that $\tilde{\mathbb{D}}_{\mathbb{J}} = \tilde{\nabla}_{\mathbf{k}}$ on density bundles. By definition, this implies $\tilde{\mathbb{D}}_{\mathbb{J}}^{\nabla} = \tilde{\nabla}_{\mathbf{k}}$ on weighted tractor bundles.

We first claim that $\mathbf{k}^b \tilde{\nabla}_b \tilde{W}_A{}^a = 0$. By (4.5), $\mathbf{k}^b \tilde{\nabla}_b \mathbb{Z}_A{}^a = -\ell^a X_A - \mathbf{k}^a Y_A$. On the other hand, using that $L_A = Y_B \mathbb{J}^B{}_A$ and $\mathbf{k}^b \tilde{\nabla}_b \mathbf{k}^a = 0$, we get $\mathbf{k}^b \tilde{\nabla}_b \mathsf{L}_A \mathbf{k}^a = -Y_A \mathbf{k}^a$. Likewise, $\mathbf{k}^b \tilde{\nabla}_b \mathsf{K}_A = -X_A$, so inserting the definition (4.12) of $\tilde{W}_A{}^a$ we obtain $\mathbf{k}^b \tilde{\nabla}_b \tilde{W}_A{}^a = -\mathsf{K}_A \mathbf{k}^b \tilde{\nabla}_b \ell^a$, which vanishes by part (4) of Proposition 4.2.

For $s^A \in \Gamma(\tilde{\mathcal{T}}_{\mathbb{C}}[1])$ we therefore get $\mathbf{k}^c \tilde{\nabla}_c s^A \tilde{W}_A{}^a = \tilde{W}_A{}^a \mathbf{k}^c \tilde{\nabla}_c s^A$. If s^A lies in the subspace $\Gamma(\mathcal{T}^\Phi(1,0))$, then $\mathbf{k}^c \tilde{\nabla}_c s^A = i s^A$ and hence $\mathbf{k}^c \tilde{\nabla}_c s^A \tilde{W}_A{}^a = i s^A \tilde{W}_A{}^a$. By Proposition 4.6 we see that $s^A \tilde{W}_A{}^a \in \Gamma(H^\alpha) \subset \Gamma(\tilde{H}^\alpha)$. Likewise, if $s^A \in \Gamma(\mathcal{T}^{\bar{\Phi}}(0,1))$, we get $s^A \tilde{W}_A{}^a \in \Gamma(H^{\bar{\alpha}})$, so we see that the holomorphic and anti-holomorphic parts of $\tilde{W}_A{}^a$ descend as required. Since the appropriate parts of the complementary projections X_A and Y_A descend to their CR analogs, these descended sections must coincide with $W_\Phi{}^\alpha$ and $W_{\bar{\Phi}}{}^{\bar{\alpha}}$.

(1) For a section ξ^a of (the complexification of) $\tilde{H}$ we compute

$$\mathbb{I}^a_c \tilde{\nabla}_b \xi^c = \tilde{W}_A{}^a \tilde{W}^A{}_c \tilde{\nabla}_b \xi^c = \tilde{W}_A{}^a \tilde{\nabla}_b \tilde{W}^A{}_c \xi^c - \tilde{W}_A{}^a \xi^c \tilde{\nabla}_b \tilde{W}^A{}_c.$$

By (4.12), $\tilde{\nabla}\tilde{W}^A{}_c - \tilde{\nabla}\mathbb{Z}^A{}_c$ is a sum of terms which contain one of the four elements K^A, L^A, $\mathbf{k}_c$, or ℓ_c undifferentiated. But the first two are killed by contraction with $\tilde{W}_A{}^a$, while the last two are annihilated by contraction into ξ^c. Finally, $\tilde{W}_A{}^a \tilde{\nabla}_b \mathbb{Z}^A{}_c = 0$ by (4.5), so we obtain

$$\mathbb{I}^a_c \tilde{\nabla}_b \xi^c = \tilde{W}_A{}^a \tilde{\nabla}_b \tilde{W}^A{}_c \xi^c.$$

If ξ^a lies in $\Gamma(H^\alpha) \subset \Gamma(\tilde{H} \otimes \mathbb{C})$, then by part (3), $\tilde{W}^A{}_c \xi^c$ lies in $\Gamma(\mathcal{T}_\Phi(-1,0)) \subset \Gamma(\tilde{\mathcal{T}}_{\mathbb{C}}[-1])$, so we can apply part (1). Together with part (3) we see that $\mathbb{I}^a_c \mathbb{I}^d_b \tilde{\nabla}_d \xi^c$ descends to $W_\Phi{}^\alpha \nabla^H W^\Phi{}_\beta \xi^\beta$. Applying the Leibniz rule, and using that the formulae in Section 4.3 show that $W_\Phi{}^\alpha \nabla^H W^\Phi{}_\beta = 0$, we conclude that $\mathbb{I}^a_c \mathbb{I}^d_b \tilde{\nabla}_d \xi^c$ descends to $\nabla^H \xi^\alpha$.

Still assuming that $\xi^a \in \Gamma(H^\alpha) \subset \Gamma(\tilde{H} \otimes \mathbb{C})$, we see that $2\mathbb{I}^a_c \ell^b \tilde{\nabla}_b \xi^c$ descends to $W_\Phi{}^\alpha \nabla_0 W^\Phi{}_\beta \xi^\beta + (i/(n+2))P\xi^\alpha$. The formulae for ∇W in Section 4.3 show that

$$W_\Phi{}^\alpha \tilde{\nabla}_0 W^\Phi{}_\beta = iP_\beta{}^\alpha - \frac{i}{n+2} P \delta^\alpha_\beta,$$

so $2\mathbb{I}^a_c \ell^b \tilde{\nabla}_b \xi^c$ descends to $\nabla_0 \xi^\alpha + iP_\beta{}^\alpha \xi^\beta$. Since we have established the formulae for densities already in part (1), this completes the proof. □

4.11. Conformal Killing fields on Fefferman spaces. We can now make the decomposition of conformal Killing fields on a Fefferman space from Theorem 3.6 explicit.

Theorem 4.9. *Let M be a CR manifold with Fefferman space $\tilde{M}$, let v^a be a conformal Killing field on $\tilde{M}$, and fix a choice of preferred scale.*

(1) *$v^b \tilde{\nabla}_b \mathbf{k}^a - \mathbf{k}^b \tilde{\nabla}_b v^a$ is a conformal Killing field on $\tilde{M}$ which inserts trivially into the tractor curvature $\Omega_{ab}{}^C{}_D$.*
(2) *The vector field $u^a := v^a - (\mathbf{k}^c \tilde{\nabla}_c v^d)\tilde{\nabla}_d \mathbf{k}^a + \mathbf{k}_b v^b \ell^a$ on $\tilde{M}$ descends to an infinitesimal CR automorphism of M. Further, $\mathbf{k}_b v^b$ descends to a smooth function on M from which this infinitesimal automorphism can be recovered by a CR-invariant differential operator.*
(3) *Define $w^a := \mathbb{I}^a_c v^c + (\mathbf{k}^b \tilde{\nabla}_b v^d)(\tilde{\nabla}_d \mathbf{k}^a)$. Then the section $w^a - i w^c \tilde{\nabla}_c \mathbf{k}^a$ of $\tilde{H} \otimes \mathbb{C}$ descends to a section w^α of $\mathcal{E}^\alpha(-1,1)$ which satisfies $\nabla_\alpha w^\beta = \delta^\beta_\alpha \nabla_\gamma w^\gamma$ as well as $\nabla^\alpha w^\beta = -\nabla^\beta w^\alpha$.*

Proof. As in the proof of Proposition 4.2 we put

$$V^B := \mathbb{Z}^B{}_a v^a - \frac{1}{2n+2} X^B \tilde{\nabla}_a v^a$$

and consider the adjoint tractor field $s_A{}^B = (1/(2n+2))D_A V^B$ associated to v^a.

(1) By Lemma 3.5, $\{s, \mathbb{J}\}$ is a parallel section of $\tilde{\mathcal{A}}$, so the underlying vector field is conformal Killing and inserts trivially into the Cartan curvature. We can compute this underlying vector field as

$$X^A \mathbb{Z}_B{}^a (s_A{}^C \mathbb{J}_C{}^B - \mathbb{J}_A{}^C s_C{}^B).$$

Using part (5) of Proposition 4.2 and formula (4.6), this expands as

$$V^C(Y_C \mathbf{k}^a + \mathbb{Z}_C{}^c \tilde{\nabla}_c \mathbf{k}^a + X_C \ell^a) - \mathbb{Z}_B{}^a \mathbf{k}^c \tilde{\nabla}_c V^B,$$

which easily leads to the required expression.

(2) By Theorem 3.6, we can form the complex linear part of $s_A{}^B$ and add an appropriate multiple of $\mathbb{J}_A{}^B$ to obtain an element of $\Gamma(\mathcal{A}) \subset \Gamma(\tilde{\mathcal{A}})$ which defines an infinitesimal CR automorphism of M. In particular, the corresponding conformal Killing field on $\tilde{M}$ descends to that infinitesimal CR automorphism on M. Since the multiple of $\mathbb{J}_A{}^B$ just contributes a multiple of $\mathbf{k}^a$ to the underlying vector field, we can ignore it in the computation. The conformal Killing field underlying the complex linear part of s_A^B can be computed as

$$\frac{1}{2} X^A \mathbb{Z}_B{}^a (s_A{}^B - \mathbb{J}_A{}^C s_C{}^D \mathbb{J}_D{}^B).$$

Using formula (4.6) and part (5) of Proposition 4.2, this is easily evaluated directly, and one obtains

$$\frac{1}{2}\left(u^a + \left(\ell_b v^b + \frac{1}{2n+2}\mathbf{k}^c \tilde{\nabla}_c \tilde{\nabla}_b v^b\right)\mathbf{k}^a\right).$$

Hence we see that u^a descends to an infinitesimal CR automorphism on M. From the definition of u we immediately see that $\mathbf{k}_a u^a = 2\mathbf{k}_a v^a$. In view of Lemma 4.5, we obtain the section of TM/H induced by this infinitesimal automorphism by multiplying the Reeb field by the function $\frac{1}{2}\mathbf{k}_a v^a$. Now an infinitesimal CR automorphism on M can be recovered by applying an invariant differential operator to its projection to TM/H, see [7, 3.4].

We can also verify the last two facts directly: Using Proposition 4.2 we get $\mathbf{k}^b \tilde{\nabla}_b \mathbf{k}_c v^c = \mathbf{k}^b \mathbf{k}^c \tilde{\nabla}_b v_c$, and this vanishes since the symmetric part of $\tilde{\nabla}_b v_c$ is pure trace and $\mathbf{k}$ is isotropic. Thus, the function $\mathbf{k}_c v^c$ descends to M. To recover the infinitesimal CR automorphism from this function, it suffices to recover $\mathbb{I}^a_b u^b = \mathbb{I}^a_b v^b - (\mathbf{k}^c \tilde{\nabla}_c v^d)\tilde{\nabla}_d \mathbf{k}^a$. To do this, we use that v is conformal Killing and $\mathbf{k}$ is Killing to compute

$$\tilde{\nabla}^c \mathbf{k}_b v^b = -\mathbf{k}^b \tilde{\nabla}_b v^c + \frac{1}{n+1}\mathbf{k}^c \tilde{\nabla}_b v^b - v^b \tilde{\nabla}_b \mathbf{k}^c,$$

which implies that $\mathbb{I}^a_b u^b = (\tilde{\nabla}^c \mathbf{k}_b v^b)(\tilde{\nabla}_c \mathbf{k}^a)$, since $\tilde{\nabla}_b \mathbf{k}^c$ is the complex structure on $\tilde{H}$ and hence $(\tilde{\nabla}_b \mathbf{k}^c)(\tilde{\nabla}_c \mathbf{k}^d) = -\mathbb{I}^d_b$.

(3) As before, let $s_A{}^B$ be the adjoint tractor field corresponding to the conformal Killing field v^a. From Theorem 3.6 we know that the conjugate linear part of $s_A{}^B$ descends to a section of $\mathcal{E}^{[\Phi\Psi]}$, and that there is a canonical projection from that bundle to its irreducible quotient $\mathcal{E}^\alpha(-1,1)$. Applying this projection to the conjugate linear part, we obtain a section in the kernel of two CR invariant operators, and from Remarks 3.7 we know that this corresponds to the two claimed properties of w^α. From the definitions, one easily concludes that the projection $\mathcal{E}^{[\Phi\Psi]} \to \mathcal{E}^\alpha(-1,1)$ is explicitly given by $Z_\Phi W^\alpha_\Psi$. Now a direct computation shows that

$$w^a = X^A \tilde{W}_B{}^a (s_A{}^B + \mathbb{J}_A{}^C s_C{}^D \mathbb{J}_D{}^B),$$

so the section of $\mathcal{E}^\alpha(-1,1)$ in question can be computed as the holomorphic part $w^a - i w^c \tilde{\nabla}_c \mathbf{k}^a$ of w^a.

Alternatively, one may verify all the claims by direct computations along the following lines: We can write

$$w^a = (\mathbf{k}^b \tilde{\nabla}_b v^d - v^b \tilde{\nabla}_b \mathbf{k}^d)(\tilde{\nabla}_d \mathbf{k}^a)$$

and differentiate this formula. To expand these derivatives, one has to use the differential consequences of the conformal Killing equation as detailed in [23]: Putting $\rho_a := -(1/(2n+2))\nabla_a \nabla_c v^c - \tilde{P}_{ac} v^c$, one has

$$\tilde{\nabla}_a \tilde{\nabla}_b v^c = g_{ab}\rho^c - \delta^c_a \rho_b - \tilde{P}_{ab} v^c + \tilde{P}_a{}^c v_b + C_b{}^{cd}{}_a v_d - \rho_a \delta^c_b - \tilde{P}_{ad} v^d \delta^c_b,$$

where $C_{ab}{}^c{}_d$ is the Weyl curvature. Moreover, in the corresponding equation for $\mathbf{k}$ instead of v, one may replace ρ_a by $-\ell_a$ and the last three summands vanish.

Using these identities, one easily verifies directly that $\mathbf{k}^b \tilde{\nabla}_b w^a = -w^b \tilde{\nabla}_b \mathbf{k}^a$. This immediately implies that the section $w^a - i w^c \tilde{\nabla}_c \mathbf{k}^a$ of $\tilde{H} \otimes \mathbb{C}$ is an eigenvector for the operator $\mathbf{k}^b \tilde{\nabla}_b$ with eigenvalue $-i$. By Proposition 4.6, this implies that it descends to a section w^α of $\mathcal{E}^\alpha(-1,1)$.

Next, one computes $\mathbb{I}^a_c \mathbb{I}^d_b \tilde{\nabla}_d w^c$ by first expanding the expression for $\nabla_d w^c$ and ignoring those terms which have a free index on either $\mathbf{k}$ or ℓ. This leads to

$$\mathbb{I}^a_c \mathbb{I}^d_b \tilde{\nabla}_d w^c = \mathbb{I}^a_c \mathbb{I}^d_b \tilde{\nabla}_d v^c + (\tilde{\nabla}_b \mathbf{k}^c)(\tilde{\nabla}_c v^d)(\tilde{\nabla}_d \mathbf{k}^a) \\ + \mathbb{I}^a_b \mathbf{k}^c \ell_d \tilde{\nabla}_c v^d - (\mathbf{k}^c \rho_c + \ell_c v^c)\tilde{\nabla}_b \mathbf{k}^a.$$

The first and second line of this formula exactly are the conjugate linear part, respectively the complex linear part of the resulting endomorphisms of $\tilde{H}$. The second line is evidently a complex multiple of the identity. On the other hand, the fact that the symmetrisation of $\tilde{\nabla}_c v^d$ is pure trace, easily implies that the first line is skew symmetric. Using part (2) of Proposition 4.8, this easily implies the result. ◻

4.12. Relating double-D's. Consider the conformal double-D-operator $\tilde{\mathbb{D}}^\nabla$ as an operator mapping sections of a weighted complex conformal tractor bundle $\tilde{\mathcal{W}}[w+w']$ to sections of $\tilde{\mathcal{A}}_{\mathbb{C}} \otimes \tilde{\mathcal{W}}[w+w']$, see Section 3.7. Hence we denote the operator by $(\tilde{\mathbb{D}}^\nabla)_A{}^B$. Having a complex conformally natural bundle $\tilde{\mathcal{W}}$, we can use the splitting of $\tilde{\mathcal{A}}_{\mathbb{C}}$ from Section 3.7 to obtain

$$(\tilde{\mathbb{D}}^\nabla)_A{}^B = \begin{pmatrix} (\tilde{\mathbb{D}}^\nabla)_\Phi{}^\Psi & (\tilde{\mathbb{D}}^\nabla)_{\bar{\Phi}}{}^\Psi \\ (\tilde{\mathbb{D}}^\nabla)_\Phi{}^{\bar{\Psi}} & (\tilde{\mathbb{D}}^\nabla)^{\bar{\Phi}}{}_{\bar{\Psi}} \end{pmatrix}. \tag{4.14}$$

The fact that $(\tilde{\mathbb{D}}^\nabla)_A{}^B$ is skew symmetric implies that the component in the bottom left corner is the negative transpose of the one in the top right corner, so the notation is consistent.

From Theorem 3.12 we know that $(\tilde{\mathbb{D}}^\nabla)_\Phi{}^\Psi$ restricts to the CR-double-D-operator

$$(\mathbb{D}^\nabla)_\Phi{}^\Psi : \Gamma(\mathcal{W}(w,w')) \to \Gamma(\mathcal{A}_{\mathbb{C}} \otimes \mathcal{W}(w,w')).$$

On the other hand, the off diagonal components descend to operators

$$(\mathbb{D}^\nabla)_{\Phi\Psi} : \Gamma(\mathcal{W}(w,w')) \to \Gamma(\mathcal{E}_{[\Phi\Psi]} \otimes \mathcal{W}(w-1,w'+1)),$$
$$(\mathbb{D}^\nabla)_{\bar{\Phi}\bar{\Psi}} : \Gamma(\mathcal{W}(w,w')) \to \Gamma(\mathcal{E}_{[\bar{\Phi}\bar{\Psi}]} \otimes \mathcal{W}(w+1,w'-1)).$$

Theorem 4.10. *Consider a weighted complex CR tractor bundle $\mathcal{W}(w,w')$. Then the operators $\mathbb{D}$ constructed above are explicitly given by*

$$\begin{aligned}
\mathbb{D}_{\Phi\Psi} f &= 2w Z_{[\Psi} Y_{\Phi]} f + 2 Z_{[\Psi} W_{\Phi]}{}^\alpha \nabla_\alpha f \\
\mathbb{D}_{\bar{\Phi}\bar{\Psi}} f &= 2w' Z_{[\bar{\Psi}} Y_{\bar{\Phi}]} f + 2 Z_{[\bar{\Psi}} W_{\bar{\Phi}]}{}^{\bar{\alpha}} \nabla_{\bar{\alpha}} f \\
\mathbb{D}_{\Phi\bar{\Psi}} f &= w Z_{\bar{\Psi}} Y_\Phi f - w' Z_\Phi Y_{\bar{\Psi}} f + Z_{\bar{\Psi}} W_\Phi{}^\alpha \nabla_\alpha f - Z_\Phi W_{\bar{\Psi}}^{\bar{\alpha}} \nabla_{\bar{\alpha}} f \\
&\qquad - Z_\Phi Z_{\bar{\Psi}} \left(i \nabla_0 f + \frac{w'-w}{n+2} P f \right).
\end{aligned}$$

Proof. By [20, Section 4], the conformal double D-operator on $\tilde{\mathcal{W}}[w+w']$ is given by

$$\tilde{\mathbb{D}}^\nabla_{AB} f = 2(w+w') X_{[B} Y_{A]} f + 2 X_{[B} \mathbb{Z}_{A]}{}^a \tilde{\nabla}_a f,$$

see also [21]. Now let us insert $\mathbb{Z}_A{}^a = \tilde{W}_A{}^a + \mathsf{L}_A \mathbf{k}^a + \mathsf{K}_A \ell^a$. From the proof of part (3) of Proposition 4.8, we know that, in a CR scale, $\tilde{\nabla}_{\mathbf{k}}$ coincides with $\tilde{\mathbb{D}}^\nabla_{\mathbb{J}}$ on weighted tractor bundles, so $\mathbf{k}^a \tilde{\nabla}_a f = i(w-w')f$. Using this, we can rewrite $\frac{1}{2} \tilde{\mathbb{D}}^\nabla_{AB} f$ as

$$(w+w')X_{[B}Y_{A]}f + X_{[B}\tilde{W}_{A]}{}^a\tilde{\nabla}_a f + i(w-w')X_{[B}\mathsf{L}_{A]}f + X_{[B}\mathsf{K}_{A]}\ell^a\tilde{\nabla}_a f.$$

For each term occurring in this decomposition, we understand explicitly the decomposition into holomorphic and anti-holomorphic parts. In particular, the components of $(w+w')Y_A + i(w-w')L_A$ descend to $(wY_\Phi, w'Y_{\bar{\Phi}})$, compare with Section 4.6. Further, the components of X_B, K_A, and $\tilde{W}_A{}^a$ descend to $(Z_\Psi, Z_{\bar{\Psi}})$, $(-iZ_\Phi, iZ_{\bar{\Phi}})$, and $(W_\Phi{}^\alpha, W_{\bar{\Phi}}{}^{\bar{\alpha}})$, respectively. Inserting this, decomposing and using Proposition 4.8, the claimed formulae follow. □

4.13. Relating tractor-D's. To complete our picture, it remains to interpret the conformal Rho-tensor $\tilde{P}$ in terms of CR-data.

Proposition 4.11. *Let $\pi : \tilde{M} \to M$ be a Fefferman space, and consider the CR scale on $\tilde{M}$ induced by a choice of contact form on M. Then the complex bilinear extension of the conformal Rho-tensor is given by*

$$\tilde{P}_{ab} = \ell_a\ell_b - \frac{1}{4}S\mathbf{k}_a\mathbf{k}_b + i\mathbf{k}_{(a}\mathbb{I}^{\alpha}_{b)}T_\alpha - i\mathbf{k}_{(a}\mathbb{I}^{\bar{\alpha}}_{b)}T_{\bar{\alpha}} \\ + \frac{i}{2}\mathbb{I}^{\alpha}_{a}\mathbb{I}^{\beta}_{b}A_{\alpha\beta} - \frac{i}{2}\mathbb{I}^{\bar{\alpha}}_{a}\mathbb{I}^{\bar{\beta}}_{b}A_{\bar{\alpha}\bar{\beta}} + \frac{1}{2}\mathbb{I}^{\bar{\beta}}_{a}\mathbb{I}^{\alpha}_{b}P_{\alpha\bar{\beta}} + \frac{1}{2}\mathbb{I}^{\beta}_{a}\mathbb{I}^{\bar{\alpha}}_{b}P_{\bar{\alpha}\beta}.$$

In particular, $\tilde{P}_a{}^a = P_\alpha{}^\alpha$.

Proof. By (4.5), we have $\tilde{P}_{ab} = \mathbb{Z}^A{}_b\tilde{\nabla}_a Y_A$. Now we can decompose

$$\tilde{\nabla}_a Y_A = (\ell_a\mathbf{k}^c + \mathbf{k}_a\ell^c + \mathbb{I}^c_a)\tilde{\nabla}_c Y_A.$$

By Proposition 4.4, $Y_A = \frac{1}{2}(Y_\Phi, Y_{\bar{\Phi}})$ and the components lie in $\Gamma(\mathcal{T}_\Phi(-1,0))$ respectively in $\Gamma(\mathcal{T}_{\bar{\Phi}}(0,-1))$.

By Proposition 4.8, $\mathbb{I}^c_a\tilde{\nabla}_c Y_A$ decomposes as $\frac{1}{2}(\nabla^H Y_\Phi, \nabla^H Y_{\bar{\Phi}})$ and

$$\begin{aligned}\ell^c\tilde{\nabla}_c Y_A &= \frac{1}{4}\left(\nabla_0 Y_\Phi - \frac{i}{n+2}PY_\Phi, \nabla_0 Y_{\bar{\Phi}} + \frac{i}{n+2}PY_{\bar{\Phi}}\right) \\ &= \frac{1}{2}\left(iT_\alpha W_\Phi{}^\alpha + \frac{i}{2}SZ_\Phi, -iT_{\bar{\alpha}}W_{\bar{\Phi}}{}^{\bar{\alpha}} - \frac{i}{2}SZ_{\bar{\Phi}}\right).\end{aligned}$$

As observed in the proof of Proposition 4.8, $\tilde{\nabla}_{\mathbf{k}}$ coincides with the double D-operator on weighted tractor bundles, so by Corollary 3.9 we get $\mathbf{k}^c\tilde{\nabla}_c Y_A = \frac{1}{2}(-iY_\Phi, iY_{\bar{\Phi}})$.

On the other hand, we have

$$\begin{aligned}\mathbb{Z}^A{}_b &= \tilde{W}^A{}_b + \mathsf{K}^A\ell_b + \mathsf{L}^A\mathbf{k}_b \\ &= (\mathbb{I}^{\alpha}_{b}W^\Phi{}_\alpha, \mathbb{I}^{\bar{\alpha}}_{b}W^{\bar{\Phi}}{}_{\bar{\alpha}}) + \ell_b(iZ^\Phi, -iZ^{\bar{\Phi}}) + \mathbf{k}_b\frac{1}{2}(iY^\Phi, -iY^{\bar{\Phi}}).\end{aligned}$$

From this, the result follows by direct evaluation. □

Using this we can now directly analyse the formula (4.6) for the tractor-D operator.

Theorem 4.12. *Let $\tilde{\mathcal{W}}_{\mathbb{C}}[w+w']$ be a weighted complex conformal tractor bundle and consider the conformal tractor-D operator D_A which maps sections of $\tilde{\mathcal{W}}_{\mathbb{C}}[w+w']$ to sections of $\mathcal{E}_A\otimes\tilde{\mathcal{W}}_{\mathbb{C}}[w+w'-1]$. Then for any $t\in\Gamma(\mathcal{W}(w,w'))\subset\Gamma(\tilde{\mathcal{W}}_{\mathbb{C}}[w+w'])$, the holomorphic and anti-holomorphic parts of D_At descend to sections $2D_\Phi t\in\mathcal{W}(w-1,w')$ and $2D_{\bar\Phi}t\in\mathcal{W}(w,w'-1)$, and the operators induced in that way coincide with the CR tractor-D operators from* [24].

Proof. In the formula (4.6) from Section 4.4, we have to replace w by $w+w'$, and then expand in a preferred scale. We only consider the holomorphic part, the anti-holomorphic part is dealt with in the same way. The holomorphic part of $(2n+2w+2w')(w+w')Y_At$ simply is $(n+w+w')(w+w')Y_\Phi t$. Next, we have to consider $2(n+w+w')\mathbb{Z}_A{}^a\tilde\nabla_at$. Inserting (4.12), this can be written as

$$2(n+w+w')(\tilde W_A{}^a+\mathsf{K}_A\ell^a+\mathsf{L}_A\mathbf{k}^a)\tilde\nabla_at.$$

Using Propositions 3.11 and 4.11 we conclude that the holomorphic part of this is given by

$$(n+w+w')\left(2W_\Phi{}^\alpha\nabla_\alpha t-iZ_\Phi\left(\nabla_0t+\frac{i(w-w')}{n+2}Pt\right)+Y_\Phi(w-w')t\right).$$

In view of Proposition 4.11, the term $-(w+w')X_A\tilde Pt$ has a contribution of $-(w+w')Z_\Phi Pt$ to the holomorphic part, and it remains to analyse the contribution of $-X_A\tilde\nabla^a\tilde\nabla_at$. Using (4.9) we get

$$\tilde\nabla^a\tilde\nabla_at=\tilde\nabla^a(\ell_a\mathbf{k}^c+\mathbf{k}_a\ell^c+\mathbb{I}_a^c)\tilde\nabla_ct. \tag{4.15}$$

Now $\tilde\nabla^a\mathbf{k}_a=0$ by Proposition 4.2. The Bianchi identity implies that $\tilde\nabla^a\tilde P_{ab}=\tilde\nabla_b\tilde P$, where $\tilde P=\tilde P_a{}^a$. Using this and Proposition 4.2 again, we get

$$\tilde\nabla^a\ell_a=\tilde\nabla^a\tilde P_{ab}\mathbf{k}^b=\mathbf{k}^b\tilde\nabla_b\tilde P+\tilde P_{ab}\tilde\nabla^a\mathbf{k}^b.$$

The first summand of this vanishes, since $\mathbf{k}$ is a Killing field in a preferred scale, and the second one vanishes by symmetry of $\tilde P_{ab}$ and skew symmetry of $\tilde\nabla^a\mathbf{k}^b$. Hence we can write (4.15) as

$$\ell^a\tilde\nabla_a\mathbf{k}^c\tilde\nabla_ct+\mathbf{k}^a\tilde\nabla_a\ell^c\tilde\nabla_ct+\tilde\nabla^a\mathbb{I}_a^c\tilde\nabla_ct.$$

Since the Lie bracket of $\mathbf{k}$ and ℓ vanishes, and $\mathbf{k}$ hooks trivially into the conformal tractor curvature, the first two summands are equal. Multiplying by $-X_A$ they together contribute

$$-i(w-w')Z_\Phi\left(\nabla_0t+\frac{i(w-w')}{n+2}Pt\right)$$

to the holomorphic part. To analyse the last remaining term, we again use (4.9) to get

$$\tilde{\nabla}^a \mathbb{I}_a^c \tilde{\nabla}_c t = (\mathbf{k}^a \ell_b + \ell^a \mathbf{k}_b + \mathbb{I}_b^a)\tilde{\nabla}^b \mathbb{I}_a^c \tilde{\nabla}_c t.$$

Since $\mathbf{k}^a \mathbb{I}_a^c = 0$ and $\ell_b \tilde{\nabla}^b \mathbf{k}^a = 0$, the first summand does not give any contribution and likewise the second summand vanishes. Thus we are left with $\mathbb{I}_b^a \tilde{\nabla}^b \mathbb{I}_a^c \tilde{\nabla}_c t$. But this is exactly the trace over $\tilde{\nabla}^{\tilde{H}} \tilde{\nabla}^{\tilde{H}} t$, which descends to the trace of $\nabla^H \nabla^H t$. This trace in turn can be written as $\nabla^\alpha \nabla_\alpha t + \nabla^{\bar{\alpha}} \nabla_{\bar{\alpha}} t$. From [24, Proposition 2.2], we obtain

$$\nabla^{\bar{\alpha}} \nabla_{\bar{\alpha}} t = \nabla^\alpha \nabla_\alpha t + \frac{(w - w')2(n+1)}{n+2} Pt - in\nabla_0 t$$

in the case that t is a density, and this formula extends to tractors by [24, formula (3.4)].

Collecting our results, we see that the holomorphic part of $D_A t$ is given by

$$\begin{aligned} &2(n + w + w')wY_\Phi t + 2(n + w + w')W_\Phi{}^\alpha \nabla_\alpha t \\ &\qquad - 2Z_\Phi \left(iw\nabla_0 t + \nabla^\alpha \nabla_\alpha + tw\left(1 + \frac{w' - w}{n+2}\right) Pt \right), \end{aligned}$$

which is exactly twice the expression for the holomorphic CR tractor-D from [24, Section 3]. □

4.14. Almost CR-Einstein structures. Following [22], we define an *almost Einstein* structure on a pseudo-Riemannian or conformal manifold to be a parallel section of the conformal standard tractor bundle. Almost Einstein structures generalise the notion of Einstein manifolds since an almost Einstein structure determines, on an open dense subset, a scale that makes the manifold Einstein on that subset. In general, a conformal structure does not admit an almost Einstein structure.

Now there is an obvious analog of this condition in CR geometry. Namely, we define an *almost CR-Einstein* structure on a CR manifold M to be a parallel section of the CR standard tractor bundle. By Proposition 3.3, there is a bijection between the spaces of parallel CR standard tractors on M and parallel conformal standard tractors on the Fefferman space $\tilde{M}$. In particular, M admits an almost CR-Einstein structure if and only if $\tilde{M}$ admits an almost Einstein structure.

If I_Φ is a (non-zero) parallel tractor on the CR manifold M, then $\sigma := Z^\Phi I_\Phi$ is non-vanishing on an open dense subspace of M; this follows because from the formula for the connection we see that, at each point x, the vanishing of the tractor covariant derivative of I implies that I depends only on the 2-jet of σ at x. Away from the points where it vanishes, σ determines a scale $\sigma\bar{\sigma}$. We say that I_Φ is a *CR-Einstein* structure on M if σ is nowhere vanishing and in this setting we will often term σ itself to be a CR-Einstein scale.

In [31], J. Lee introduced the notion of being *pseudo-Einstein* for pseudo-Hermitian structures on a CR manifold M. This condition says that the Webster-Ricci tensor has vanishing tracefree part. Motivated by ideas from tractor calculus, F. Leitner introduced in [33] the name TSPE ("transversally symmetric pseudo-Einstein") for pseudo-Hermitian structures which are pseudo-Einstein and define a transverse symmetry. The last condition means that the Reeb vector field corresponding to the pseudo-Hermitian structure is an infinitesimal automorphism of the CR structure. Leitner also showed that there are interesting relations between TSPE structures and Kähler-Einstein metrics, and in particular, that there are many examples of TSPE-structures.

Proposition 4.13. *A contact form on a CR manifold M induces a TSPE structure if and only if the corresponding scale is a CR-Einstein scale.*

Proof. Using some scale, suppose that $I_\Phi = \sigma Y_\Phi + W_\Phi{}^\alpha \tau_\alpha + \rho Z_\Phi$ is parallel. From the formula for the connection we easily deduce that we have the equations

$$\nabla_{\bar\beta}\sigma = 0 \quad \text{and} \quad \nabla_\alpha\nabla_\beta\sigma + i\sigma A_{\alpha\beta} = 0, \tag{4.16}$$

which are valid in any scale and hence CR invariant. Note that, using the formula for the covariant commutator for $f \in \mathcal{E}(w, w')$

$$\nabla_\alpha\nabla_{\bar\beta}f - \nabla_{\bar\beta}\nabla_\alpha f = (w - w')P_{\alpha\bar\beta}f + \frac{w - w'}{n+2}Ph_{\alpha\bar\beta}f - ih_{\alpha\bar\beta}\nabla_0 f, \tag{4.17}$$

from (2.4) of [24], the system (4.16) implies

$$\nabla_{\bar\beta}\nabla_\alpha\sigma + P_{\alpha\bar\beta}\sigma + h_{\alpha\bar\beta}\rho = 0$$

for a density ρ in $\mathcal{E}(0,-1)$. This is another equation from the system expressing that I is covariantly parallel.

If we now suppose that σ is non-vanishing and calculate using the pseudo-Hermitian connection ∇ determined by the scale $\sigma\bar\sigma$, then $\nabla_\alpha\sigma\bar\sigma = 0$ and this together with $\nabla_\alpha\bar\sigma = \overline{\nabla_{\bar\alpha}\sigma}$ and the above shows that $\nabla_\beta\sigma = 0$. Thus the system (4.16) implies that, in the scale $\sigma\bar\sigma$, we have

$$A_{\alpha\beta} = 0 \quad \text{and} \quad P_{\alpha\bar\beta} + h_{\alpha\bar\beta}\rho/\sigma = 0.$$

The first condition is well known to be equivalent to a transverse symmetry, i.e., the fact that the Reeb field is an infinitesimal CR automorphisms, while the second is the equation for a pseudo-Einstein structure in the sense of [31].

Conversely suppose that we have a contact form θ such that $P_{\alpha\bar\beta}$ is a multiple of $h_{\alpha\bar\beta}$. Now recall from Section 2.3 that the relation between θ and the possible choices of σ such that $\sigma\bar\sigma$ gives the CR scale corresponding to θ is given by the fact that $\sigma^{n+2} \in \Gamma(\mathcal{E}(-n-2, 0))$ is volume normalised with respect to θ, and this determines σ^{n+2} up to a phase factor. Now $\mathcal{E}(-n-2,0)$ is the canonical bundle, so σ^{n+2} is an $(n+1,0)$-form, and in [31, Theorem 4.2] it is shown that this phase factor can be adjusted in such a way that the resulting form is closed. This in particular implies that $\nabla_{\bar{\alpha}}\sigma = 0$.

Calculating in the pseudo-hermitian scale $\sigma\bar{\sigma}$ we obtain $\nabla_\alpha\sigma = 0$ as above, so if we in addition require that $A_{\alpha\beta} = 0$, we see that σ solves (4.16). From (4.17) we further get $i\nabla_0\sigma = (2(n+1)/(n(n+2)))P\sigma$, and hence $i\nabla_\alpha\nabla_0\sigma = (2(n+1)/(n(n+2)))\sigma\nabla_\alpha P$.

On the other hand, by (2.4) of [24] the vanishing of $A_{\alpha\beta}$ implies that ∇_α and ∇_0 commute on densities, so $\nabla_\alpha\nabla_0\sigma = \nabla_0\nabla_\alpha\sigma = 0$. Thus $\nabla_\alpha P = 0$. By a similar argument we get $\nabla_{\bar{\beta}}P = 0$. Since P is a function on M it follows that P is constant. Using this and the formula for the tractor connection it is easily verified that

$$I_\Phi := \frac{1}{n+1}D_\Phi\sigma = \sigma Y_\Phi - \frac{1}{n}P\sigma Z_\Phi$$

is annihilated by ∇_α and hence by $\nabla_{\bar{\beta}}$ and ∇_0. Thus I_Φ is parallel and since σ is non-vanishing this is a CR-Einstein structure on M. □

For the reverse implication in the proof we could also use that [33] establishes that the TSPE system implies the Fefferman space is almost Einstein.

We have noted above that a CR-Einstein structure on M induces an almost Einstein structure on the Fefferman space $\tilde{M}$. However, there is never a global Einstein metric on $\tilde{M}$:

Theorem 4.14. *On a Fefferman space there is no Einstein metric in the conformal class.*

Proof. Suppose that $\tilde{M}$ is Einstein. Then it has a parallel tractor I_A with X^AI_A non-vanishing. That is, writing σ for the section $Z^\Phi I_\Phi$ of $\mathcal{E}(1,0)$, we have that $\sigma + \bar{\sigma}$ is non-vanishing. By Corollary 3.9 we have $\tilde{\nabla}_{\mathbf{k}}\sigma = i\sigma$ in a preferred scale. Hence under the fibrewise action ρ of S^1 on $\tilde{M}$ we have $\rho_s^*\sigma = e^{is}\sigma$ and so, at any fixed point of $\tilde{M}$, obviously there is $s \in (0, 2\pi]$ so that the real part of $\rho_s^*\sigma$ vanishes, and this is a contradiction. □

Remarks 4.15. (1) The theorem is different and essentially stronger than that of Lee [30, Theorem 6.6]. In terms of our current language the theorem of Lee shows that the metrics determined by CR scales are never Einstein, and in this case, there is a simpler proof: For a parallel section I_A of the complexified standard tractor bundle, the holomorphic and anti-holomorphic parts are parallel, too. For an Einstein scale $\alpha \in \Gamma(\tilde{\mathcal{E}}[1])$, the tractor $I_A = D_A\alpha$ is parallel, see [19, 25]. But for a CR scale, we have $\alpha \in \mathcal{E}(\frac{1}{2},\frac{1}{2}) \subset \tilde{\mathcal{E}}_{\mathbb{C}}[1]$, so the components of $D_A\alpha$ are $D_\Phi\alpha \in \Gamma(\mathcal{E}(-\frac{1}{2},\frac{1}{2}))$ and $D_{\bar{\Phi}}\alpha \in \Gamma(\mathcal{E}(\frac{1}{2},-\frac{1}{2}))$ by Theorem 4.12. By Corollary 3.9, none of the two sections can be annihilated by $\tilde{D}_{\mathbb{J}}^\nabla$, and the latter operator coincides with $\tilde{\nabla}_{\mathbf{k}}$ in a CR scale. Hence I_A cannot be parallel.

(2) In the special case that $\dim(M) = 3$, it was proved in [35] that there are (locally) no non-flat Einstein metrics in the Fefferman conformal class. This follows from the proposition above and [33], as in this dimension the corresponding Kähler-Einstein manifold has dimension 2 (and is thus flat).

(3) Given a CR-Einstein structure I_Φ on M, one can now imitate the developments of [23] to define operators with principal part a power of the sub-Laplacian, compare these operators to the ones obtained in [24] and use this to prove factorisation results. This will be taken up elsewhere.

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