The Riemannian deformation sequence revisited

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- Besides its obvious relevance for Riemannian geometry, the Riemannian deformation sequence also plays an important role in applied mathematics under the name "fundamental complex of linear elasticity".
- The usual construction goes back to a 1961 article of E. Calabi which mainly discusses the case of constant sectional curvature, in which one gets a complex. In the late 1990's it was observed that this is a special case of a projective BGG sequence, which was one of the starting points of metric projective geometry.
- While this is a nice conceptual construction, a relation to deformations can only be obtained on a computational level.
- In my talk, I will discuss a construction based on the Cartan geometry description of Riemannian manifolds and and analog of the BGG machinery. In this version the relation to deformations is manifest from the Cartan picture.

Contents

Cartan geometries

The starting point here is the *homogeneous model*, which is Euclidean space $\mathbb{R}^n$ viewed as a homogeneous space of $G := \mathrm{Euc}(n)$, the group of rigid motions. Hence $\mathbb{R}^n = G/H$, where $H = O(n) \subset \mathrm{Euc}(n)$ the subgroup of motions fixing $0 \in \mathbb{R}^n$.

A *Cartan geometry* of type (G, H) on an n-manifold M is a pair $(\mathcal{G} \to M, \omega)$ of a principal H-bundle and a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, where $\mathfrak{g} = \mathfrak{euc}(n)$ is the Lie algebra of G. So ω induces an H-equivariant trivializtion $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$ and reproduces the generators of fundamental vector fields.

Observe that by definition, the difference of two Cartan connections lies in $\Omega^1_h(\mathcal{G}, \mathfrak{g})^H$ (horizontal and H-equivariant $\mathfrak{g}$-valued 1-forms). More precisely, Cartan connections form an open subset of an affine space modelled on $\Omega^1_h(\mathcal{G}, \mathfrak{g})^H$.

Normalization

The curvature $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of the Cartan connection ω is defined by $K(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$. The defining properties of ω easily imply that K is horizontal and H-equivariant, i.e. $K \in \Omega^2_h(\mathcal{G}, \mathfrak{g})^H$. We call ω *torsion-free* if K has values in $\mathfrak{o}(n) \subset \mathfrak{g}$.

Theorem

There is a categorical equivalence between n-dimensional Riemannian manifolds (M, g) and torsion-free Cartan geometries $(\mathcal{G} \to M, \omega)$ of type (G, H).

Sketch of proof: $\mathrm{Euc}(n) = O(n) \ltimes \mathbb{R}^n$, so $\mathfrak{euc}(n) \cong_{O(n)} \mathbb{R}^n \oplus \mathfrak{o}(n)$. Given $(\mathcal{G} \to M, \omega)$ and decomposing $\omega = \theta \oplus \gamma$ accordingly, θ and γ are $O(n)$-equivariant. θ is strictly horizontal and hence identifies $\mathcal{G}$ as a reduction of the frame bundle of M to the structure group $O(n) \Leftrightarrow$ Riemannian metric g on M. γ is a principal connection on $\mathcal{G}$ and hence induces ∇ on TM with $\nabla g = 0$.

Sketch of proof (continued)

By definition, the components of K in $\mathbb{R}^n$ and $\mathfrak{o}(n)$ encode the torsion and the curvature of this connection, so if ω is torsion-free, γ is the Levi-Civita connection of g.

Conversely, given (M, g), define $\mathcal{G} := \mathcal{O}M$, the orthonormal frame bundle of M with respect to g. This comes with a soldering form $\theta \in \Omega^1(\mathcal{O}M, \mathbb{R}^n)$, and the Levi-Civita connection of g is induced by a principal connection $\gamma \in \Omega^1(\mathcal{O}M, \mathfrak{o}(n))$. Putting $\omega := \theta \oplus \gamma$, one obtains a torsion-free Cartan geometry as required. □

The Cartan setup suggests looking at $\Omega^k_h(\mathcal{G}, \mathfrak{g})^H$. For $k = 1$, 2, we can directly interpret these as infinitesimal changes of Cartan connections and curvatures by the above observations. For $k = 0$, $f : \mathcal{G} \to \mathfrak{g}$ corresponds to $\xi \in \mathfrak{X}(\mathcal{G})$ via $\omega(\xi(u)) = f(u)$. H-equivariancy of f is equivalent to ξ being H-invariant. So here we obtain infinitesimal principal bundle automorphisms.

Cohomology

$[\ ,\ ]:\mathbb{R}^n\times\mathfrak{g}\to\mathfrak{g}$ is an $O(n)$-equivariant representation of the Abelian Lie algebra $\mathbb{R}^n$ on the vector space $\mathfrak{g}$. This induces $O(n)$-equivariant maps $\partial:\Lambda^k\mathbb{R}^{n*}\otimes\mathfrak{g}\to\Lambda^{k+1}\mathbb{R}^{n*}\otimes\mathfrak{g}$ computing $H^*(\mathbb{R}^n,\mathfrak{g})$. These cohomology spaces are representations of $O(n)$ and thus define natural bundles on Riemannian n-manifolds.

$[\mathbb{R}^n,\mathbb{R}^n]=0$ and $[\mathbb{R}^n,\mathfrak{o}(n)]\subset\mathbb{R}^n$ (standard action of $\mathfrak{o}(n)$ on $\mathbb{R}^n$), and similarly for ∂. In degree 1, one thus obtains a map $\mathbb{R}^{n*}\otimes\mathfrak{o}(n)\to\Lambda^2\mathbb{R}^{n*}\otimes\mathbb{R}^n$. This is the classical Spencer differential which is a linear isomorphism ($\Leftrightarrow$ existence and uniqueness of the Levi-Civita connection).

One easily shows that the spaces $H^k(\mathbb{R}^n,\mathfrak{g})$ are $\mathbb{R}^n\subset\mathfrak{g}$ for $k=0$, $\mathcal{S}(\mathbb{R}^n)\subset\mathbb{R}^{n*}\otimes\mathbb{R}^n$ (symmetric endomorphisms) for $k=1$, and $\ker(\mathrm{Alt})\subset\Lambda^k\mathbb{R}^{n*}\otimes\mathfrak{o}(n)$ for $k\geqslant 2$.

Remark

One can realize Euc(n) as a Lie subgroup of $GL(n+1,\mathbb{R})$, thus obtaining a natural representation on $\mathbb{R}^{n+1}$ and hence a representation on $\mathbb{V} := \Lambda^2\mathbb{R}^{(n+1)*}$. Playing the same game as before, we get an action of $\mathbb{R}^n$ on $\mathbb{V}$ which is $O(n)$-equivariant and hence $\partial : \Lambda^k\mathbb{R}^{n*} \otimes \mathbb{V} \to \Lambda^{k+1}\mathbb{R}^{n*} \otimes \mathbb{V}$ and $H^*(\mathbb{R}^n, \mathbb{V})$.

While $\mathfrak{g}$ and $\mathbb{V}$ are not isomorphic as representations of Euc(n), there is an $O(n)$-equivariant isomorphism of representations of $\mathbb{R}^n$ between them. Hence $H^k(\mathbb{R}^n, \mathfrak{g})$ and $H^k(\mathbb{R}^n, \mathbb{V})$ are isomorphic as representations of $O(n)$ for each k. But the differential ∂ for $\mathbb{V}$ is exactly the one arising in the BGG construction in projective geometry that produces the Calabi sequence, so it is equivariant for the natural action of $SL(n,\mathbb{R})$. Hence each $H^k(\mathbb{R}^n, \mathfrak{g})$ is isomorphic to a representation of $SL(n,\mathbb{R})$ (which is irreducible by Kostant's theorem).

Two natural operations $C^\infty(\mathcal{G},\mathfrak{g})^H \cong \mathfrak{X}(\mathcal{G})^H \to \Omega^1_h(\mathcal{G},\mathfrak{g})^H$:

1. For $f:\mathcal{G}\to\mathfrak{g}$ consider $d^\omega f(\eta) := \eta\cdot f + [\omega(\eta), f]$.
2. For $\xi\in\mathfrak{X}(\mathcal{G})^H$ consider $\tilde{d}^\omega\xi := \mathcal{L}_\xi\omega$ (Lie derivative). One easily computes that $\tilde{d}^\omega\xi = d^\omega\xi + K(\xi,\ )$.

$\Omega^k_h(\mathcal{G},\mathfrak{g})^H \cong \Omega^k(M,\mathcal{A}M)$, where $\mathcal{A}M := \mathcal{G}\times_H\mathfrak{g}$ is the *adjoint tractor bundle*. d^ω and $\tilde{d}^\omega$ induce linear connections $\nabla^{\mathcal{A}}$ and $\tilde{\nabla}^{\mathcal{A}}$ on $\mathcal{A}M$. Since $\mathfrak{g} = \mathbb{R}^n\oplus\mathfrak{o}(n)$ as a representation of H, we get $\mathcal{A}M = TM\oplus\mathfrak{o}(TM)$. Here $\mathfrak{o}(TM)$ is the space of endomorphisms of TM that are skew symmetric with respect to g. We write sections of $\mathcal{A}M$ as $\binom{\eta}{\Phi}$, $\eta\in\mathfrak{X}(M)$ and $\Phi\in\Gamma(\mathfrak{o}(TM))$.

Theorem

$$\nabla^{\mathcal{A}}_\xi\begin{pmatrix}\eta\\ \Phi\end{pmatrix} = \begin{pmatrix}\nabla_\xi\eta - \Phi(\xi)\\ \nabla_\xi\Phi\end{pmatrix} \qquad \tilde{\nabla}^{\mathcal{A}}_\xi\begin{pmatrix}\eta\\ \Phi\end{pmatrix} = \begin{pmatrix}\nabla_\xi\eta - \Phi(\xi)\\ \nabla_\xi\Phi - R(\xi,\eta)\end{pmatrix}$$

From these formulae, it is easy to compute the curvatures $R^{\mathcal{A}}$ of $\nabla^{\mathcal{A}}$ and $\tilde{R}^{\mathcal{A}}$ of $\tilde{\nabla}^{\mathcal{A}}$. $R^{\mathcal{A}}$ is just the row-wise action of the Riemann curvature, but for $\tilde{R}^{\mathcal{A}}$ the result is very surprising.

Let $\bullet$ be the bundle map obtained from the infiniresimal action of $\mathfrak{o}(n)$ on $\Lambda^2\mathbb{R}^{n*}\otimes\mathbb{R}^{n*}\otimes\mathbb{R}^n$. Then

$$\tilde{R}^{\mathcal{A}}(\xi_1,\xi_2)\begin{pmatrix}\eta\\ \Phi\end{pmatrix}=\begin{pmatrix}0\\ (\nabla_\eta R)(\xi_1,\xi_2)-(\Phi\bullet R)(\xi_1,\xi_2)\end{pmatrix}.$$

In particular, $\tilde{\nabla}^{\mathcal{A}}$ is flat iff g has constant sectional curvature.

Extending to operations on higher degree forms is straightforward in both picutures, and the tensorial part in $d^{\nabla^{\mathcal{A}}}$ is induced by ∂. In degree zero, the kernel of $\tilde{d}^{\omega}$ by definition are the infinitesimal automorphisms of $(\mathcal{G}\to M,\omega)$. In degree 1, up to a subtlety in interpretation, one easily shows that $\tilde{d}^{\omega}$ computes the infinitesimal change of K caused by an infinitesimal change of ω. The subtlety has a counterpart ($R_{ijk\ell}$ vs. $R_{ij}{}^k{}_\ell$)in the picture of (M,g).

From here on, one follows the BGG machinery in a simplified setting. The bundles induced by $H^k(\mathbb{R}^n, \mathfrak{g})$ sit in $\Lambda^k T^*M \otimes \mathcal{A}M$ as $\mathcal{H}^0 := TM \subset \mathcal{A}M$, $\mathcal{H}^1 := \mathcal{S}(TM) \subset T^*M \otimes TM$, and $\mathcal{H}^k := \ker(\mathrm{Alt}) \subset \Lambda^k T^*M \otimes \mathfrak{o}(TM)$ for $k \geqslant 2$. Two notions:

Interpret $\binom{\psi}{\Psi} \in \Omega^1(M, \mathcal{A}M)$ (so $\psi \in \Gamma(T^*M \otimes TM)$) as an infintesimal deformation of ω.

- The induced infinitesimal deformation of g is given by $\psi + \psi^t$. Call $\binom{\psi}{\Psi}$ *symmetric* if $\psi^t = \psi$.
- Call $\binom{\psi}{\Psi}$ *torsion-free* if the resulting infinitesimal deformation of torsion vanishes, i.e. $d^{\tilde{\nabla}^{\mathcal{A}}}\binom{\psi}{\Psi} = \binom{0}{*}$.

Given $\eta \in \mathfrak{X}(M)$, $\exists! \Phi \in \Gamma(\mathfrak{o}(TM))$ such that, for $L(\eta) := \binom{\eta}{\Phi}$, $d^{\tilde{\nabla}^{\mathcal{A}}} L(\eta)$ has top component in $\mathcal{H}^1$. Call this $D(\eta) \in \Gamma(\mathcal{H}^1)$.Thus, any vector field on M has a unique H-invariant lift to $\mathcal{G}$ which induces a symmetric infinitesimal deformation of ω.

For $h \in \Gamma(\mathcal{H}^1)$, $\exists! \Psi \in \Omega^1(M, \mathfrak{o}(TM))$ such that, for $L(h) := \binom{h}{\Psi}$, $d^{\tilde{\nabla}^{\mathcal{A}}} L(h) = \binom{0}{*}$. The bottom component of this always lies in $\Gamma(\mathcal{H}^2)$, call it $D(h)$. Thus any (symmetric) infinitesimal deformation of g uniquely lifts to a torsion-free deformation of ω and D computes the resulting change of Riemann curvature.

For $k \geqslant 2$, one defines L to be the inclusion $\Gamma(\mathcal{H}^k) \to \Omega^k(M, \mathcal{A}M)$ and shows that $D := d^{\tilde{\nabla}^{\mathcal{A}}} \circ L$ has values in $\Gamma(\mathcal{H}^{k+1})$. Further:

- $d^{\tilde{\nabla}^{\mathcal{A}}} \circ L = L \circ D$ in all degrees, and in degree 0, L induces an isommorphism between $\ker(D)$ and sections of $\mathcal{A}M$, which are parallel for $\tilde{\nabla}^{\mathcal{A}}$.
- constant curvature $\Rightarrow (d^{\tilde{\nabla}^{\mathcal{A}}})^2 = 0 \Rightarrow D^2 = 0$, and then L induces an isomorphism in cohomology.

In degree 0, D is the Killing operator, in degree 1, the explicit formula is easy to compute (and agrees with Calabi's result for the deformation of curvature).

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