Projective Compactness

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- This talk reports on joint work in progress with Rod Gover (Auckland).
- The notion of a projectively compact affine connection is an analog of a conformally compact Riemannian metric. Given a smooth manifold $\bar{M}$ with boundary ∂M and interior M, a linear connection ∇ on TM is called projectively compact if its projective equivalence class smoothly extends to $\bar{M}$.
- It turns out that one may involve a parameter $\alpha \in (0, 2]$ in the picture which is called the *order* or projective compactness.
- If the connection preserves a volume density, then the power of this which is a section of $\mathcal{E}(\alpha)$ is a natural defining density for the boundary.
- For $\alpha = 1, 2$ there are projectively invariant overdetermined systems defined on sections of $\mathcal{E}(\alpha)$ and we study the consequences of the defining density being a solution respectively a normal solution of this system.

Contents

Conformal compactness

Let $\bar{M}$ be a smooth manifold of dimension $n+1$ with boundary ∂M and interior M. Recall that a *local defining function* for ∂M is a smooth function $r : U \to \mathbb{R}_{\geq 0}$, where $U \subset \bar{M}$ is open, such that $r^{-1}(\{0\}) = U \cap \partial M$ and dr is nowhere zero on $U \cap \partial M$.

A Riemannian metric g on M is called *conformally compact* iff locally around each $x \in \partial M$ there is a local defining function $r : U \to \mathbb{R}_{\geq 0}$ such that the metric $r^2 g$ on $U \cap M$ smoothly extends to all of U.

- Changing the defining function, the metric again extends, and one gets a conformally related metric on ∂M. This leads to the notion of *conformal infinity*.
- If one requires in addition that g is negative Einstein, then one arrives at the concept of a *Poincaré–Einstein manifold*.

Relation to conformal holonomy

It is well known that Einstein metrics in a conformal class are related to solutions of a second order overdetermined operator defined on $\mathcal{E}[1]$. Via the volume density, any metric in the conformal class determines a nowhere vanishing section of $\mathcal{E}[1]$.

- This section is a solution if and only if the metric is Einstein.
- Conversely, a solution σ determines an Einstein metric outside its zero set $\mathcal{Z}(\sigma)$.

Via parallel sections of the standard tractor bundle, solutions correspond to reductions of conformal holonomy. Using this, R. Gover showed that in the negative Einstein case $\mathcal{Z}(\sigma)$ is an embedded hypersurface for which σ becomes a defining density. This leads to all Poincaré–Einstein structures.

A projective analog

There is an analog in projective geometry. Here one considers reductions of projective holonomy to an appropriate orthogonal group. These are equivalent to normal solutions σ of a natural third order operator defined on $\mathcal{E}(2)$, and outside of $\mathcal{Z}(\sigma)$, one obtains the Levi–Civita connection of an Einstein metric in the projective class. $\mathcal{Z}(\sigma)$ again is an embedded hypersurface, which inherits a natural conformal structure.

Analyzing the structure of such solutions (which can be done on the homogeneous model of the geometry), one is led to the general notion of projective compactness of order 2 as defined below.

A major motivation for the whole project is that the holonomy approaches to conformal and projective compactness both admit an analog in which the boundary has a natural CR structure rather than a conformal structure. Work in this direction is in progress.

Projective structures

For two affine connections ∇ and $\hat{\nabla}$ on a smooth manifold N, the following conditions are equivalent:

- ∇ and $\hat{\nabla}$ have the same geodesic paths
- There is a one–form $\Upsilon \in \Omega^1(N)$ such that for all $\xi, \eta \in \mathfrak{X}(N)$ we have $\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$. We will indicate this relation symbolically by $\hat{\nabla} = \nabla + \Upsilon$.

definition

These conditions define *projective equivalence* of ∇ and $\hat{\nabla}$. A projective equivalence class of torsion free connections is called a *projective structure* on N.

Similarly to the conformal case, one defines the projective density bundles $\mathcal{E}(w)$. Adding "(w)" to the name of a bundle will indicate a tensor product with $\mathcal{E}(w)$.

Definition of projective compactness

Let $\bar{M}$ be a smooth manifold with boundary ∂M and interior M and let α be a real number with $0 < \alpha \leq 2$.

definition

A torsion free affine connection ∇ on M is called *projectively compact of order* α if and only if for each $x \in \partial M$ there is an open subset $U \subset \bar{M}$ with $x \in U$ and a local defining function $r : U \to \mathbb{R}_{\geq 0}$ for ∂M such that the affine connection $\nabla + \frac{dr}{\alpha r}$ on $U \cap M$ extends smoothly to all of U.

Given one defining function $r : U \to \mathbb{R}_{\geq 0}$, any other defining function can be written as $\tilde{r} = e^f r$ for a smooth function $f : U \to \mathbb{R}$. Consequently, $\frac{d\tilde{r}}{\alpha\tilde{r}} = \frac{dr}{\alpha r} + \frac{1}{\alpha} df$. This shows that our definition does not depend on the choice of r and that ∇ gives rise to a projective structure on $\bar{M}$.

Completeness

The last observation shows that a projectively compact connection ∇ gives rise to a distinguished family of paths on all of $\bar{M}$. Our first result is related to completeness of ∇.

Proposition

Consider a distinguished path in $\bar{M}$ which meets the boundary ∂M transversally in a point x_0. Then a part of this path can be parametrized as a geodesic for ∇ in the form $c : [0, \infty) \to M$ such that $\lim_{t\to\infty} c(t) = x_0$.

Sketch of proof: Take some local defining function r defined around x_0 and parametrize a part of the path as a geodesic $\tilde{c}$ for $\hat{\nabla} = \nabla + \frac{dr}{\alpha r}$. Then there is a reparametrization $\varphi : [0, b) \to [0, t_0)$ such that $c = \tilde{c} \circ \varphi$ is a geodesic for ∇. This implies that $(r \circ c)^{-1}$ satisfies an ODE, which implies the claimed behavior since $\alpha \leq 2$.

Volume asymptotics

Given a hypersurface in a manifold N, the notion of a local defining density (of some fixed weight) makes sense. Given $U \subset N$ open, choose a nowhere vanishing density τ over U. Call σ a *defining density* if the uniquely determined function $r : U \to \mathbb{R}$ such that $\sigma = r\tau$ is a defining function. This is independent of τ.

Assume next that ∇ admits a parallel volume density vol. Then for each $w \in \mathbb{R}$, $\mathrm{vol}^{\frac{-w}{n+2}}$ is a section of $\mathcal{E}(w)$ which is parallel for ∇. This pins down ∇ within its projective class.

Proposition

Suppose that ∇ admits a parallel volume density vol and is projectively compact of order α. Then the section of $\mathrm{vol}^{\frac{-\alpha}{n+2}}$ of $\mathcal{E}(\alpha)|_M$ extends by zero to a defining density for ∂M.

Projectively compact metrics

We call a pseudo–Riemannian metric g on M projectively compact of order $\alpha \in (0,2]$ if its Levi Civita connection is projectively compact of that order. This seems to be mainly interesting in the case $\alpha = 2$. In this case, be can derive the following sufficient condition.

Proposition

Suppose that around each $x_0 \in \partial M$, we can find a local defining function $r : U \to \mathbb{R}_{\geq 0}$ and a non–zero constant C such that the $\binom{0}{2}$–tensor field $h := rg - C\frac{dr \odot dr}{r}$ on $U \cap M$ extends smoothly to all of U in such a way that the restriction to the boundary is non–degenerate on $T(U \cap \partial M)$. Then g is projectively compact of order 2.

It is easy to see that if this condition is satisfied for one defining function then it is satisfied for any local defining function.

Suppose that ∇ is an affine connection on M which is projectively compact of order α and admits a parallel volume density (so this is always true in the metric case). Then we obtain a canonical defining density for ∂M, namely $\sigma := \mathrm{vol}^{\frac{-\alpha}{n+2}} \in \Gamma(\mathcal{E}(\alpha))$.

For $\alpha = 1$ and $\alpha = 2$, the machinery of BGG sequences provides a projectively invariant differential operator defined on $\Gamma(\mathcal{E}(\alpha))$ and a relation to sections of some tractor bundle. This operator defines an overdetermined system of PDE. Apart from general solutions, there is the subclass of so called *normal solutions* which are in bijective correspondence to parallel sections of the tractor bundle in question.

Hence for $\alpha = 1, 2$ one can single out special projectively compact connections of order α by requiring that the defining density is a solution respectively a normal solution of the appropriate BGG operator.

Tractor bundles

For projective structures, all tractor bundles can be obtained by tensorial constructions from the *standard cotractor bundle* $\mathcal{T}^* := J^1\mathcal{E}(1)$. This comes with a composition series derived from the jet exact sequence $0 \to \mathcal{E}_a(1) \to \mathcal{T}^* \to \mathcal{E}(1) \to 0$, and the projective structure induces a canonical linear connection on $\mathcal{T}^*$. Applying tensorial constructions, one gets:

Properties of tractor bundles

Any tractor bundle $\mathcal{V}$ comes with

- a canonical *tractor connection* $\nabla^{\mathcal{V}}$
- a natural quotient $\mathcal{H}_0$ which is a tensor bundle

From the jet exact sequence above, one obtains an evident mapping $\mathcal{E}_a \otimes \mathcal{T}^* \to \mathcal{T}^*$. This extends to a sequence of bundle maps $\partial^* : \mathcal{E}_{[a_1 \dots a_k]} \otimes \mathcal{T}^* \to \mathcal{E}_{[b_1 \dots b_{k-1}]} \otimes \mathcal{T}^*$ such that $\partial^* \circ \partial^* = 0$.

The first BGG operator

Given a tractor bundle $\mathcal{V}$, we have the projection $\Pi : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{H}_0)$, the operator $\nabla^{\mathcal{V}} : \Gamma(\mathcal{V}) \to \Omega^1(N, \mathcal{V})$ and the maps $\Omega^2(N, \mathcal{V}) \stackrel{\partial^*}{\to} \Omega^1(N, \mathcal{V}) \stackrel{\partial^*}{\to} \Gamma(\mathcal{V})$. It turns out that $\ker(\partial^*)/\operatorname{im}(\partial^*) = \Gamma(\mathcal{H}_1)$ for a certain tensor bundle $\mathcal{H}_1$.

- For $\sigma \in \Gamma(\mathcal{H}_0)$ there is a unique section $s =: L(\sigma) \in \Gamma(\mathcal{V})$ such that $\Pi(s) = \sigma$ and $\partial^*(\nabla^{\mathcal{V}} s) = 0$.
- This defines a projectively invariant differential operator $L : \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{V})$, the *splitting operator*.
- For the first BGG operator $D : \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{H}_1)$, $D(\sigma)$ is given by projecting $\nabla^{\mathcal{V}} L(\sigma)$ to $\Gamma(\mathcal{H}_1)$.
- If $s \in \Gamma(\mathcal{V})$ satisfies $\nabla^{\mathcal{V}} s = 0$, then $s = L(\Pi(s))$ and $D(\Pi(s)) = 0$ ("normal solutions").

Projective compactness of order one

In this case, we have $\mathcal{V} = \mathcal{T}^*$ and $\mathcal{H}_0 = \mathcal{E}(1)$. The first BGG operator $D : \Gamma(\mathcal{E}(1)) \to \Gamma(\mathcal{E}_{(ab)}(1))$ is given by $D = \nabla_{(a}\nabla_{b)} + \mathrm{P}_{(ab)}$. The splitting operator $L : \Gamma(\mathcal{E}(1)) \to \Gamma(\mathcal{T}^*)$ is of order one and any solution σ of D is automatically normal, i.e. $L(\sigma)$ is parallel if $D(\sigma) = 0$.

Theorem

Let ∇ be projectively compact of order one with induced canonical defining density $\sigma \in \Gamma(\mathcal{E}(1))$ for ∂M. Then

- $L(\sigma) \in \Gamma(\mathcal{T}^*)$ is nowhere vanishing on $\bar{M}$
- $\mathrm{P}_{ab}\sigma \in \Gamma(\mathcal{E}_{(ab)}(1)|_M)$ extends smoothly to $\bar{M}$ and restricts to a projectively invariant second fundamental form on ∂M.
- $D(\sigma) = 0$ iff ∇ is Ricci flat. In this case, ∂M is totally geodesics and thus inherits a projective structure.

Projective compactness of order two

Here the BGG operator $D : \Gamma(\mathcal{E}(2)) \to \Gamma(\mathcal{E}_{(abc)}(2))$ is given by

$$D(\sigma) = \nabla_{(a}\nabla_{b}\nabla_{c)}\sigma + 4P_{(ab}\nabla_{c)}\sigma + 2\big(\nabla_{(a}P_{bc)}\big)\sigma.$$

The tractor bundle involved is $\mathcal{V} = S^2\mathcal{T}^*$, so sections are (possibly degenerate) bundle metrics on the standard tractor bundle $\mathcal{T}$. The splitting operator L has order two in this case.

Theorem

Let ∇ be projectively compact of order two with Ricci curvature Ric_{ab} and induced canonical defining density $\sigma \in \Gamma(\mathcal{E}(2))$ for ∂M.

- $D(\sigma) = 0$ if and only if $\nabla_{(a}\mathrm{Ric}_{bc)} = 0$.
- σ is a normal solution of D iff $\nabla_a \mathrm{Ric}_{bc} = 0$. If Ric_{ab} is non–degenerate then it defines an Einstein metric on M with Levi–Civita connection ∇.

Klein–Einstein metrics

Let g be a negative Einstein metric on M which is projectively compact of order 2 and let $\sigma \in \Gamma(\mathcal{E}(2))$ be the canonical defining density for ∂M. Then we have

Proposition

- $L(\sigma) \in \Gamma(S^2\mathcal{T}^*)$ is a parallel, non–degenerate bundle metric on $\mathcal{T}$ on all of $\bar{M}$.
- For a local defining function r for ∂M, the symmetric $\binom{0}{2}$–tensor field $r\frac{2nJ}{n+1}g + \frac{dr\odot dr}{r}$ extends to the boundary, and the boundary value is non–degenerate on $T\partial M$.

Note that dividing by the non–zero constant $\frac{2nJ}{n+1}$ the last part of this result give the condition for projective compactness derived before.