Projective BGG equations, polynomial systems, and compactifications of Einstein manifolds

Andreas Čap

University of Vienna
Faculty of Mathematics

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- based on recent joint work with A.R. Gover and M. Hammerl, available as arXiv:1005.2246.
- The machinery of BGG sequences gives a construction of a large number of invariant differential operators associated to a family of geometric structures called parabolic geometries.
- The first operator in each BGG sequence defines a geometric overdetermined system of PDEs, so existence of solutions for each of these systems defines an interesting subclass of geometries.
- My talk will be devoted to the study of solutions of such systems for the simplest example of a parabolic geometry, namely classical projective structures. This leads to surprising relations to algebraic geometry and compactifications of Einstein manifolds. Many of these considerations have analogs for the other parabolic geometries.

Conformal BGG equations

- The overdetermined systems produced by the machinery of BGG sequences on conformal structures include the equations for twistor spinors, the conformal Killing equations on vector fields, differential forms, and on symmetric tensor fields, and the equation of almost Einstein scales.
- For many of these equations, there are interesting results on possible zeros (or higher order zeros) of solutions.
- Let us discuss the **almost Einstein equation** in a bit more detail:

Let g_{ab} be a metric in the conformal class and let ∇ be its Levi Civita connections and P_{ab} its Schouten tensor. Look for functions σ such that

Almost Einstein equation

$\nabla_a \nabla_b \sigma + \mathsf{P}_{ab}\sigma = f g_{ab}$ for some smooth function f

It turns out that for any solution σ of this equation the subset $U := \{x : \sigma(x) \neq 0\}$ is open and dense, and the equation is equivalent to the fact that the rescaling $\frac{1}{\sigma^2} g_{ab}$ defines an Einstein metric on U. From this interpretation it follows easily that the equations satisfies a conformal covariance property.

Via the BGG machinery, solutions to the almost Einstein equation are in bijective correspondence with sections of the standard tractor bundle, which are parallel for the canonical linear connection on this bundle.

Using this correspondence, R. Gover analyzed the possible zeros of solutions. It turns out that zeros are either isolated or form embedded hypersurfaces, and locally around these, one obtains Poinceré–Einstein metrics (i.e. conformal compactifications of Einstein manifolds). Conversely, any such compactification is obtained in this way.

Basic definitions

Let M be a smooth manifold of dimension $n \geq 2$.

- Two torsion free linear connections ∇ and $\hat{\nabla}$ on the tangent bundle TM are called *projectively equivalent* if their difference is given by $\hat{\nabla}_\xi \eta - \nabla_\xi \eta = \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$ for some $\Upsilon \in \Omega^1(M)$ and all $\xi, \eta \in \mathfrak{X}(M)$.
- This can be equivalently characterized geometrically as ∇ and $\hat{\nabla}$ having the same geodesics up to parametrization.
- Denoting by $R_{ab}{}^c{}_d$ the curvature of a connection ∇ on TM, one puts $R_{ab} := R_{ca}{}^c{}_b$ and defines the *Rho–tensor* and the *Weyl curvature* by $\mathsf{P}_{ab} = \frac{-1}{(n-1)(n+1)}(nR_{ab} + R_{ba})$ and

$$W_{ab}{}^c{}_d = R_{ab}{}^c{}_d + \delta^c_a \mathsf{P}_{bd} - \delta^c_b \mathsf{P}_{ad} - \mathsf{P}_{ab}\delta^c_d + \mathsf{P}_{ba}\delta^c_d$$

Now one defines a *projective structure* on M as a projective equivalence class $[\nabla]$ of torsion free linear connections on TM. It turns out that all connections in the class have the same Weyl–curvature, which is the basic invariant of the structure if $n \geq 3$. (For $n = 2$, the symmetries of W imply its vanishing, the basic invariant then is the tensor $\nabla_{[a}\mathsf{P}_{b]c}$.)

A projective structure on M gives rise to a family of distinguished unparametrized curves, with one curve through each point in each direction. It turns out that a local diffeomorphism between two manifolds with projective structures is a morphism of projective structures (in the evident sense) if and only if it is compatible with these families. In this way, projective structures are closely related to the geometry of systems of second order ODEs.

The homogeneous model

The basic example of a projective structure is given by projective space $\mathbb{R}P^n$ with the family of projective lines as distinguished curves. Locally on each affine chart, these can be realized as the geodesics of the flat connection on $\mathbb{R}^n$ which describes the projective class in terms of connections. The group $G := PSL(n+1,\mathbb{R})$ acts transitively on $\mathbb{R}P^n$ by automorphisms of the projective structure, and it turns out that these are all automorphisms. Hence we can identify $\mathbb{R}P^n$ with G/P, where $P \subset G$ is the stabilizer of a line.

It turns out that $P \cong G_0 \ltimes P_+$, where $P_+ \cong \mathbb{R}^{n*}$ is an Abelian normal subgroup and $P/P_+ = G_0 \cong GL(n,\mathbb{R})$. In particular, the natural projection $G/P_+ \to G/P$ is a principal bundle with structure group G_0. It turns out that this can be naturally identified with the linear frame bundle of $\mathbb{R}P^n$.

The Cartan description

We want to obtain an analogous description for $(M, [\nabla])$.

- Let $p_0 : \mathcal{G}_0 \to M$ be the linear frame bundle of M, which has structure group $G_0 = GL(n, \mathbb{R})$.
- Attach to each point $u_0 \in \mathcal{G}_0$ the values of the connections in $[\nabla]$ in the point $x = p_0(u_0)$, thus defining $p : \mathcal{G} \to M$. This can be made into a principal P–bundle.
- The Lie algebra $\mathfrak{g}$ of G can be identified with $\mathbb{R}^n \oplus \mathfrak{g}_0 \oplus \mathbb{R}^{n*}$ with the last two summands forming the Lie algebra $\mathfrak{p}$ of P. Construct a tautological one form $\underline{\omega} \in \Omega^1(\mathcal{G}, \mathbb{R}^n \oplus \mathfrak{g}_0)$ from the soldering form and the connection forms of the connections in the projective class.
- This can be canonically extended to a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ which trivializes $T\mathcal{G}$ in a P–equivariant way.

Tensor bundles

Via the adjoint action, $\mathfrak{g}$ is a representation of P and $\mathfrak{p} \subset \mathfrak{g}$ is a P–invariant subspace, so there is a natural P–action on $\mathfrak{g}/\mathfrak{p}$. It follows directly from the definition of a Cartan connection that, via ω, the associated bundle $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$ can be identified with the tangent bundle TM.

Forming duals, tensor products, and natural subbundles in there, we see that any tensor bundle over M is associated to the Cartan bundle $\mathcal{G} \to M$. The Cartan connection ω does *not* induce linear connections on general associated vector bundles. On each tensor bundle one only has the family of preferred linear connections induced by the connections in the projective class.

Standard tractors

Let $\mathbb{R}^{n+1}$ be the standard representation of G, restrict it to P and put $\mathcal{T} := \mathcal{G} \times_P \mathbb{R}^{n+1}$, the *standard tractor bundle*. This contains a line subbundle $\mathcal{T}^1 =: \mathcal{E}(-1)$ (coming from the line stabilized by P) such that $\mathcal{T}/\mathcal{T}^1 \cong TM(-1) := TM \otimes \mathcal{E}(-1)$. It turns out that $\mathcal{E}(-1)$ is the bundle of $\frac{1}{n}$–densities, and we denote by $\mathcal{E}(w)$ the bundle of $(-\frac{w}{n})$–densities. The Cartan connection ω induces a linear connection $\nabla^{\mathcal{T}}$ on $\mathcal{T}$.

On $\mathbb{R}P^n$, $\mathcal{T}$ is trivial, $\nabla^{\mathcal{T}}$ is flat, $\mathcal{T}^1 \subset \mathcal{T}$ is the tautological bundle, and the description of $\mathcal{T}/\mathcal{T}^1$ just encodes the well known description of $T\mathbb{R}P^n$ as $L(\mathcal{T}^1, \mathcal{T}/\mathcal{T}^1)$.

The Cartan bundle $\mathcal{G}$ and ω can be recovered from $(\mathcal{T}, \mathcal{T}^1, \nabla^{\mathcal{T}})$ as an adapted frame bundle.

The cone description

For projective structures, there is a particularly simple way to equivalently encode the standard tractor bundle. Let $M_\#$ be the frame bundle of $\mathcal{E}(-1)$, so $\pi : M_\# \to M$ is a principal $\mathbb{R}_+$–bundle, and let $\zeta \in \mathfrak{X}(M_\#)$ be the infinitesimal generator of the $\mathbb{R}_+$–action.

- $M_\# = \mathcal{G}/Q = \mathcal{G} \times_P (P/Q)$ for a subgroup $Q \subset P$
- ω induces a linear connection $\nabla^\#$ on $TM_\#$, whose geodesics project to the distinguished paths on M
- $\mathcal{T}$ can be recovered as $TM_\#/\mathbb{R}_+$, where the action is chosen in such a way that sections of $\mathcal{T}$ correspond to vector fields homogeneous of degree -1
- $\mathcal{T}^1$ comes from the vertical bundle and $\nabla^{\mathcal{T}}$ is induced by $\nabla^\#$

Tractor bundles

The dual $\mathcal{T}^* \to M$ of $\mathcal{T}$ is the *standard cotractor bundle*. One obtains an exact sequence

$$0 \to T^*M \otimes \mathcal{E}(1) \to \mathcal{T}^* \to \mathcal{E}(1) \to 0$$

Applying tensorial constructions to $\mathcal{T}$ and $\mathcal{T}^*$, one obtains *tractor bundles* $\mathcal{V} \to M$, which all inherit canonical linear connections from $\nabla^{\mathcal{T}}$. Any such tractor bundle $\mathcal{V}$ has a composition series induced by the composition series of $\mathcal{T}$ and $\mathcal{T}^*$, and we denote by H_0 the canonical quotient of $\mathcal{V}$, which is a tensor bundle.

A tractor bundle $\mathcal{V} \to M$ corresponds to a tensor bundle $V_\# \to M_\#$ and sections correspond to tensor fields which are homogeneous of appropriate degree. This identification is compatible with the natural connections on both bundles, so parallel tractor fields on M correspond to parallel tensor fields on $M_\#$.

For any tractor bundle $\mathcal{V} \to M$, one has the tractor connection $\nabla^{\mathcal{V}}$. The machinery of BGG–sequences uses this to construct higher order operators, which are intrinsic to the projective structure. These operators map between certain tensor bundles, which are subquotients of the bundles of $\mathcal{V}$–valued differential forms. We only need a special case:

- $\nabla^{\mathcal{V}} : \Gamma(\mathcal{V}) \to \Omega^1(M, \mathcal{V})$ induces an invariant operator $D^{\mathcal{V}} : \Gamma(H_0) \to \Gamma(H_1)$, where H_0 is the canonical quotient of $\mathcal{V}$ and H_1 is a certain subquotient of $T^*M \otimes \mathcal{V}$
- $D^{\mathcal{V}}(\sigma) = 0$ is an overdetermined system of PDEs, which has no solutions on generic projective structures
- if $s \in \Gamma(\mathcal{V})$ satisfies $\nabla^{\mathcal{V}} s = 0$, then it projects to $\sigma \in \Gamma(H_0)$ satisfying $D^{\mathcal{V}}(\sigma) = 0$ ("*normal solutions*")

Normal charts and generalized homogeneous coordinates

Choose a point u_0 in $M_\#$ and a basis $\{e_0, \dots, e_n\}$ of $T_{u_0}M_\#$, such that $e_0 = \zeta(u_0)$, the infinitesimal generator of the $\mathbb{R}_+$–action.

- Consider the span of $\{e_1, \dots, e_n\}$ in $T_{u_0}M_\#$. Use the affine exponential map of $\nabla^\#$ to identify an open neighborhood of zero with a subset in $M_\#$ which diffeomorphically projects onto an open subset $U \subset M$.
- Apply the same construction to $\mathbb{R}^{n+1} \setminus \{0\} = (\mathbb{R}P^n)_\#$ with u_0 the first vector in the standard basis, and all standard basis vectors for the e_i.
- This defines an $\mathbb{R}_+$–equivariant diffeomorphism Φ from an invariant open subset of $\mathbb{R}^{n+1} \setminus \{0\}$ to $\pi^{-1}(U) \subset M_\#$, which descends to a diffeomorphism φ from an open subset $U' \subset \mathbb{R}P^n$ onto U, which is compatible with distinguished curves through u_0.

Generalized homogeneous coordinates and normal frames

- Pulling back the standard coordinates x^i on $\mathbb{R}^{n+1}$ via Φ one obtains densities $X^0, \dots, X^{n+1} \in \Gamma(\mathcal{E}(1))$ on M ("generalized homogeneous coordinates").
- Next, transport the tangent vectors $e_1, \dots, e_n \in T_{u_0}M_\#$ parallely along geodesics in directions spanned by these vectors.
- Then extend $e_1, \dots, e_n$ homogeneous of degree -1 for the $\mathbb{R}_+$–action. Together with $e_0 = \zeta$, this defines a frame for $TM_\#$ on $\pi^{-1}(U)$.
- This corresponds to a local adapted frame for $\mathcal{T}$, while $\{e_1, \dots, e_n\}$ projects to a local frame of TM. This induces local frames of all tractor and tensor bundles, called *normal frames*. On $(\mathbb{R}P^n)_\#$, the normal frame is $\{\sum x^i\partial_i, \partial_1, \dots, \partial_n\}$.

Putting $f_0 := e_0 - \sum_{j=1}^{n} \frac{X^j}{X^0} e_i$ and $f_i = e_i$ for $i = 1, \dots, n$ on obtains a local frame on $\pi^{-1}(U)$ which is parallel along horizontal geodesics through u_0. In particular, any parallel section of a tractor bundle must have constant coordinate functions with respect to the frame induced by $\{f_i\}$.

This also holds on $\mathbb{R}P^n$, where one in addition knows that any tractor bundle is trivial, so any element in the distinguished fiber of a tractor bundle uniquely extends to a parallel section. Passing back to the frames e_i and induced frames of H_0, we conclude:

Theorem

For any normal solution σ of a first BGG operator on M, the coefficients of σ with respect to a local normal frame of the bundle H_0 are pulled back via φ to the coefficients of a solution on of the same BGG operator on $\mathbb{R}P^n$. On $\mathbb{R}P^n$, these solutions are certain polynomial systems, which can be described explicitly.

In particular, this implies that $\varphi : U' \to U$ restricts to a bijection between the zero sets of the two solutions, and on $\mathbb{R}P^n$ this zero set is an algebraic variety. Moreover, since φ is compatible with certain distinguished curves one can carry over statements about subsets being totally geodesic, or open subsets being geodesically complete for some representatives of the projective class from $\mathbb{R}P^n$ to M.

Example 1: Ricci flatness

The first BGG equation corresponding to the standard cotractor bundle $\mathcal{T}^*$ is the equation $\nabla_a \nabla_b \sigma + \mathsf{P}_{ab}\sigma = 0$ on $\sigma \in \Gamma(\mathcal{E}(1))$. Outside its zero set, a solution σ determines a connection ∇ in the projective class with $R_{ab} = 0$.

On $\mathbb{R}P^n$, solutions of this equation simply correspond to linear functionals $\lambda \in \mathbb{R}^{(n+1)*}$ and the zero set of the corresponding density is the projective hyperplane corresponding to the kernel of λ. In particular, this is totally geodesic, and we conclude:

If $\sigma \in \Gamma(\mathcal{E}(1))$ satisfies $\nabla_a \nabla_b \sigma + \mathsf{P}_{ab}\sigma = 0$, then its zero set is either empty or a totally geodesic hypersurface of M. On the dense open subset where $\sigma \neq 0$, one obtains a Ricci flat affine connection in the projective class.

Example 2: Klein–Einstein structures

The first BGG equation corresponding to $S^2\mathcal{T}^*$ is the equation

$$\nabla_{(a}\nabla_b\nabla_{c)}\sigma + 4\mathrm{P}_{(ab}\nabla_{c)}\sigma + 2(\nabla_{(a}\mathrm{P}_{bc)})\sigma = 0 \qquad (*)$$

on a density $\sigma \in \Gamma(\mathcal{E}(2))$. Outside its zeros, σ determines a connection ∇ in the projective class such that $\nabla_a \mathrm{P}_{bc} = 0$. If P_{ab} is non–degenerate, it defines a pseudo–Riemannian metric, for which ∇ is the Levi–Civita connection and which is automatically Einstein. If M is compact, the closure of such a set can thus be viewed as a compactification of an Einstein manifold.

On $\mathbb{R}P^n$, solutions correspond to symmetric bilinear forms on $\mathbb{R}^{n+1}$ and in the non–degenerate case, the zero set of the corresponding density is a quadric in $\mathbb{R}P^n$, which canonically inherits a conformal structure. Outside the zero set, P_{ab} is non–degenerate and connected components of this open subset are geodesically complete. Thus we obtain:

Let σ be a solution of $(*)$ which corresponds to a non–degenerate bilinear form on $\mathcal{T}$. Then the zero set of σ is either empty or an embedded hypersurface, which inherits a conformal structure from the projective structure on M. On each connected component of of $\{x \in M : \sigma(x) \neq 0\}$, one obtains a pseudo–Riemannian Einstein metric whose Levi–Civita connection lies in the projective class.

If M is compact, these Einstein metrics are geodesically complete. Taking the closure of such a component defines a compactification of the Einstein manifold, in which one adds a boundary at infinity which carries a conformal structure. This is not a conformal compactification, however.