The Rumin complex of a domain in $\mathbb{R}^3$

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- Many of the complexes that are important in applications arise as BGG (Berstein-Gelfand-Gelfand) sequences. The most prominent example here is the fundemantal complex of linear elasticity (or Calabi complex).
- In my talk, I will discuss an instance of the BGG construction, which in several respects is partciularly simple. It leads to a complex that computes the de Rham cohomology, but with smaller spaces than the usual de Rham complex. There is a geometric structure in the background, but still an infinite dimensional group of symmetries acts on the complex.
- To put things into perspective, I'll start by reviewing several aspects of the BGG construction in the beginning of my talk.
- If time permits, I will briefly outline generalizations (i.e. smaller complexes that compute de Rham cohomology) in the end of the talk.

Contents

I will review here those cases of the BGG construction that have their origin in representation theory. Moreover, I will restrict to the case of smooth objects on an open subset $U \subset \mathbb{R}^n$ and ingnore versions for manifolds or in lower regularity.

The basic pattern of the construction looks as follows:

1. Consider differential forms on U with values in vector space $V = V_1 \oplus \cdots \oplus V_N$, so $\Omega^k(U, V) = \oplus_{i=1}^N \Omega^k(U, V_i)$.
2. Introduce tensorial (point-wise) operations $S : \Omega^k(U, V_i) \to \Omega^{k+1}(U, V_{i-1})$, which are chosen in such a way that $d - S$ computes the same cohomology as d.
3. Use the operators S to define subspaces $H^k \subset \Omega^k(U, V)$ and d and S to define operators $D : H^k \to H^{k+1}$ such that (H^*, D) is a complex that computes the same chomology as $(\Omega^*(U, V), d - S)$.

A major difficulty is that in general situations finding the "right" choices for V and S and computing the subspaces H^k and the operators D requires a substantial amount of representation theory. The nature of this theory implies that there is a fixed list of cases for which the construction works, and finding "small variations" may turn out to be unexpectedly difficult.

The interpretation as V-valued differential forms is misleading, since the S-operators mix the form-part with the values. For elasticity on $U \subset \mathbb{R}^3$, one usually takes $V_1 = V_2 = \mathbb{R}^3$, but one should actually view them as $\mathbb{R}^{3*}$ and $\Lambda^2\mathbb{R}^{3*}$, respectively. A function $U \to V_2$ then can be viewed a 2-form on U and hence a special (skew-symmetric) instance of a 1-form with values in $\mathbb{R}^{3*}$. This defines $S : C^\infty(U, V_2) \to \Omega^1(U, V_1)$.

This is important, since identifying $C^\infty(U, \mathbb{R}^{3*})$ with $\Omega^1(\mathbb{R}^3)$ uses the trivialization of the cotangent bundle or equivalently the flat connection on U coming from $\mathbb{R}^3$.

This trivialization is not preserved by general diffeomorphisms, but only by restrictions of affine transformations. Hence, although one starts from de Rham complexes (which are invariant under diffeomorphisms), the resulting BGG sequence is only invariant under (restrictions of) affine transformations.

Moreover, the "right" action of affine transformations on the spaces in question can only be found using the correct geometric interpretation and looks quite unexpected. For the elasticity complex, this is discussed in [1] for volume preserving affine transformations, the general formulae are even more complicated.

The BGG sequences discussed above come in two families. For one of those, one obtains an action of affine transformations (with simpler formulae in the volume-preserving case). For the other family, conformal transformations act, with simpler formulae for isometries, i.e. restrictions of rigid motions.

The Rumin complex that I will discuss now does not belong to the two families discussed above, it is related to a different geometric structure. From the current perspective, I would like to emphasize the following aspects:

1. The starting point is the (scalar) de Rham complex on a domain $U \subset \mathbb{R}^3$ and the resulting complex will also compute the de Rham cohomology of U.
2. The S-operators arise in a natural way from choosing a so-called *contact form* α on $\mathbb{R}^3$. One can view the initial change from the de Rham complex as "sorting things differently", a more conceptual intrepretation will be discussed later.
3. The resulting complex admits a natural action of the group of all diffeomorphisms that preserve α up to multiplication by a nowhere-vanishing function. This group turns out to be infinite dimensional.

For $U \subset \mathbb{R}^3$ open, the de Rham complex can be viewed as $C^\infty(U,\mathbb{R}) \xrightarrow{grad} C^\infty(U,\mathbb{R}^3) \xrightarrow{curl} C^\infty(U,\mathbb{R}^3) \xrightarrow{div} C^\infty(U,\mathbb{R})$.

Via the following identifictaions, these are instances of the exterior derivative on differential forms:

$$\begin{aligned}(f_1,f_2,f_3) &\leftrightarrow f_1dx + f_2dy + f_3dz\\ (f_1,f_2,f_3) &\leftrightarrow f_1dy\wedge dz - f_2dx\wedge dz + f_3dx\wedge dy\\ f &\leftrightarrow fdx\wedge dy\wedge dz\end{aligned}$$

To pass to the Rumin complex, we replace one of the coordinate forms (here dz) by a *contact form* α. We use a symmetric choice, namely $\alpha := dz + \frac{1}{2}xdy - \frac{1}{2}ydx$. This satisfies $d\alpha = dx\wedge dy$ and hence $d\alpha\wedge\alpha = \alpha\wedge d\alpha = \mathrm{vol} := dx\wedge dy\wedge dz$. Clearly, any 1-form can be expanded in terms of dx, dy, and α and hence any 2-form in terms of $dy\wedge\alpha$, $-dx\wedge\alpha$, and $d\alpha = dx\wedge dy$.

We can simply compute the exerior derivatives in terms of this new "coframe" $\{dx, dy, \alpha\}$ to get an isomorphic copy of the de Rham complex. The spaces in this complex are again $C^\infty(U, \mathbb{R})$ and $C^\infty(U, \mathbb{R}^3)$ but with different interpretations as differential forms.

This is conveniently encoded by introducing differential operators $\tilde{\partial}_x := \partial_x + \frac{1}{2}y\partial_z$ and $\tilde{\partial}_y := \partial_y - \frac{1}{2}x\partial_z$. (Viewed as vector fields, these span the kernel of α.) Corresponding to the difference between dz and α, these do not commute, but $[\tilde{\partial}_x, \tilde{\partial}_y] = -\partial_z$.

This exhibts a first key feature: If $\tilde{\partial}_x f = 0$ and $\tilde{\partial}_y f = 0$, then $\partial_z f = 0$ and hence $df = 0$. (This corresponds to the so–called "Hörmander condition" in analysis.) In fact,

$$df = \partial_x f dx + \partial_y f dy + \partial_z f dz = \tilde{\partial}_x f dx + \tilde{\partial}_y f dy + \partial_z f \alpha,$$

and this is the motivation for the definition of $\tilde{\partial}_x$ and $\tilde{\partial}_y$.

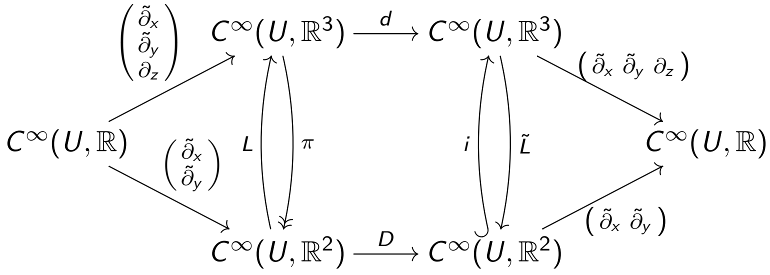
The exterior derivative from 2-forms to 3-forms is also easily computed directly:
$d(f_1 dy \wedge \alpha - f_2 dx \wedge \alpha + f_3 d\alpha) = \left(\tilde{\partial}_x f_1 + \tilde{\partial}_y f_2 + \partial_z f_3\right) \mathrm{vol}.$
The main features occur in degree one, with key computation
$d(f\alpha) = (\tilde{\partial}_x f dx + \tilde{\partial}_y f dy + \partial_z f \alpha) \wedge \alpha + f d\alpha.$
More generally,one computes that $d(f_1 dx + f_2 dy + f_3 \alpha)$ is given by

$$(\tilde{\partial}_y f_3 - \partial_z f_2) dy \wedge \alpha - (-\tilde{\partial}_x f_3 + \partial_z f_1) dx \wedge \alpha + (-\tilde{\partial}_y f_1 + \tilde{\partial}_x f_2 + f_3) d\alpha,$$

and the un-differentiated f_3 in the last summand is the tensorial component of d created by the different coframe. Hence

1. If $d(f_1 dx + f_2 dy + f_3 \alpha) = 0$, then $f_3 = \tilde{\partial}_y f_1 - \tilde{\partial}_x f_2$.
2. If $\psi = g_1 dy \wedge \alpha - g_2 dx \wedge \alpha + g_3 d\alpha$, then $\psi - d(g_3 \alpha)$ has vanishing $d\alpha$-component.

These two observations directly lead to the definition of the Rumin complex and to its relation to the de Rham complex.



$L(f_1,f_2)=(f_1,f_2,\tilde\partial_y f_1-\tilde\partial_x f_2)$, $\tilde L(g_1,g_2,g_3)=(g_1-\tilde\partial_y g_3, g_2+\tilde\partial_x g_3)$.

$$d=\begin{pmatrix}0 & -\partial_z & \tilde\partial_y\\ \partial_z & 0 & -\tilde\partial_x\\ -\tilde\partial_y & \tilde\partial_x & 1\end{pmatrix}\quad D=\begin{pmatrix}\tilde\partial_y^2 & -\partial_z-\tilde\partial_y\tilde\partial_x\\ \partial_z-\tilde\partial_x\tilde\partial_y & \tilde\partial_x^2\end{pmatrix}$$

Now $d\varphi=0\Rightarrow\varphi=L(\pi(\varphi))$, $d\circ L=i\circ D$, so L and i are a chain map. The induced map in chomology is an isomorphism in degrees 0 and 1.
Also, $i\circ\tilde L(g_1,g_2,g_3)=(g_1,g_2,g_3)-dg_3\alpha$ and this easily implies that one also obtains isomorphisms in degerees 2 and 3.

Conceptual interpretation

The most relevant structure is the familiy $H = \{H_x := \ker(\alpha(x))\}$ of planes in the tangent spaces at $x \in U$ ("contact structure"). The condition that $\alpha \wedge d\alpha$ is nowhere vanishing says that these are as far from tangent planes to embedded surfaces as possible. All contact structures are locally isomorphic ("Pfaff theorem").

The two middle spaces in the Rumin complex should be interpreted as restrictions of 1-forms to H and as 2-forms that vanish if both their entries are from H. If Φ is a diffeomorphism, such that $\Phi^*\alpha = h\alpha$ with $h(x) \neq 0 \quad \forall x$, then $D\Phi(x)$ maps H_x to $H_{\Phi(x)}$. One shows that the Rumin complex is functorial for such diffeomorphisms, which form an infinite dimensional group.

Contact structures have a nice interpretation as describing constrained mechanical systems, for example describing the motion of a hand truck and of a unicycle.

Generalizations

Contact forms and contact structures exist in all odd dimensions and there always is an associated Rumin complex. The ranks of the involved bundles drop more heavily than for $n = 3$ (total rank 6 instead of 8), e.g. 20 instead of 32 for $n = 5$ and 84 instead of 128 for $n = 7$.

For $n = 4$, there are so-called Engel distributions of rank 2. These again are all locally isomorphic and have infinite dimensional automorphisms. There is a BGG complex with bundles of ranks 1, 2, 2, 2 and 1 (instead of 1, 4, 6, 4 and 1 for de Rham).

In higher dimensions, there are examples of generic distributions with (rank,dimension)$=(2,5)$, $(3,6)$, $(4,7)$, $(4,8)$, $(n, \frac{n(n+1)}{2})$, each of which leads to an associated BGG complex with much smaller bundles than de Rham. However, these distributions themselves have local invariants and finite dimensional automorphism groups.

A. Čap, A., K. Hu, BGG sequences with weak regularity and applications, Found. Comput. Math. **24** (2024), 1145-1184.