Induced almost para-Kähler-Einstein metrics on cotangent bundles

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CaLISTA training school, Brno, September 7, 2023

- This talk reports on joint work with T. Mettler (Uni Distance, Brig, Switzerland) available as arXiv:2301.03217 and building on arXiv:1908.10325.
- A torsion-free affine connection on a smooth manifold M gives rise to a split signature metric on T^*M, called the Patterson-Walker metric. For Weyl connections of a torsion-free AHS structure, this construction can be modified in a universal way to produce almost para-Kähler-Einstein metrics that depend (up to isomorphism) only on the underlying AHS-structure.
- I will first explain this for projective, conformal and $(2, m)$ Grassmannian structures.
- Then I will give a uniform description and explain the Cartan geometric origins of the construction.

Contents

Let M be a smooth n-manifold and let ∇ be a torsion-free linear connection on TM. This also induces a connection on T^*M and hence a horizontal subbundle $H \subset T(T^*M)$ which is complementary to the vertical subbundle V. For $x \in M$ and $\alpha \in T^*_xM$, we can identify H_α with T_xM and V_α with T^*_xM.

Viewing the dual pairing as a map $T_\alpha T^*M \times T_\alpha T^*M$ and symmetrizing this, one obtains a pseudo-Riemannian metric h_∇ on T^*M for which the subbundle V and H are isotropic.

In the setting of 2-dimensional projective structures, M. Dunajski and T. Mettler proposed a modification of h_∇ involving the Ricci-curvature of ∇ and the tautological one-form on T^*M. They showed that together with the canonical symplectic form (slightly modified if ∇ is not volume preserving), this modification defines a self-dual almost para-Kähler-Einstein metric on the 4-manifold T^*M.

In a general dimension n consider a connection ∇ with Ricci curvature R_{ij} and let $\mathrm{P}_{ij} = \frac{1}{n-1} R_{(ij)} + \frac{1}{n+1} R_{[ij]}$ be its (projective) Schouten tensor. Denote by $\tau \in \Omega^1(T^*M)$ the tautological one-form on T^*M. Define $\tilde{h}_\nabla := h_\nabla - p^*\mathrm{P}_{(ij)} - \tau \otimes \tau$, where $p : T^*M \to M$ is the projection.

Theorem

The split-signature metric $\tilde{h}_\nabla$ on T^*M is Einstein. Together with the two-form $-d\tau + p^*(\mathrm{P}_{[ij]})$ (which coincides with the canonical two-form if ∇ preserves a volume form) it defines an almost para-Kähler-Einstein metric on T^*M. The metrics associated to projectively equivalent connections are isometric.

Observe that the two terms used to modify h_∇ are of different nature. The first term is a pullback and hence constant along the fibers. The canonical one-form τ is of course homogeneous of degree one along the fibers of T^*M, and hence the second term is homogeneous of degree two along the fibers.

Next, suppose that (M, g) is a pseudo-Riemannian n-manifold and that ∇ is a torsion-free Weyl connection for g, i.e. that ∇g is conformal to g. Let R_{ij} be the Ricci curvature of ∇ and $\mathsf{P}_{ij} = \frac{1}{n-2} R_{(ij)_0} + \frac{1}{n} R_{[ij]} + \frac{1}{n(n-2)} R g_{ij}$ its conformal Schouten tensor. The metric g on M induces a natural symmetric $\binom{0}{2}$-tensor field $|\_|^2 p^* g$ on T^*M, which sends $X, Y \in T_\alpha T^*M$ to $g^{-1}(\alpha, \alpha) g(T_\alpha p(X), T_\alpha p(Y))$. Observe that this is homogeneous of degree two along the fibers and depends only on the conformal class of g. Now define $\tilde{h}_\nabla := h_\nabla - p^* \mathsf{P}_{(ij)} - \tau \otimes \tau + \frac{1}{2} |\_|^2 p^* g$.

Theorem

The metric $\tilde{h}_\nabla$ on T^*M is Einstein. Together with the two-form $-d\tau + p^*(\mathsf{P}_{[ij]})$ it defines an almost para-Kähler-Einstein metric on T^*M. The metrics associated to different Weyl connections are isometric.

There is a parallel story for $(2, m)$ Grassmannian structures with $n = 2m$. Basically, these are given by isomorphisms $TM \cong E^* \otimes F$ and $\Lambda^2 E^* \cong \Lambda^m F$ for auxiliary bundles $E, F \to M$ of rank 2 and m, respectively. Compatible connections on TM are induced by connections on E and F, and there is a natural notion of the Schouten tensor for such connections.

The other part of the modification is more subtle. We get $T^*M \cong E \otimes F^*$, so there is a “composition map” that sends $X \in T_xM$ and $\alpha \in T_x^*M$ to $\alpha \circ X : E_x \to E_x$. This leads to an analog $\tau^G \in \Omega^1(T^*M, L(p^*E, p^*E))$ of the tautological one-form τ. Explicitly, $\tau^G(\alpha)(X) := \alpha \circ T_\alpha p(X)$ and $\tau = \operatorname{tr}(\tau^G)$, the (point-wise) trace of τ^G. Via composition, we define a symmetric two-tensor $\tau^G \otimes \tau^G$ with values in $L(p^*E, p^*E)$.

The point-wise trace of this defines a symmetric tensor field on T^*M and for torsion-free ∇, $\tilde{h}_\nabla = h_\nabla - p^*\mathsf{P}_{(ij)} - \operatorname{tr}(\tau^G \otimes \tau^G)$ is an almost para-K.-E. modification of the Patterson-Walker metric.

Towards a uniform interpretation

Conformal structures and $(2, m)$ (almost) Grassmannian structures can be described as G_0-structures for appropriate subgroups $G_0 \subset GL(n, \mathbb{R})$. On the Lie algebra level, these arise via a grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}$ is simple and the representation of G_0 on $\mathbb{R}^n \cong \mathfrak{g}_{-1}$ comes from the adjoint action. Also, $\mathfrak{g}_1 \cong \mathfrak{g}_{-1}^*$ as a representation of G_0 via the Killing form B of $\mathfrak{g}$. For projective structures there is a similar underlying picture, in which $G_0 = GL(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R})$.

This Lie algebraic picture leads to a *fundamental tensor field* on M, which is of type $\binom{2}{2}$. This is induced by the map $(Z, W, X, Y) \mapsto \frac{1}{2} B(W, [[X, Z], Y])$ for $X, Y \in \mathfrak{g}_{-1}$ and $Z, W \in \mathfrak{g}_1$. One immediately verifies that this is symmetric both in (X, Y) and in (Z, W), so it induces a tensor field $q \in \Gamma(S^2 TM \otimes S^2 T^* M)$.

One can view the tautological one-form on T^*M as a "lift" of the canonical $\binom{1}{1}$-tensor field $\mathrm{id}_{TM} \in \Gamma(TM \otimes T^*M)$ to a $\binom{0}{1}$-tensor field on T^*M, which is homogeneous of degree one along the fibers. In the same way, the tensor field q on M defines a symmetric $\binom{0}{2}$-tensor field $\tilde{q}$ on T^*M. Explicitly, for $\alpha \in T^*_xM$ and $X, Y \in T_\alpha T^*M$, one defines $\tilde{q}(\alpha)(X, Y) := q(x)(\alpha, \alpha, T_\alpha p(X), T_\alpha p(Y))$.

In particular, $\tilde{q}$ is always homogeneous of degree two along the fibers of $T^*M \to M$. It turns out that in each of the examples above, the modification of the Patterson-Walker metric is given by $-\mathrm{Symm}(\mathrm{P}) + \tilde{q}$.

The Lie algebraic approach also provides a uniform description of the Schouten tensor, which is closely related to the fundamental tensor field q as follows.

Given a $\binom{0}{2}$-tensor field V, one defines a $\binom{1}{3}$-tensor field ∂V with curvature type symmetries via

$$Z(\partial V(X_1, X_2)(X_3)) = q(Z, V(X_1), X_2, X_3) - q(Z, V(X_2), X_1, X_3).$$

Then the Schouten tensor of a torsion-free compatible connection with curvature R is the unique $\binom{0}{2}$-tensor P such that $R - \partial\mathrm{P}$ has vanishing Ricci-type contraction.

At this point we have described both ingredients for the modification of the Patterson-Walker metric in terms of the G_0-structure and the data coming from the grading of the simple Lie algebra defining the structure. It should then be possible to directly verify the properties of the modified Patterson-Walker metric using this Lie algebraic input.

The is a much neater way towards this, which is also how these metrics originally arose. This is based on the Cartan geometric description of projective, conformal and Grassmannian structures.

The G_0-structures discussed above are given by a G_0-principal bundle $\mathcal{G}_0 \to M$. A fundamental fact about these structures is that there is a canonical extension of the bundle to a principal bundle $p : \mathcal{G} \to M$ with structure group $P \supset G_0$ that has Lie algebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$. This bundle can be endowed with a canonical Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ whose curvature satisfies a normalization condition that can be expressed in Lie algebraic terms.

Consider $A := \mathcal{G}/G_0 \cong \mathcal{G} \times_P (P/G_0) \to M$. Then $A \to M$ is a natural fiber bundle and the obvious projection $\mathcal{G} \to A$ is a principal fiber bundle with structure group G_0. The canonical Cartan connection ω makes this into a Cartan geometry with homogeneous model G/G_0. The decomposition of $\mathfrak{g}$ by the grading readily shows that G/G_0 is a para-Hermitian symmetric space.

Large parts of this structure on the homogeneous model carry over to A via the Cartan geometry.

- $TA \cong \mathcal{G} \times_{G_0} (\mathfrak{g}/\mathfrak{g}_0)$, so $TA = L^- \oplus L^+$, with $L^\pm = \mathcal{G} \times_{G_0} \mathfrak{g}_{\pm 1}$
- $L^+ \cong (L^-)^*$ and the pairing gives rise to $h \in \Gamma(S^2 T^* A)$ and to $\Omega \in \Omega^2(A)$ by symmetrizing and alternating
- the $\mathfrak{g}_0$-component of ω is a principal connection on $\mathcal{G} \to A$ for which $L^\pm$, h and Ω are parallel

All this follows from first principles, but understanding the properties of these structures is much more subtle. It mainly requires understanding the consequences of normality of ω for the geometric structures induced on A.

Theorem

- the canonical connection on TA is torsion-free iff ω is flat
- $d\Omega = 0$ iff the original geometry on M is torsion-free
- if the geometry on M is torsion-free, then the split signature metric h is almost para-K.-E. with fundamental 2-form Ω.

A as the bundle of Weyl structures

Recall that $A = \mathcal{G} \times_P (P/G_0) \to M$. By principal bundle theory, sections of $\pi : A \to M$ parametrize reductions of $p : \mathcal{G} \to M$ to the structure group G_0. These are equivalent to G_0-equivariant sections $\mathcal{G}_0 \to \mathcal{G}$, which are known as *Weyl-structures* and well understood. In particular, they parametrize the Weyl connections.

One obtains a direct relation between natural bundles over A and over M, in particular $L^- = \pi^* TM$ and hence $L^+ \cong \pi^* T^* M$. Given a section $s : M \to A$, one can obtain the corresponding Weyl connection and its Schouten tensor as pullbacks of objects on A. In particular, viewed as an element of $\Omega^1(A, L^+)$, the projection $TA \to L^+$ pulls back to P along s.

To close the circle, we have to relate $A \to M$ to $T^* M$. This is the interpretation on A of the well-known fact that Weyl structures form an affine space modelled on $\Omega^1(M)$.

Proposition

A smooth section $s : M \to A$ canonically lifts to an isomorphism $\Psi_s : T^*M \to A$ of fiber bundles that sends the zero-section of T^*M to s. In particular, a Weyl connection ∇ with section s_∇ leads to the almost para-Kähler-Einstein metric $(\Psi_{s_\nabla})^* h$ on T^*M.

The diffeomorphism Ψ_s can be made explicit using that $T^*M = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_1$. Denoting by $\sigma : \mathcal{G}_0 \to \mathcal{G}$ the section determined by s, it is induced by $(u, Z) \mapsto \sigma(u) \cdot \exp(Z) G_0$. To compute $(\Psi_{s_\nabla})^* h$ and $(\Psi_{s_\nabla})^* \Omega$ mainly needs computing the pullbacks of the components of ω in $\mathfrak{g}_{\pm 1}$ via the map $(u, Z) \mapsto \sigma(u) \cdot \exp(Z)$.

This leads to the universal formula $(\Psi_{s_\nabla})^* h = h_\nabla - \mathrm{Symm}(\mathrm{P}) + \tilde{q}$, with P coming from the pullback of the $\mathfrak{g}_1$ component ω_1, while the double brackets producing $\tilde{q}$ come from computing the $\mathfrak{g}_1$-component of $\mathrm{ad}(\exp Z) \circ \omega_{-1}$. Specializing to the individual geometries is then a matter of direct computations.