The bundle of Weyl structures

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- This talk reports on joint work in progress with T. Mettler (Frankfurt) that studies the "bundles of Weyl structures" associated to parabolic geometries.
- These bundles were introduced by M. Herzlich to study distinguished curves. With an improved construction of the canonical connection, we obtain a geometric approach to all elements of the theory of Weyl structures in this picture.
- For torsion-free AHS structures, it turns out that the bundle of Weyl structures carries a nice intrinsic geometric structure, which includes a split-signature Einstein metric. This can be studied using the relation to Weyl structures that leads to an efficient calculus.
- The motivation for studying this structure comes from a connection to fully nonlinear PDE that are naturally associated to the initial AHS structure. In the case of projective structure in dimension two, this connects to the theory of convex projective structures, representation varieties, etc.

Contents

To define a type of parabolic geometries, one needs a grading on a semisimple Lie algebra $\mathfrak{g}$ of the form $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$, which we write as $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$. Putting $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$, we choose a group G, a (parabolic subgroup) P and take the Levi decomposition $P = G_0 \ltimes P_+$. A parabolic geometry of type (G, P) is then a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of that type. Defining $\mathcal{G}_0 := \mathcal{G}/P_+$, one obtains a G_0-principal bundle over M that defines a filtered analog of a G_0-structure on M.

The best known examples of such structures are conformal and projective structures and Levi-non-degenerate CR structures of hypersurface type. The G_0-bundle $\mathcal{G}_0 \to M$ provides a simple encoding of this geometry, the Cartan geometry is an equivalent encoding which is much more involved but leads to a large number of efficient tools.

A parabolic geometry never determines a distinguished connection on TM, but there is always a class of distinguished connections. This is captured by the theory of Weyl structures. By definition, a Weyl structure is a G_0–equivariant section of the projection $\mathcal{G} \to \mathcal{G}/P_+ = \mathcal{G}_0$, so in particular, this defines a reduction of $\mathcal{G}$ to the structure group G_0. The associated geometric objects, which include a principal connection on $\mathcal{G}_0$, are obtained from pulling back the Cartan connection.

As observed by M. Herzlich, any reduction of $\mathcal{G}$ to the structure group G_0 is obtained from some Weyl structure. Hence Weyl structures can be identified with sections of the bundle $\mathcal{G} \times_P (P/G_0) =: A \to M$, the *bundle of Weyl structures*. Herzlich gave a rather intricate construction of a connection on TA and used the geodesics of this connection to study distinguished curves of the underlying parabolic geometry.

One easily shows that multiplication from the right makes P/G_0 into a principal homogeneous space of P_+ and $\exp : \mathfrak{p}_+ \to P_+$ is a diffeomorphism. This implies that $\Gamma(A)$ is an affine space modeled on $\Gamma(\mathrm{gr}(T^*M))$.

By construction, $A = \mathcal{G}/G_0$, so the canonical projection $\mathcal{G} \to A$ is a principal G_0-bundle and $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ defines a Cartan connection on that bundle. Since $\mathfrak{g}$ admits a G_0-invariant decomposition as $\mathfrak{g}_0 \oplus (\mathfrak{g}_- \oplus \mathfrak{p}_+)$ we see that

- ω_0 defines a principal connection on $\mathcal{G} \to A$ and thus induces linear connections D on all associated vector bundles.
- TA decomposes as $L^+ \oplus L^-$ (parallel for D) where L^+ is the vertical bundle of $\pi : A \to M$ and $L^- \cong \pi^* TM$.
- Correspondingly, we can decompose D into a sum $D^+ \oplus D^-$ of partial connections, which only differentiate in the directions in one of the subbundles.

Natural vector bundles on M are of the form $\mathcal{V}M = \mathcal{G} \times_P \mathbb{V}$ for representations $\mathbb{V}$ of P. Restricting to $G_0 \subset P$, we can also form $\mathcal{V}A := \mathcal{G} \times_{G_0} \mathbb{V} \to A$. Recall that $\mathbb{V}$ carries a canonical P-invariant filtration and that the associated graded $\mathrm{gr}(\mathbb{V})$ is isomorphic to $\mathbb{V}$ as a G_0-module. Now we obtain:

- A natural inclusion $\Gamma(\mathcal{V}M \to M) \hookrightarrow \Gamma(\mathcal{V}A \to A)$, which we write as $\sigma \mapsto \tilde{\sigma}$. The image is characterized by $D^+_\varphi \tau = \varphi \bullet \tau$, so for irreducible $\mathbb{V}$, one gets D^+-parallel sections.
- A pullback operation $s^* : \Gamma(\mathcal{V}A) \to \Gamma(\mathrm{gr}(\mathcal{V}M))$ associated to each section s of $\pi : A \to M$ (i.e. any Weyl structure). Similarly, this works for forms with values in $\mathcal{V}A$.

Combining these, $\sigma \mapsto s^*(\tilde{\sigma})$ defines a map $\Gamma(\mathcal{V}M) \to \Gamma(\mathrm{gr}(\mathcal{V}M))$ which recovers the isomorphism $\mathcal{V}M \to \mathrm{gr}(\mathcal{V}M)$ induced by a Weyl structure. The dependence of this on the Weyl structure can be easily recovered from the differential equation satisfied by $D^+\tilde{\sigma}$.

Weyl connection and Rho tensor

For $\sigma \in \Gamma(\mathcal{V}M)$ consider $D\tilde{\sigma} \in \Omega^1(A, \mathcal{V}A)$. On the other hand, projection to the L^+-component in $TA = L^- \oplus L^+$ can be viewed as $\mathsf{P} \in \Omega^1(A, L^+)$. Given a smooth section s of $\pi : A \to M$, we can pull these back to $s^*(D\tilde{\sigma}) \in \Omega^1(M, \mathrm{gr}(\mathcal{V}M))$ and to $s^*\mathsf{P} \in \Omega^1(M, \mathrm{gr}(T^*M))$, respectively.

Theorem

- Denoting by ∇^s the Weyl connection on $\mathcal{V}M$ determined by s, $s^*(D\tilde{\sigma})$ coincides with the image of $\nabla^s\sigma$ under the isomorphism $\mathcal{V}M \to \mathrm{gr}(\mathcal{V}M)$ induced by s.
- For $\xi \in \mathfrak{X}(M)$ the pullback $s^*(D^-_{\tilde{\xi}}\tilde{\sigma}) \in \Gamma(\mathrm{gr}(\mathcal{V}M))$ recovers the *Rho-corrected derivative* induced by s.
- The form $s^*\mathsf{P}$ coincides with the Rho-tensor associated to the Weyl structure s.

Cartan curvature

The curvature of ω is the horizontal, P-equivariant form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ defined by $K(\xi,\eta) = d\omega(\xi,\eta) + [\omega(\xi),\omega(\eta)]$. This can be equivalently interpreted as $\kappa \in \Omega^2(M, \mathcal{A}M)$. On the level of A, we get a further splitting into $\kappa_\pm \in \Gamma(\Lambda^2(L^-)^* \otimes L^\pm)$ and $\kappa_0 \in \Gamma(\Lambda^2(L^-)^* \otimes (\mathcal{G} \times_{G_0} \mathfrak{g}_0))$. Pulling back along $s : M \to A$, one recovers the torsion, the Weyl curvature, and the Cotton-York tensor associated to a Weyl structure.

On A, let τ and ρ be the torsion and the curvature of D. Let $\{\ ,\ \} \in \Omega^2(A, TA)$ and $\{\ ,\ \}_0 \in \Omega^2(A, \mathcal{G} \times_{G_0} \mathfrak{g}_0)$ be induced by appropriate restrictions of the Lie bracket on $\mathfrak{g}$. Then we have:

- $\tau + \{\ ,\ \} \in \Omega^2(A, TA)$ vanishes upon insertion of one section of L^+ and restricts to $\kappa_- \oplus \kappa_+$ on $\Lambda^2(L^-)^*$.
- $\rho + \{\ ,\ \}_0$ vanishes upon insertion of one section of L^+ and restricts to κ_0 on $\Lambda^2(L^-)^*$.

The two summands in the decomposition $TA = L^- \oplus L^+$ are dual to each other (via the Killing form of $\mathfrak{g}$). Extending this pairing to a skew-symmetric respectively to a symmetric bilinear form on TA, we obtain $\Omega \in \Omega^2(A)$ and $h \in \Gamma(S^2 T^* A)$ which both are non-degenerate. Thus A carries an almost bi-Lagrangean structure.

Theorem

We get $d\Omega = 0$ and thus a bi-Lagrangian structure on A iff the parabolic geometry $(p : \mathcal{G} \to M, \omega)$ is associated to a $|1|$-grading of $\mathfrak{g}$ and torsion-free.

Idea of proof: By construction $D\Omega = 0$, which allows computing $d\Omega$ via the alternation of $i_\tau \Omega$. Decomposing this into $\pm$-types, $\kappa_- + \{\ ,\ \}_-$ can be recovered from one of the components, which implies necessity of the condition. To prove sufficiency, one has to show that the complete alternation of κ_+ vanishes, which is shown using the Bianchi identity.

Hence we restrict to torsion-free AHS-structures from now on. For some geometries, this implies local flatness, but the locally flat case is of interest anyway.

By construction, $Dh = 0$, so one may compute the Levi-Civita connection ∇^h of h from D and its torsion. This in turn also relates the curvatures of D and of ∇^h. These computations can be done explicitly using that local frames for TM and T^*M induce local frames for TA and the relation to Weyl structures. This gives:

Theorem

For a torsion-free AHS structure, consider the induced metric h (of split signature) and the canonical connection D on A. Then

- The Ricci-type contraction of the curvature R^D of D is proportional to h.
- The metric h is Einstein.

Any irreducible representation of $\mathfrak{g}$ has a canonical $\mathfrak{p}$-irreducible quotient. There is a unique fundamental representation $\mathbb{V}$ of $\mathfrak{g}$, for which that quotient is one-dimensional, and we assume that $\mathbb{V}$ integrates to G. Restricting $\mathbb{V}$ to P, one obtains the *basic tractor bundle* $\mathcal{V} := \mathcal{G} \times_P \mathbb{V} \to M$ and the irreducible quotient induces the *bundle of 1-densities* $\mathcal{E}[1] \to M$. The bundle $\mathcal{V}$ carries a natural decreasing filtration by smooth subbundles $\{\mathcal{V}^i\}$. Using this, the bundle $\pi : A \to M$ can be equivalently described as:

- The open subbundle in the projectivization $\mathcal{P}(\mathcal{V}/\mathcal{V}^2)$ consisting of lines that are transversal to the hyperplane subbundle $\mathcal{V}^1/\mathcal{V}^2$.
- The bundle $Q\mathcal{E}[1] \to M$ of linear connections on the bundle of 1-densities.

The latter description nicely corresponds to the parametrization of Weyl structures by connections on a bundle of scales.

A local non-vanishing section of $\mathcal{E}[1]$ determines a flat connection on $\mathcal{E}[1]$ and thus a local smooth section s of $\pi : A \to M$. Now it turns out that there is a natural fully non-linear PDE on nowhere vanishing sections of $\mathcal{E}[1] \to M$:

- Since $\mathbb{V}$ is a fundamental representation, the corresponding first BGG operator H has order 2 and maps $\Gamma(\mathcal{E}[1])$ to $\Gamma(\circledcirc^2 T^*M \otimes \mathcal{E}[1])$ (invariant Hessian).
- The determinant induces a (non-linear) natural bundle map $S^2T^*M \to S^2\Lambda^n T^*M$, and the latter turns out to be some power $\mathcal{E}[-N]$ of $\mathcal{E}[-1] := \mathcal{E}[1]^*$.
- Thus $\sigma \mapsto \det(H(\sigma))$ is a non-linear invariant operator $\Gamma(\mathcal{E}[1]) \to \Gamma(\mathcal{E}[-(N-n)])$, and $\det(H(\sigma)) = \sigma^{-(N-n)}$ is an invariant PDE of Monge-Ampère type.

Now this nicely connects to the geometry of Weyl structures. In the picture of sections $s : M \to A$, we call a Weyl-structure *Lagrangean* if $s^*\Omega = 0$ and *non-degenerate* if s^*h is non-degenerate. Equivalently, the Rho tensor of s has to be symmetric and non-degenerate. The main connection to the invariant Monge-Ampère equation is

Theorem

Consider a locally flat AHS structure on M. Then a nowhere vanishing sections of $\Gamma(\mathcal{E}[1])$ satisfies the invariant Monge-Ampère equation if and only if the corresponding section $s : M \to A$ has the property that $s(M) \subset A$ is a minimal submanifold.

In projective geometry, solutions of the invariant Monge-Ampère equation are related to convex projective structures, which play an important role in the study of representation varieties.

Thank you for your attention!