7.8. Good pairs. We start the discussion of cellular homology by discussing a result which also is of independent interest. Under certain assumptions, it provides a relation between absolute and relative homology.

Definition 7.8. A topological pair (X, A) is called a *good pair* if A is closed in X and there is an open subset $U \subset X$ with $A \subset U$ such that A is a strong deformation retract in U.

Example 7.8. (1) The pair (B^n, S^{n-1}) is a good pair for each n. We can simply take $U := B^n \setminus \{0\}$ (or inded the complement of any closed ball of radius less than one).

(2) Let Y be any topological space and for a continuous function $f : S^{n-1} \to Y$ put $X := Y \cup_f B^n$. Then as in (1) we can take the complement of $0 \in B^n$ to see that (X, Y) is a good pair.

(3) In the situation of (2), we may as well glue an arbitrary number of cells. This shows that for any CW complex X with n–Skeleton X^n, the pair (X^n, X^{n-1}) is a good pair.

(4) Via so–called tubular neighborhoods, one shows that for any smooth manifold M and closed submanifold $N \subset M$, the pair (M, N) is a good pair.

Proposition 7.8. *Let (X, A) be a good pair. Viewing the projection to the quotient as a map $p : (X, A) \to (X/A, [A])$ of pairs, it induces isomorphisms $p_\# : H_q(X, A) \to H_q(X/A, [A])$ for each $q \geq 0$. In particular, $H_q(X, A) \cong H_q(X/A)$ for all $q > 0$.*

Proof. Let $U \subset X$ be an open subset as in the definition of a good pair. The we can consider the pairs (X, A), (X, U), $(X/A, [A])$ and $(X/A, U/A)$. From natural inclusions and projections we get the following commutative diagram of continuous maps of pairs and the induced diagram in homology:

$$\begin{tikzcd} (X,A) \arrow[r] \arrow[d] & (X,U) \arrow[d] \\ (X/A,[A]) \arrow[r] & (X/A,U/A) \end{tikzcd} \qquad \begin{tikzcd} H_q(X,A) \arrow[r] \arrow[d] & H_q(X,U) \arrow[d] \\ H_q(X/A,[A]) \arrow[r] & H_q(X/A,U/A) \end{tikzcd}$$

We want to prove that the left vertical arrow in the right diagram is an isomorphism. Now since $A \subset U$ is a deformation retract, example (4) of 7.1 shows that $H_q(U, A) = 0$ for all $q \geq 0$. Feeding this information into the long exect sequence of the triple (X, U, A) from Theorem 7.2, we conclude that the top horizontal arrow in the right diagram is an isomorphism.

A retraction $r : U \to A$ of course desends to a retraction $\underline{r} : U/A \to \{[A]\}$. Since $A \subset U$ is a strong deformation retract, viewing r as a map $U \to U$, it is homotopic relative to A to the identity. But such a homotopy descends to a map $U/A \times I \to U/A$, which then shows that $\underline{r}$ is homotopic to the identity. Hence $\{[A]\} \subset U/A$ is a strong deformation retract, and as above, we conclude that the bottom horizontal arrow in the right diagram is an isomorphism. Hence we can prove the first part of the Proposition by showing that $H_q(X, U) \cong H_q(X/A, U/A)$ for all q.

But now we get a second set of diagrams induced by canonical inclusions and projections:

$$\begin{tikzcd} H_q(X \setminus A, U \setminus A) \arrow[r] \arrow[d] & H_q(X,U) \arrow[d] \\ H_q(X/A \setminus [A], U/A \setminus [A]) \arrow[r] & H_q(X/A,U/A) \end{tikzcd}$$

We have to show that the right vertical arrow in the right diagram is an isomorphism. But since $A = \bar{A} \subset U^o = U$, the top horizontal arrow in the right diagram is an isomorphism by excision, see Theorem 7.4. The same reasoning applies to the bottom horizontal arrow, so this is an isomorphism, too. By definition of the quotient, the projection restricts to a homeomorphim $(X \setminus A, U \setminus A) \to (X/A \setminus [A], U/A \setminus [A])$ of pairs, so the left vertical arrow in the right diagram is an isomorphism, too. This completes the proof that $H_q(X, A) \cong H_q(X/A, [A])$ and the last statement then follows from Example (3) of 7.1. □

7.9. On the homology of CW–complexes. Let X be a CW–complex and for each $k \in \mathbb{N}$ let X^k be the k–skeleton of X.

PROPOSITION 7.9. *(1) For each k, the relative homology group $H_q(X^k, X^{k-1})$ vanishes for $q \neq k$, while $H_k(X^k, X^{k-1}) = \oplus_\alpha \mathbb{Z}$, where α runs through the set of k–cells of X.*

(2) The homomorphism $H_q(X^k) \to H_q(X)$ induced by the inclusion $X^k \hookrightarrow X$ is bijective for $q < k$ and surjective for $q = k$.

(3) If for some $q > 0$, X has no cells in dimension q, the $H_q(X) = 0$. In particular, if X is n–dimensional, then $H_q(X) = 0$ for $q > n$.

PROOF. (1) For each k, we know from Section 4.18 that there is a relative homeomorphism $F : (\sqcup_\alpha B^k_\alpha, \sqcup_\alpha S^{k-1}_\alpha) \to (X^k, X^{k-1})$. Now the quotient space $(\sqcup_\alpha B^k_\alpha)/(\sqcup_\alpha S^{k-1}_\alpha)$ clearly is a wedge $\vee_\alpha S^k_\alpha$ and from the construction it is evident that F induces a homeomorphism from this wedge onto X^k/X^{k-1}. In view of Proposition 7.8, we conclude that F induces an isomorphism $H_q(\sqcup_\alpha B^k_\alpha, \sqcup_\alpha S^{k-1}_\alpha) \cong H_q(X^k, X^{k-1})$ for each $q \geq 0$.

Now from Example (3) of Section 6.6, we conclude that $S_q(\sqcup_\alpha B^k_\alpha) = \oplus_\alpha S_q(B^k_\alpha)$ and likewise for the spheres. Thus we see that $S_q(\sqcup_\alpha B^k_\alpha, \sqcup_\alpha S^{k-1}_\alpha) = \oplus_\alpha S_q(B^k_\alpha, S^{k-1}_\alpha)$. Of course, this decomposition is compatible with the boundary operators, so the relative homology groups decompose as direct sums. Using this, (1) follows from Proposition 7.6.

(2) The long exact sequence of the pair (X^k, X^{k-1}) contains the parts

$$\dots \to H_{q+1}(X^k, X^{k-1}) \to H_q(X^{k-1}) \to H_q(X^k) \to H_q(X^k, X^{k-1}) \to \dots$$

For $q \neq k, k-1$ the relative homologies both vanish by part (1) and hence $X^{k-1} \hookrightarrow X^k$ induces an isomorphism $H_q(X^{k-1}) \cong H_q(X^k)$. If $q = k-1$, then the right hand relative homology still is zero, so the induced homomorphism $H_{k-1}(X^{k-1}) \to H_{k-1}(X^k)$ is surjective.

Now consider the inclusions $X^k \hookrightarrow X^{k+1} \hookrightarrow X^{k+2} \hookrightarrow \dots$ and the induced maps $H_q(X^k) \to H_q(X^{k+1}) \to H_q(X^{k+2}) \to \dots$. For $q < k$, they are all isomorphisms, while for $q = k$ the first is surjective and all others are isomorphisms. This already implies (2) in case that X is finite dimensional, since then $X = X^N$ for sufficiently large N.

In general, we claim that each $c \in S_q(X)$ lies in the image of the inclusion $S_q(X^N) \to S_q(X)$ for sufficiently large N. This simply follows since any singular simplex in X is compact subset and hence meets only finitely many cells, see Proposition 4.17. Hence there is a maximal dimension of cells met by a singular simplex and taking the maximum over the finitely many singular simplices showing up in c, we get the required number N. Now any class in $H_q(X)$ can be written as $[c]$ for some chain c, so it lies in the image of the homomorphism $H_q(X^N) \to H_q(X)$ and hence also in the image of $H_q(X^q) \to H_q(X)$. Likewise, suppose that for $\tilde{c} \in S_q(X^{q+1})$ we have $0 = [\tilde{c}] \in H_q(X)$. Then there is a chain $c \in S_{q+1}(X)$ such that $dc = \tilde{c}$ and as above c sctually lies in $S_{q+1}(X^N)$ for sufficiently large n. But this says that $0 = [c] \in H_q(X^N)$ and hence also in $H_q(X^{q+1})$. So the proof of (2) is complete.

(3) The above observations also show that $H_q(X^{q-1}) \cong H_q(X^{q-2}) \cong \ldots \cong H_q(X_0)$, and the latter group vanishes, since X^0 is discrete and $q > 0$. But if X has no celles of dimension q, then $X^q = X^{q-1}$, so $H_q(X^q) = \{0\}$. But from (2) we know that $H_q(X^q)$ surjects onto $H_q(X)$. □

7.10. The cellular complex. Let X be a CW–complex and for $k \in \mathbb{N}$ let X^k be the k skeleton of X. Then for each $q \geq 0$ define $W_q(X)$ to be the relative homology group $H_q(X^q, X^{q-1})$. So part (1) of Proposition 7.9 shows that $W_q(X)$ is the free group generated by the q–cells of X. Next, consider the long exact sequence of the triple (X^q, X^{q-1}, X^{q-2}), see Theorem 7.2. Its connecting homomorphism in degree q maps $H_q(X^q, X^{q-1})$ to $H_{q-1}(X^{q-1}, X^{q-2})$, so this defines a homomorphism $\partial = \partial_q : W_q(X) \to W_{q-1}(X)$.

In section 7.2, we have observed that ∂_q can be written as a composition of two homomorphisms as in $H_q(X^q, X^{q-1}) \to H_{q-1}(X^{q-1}) \to H_{q-1}(X^{q-1}, X^{q-2})$. Here the first homomorphism is the connecting homomorphism in the long exact sequence of the pair (X^q, X^{q-1}), while the second homomorphism is induced by the inclusion $(X^{q-1}, \emptyset) \hookrightarrow (X^{q-1}, X^{q-2})$, so this is part of the long exact sequence of the pair (X^{q-1}, X^{q-2}). Doing the same for ∂_{q-1}, we get an expression for $\partial_q \circ \partial_{q-1}$ as a composition of four homomorphisms. But the middle two of these homomorphisms are two subsequent homomorphisms in the long exact sequence of the pair (X^{q-1}, X^{q-2}), so already their composition is zero.

Hence we conclude that $(W_*(X), \partial)$ is a chain complex, which is called the *cellular complex* of X. The homology groups of this complex are called the *cellular homology groups* of X. We will describe the differential ∂ in this complex more explicitly later. First we prove that the cellular complex can be used to compute the singular homology of a CW complex.

Theorem 7.10. *The cellular homology groups of a CW complex X are isomorphic to the singular homology groups $H_*(X)$.*

Proof. Fix a degree $q \geq 0$ and let X^q be the q–skeleton of X. Then we know that the inclusion $X^q \hookrightarrow X$ induces a surjection $H_q(X^q) \to H_q(X)$. Now the long exact sequence of the pair (X^q, X^{q-1}) contains the part

$$\cdots \to H_q(X^{q-1}) \to H_q(X^q) \xrightarrow{j_\#} H_q(X^q, X^{q-1}) \xrightarrow{\delta} H_{q-1}(X^{q-1}) \to \ldots$$

Now part (3) of Proposition 7.9 shows that $H_q(X^{q-1}) = 0$, so $j_\#$ injects $H_q(X^q)$ into $H_q(X^q, X^{q-1}) = W_q(X)$. The same argument in degree $q-1$ shows that $j_\# : H_{q-1} \to W_{q-1}(X)$ is injective and from above we know that its composition with δ coincides with ∂_q. But this shows that $\ker(\partial_q) \subset W_q(X)$ coincides with $\ker(\delta)$ and thus with the image of $j_\# : H_q(X^q) \to W_q(X)$. Hence $j_\#$ identifies $H_q(X^q)$ with the q–dimensional cycles in the cellular complex.

To complete it suffices to show that for a homology class $[c] \in H_q(X^q)$, the image $j_\#([c])$ lies in the image of ∂_{q+1} if and only if $[c]$ lies in the kernel of the map $H_q(X^q) \to H_q(X)$ induced by the inclusion. From the proof of Proposition 7.9 we see that this kernel coincides with the kernel of the map $H_q(X^q) \to H_q(X^{q+1})$ induced by the inclusion. The long exact sequence of the pair (X^{q+1}, X^q) then shows that this kernel coincides with the image of the connecting homomorphism $\delta : H_{q+1}(X^{q+1}, X^q) \to H_q(X^q)$. But as we have noticed above, $\partial_{q+1} = j_\# \circ \delta$, so the proof is complete. □