Proseminar "Analysis on manifolds"

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(1) Recall the definitions of second countability and of separability for topological spaces. Prove that any second countable space is separable and that a subspace of a second countable space is second countable, too. Is a subspace of a separable space automatically separable?

(2) Prove that a second countable topological space X is a Lindelöff space, i.e. that for any open covering of X, there is a countable subcovering.

(3) For a continuous function $f : X \to \mathbb{R}$ on a topological space X define the support $\operatorname{supp}(f)$ of f to be the closure of the subset $\{x \in X : f(x) \neq 0\}$. Suppose that $\{f_i : i \in I\}$ is a collection of continuous functions $f_i : X \to \mathbb{R}$ such that for each $x \in X$, there is a neighborhood U of x in X such that U intersects only finitely many of the sets $\operatorname{supp}(f_i)$. Show that $f(x) := \sum_{i\in I} f_i(x)$ defines a continuous function $f : X \to \mathbb{R}$ such that $\operatorname{supp}(f) \subset \cup_{i\in I} \operatorname{supp}(f_i)$.

Hint: Argue why it suffices to show that each point x has an open neighborhood U such that $f|_U$ is continuous to conclude continuity of f. For the claim about the support, show that $\cup_{i\in I} \operatorname{supp}(f_i)$ is closed in X.

(4) Let X be a second countable Hausdorff space which is locally Euclidean in the sense that any point $x \in X$ has an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. Show that X is locally compact and use this and exercise (2) to prove that there is a family $\{K_n : n \in \mathbb{N}\}$ of compact subsets of X such that $X = \cup_{n\in\mathbb{N}} K_n$. ("$X$ is σ-compact and admits a compact exhaustion").

(5) Let $U \subset \mathbb{R}^n$ be an open subset and let $f : U \to \mathbb{R}^m$ be a continuously differentiable function. Suppose that for some point $x \in U$, the derivative $Df(x) : \mathbb{R}^n \to \mathbb{R}^m$ has rank k (as a linear map). Prove that there is an open neighborhood V of x in U such that for all $y \in V$, the rank of $Df(y)$ is at least k.

(6) Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension k and let $W \subset \mathbb{R}^n$ be an open set with $M \subset W$. Show that for a diffeomorphism F from W onto another open subset $\tilde{W} \subset \mathbb{R}^n$, also $F(M)$ is a smooth submanifold of $\mathbb{R}^n$.

(7) ("Stereographic projection") Take the unit sphere $S^n \subset \mathbb{R}^{n+1}$ and let $N := (0, \dots, 0, 1)$ be the "north pole". Identify $\mathbb{R}^n$ with the affine hyperplane orthogonal to N through the point $-N$ (so this is tangent to S^n). Define a map $f : \mathbb{R}^n \to S^n$ by sending each point x in that hyperplane to the intersection of the line segment connecting x to N with S^n. Prove that f defines a diffeomorphism from $\mathbb{R}^n$ to $S^n \setminus \{N\}$.

(8) Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension k. Suppose that there is a local trivialization $\Phi : U \to V$ for M and consider the open subset $U \cap M \subset M$. Prove that there are smooth functions $\xi_1, \dots, \xi_k : U \cap M \to \mathbb{R}^n$ such that for each $x \in U \cap M$ the vectors $\xi_1(x), \dots, \xi_k(x)$ form a basis for the tangent space T_xM.

(9) Let $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^m$ be submanifolds of dimensions k and ℓ, respectively. Show that $M \times N \subset \mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$ is a submanifold of dimension $k + \ell$ and that the projections $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are smooth. Also show that $T_{(x,y)}(M \times N) = \{(X, Y) : X \in T_x M, Y \in T_y N\}$.

(10) Find a description of a torus as a submanifold of $\mathbb{R}^3$ and prove that it is diffeomorphic to $S^1 \times S^1 \subset \mathbb{R}^4$.

(11) Find a description of an (open) Möbius strip as a submanifold of $\mathbb{R}^3$. Explain why this cannot be globally realized as a regular zero set of a smooth function.

(12) View the space $M_n(\mathbb{R})$ of real $n \times n$-matrices as $\mathbb{R}^{n^2}$. Show that $M := \{A \in M_n(\mathbb{R}) : \det(A) = 1\}$ is a smooth submanifold of $\mathbb{R}^{n^2}$ and determined the tangent space $T_{\mathbb{I}}M$ at the identity matrix $\mathbb{I}$.

(13) ("The line with two origins", see [Lee, Problem 1-1.]) On $X := \{(x, y) \in \mathbb{R}^2 : y = \pm 1\}$ consider the equivalence relation generated by $(x, 1) \sim (x, -1)$ for $x \neq 0$. Let M be the set of equivalence classes endowed with the quotient topology. Show that M satisfies all defining properties of a topological manifold except for the Hausdorff property.

(14) Use local charts for hemispheres in S^n to make the space $\mathbb{R}P^{n-1}$ discussed in the end of Section 1.5 of the course into a smooth manifold.

(15) On $\mathbb{C}^n \setminus \{0\}$ consider the equivalence relation defined by $x \sim y$ if there is $z \in \mathbb{C}$ such that $y = zx$ and let $\mathbb{C}P^{n-1}$ be the space of equivalence classes (endowed with the qoutient topology). Denote the equivalence class of a point $x = (x^1, \dots, x^n)$ by $[x^1 : \dots : x^n]$. Show that one obtains a finite atlas by putting $U_i := \{[x^1 : \dots : x^n] : x^i \neq 0\}$ and defining $u_i : U_i \to \mathbb{C}^{n-1}$ by

$$u_i([x^1 : \dots : x^n]) := (\tfrac{x^1}{x^i}, \dots, \tfrac{x^{i-1}}{x^i}, \tfrac{x^{i+1}}{x^i}, \dots, \tfrac{x^n}{x^i}).$$

Observe that this even makes $\mathbb{C}P^{n-1}$ into a complex manifold.

(16) Prove that $\mathbb{C}P^1$ is diffeomorphic to the sphere S^2, e.g. by using example (7).

(17) In the setting of example (15) consider the map $q : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}P^{n-1}$ which sends each point to its equivalence class. Consider the sphere S^{2n-1} as the unit sphere in $\mathbb{C}^n$ and define $\underline{q} := q|_{S^{2n-1}}$ ("Hopf fibration"). Show that q and $\underline{q}$ are smooth and surjective, and that for each $y \in \mathbb{C}P^{n-1}$ the preimage $\underline{q}^{-1}(y)$ is a smooth submanifold of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ diffeomorphic to S^1.

(18) In the case $n = 2$, describe the Hopf fibration $\underline{q}$ from Example (17) explicitly as a map from $S^3 = \{(z, w) \in \mathbb{C}^2 : z\bar{z} + w\bar{w} = 1\}$ to $\mathbb{R}^3$. (This needs making the diffeomorphism from Example (16) explicit as a map to $\mathbb{R}^3$.)

(19) Show that the map $q : \mathcal{V}(k, n) \to Gr(k, n)$ introduced in Example 1.7. (6) of the course is smooth. Prove that in the picture of $n \times k$-matrices two elements $A, B \in \mathcal{V}(k, n)$ satisfy $q(A) = q(B)$ if and only if there is an invertible $k \times k$-matrix C such that $A = BC$.

(20) For $k < n$, let $\mathcal{O}(k,n)$ be the space of k-tuples of orthonormal vectors in $\mathbb{R}^n$ with the structure of a submanifold as introduced in Section 1.3 of the lecture course. Show that sending a tuple to the subspace it spans defines a surjective, smooth map q from $\mathcal{O}(k,n)$ to the Grassmann manifold $Gr(k,n)$. Show that in the picture of matrices, one has $q(A) = q(B)$ if and only if there is an orthogonal $k \times k$-matrix C such that $A = BC$.

(21) Prove the smooth version of Urysohn's lemma for smooth manifolds: Let $A, B \subset M$ be closed subsets with $A \cap B = \emptyset$. Then there is a smooth function $f : M \to \mathbb{R}$ with values in $[0,1]$ which is identically one on A and identically zero on B.

Hint: This is similar to the proof of Corollary 1.9, starting from the open covering of M formed by $U := M \setminus A$ and $V := M \setminus B$.

(22) Prove that the tangent bundle TS^1 is diffeomorphic to the the manifold $S^1 \times \mathbb{R}$ in such a way that $p : TS^1 \to S^1$ corresponds to the first projection in the product.

(23) Let M be a smooth manifold $N \subset M$ a submanifold, and let (U,u) be a chart for M (not a submanifold chart for N in general) such that $U \cap N \neq \emptyset$. Show that $u(U \cap N)$ is a smooth submanifold of $\mathbb{R}^n$.

Hint: Keep in mind that this can be verified locally and look for a simple solution, e.g. via regular zero sets.

(24) Prove part (1) of Proposition 1.19 of the course.

(25) *For this exercise, please take the real analog of exercise* (15) *as granted: We have* $q : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}P^{n-1}$, *use homogeneous coordinates* $[x^1 : \dots : x^n]$ *and get a finite atlas* (U_i, u_i) *by exactly the same constructions as in exercise* (15).

Put $E := \{(q(x), \lambda x) : x \in \mathbb{R}^n \setminus \{0\}, \lambda \in \mathbb{R}\}$ and let $p : E \to \mathbb{R}P^{n-1}$ be the projection to the first component. Show that

- E is a smooth submanifold of $\mathbb{R}P^{n-1} \times \mathbb{R}^n$ and $p : E \to \mathbb{R}P^{n-1}$ is smooth.
- For each i, there is a diffeomorphism $p^{-1}(U_i) \to U_i \times \mathbb{R}$ whose first component coincides with p.

Hint: Construct a diffeomorphism $U_i \times \mathbb{R}^n \to u_i(U_i) \times \mathbb{R}^n$ whose components on $(q(x), y)$ are $u_i(q(x))$, y^i and $y^j - y^i \frac{x^j}{x^i}$ for $j \neq i$. This provides both a submanifold chart for E and (via leaving out the last $n-1$ components) the required diffeomorphism.

(26) In the setting of the last exercise, consider $E_0 := E \cap (\mathbb{R}P^{n-1} \times \{0\})$ and the open subset $E \setminus E_0$. Show that $x \mapsto (q(x), x)$ defines a diffeomorphism $\mathbb{R}^n \setminus \{0\} \to E \setminus E_0$.
Remark: This essentially shows that E cannot be isomorphic to a product $\mathbb{R}P^{n-1} \times \mathbb{R}$ in an appropriate sense (since then $E \setminus E_0$ would be isomorphic to $\mathbb{R}P^{n-1} \times (\mathbb{R} \setminus \{0\})$, which is not connected). Indeed, for $n = 2$, E is diffeomorphic to a Möbius band.

(27) Let $M_n(\mathbb{R})$ be the vector space of real $n \times n$-matrices and let G be the open subset of invertible matrices. For fixed $X \in M_n(\mathbb{R})$, show that the map $L_X : G \to M_n(\mathbb{R})$, $L_X(A) := AX$ defines a vector field on G. Show that $(A, X) \mapsto (A, L_X(A))$ defines a diffeomorphism $G \times M_n(\mathbb{R}) \to TG$ and explain why/how this differs from the identification of TG with $G \times M_n(\mathbb{R})$ coming from the fact that G is an open subset in $\mathbb{R}^{n^2}$.

(28) Recall from Section 1.3 of the course that the group $O(n)$ of orthogonal $n \times n$-matrices is a smooth submanifold of $M_n(\mathbb{R})$ contained in G. Use the results there to determine the tangent space $\mathfrak{o}(n) := T_{\mathbb{I}}O(n)$ to this submanifold at the unit matrix $\mathbb{I}$. Show that for $X \in \mathfrak{o}(n)$ and $A \in O(n)$, $L_X(A) := AX$ lies in $T_AO(n)$ and conclude that one obtains a vector field L_X on $O(n)$ as well as a diffeomorphism $O(n) \times \mathfrak{o}(n) \to TO(n)$.

(29) For a smooth submanifold M of $\mathbb{R}^n$ take two vector fields $\xi, \eta \in \mathfrak{X}(M)$. View them also as $\mathbb{R}^n$-valued functions to form the directional derivatives $\xi \cdot \eta, \eta \cdot \xi : M \to \mathbb{R}^n$. Show that for each $x \in M$, one has $(\xi \cdot \eta)(x) - (\eta \cdot \xi)(x) \in T_xM$ and that $\xi \cdot \eta - \eta \cdot \xi = [\xi, \eta]$.

(30) Use the last exercise to show that in the setting of exercises (27) and (28) one has $[L_X, L_Y] = L_Z$, where $Z = XY - YX \in M_n(\mathbb{R})$. Observe that for $X, Y \in \mathfrak{o}(n)$, one always gets $Z \in \mathfrak{o}(n)$.

(31) In the setting of exercises (27) and (28), let G be the set of invertible matrices respectively $G = O(n)$. Show that for any $A \in G$, the map $\lambda_A(B) := AB$ defines a diffeomorphism $\lambda_A : G \to G$. Then show that a vector field $\xi \in \mathfrak{X}(G)$ is of the form L_X for some $X \in T_{\mathbb{I}}G$ if and only if it satisfies $(\lambda_A)^*\xi = \xi$ for all $A \in G$. Use this as an alternative argument for the fact that for $X, Y \in T_{\mathbb{I}}G$, we must have $[L_X, L_Y] = L_Z$ for some $Z \in T_{\mathbb{I}}G$.

(32) Explain how the Lie bracket can be obtained as a well defined operation on vector fields based on the coordinate formula (2.3) from Theorem 2.4 (i.e. without using the action on smooth functions). Carry out the necessary verifications and give an alternative direct proof or part (3) of Theorem 2.4 in this setting.

(33) In the coordinate based approach to the Lie bracket from the last exercise (i.e. again without using the action on functions), verify compatibility of the Lie bracket with the pullback along local diffeomorphisms.

Hint: Given a local diffeomorphism $F : M \to N$ and $x \in M$, use a chart (V, v) for N around $F(x)$ and F to construct a chart $(U.u)$ for M around x. Describe the relation between the components of $\eta \in \mathfrak{X}(N)$ with respect to (V, v) to the components of $F^*\eta$ with respect to (U, u) and use this to establish the result.

(34) Let $U \subset \mathbb{R}^2$ be an open subset and $f : U \to \mathbb{R}$ a smooth function. On $U \times \mathbb{R} \subset \mathbb{R}^3$ consider the vector fields $\xi := \frac{\partial}{\partial x^1} + \frac{\partial f}{\partial x^1}\frac{\partial}{\partial x^3}$ and $\eta := \frac{\partial}{\partial x^2} + \frac{\partial f}{\partial x^2}\frac{\partial}{\partial x^3}$. Call a smooth curve $c : I \to \mathbb{U} \times \mathbb{R}$ *admissible* if for each $t \in I$, $c'(t)$ can be written as a linear combination of $\xi(c(t))$ and $\eta(c(t))$.

Prove that a smooth curve $c = (c^1, c^2, c^3)$ is admissible if and only if $c^3(t) - f(c^1(t), c^2(t))$ is constant and give a geometric explanation for this fact.

(35) Consider the vector fields $\xi := \frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^3}$ and $\eta = \frac{\partial}{\partial x^2} - x^1\frac{\partial}{\partial x^3}$ on $\mathbb{R}^3$. Show that for any $a \in \mathbb{R}$, there is an admissible curve c in the sense of the last example that starts in 0 and ends in $(0, 0, a)$. If you are ambitious, you can try to deduce from this fact that any point in $\mathbb{R}^3$ can be reached from the origin by piecing together finitely many (in fact, three or even two are easily seen to be enough) admissible curves.

Hint: Make an ansatz $c(t) = (r(\cos t - 1), r\sin t, x^3(t))$ and prove that for each r, there is a unique choice of a function $x^3(t)$ for which c is admissible. Then compute $x^3(2\pi)$.

(36) Compute the flow of the vector field $\xi := -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}$ on $\mathbb{R}^2$.

(37) Let M be a smooth manifold and let $\xi \in \mathfrak{X}(M)$ be a vector field. Suppose that there is a point x_0 in M and for each $y \in M$ there is a diffeomorphism $F_y : M \to M$ such that $F_y(x_0) = y$ and such that $(F_y)^*\xi = \xi$. Prove that the vector field ξ is complete and apply this and exercise (31) to deduce completeness of the vector fields L_X from exercises (27) and (28).

(38) Recall the definition of exponential function for matrices and show that $e^{(t+s)X} = e^{tX}e^{sX}$ and $\frac{d}{dt}|_{t=0}e^{tX} = X$. (If you need analytical results on the exponential series just state them, there is not need to prove those.) Use these properties to prove that the flow of the vector field L_X from exercise (27) is given by $\mathrm{Fl}_t^{L_X}(A) = Ae^{tX}$.

(39) For two finite dimensional vector spaces V and W over $\mathbb{R}$, prove that the space $\mathcal{B}(V, W)$ of bilinear maps $V \times W \to \mathbb{R}$ is a vector space of dimension $\dim(V)\dim(W)$. Using only the universal property of the tensor product, show that $V \otimes W$ is isomorphic to the dual space $\mathcal{B}(V, W)^*$ and thus has dimension $\dim(V)\dim(W)$, too. Describe the map $(v, w) \mapsto v \otimes w$ in this picture.

(40) In the setting of exercise (39) show that $V^* \otimes W$ is naturally isomorphic to $L(V, W)$ and to $L(W^*, V^*)$, and describe the resulting natural isomorphism $L(V, W) \to L(W^*, V^*)$ explicitly. Show that an element of $V^* \otimes W$ can be written in the form $\lambda \otimes w$ for $\lambda \in V^*$ and $w \in W$ if and only if the corresponding linear map $V \to W$ has rank one.

(41) In the setting of exercise (34) (with the correct + signs in the definitions of the vector fields), let $p : U \times \mathbb{R} \to U$ be the first projection. Compute the one-form $\omega := dx^3 - p^*df \in \Omega^1(U \times \mathbb{R})$ and show that a smooth curve $c : I \to U \times \mathbb{R}$ is admissible in the sense of exercise (34) if and only if $c^*\omega = 0$.

(42) Let $M \subset \mathbb{R}^{n+1}$ be a smooth submanifold of dimension n. Show that there are local unit normals for M, i.e. that for each $x \in M$, there is an open subset $U \subset M$ with $x \in U$ and smooth function $\mathfrak{n} : U \to S^n \subset \mathbb{R}^{n+1}$ such that $\mathfrak{n}(y) \perp T_yM$ for each $y \in U$. Then show that $T_{\mathfrak{n}(y)}S^n = T_yM$ and hence $T_y\mathfrak{n}$ can be interpreted as a linear map $L_y : T_yM \to T_yM$. Finally, show that $y \mapsto L_y$ defines a smooth $\binom{1}{1}$-tensor field L on U.

Hint: To construct $\mathfrak{n}$, realize $M \cap U$ a regular zero set $F^{-1}(\{0\})$ and then form the normed gradient of F.

(43) In the setting of the last exercise, show that restricting the inner product on $\mathbb{R}^{n+1}$ to the tangent spaces of M defines a Riemannian metric g on M. Prove that on an open subset U as in the last exercise $II_y(X, Y) := g_y(X, L_y(Y))$ defines a symmetric bilinear form on T_yM for each $y \in U$. Finally, show that $y \mapsto II_y$ defines a $\binom{0}{2}$-tensor field on U and interpret this as a contraction of $(g|_U) \otimes L$. Is it possible to obtain L as a contraction of $g|_U \otimes II$?

(44) Let $A \in \mathcal{T}^1_k(M)$ be a smooth $\binom{1}{k}$-tensor field on a smooth manifold M. Show that for each $x \in M$ the value A_x can be interpreted as a k-linear map $(T_xM)^k \to T_xM$. Use this to associate to vector fields $\xi_1, \dots, \xi_k \in \mathfrak{X}(M)$ a vector field $A(\xi_1, \dots, \xi_k)$. Using Lemma 3.3, show that $\mathcal{T}^1_k(M)$ can be identified with the space of k-linear operators $(\mathfrak{X}(M))^k \to \mathfrak{X}(M)$ which are linear over smooth functions in each argument.

Hint: Try to keep things simple. It is easier to reduce the claim you want to prove to the statement of Lemma 3.3 than to redo the proof of that Lemma. (If you prefer to redo the proof, just outline it.)

(45) An *affine connection* on a smooth manifold M is a bilinear operator $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ written as $(\xi, \eta) \mapsto \nabla_\xi \eta$ such that for any $\xi, \eta \in \mathfrak{X}(M)$ and any $f \in C^\infty(M, \mathbb{R})$, one has $\nabla_{f\xi}\eta = f\nabla_\xi \eta$ and $\nabla_\xi(f\eta) = \xi(f)\eta + f\nabla_\xi\eta$. (Don't worry about existence here, there are many important examples.)

Use exercise (44) to prove the following:

(i) For an affine connection ∇ on M and $A \in \mathcal{T}^1_2(M)$ also $(\xi, \eta) \mapsto \nabla_\xi\eta + A(\xi, \eta)$ is a linear connection on M.

(ii) For affine connections ∇ and $\tilde{\nabla}$ on M, the expression $(\xi, \eta) \mapsto \tilde{\nabla}_\xi\eta - \nabla_\xi\eta$ defines a $\binom{1}{2}$-tensor field on M.

(46) Suppose that ∇ is an affine connection on a smooth manifold M as in the last exercise. Using exercise (44) prove that

(i) For fixed $\eta \in \mathfrak{X}(M)$ one can interpret $\xi \mapsto \nabla_\xi\eta$ as a $\binom{1}{1}$-tensor field $\nabla\eta$ on M.

(ii) There is a unique $\binom{1}{2}$-tensor field T on M such that $T(\xi, \eta) = \nabla_\xi\eta - \nabla_\eta\xi - [\xi, \eta]$.

(iii) (only if you are ambitious) There is a unique $\binom{1}{3}$-tensor field R on M such that $R(\xi, \eta, \zeta) = \nabla_\xi\nabla_\eta\zeta - \nabla_\eta\nabla_\xi\zeta - \nabla_{[\xi,\eta]}\zeta$.

(47) Use Lemma 3.3 to verify that for a vector field $\eta \in \mathfrak{X}(M)$ and a one-form $\omega \in \Omega^1(M)$ formula (3.9) of the course defines a one-form $\mathcal{L}_\eta\omega$. Use that formula to compute $\mathcal{L}_\eta(f\omega)$ and $\mathcal{L}_{f\eta}\omega$ for $f \in C^\infty(M, \mathbb{R})$.

(48) Use formula (3.9) to compute the local coordinate expression of $\mathcal{L}_\eta\omega$ with respect to a chart (U, u) on M from the local coordinate expressions of η and ω.

(49) In the spirit of exercise (44), view $\Phi \in \mathcal{T}^1_1(M)$ also as an operator $\mathfrak{X}(M) \to \mathfrak{X}(M)$, which is linear over smooth functions. Use formula (3.10) of the course to prove that in this picture, the Lie derivative $\mathcal{L}_\eta\Phi$ of Φ along a vector field $\eta \in \mathfrak{X}(M)$ is characterized by $(\mathcal{L}_\eta\Phi)(\xi) = [\eta, \Phi(\xi)] - \Phi([\eta, \xi])$. Use this to compute to local coordinate expression of $\mathcal{L}_\eta\Phi$ for a local chart (U, u) from the local coordinate expressions of η and Φ.

Hint: Use different notations for the operators $\mathfrak{X}(M) \to \mathfrak{X}(M)$ and $\mathfrak{X}(M) \times \Omega^1(M) \to C^\infty(M, \mathbb{R})$ induced by Φ and $\mathcal{L}_\eta\Phi$ in order to avoid confusion.

(50) Verify explicitly that for $\varphi \in \Omega^1(M)$, $\omega \in \Omega^k(M)$ and vector fields $\xi_0, \dots, \xi_k \in \mathfrak{X}(M)$, one has

$$(\varphi \wedge \omega)(\xi_0, \dots, \xi_k) = \textstyle\sum_{i=0}^k (-1)^i \varphi(\xi_i)\omega(\xi_0, \dots, \widehat{\xi_i}, \dots, \xi_k).$$

Similarly, show that for $\omega \in \Omega^2(M)$ and $\xi_1, \xi_2, \xi_3, \xi_4 \in \mathfrak{X}(M)$, $(\omega \wedge \omega)(\xi_1, \xi_2, \xi_3, \xi_4)$ is a non-zero constant multiple of

$$\omega(\xi_1, \xi_2)\omega(\xi_3, \xi_4) - \omega(\xi_1, \xi_3)\omega(\xi_2, \xi_4) + \omega(\xi_1, \xi_4)\omega(\xi_2, \xi_3).$$

(51) For an open subset $U \subset \mathbb{R}^n$, view a k-form $\omega \in \Omega^k(U)$ as a smooth function $\omega : U \to L^k_a$ as discussed in Section 3.6 of the course. For smooth functions $\xi_i : U \to \mathbb{R}^n$ with $i = 0, \dots, k$, compute the derivative of $\omega(\xi_1, \dots, \xi_k)$ in direction of ξ_0 in terms of the derivatives of ω and of the ξ_i. Use this and the fact that $D(\xi_j)(\xi_i) - D(\xi_i)(\xi_j) = [\xi_i, \xi_j]$ to show that formula (3.17) is obtained from evaluating the function $d\omega : U \to L^{k+1}_a$ (constructing via alternating $D\omega$ as described in Section 3.6) on the ξ_i.

(52) Using formulae (3.17), (3.18) and (3.19) as the definitions of the operators verify Cartan's magic formula $\mathcal{L}_\eta \omega = d(i_\eta \omega) + i_\eta d\omega$ in the case of a two-form $\omega \in \Omega^2(M)$. If you are ambitious, you can try to prove the statement for k-forms along the same lines.

(53) As in exercise (42) consider a smooth submanifold $M \subset \mathbb{R}^{n+1}$ of dimension n and the notion of local unit normals defined there. Show that there exists a globally defined smooth unit normal $\mathfrak{n} : M \to S^n$ if and only if there exists a form $\omega \in \Omega^n(M)$ such that $\omega(x) \neq 0$ for all $x \in M$.

Hint: Given a local unit normal $\mathfrak{n} : U \to \mathbb{R}^{n+1}$, insert $\mathfrak{n}(x)$ into the determinant to obtain a non-zero n-linear, alternating map $(T_xM)^n \to \mathbb{R}$ and show that this defines a nowhere-vanishing n-form on U. For $U = M$, this implies one direction of the claim.

Conversely, if $\omega \in \Omega^n(M)$ is nowhere vanishing, select one of the two possible unit normals on a connected subset $U \subset M$ by the fact that the form constructed above is a positive multiple of $\omega|_U$. Show that these fit together to define a global smooth unit normal.

(54) For an open subset $U \subset \mathbb{R}^3$ show that $\Omega^3(U)$ can be identified with $C^\infty(M, \mathbb{R}) = \Omega^0(U)$ via sending $f \in C^\infty(M, \mathbb{R})$ to $f \det$. Further show that sending ξ to $X_x \mapsto \langle \xi(x), X \rangle$ and to $(X_x, Y_x) \mapsto \det(\xi(x), X_x, Y_x)$ defines isomorphisms of $\mathfrak{X}(U)$ with $\Omega^1(U)$ and $\Omega^2(U)$, respectively.

Show that via these identifications, the exterior derivative in different degrees gives gives rise to the operations of gradient, rotation, and divergence from vector analysis.

Hint: Express the identifications in terms of coordinate one forms to simplify the computations of exterior derivatives.