On sub-Riemannian structures (in low dimensions)

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- I will start by discussing sub-Riemannian structures from the point of view of G-structures, their filtered analogs, and of Cartan geometries.
- The main point here will be the different possible situations depending on the underlying distribution.
- I will then focus on the case that the underlying distribution is a contact distribution, in which a G-structure interpretation is difficult in general.
- Things are simpler in the case of dimension three, in which there is a relation to CR geometry and pseudo-Hermitian structures. This provides a connection to the CR-version of BGG sequences, for which a simplified picture is available in the pseudo-Hermitian setting.

Contents

Basic idea: endow all tangent spaces of an n-manifold M with some "structure" which is unique up to isomorphism and let $G \subset GL(n, \mathbb{R})$ the the group of linear automorphisms of a "model structure" on $\mathbb{R}^n$. Such structures admit compatible connections and the theory of G-structures uses these and algebraic tools to obtain fundamental inveriants.

Example 1: Riemannian metric g on an n-manifold M

- inner product on each tangent space, so $G = O(n)$
- Vanishing of intrinsic torsion
- Uniqueness of compatible torsion-free connection — Riemann curvature

Example 2: M^n with a distinguished distribution $H \subset TM$ of rank k. $G = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\} \subset GL(n, \mathbb{R})$. Compatible connections are those that leave H parallel, so $\eta \in \Gamma(H)$ implies $\nabla_\xi \eta \in \Gamma(H)$ for any ξ.

Intrisic torsion can be obtained directly: The Lie bracket of vector fields induces a bilinear bundle map $\mathcal{L} : \Lambda^2 H \to Q := TM/H$, which is a fundamental invariant. For fixed k and n, the behaviour can be drastically different depending on $\mathcal{L}$: If $\mathcal{L} = 0$, then $H \subset TM$ is involutive and hence (locally) has infinite dimensional groups of automorphism and no local invariants.

Consider the case $k = 3$, $n = 6$ (R. Bryant's thesis). Here $rk(H) = rk(Q) = 3$, and there are (generic) cases in which $\mathcal{L} : \Lambda^2 H \to Q$ is a linear isomorphism in each point. It turns out that such generic distributions admit a canonical Cartan connection, so they have local invariants (curvature) and automorphisms form Lie groups of dimension ≤ 21.

This is a parabolic geometry, similarly for $k \geq 4$, $n = \frac{1}{2}k(k+1)$. This also works for other generic distributions, e.g. $(2, 5)$, $(4, 7)$, and $(4, 8)$.

"constant intrinsic torsion"

In the $(3,6)$-case, $\mathcal{L}$ always being injective means that, via $\mathcal{L}_x$, $Q_x = T_xM/H_x \cong \Lambda^2 H_x$ for each $x \in M$. So here $(Q_x \oplus H_x, \mathcal{L}_x)$ isomorphic to a fixed model $(\Lambda^2\mathbb{R}^3 \oplus \mathbb{R}^3, \mathcal{L}_0)$ in each point. This leads to a natural frame bundle for $Q \oplus H$ with structure group $GL(3,\mathbb{R})$. (The Cartan bundle is an extension of this.)

This generalizes to filtered manifolds, but we will restrict to the case of one step bracket generating distirubtions. So we'll assume that, for each $x \in M$, $(Q_x \oplus H_x, \mathcal{L}_x)$ is isomorphic to a fixed 2-step nilpotent Lie algebra $(\mathfrak{m} = \mathfrak{m}_{-2} \oplus \mathfrak{m}_{-1}, [\ ,\ ])$. For $\mathfrak{m}$ a Heisenberg algebra, this is equivalent to a contact structure.

One obtains a natural frame bundle (for $Q \oplus H$) with structure group $\mathrm{Aut}_{gr}(\mathfrak{m})$. For contact structures in dimension $2n+1$ this group is $CSp(2n,\mathbb{R})$. "Better" compatible connections: $\nabla^Q_\xi \mathcal{L}(\eta,\zeta) = \mathcal{L}(\nabla^H_\xi \eta,\zeta) + \mathcal{L}(\eta,\nabla^H_\xi \zeta)$.

Filtered G-structures

Imitate G-structures using this frame bundle: Put some structure on each $Q_x \oplus H_x$ which is unique up to isomorphism compatible with $\mathcal{L}$!!. Automorphisms of a model structure on $\mathfrak{m}$ define $G_0 \subset \mathrm{Aut}_{gr}(\mathfrak{m})$. Sub-Riemannian structure: $g \in G_0$ iff action on $\mathfrak{m}_{-1}$ is orthgonal for a given inner product.

The key to understanding the possibilities thus is to understand the action of $\mathrm{Aut}_{gr}(\mathfrak{m})$ on the space of inner products on M. For generic $(3,6)$ distributions, we get $\mathrm{Aut}_{gr}(\mathfrak{m}) = GL(\mathfrak{m}_{-1})$, so in this case any sub-Riemannian structure defines a filtered G-structure.

By general results of T. Morimoto, sub-Riemannian structures that are filtered G-structures determine a canonical Cartan connection. Equivalently, one gets a canonical complement to H in TM and a canonical connection on TM.

The contact case

For $n \geq 2$, inner products on $\mathbb{R}^{2n}$ have continuous moduli with respect to $CSp(2n, \mathbb{R})$. Hence contact sub-Riemannian structures have point-wise invariants and define filtered G-strucutures only if these invariants are constant. First, splitting the filtration is easy:

Given an oriented contact sub-Riemannian structure (M, H, g) then for $x \in M$, $\mathcal{L}_x : \Lambda^2 H_x \to Q_x$ is non-degerate and g_x determines a volume element on H_x. Hence there is a unique linear isomorphism $\theta_x : Q_x \to \mathbb{R}$ such that $\theta_x \circ \mathcal{L}_x$ determineds the same volume element. These fit together to define a distinguished contact form θ, which identifies $TM \cong Q \oplus H$.

This gives a reduction to $O(2n) \subset GL(2n+1, \mathbb{R})$. But we still have $d\theta(x) : \Lambda^2 H_x \to \mathbb{R}$ for each $x \in M$. This leads to eigenvalues, which give rise to point-wise invariants.

There is an endomorphism $J_x : H_x \to H_x$ such that $d\theta(x)(X, Y) = g_x(X, J_x(Y))$ and these fit together to a bundle map $J : H \to H$, which is skew symmetric with respect to g. Hence the n (negative) eigenvalues a_j of the symmetric map $J_x^2 = J_x \circ J_x$ give point wise invariants. Volume condition implies $|\prod a_j| = 1$.

The pseudo–Hermitian case

The simplest case is $J^2 = -\operatorname{id}$ (and this is the only possibility if $n = 1$!). In this case, J makes (M, H) into a stricly pseudo–convex (partially integrable almost) CR structure, and θ defines a pseudo-Hermitian structure on (M, H, J), which equivalently encodes g. This reduces the structure group to $U(n)$, and one obtains a canoncial connection (and a Cartan connection coming from the underlying CR structure).

More interestingly, one may also look at the family $\{t\theta : t \in \mathbb{R}\}$ of contact forms, corresponding curvature quantities κ_t and look for limits for $t \to \infty$ of those.

In this simplest case, one thus obtains a relation to pseudo-Hermitian structures which have been studied intensively, both from geometric and from analytical points of view. This is always availably in dimension 3, which is also of particular interest in the study of pseudo-Hermitian structures.

Remarks

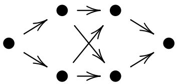
(1) While the distinguished connections on TM may not directly be what one is looking for in sub-Riemannian geometry, they are there and there are general results that they can be used to obtain all local invariants, etc.
(2) It is possible to reduce the structure group further in more general situations, say assuming that the multiplicities of eigenvalues of J^2 remain constant. But unless the eigenvalues are actually constant, the resulting G-structure will not be an equivalent encoding of the sub-Riemannian metric and one still has point-wise invariants that are functions.

The Rumin complex

I'll restrict to the three dimensional case here. Similar developments are possible for higher-dimensional pseudo-Hermitian structures but the resulting sequnces have a different form. The simplest instance of the BGG construction is the Rumin complex that is related to the $\mathbb{C}$-valued de Rham complex $(\Omega^*(M, \mathbb{C}), d)$.

Via $Q^* \hookrightarrow T^*M \twoheadrightarrow H^*$ and $Q^* \otimes H^* \hookrightarrow \Lambda^2 T^*M \twoheadrightarrow \Lambda^2 H^*$, d induces $D_0 : \Omega^0(M) \to \Gamma(H^*)$ and $D_2 : \Gamma(Q^* \otimes H^*) \to \Omega^3(M)$. The operator $\Gamma(Q^*) \to \Gamma(\Lambda^2 H^*)$ induced by d is $\alpha \mapsto -\alpha \circ \mathcal{L}$ and hence an isomorphism. Some diagram chasing then defines a second order operator $D_1 : \Gamma(H^*) \to \Gamma(Q^* \otimes H^*)$ such that the D_i form a complex that computes the de Rham cohomology of M.

Passing to values in $\mathbb{C}$, the bundles in degree 1 and 2 split into $(1, 0)$ and $(0, 1)$-parts and the operators split into components.

The resulting pattern  is typical for all BGG sequences for 3-dimensional CR structures. To define a BGG sequence, one has to choose a representation $\mathbb{V}$ of $G := SU(2,1)$. There is a specific embedding $U(1) \hookrightarrow G$, via which $\mathbb{V}$ defines a natural bundle $\mathcal{V}M$ over each sub-Riemannian contact 3-manifold.

The distinguished connection together with some algebraic ingredients defines a natural connection on $\mathcal{V}M$. Coupling this to d, one obtains operations on $\mathcal{V}M$-valued differential forms. The BGG machinery then compresses this to a sequence of higher order operators on complex line bundles in the form of the pattern above.

The bundles in the sequence and the orders of the operators can be determined in advance. By CR invariance the behaviour under $\theta \mapsto \theta_t$ is easily understood. If the underlying CR structure is spherical, we obtain a complex with cohomology $\cong H^*_{dR}(M) \otimes \mathbb{V}$.