BGG sequences in applied mathematics

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- This talk reports on joint work with Kaibo Hu (University of Oxford), arXiv:2203.01300 and in preparation.
- The Riemannian deformation complex for the flat metric on a bounded domain in $\mathbb{R}^3$ plays an important role in applied mathematics under the name "fundamental complex of linear elasticity".
- This triggered an interest in the BGG construction and led to several attempts for generalizations with a view towards applications in numerical analysis.
- These need the complexes in low (Sobolev) regularity, and I will outline how the construction of BGG sequences via geometry and representation theory can contribute to these developments.
- Our procedure is universal and algorithmic, provided that a certain algebraic background is given. This background can usually be derived from representation theory.

The setup

Recall that for a Riemannian manifold (M, g) the Riemannian deformation sequence starts as

$$\mathfrak{X}(M) \xrightarrow{K} \Gamma(S^2T^*M) \xrightarrow{\mathcal{R}} \Gamma(\mathcal{C}M) \to \dots$$

Here K is the Killing operator, $\mathcal{R}$ is the linearized curvature operator and $\mathcal{C}M$ is the bundle of curvature tensors.

This can be obtained as a BGG sequence associated to the projective structure defined by $\nabla = \nabla^g$. The BGG construction starts from the twisted de-Rham complex associated to the tractor connection on a tractor bundle $\mathcal{V}M \to M$. Fixing g, we get $\mathcal{V}M \cong T^*M \oplus \Lambda^2T^*M$.

Let $\mathcal{W}$ be the natural bundle induced by a representation $\mathbb{W}$ of $O(n)$, e.g. $\mathbb{W} = \mathbb{R}^{n*}$ for T^*M. Then $\mathcal{W}$ carries a Levi-Civita connection ∇, which can be coupled to d to obtain an operation d^∇ on $\Omega^*(M, \mathcal{W}M)$. These naturally occur as components in tractor connections, so we can view this as a starting point.

In the case of an open subset $U \subset \mathbb{R}^n$ with the flat metric, we can trivialize $\mathcal{W}U \cong U \times \mathbb{W}$ by parallel sections. This identifies $\Omega^k(U, \mathcal{W}U) \cong \Omega^k(U) \otimes \mathbb{W}$ and d^∇ with $d \otimes \mathrm{id}$. The latter space is $C^\infty(U, \Lambda^k \mathbb{R}^{n*} \otimes \mathbb{W})$, so in particular, it is easy to pass to Sobolev spaces of forms in this picture.

Take a representation $\mathbb{V} = \oplus_{i=0}^N \mathbb{V}_i$ of $O(n)$. A simplified version of BGG can be viewed as starting from $(\Omega^*(U, \mathcal{V}U), d^\nabla)$ and

1. Modify d^∇ by a tensorial term to obtain a complex that computes the same cohomology.
2. "Remove" the parts of the sequence that get identified by the tensorial terms, since they cannot contribute to cohomology.

We formulate this in an abstract setting starting with bounded Hilbert complexes $(Z^{*,j}, d^{*,j})$ for $j = 0, \dots, N$, each of length n. For step 1, we need bounded operators $K = \{K^{i,j}\}$ from which we construct bounded operators $S = \{S^{i,j}\}$ via $S = dK - Kd$ as in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & Z^{i-1,j} \arrow[r, "d"] & Z^{i,j} \arrow[r, "d"] & Z^{i+1,j} \arrow[r, "d"] & \cdots \\
\cdots \arrow[r, "d"] & Z^{i-1,j+1} \arrow[r, "d"] \arrow[u, "K"] \arrow[ur, "S"] & Z^{i,j+1} \arrow[r, "d"] \arrow[u, "K"] \arrow[ur, "S"] & Z^{i+1,j+1} \arrow[r, "d"] \arrow[u, "K"] & \cdots
\end{tikzcd}$$

Define $S = dK - Kd$ and require that $KS = SK$. Using $d^2 = 0$, these imply that $dS = -Sd$, which in turn implies $S^2 = 0$.

Theorem 1

The operator $d_V := d - S$ defines a differential on $Z^* := \oplus_j Z^{*,j}$. Moreover, $F := \exp(K)$ defines a bounded isomorphism of complexes from (Z^*, d_V) to $(Z^*, d) = \oplus_j (Z^{*,j}, d^{*,j})$.

This completes step 1. For step 2, we will not need the operators K any more, so we will only assume that we have given bounded operators $S = \{S^{i,j}\}$ such that $dS = -Sd$ and $S^2 = 0$. We have to make an additional assumption at this stage however, namely that for each i, j, the range $\mathcal{R}(S^{i,j})$ is a closed subspace of $Z^{i+1,j-1}$. Observe that $\mathcal{R}(S^{i,j})$ lies in the kernel $\mathcal{N}(S^{i+1,j-1})$.

Defining the *total degree* as $i+j$ on $Z^{i,j}$, we see that S preserves the total degree, while d raises it. Hence the "lowest homogeneous part" of d_V is tensorial and we can use this to "reduce" the complex (smaller spaces, but more complicated operators) without changing its cohomology.
By assumption $S^{i,j}$ restricts to a bounded linear isomorphism form $\mathcal{N}(S)^\perp \subset Z^{i,j}$ to $\mathcal{R}(S) \subset Z^{i+1,j-1}$ and we denote the bounded inverses by T. Define $\Upsilon^{i,j} \subset Z^{i,j}$ to be the closed subspace $\mathcal{N}(S) \cap \mathcal{R}(S)^\perp$ (homologies of S).

Now Td maps $Z^{i,j}$ to $Z^{i,j+1}$ and hence is nilpotent. Thus we can define a bounded operator $G : Z^{i,*} \to Z^{i,*}$ by $\sum_{k=0}^{\infty}(Td_V)^k T$ for each i. Then define $A : \Upsilon^{i,*} \to Z^{i,*}$ by $A := \mathrm{Id} - Gd_V$ and $D : \Upsilon^{i,*} \to \Upsilon^{i+1,*}$ as $D := Pd_V A$. Here we denote by P the projection onto Υ.

Properties of the splitting operators A

One verifies explicitly that for $\alpha \in \Upsilon^{i,*}$ one gets

- $A\alpha \in \mathcal{R}(S)^\perp$ and $PA\alpha = \alpha$
- $d_v A\alpha \in \mathcal{R}(S)^\perp$

and these two properties characterize the operator A.

Using this characterization and $d_V^2 = 0$, one easily verifies that $d_V \circ A = A \circ D$ and in turn $D \circ D = 0$. Hence (Υ^*, D) is a complex and A defines a chain map from this complex to (Z^*, d_V). Direct arguments then show:

Theorem 2

Under the assumptions on $S = \{S^{i,j}\}$ we have imposed, the chain map A induces an isomorphism in cohomology. In particular, by Theorem 1, (Υ^*, D) computes the same cohomology as (Z^*, d).

Getting examples from representation theory

This starts from Lie algebra gradings $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\mathfrak{o}(n) \subset \mathfrak{g}_0$ and $\mathfrak{g}_{-1} \cong \mathbb{R}^n$ and $\mathfrak{g}_1 = \mathbb{R}^{n*}$. These are available for $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{R})$ ("projective case") and for $\mathfrak{g} = \mathfrak{so}(n+1,1)$ ("conformal case"). An irreducible representation $\mathbb{V}$ of $\mathfrak{g}$ naturally decomposes as $\mathbb{V} = \oplus_{j=0}^N \mathbb{V}_j$ such that $\mathfrak{g}_i \cdot \mathbb{V}_j \subset \mathbb{V}_{i+j}$.

Now $\mathbb{V}$ is a representation of $\mathfrak{g}_{-1}$, and this gives rise to a $\mathfrak{g}_0$-equivariant *Lie algebra cohomology differential* $\partial : \Lambda^k\mathbb{R}^{n*} \otimes \mathbb{V} \to \Lambda^{k+1}\mathbb{R}^{n*} \otimes \mathbb{V}$ which maps the $\mathbb{V}_i$-component to the $\mathbb{V}_{i-1}$-component and satisfies $\partial \circ \partial = 0$.

For $U \subset \mathbb{R}^n$ open and each i and j, composing with ∂ defines a tensorial map $S : \Omega^i(U) \otimes \mathbb{V}_j \to \Omega^{i+1}(U) \otimes \mathbb{V}_{j-1}$. This of course extends to Sobolev spaces provided that the regularity of the source is at least the regularity of the target.

In all cases that we looked at, the operator S can be written as $dK - Kd$ for operators $K : \Omega^i(U) \otimes \mathbb{V}_j \to \Omega^i(U) \otimes \mathbb{V}_{j-1}$. These are tensorial, but not natural, involving e.g. multiplications by coordinate functions. They are obtained via transforming the d-parallel (standard) frame into a d_V-parallel frame. Assuming that $U \subset \mathbb{R}^n$ is bounded, the operators K also make sense in a Sobolev setting, provided that the regularity of the source is at least the regularity of the target.

Fixing $q \in \mathbb{R}$ and denoting Sobolev spaces by H^s, we have all ingredients to do step 1 with $Z^{i,j} = H^{q-i}(U, \Lambda^i \mathbb{R}^{n*} \otimes \mathbb{V}_j)$. While the operators S are bounded in this setting, they don't have closed range, since they map H^{q-i} to H^{q-i-1}.

Assuming that U has Lipschitz boundary, one can use results on Sobolev-de-Rham complexes to prove that the inclusion induces an isomorphism in cohomology for $(Z^{i,*}, d_V)$ and $(\hat{Z}^{i,*}, d_V)$ where $\hat{Z}^{i,j} := H^{q-i-j}(U, \Lambda^i \mathbb{R}^{n*} \otimes \mathbb{V}_j)$, for which the operators S have closed range. (But the K's are no more available.)

Hence we can do step 2 and arrive at a Sobolev-BGG complex. The spaces in this complex can be obtained from explicitly analyzing $\mathcal{N}(S^{i,j})/\mathcal{R}(S^{i+1,j-1})$ or via computing Lie algebra cohomology via Kostant's theorem. BGG operators defined on $\Upsilon^{i,j}$ can only map to $\Upsilon^{i+1,k}$ with $k \geq j$ and then the order is $k - j + 1$, which also shows that one gets the "right" Sobolev regularity in each case.

Apart from computing the cohomology of Sobolev-BGG complexes, we get direct applications to inequalities including a new 2-dimensional analog of the conformal Korn inequality and a construction of bounded Poincaré operators (using available results for Sobolev-de-Rham complexes).

Example: Conformal Hessian (standard tractors) in 3d

Put $N = 2$, $\mathbb{V}_0 = \mathbb{V}_2 = \mathbb{R}$, $\mathbb{V}_1 := \mathbb{R}^3$. For $(\psi_1, \psi_2, \psi_3) \in \Omega^i(U, \mathbb{V}_1)$ and $\omega \in \Omega^i(U, \mathbb{V}_2)$ define
$K(\psi_1, \psi_2, \psi_3) := \sum x^\ell \psi_\ell \qquad K(\omega) := (x^1\omega, x^2\omega, x^3\omega)$.

Defining S via $S = dK - Kd$ we obtain
$S(\psi_1, \psi_2, \psi_3) := \sum dx^\ell \wedge \psi_\ell \qquad S(\omega) := (dx^1 \wedge \omega, dx^2 \wedge \omega, dx^3 \wedge \omega)$.

To apply the machinery, we only need to check that $KS(\omega) = SK(\omega)$ and obviously both sides give $\sum_\ell x^\ell dx^\ell \wedge \omega$.
The pattern of ∂'s has the form

$$\begin{tikzcd}
\mathbb{R} & \mathbb{R}^{3*} & \Lambda^2\mathbb{R}^{3*} & \Lambda^3\mathbb{R}^{3*} \\
\mathbb{R}^3 \arrow[ur] & \mathbb{R}^{3*} \otimes \mathbb{R}^3 \arrow[ur] & \Lambda^2\mathbb{R}^{3*} \otimes \mathbb{R}^3 \arrow[ur] & \Lambda^3\mathbb{R}^{3*} \otimes \mathbb{R}^3 \\
\mathbb{R} \arrow[ur] & \mathbb{R}^{3*} \arrow[ur] & \Lambda^2\mathbb{R}^{3*} \arrow[ur] & \Lambda^3\mathbb{R}^{3*}
\end{tikzcd}$$

Example (continued)

This shows that non-zero homologies are $\mathbb{R}$ in degrees $(0,0)$ and $(3,2)$ and $S^2_0\mathbb{R}^{3*}$ in degrees $(1,1)$ and $(2,1)$. Hence the resulting Sobolev BGG complex gets the form

$$H^q(U,\mathbb{R}) \to H^{q-2}(U,S^2_0\mathbb{R}^{3*}) \to H^{q-3}(U,S^2_0\mathbb{R}^{3*}) \to H^{q-5}(U,\mathbb{R})$$

with operators of order 2, 1 and 2. The first operator is the tracefree part of the Hessian.

If U is bounded with Lipschitz boundary, then the cohomology of this complex is given (in any regularity q) by the tensor product of the de-Rham cohomology $\mathcal{H}^*(U)$ with $\mathbb{R}^5$.