G-structures

(lecture notes)

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CHAPTER 1

Structures on a vector space

The aim of this course is to discuss a general approach to a very broad notion of "geometric structures" in the setting of differential geometry. The model example of a geometric structure in this sense is provided by a Riemannian metric on a manifold. But there are other examples that typically occur already in a course on analysis on manifolds which are of quite different nature, for example an orientation on a manifold or a symplectic structure. As a motivation for the further developments, we start by outlining the fundamental differences between Riemannian metrics and (almost) symplectic structures. We then switch to the "point-wise" version of a discussion of geometric structures, which discusses "structures" on a vector space.

Motivation: Riemannian and almost symplectic structures

1.1. Some remarks on diffeomorphisms. As a first step we have to recall how soft and flexible diffeomorphisms of a manifold M are. Of course, for any smooth manifold M, the diffeomorphisms $\Phi : M \to M$ form a group $\mathrm{Diff}(M)$ under composition. Further, for any vector field $\xi \in \mathfrak{X}(M)$ with compact support, we get a globally defined flow $\mathrm{Fl}^\xi_t : M \to M$ for each $t \in \mathbb{R}$ which is a diffeomorphism with inverse Fl^ξ_{-t}. In particular, taking the open unit ball $B^n_1 \subset \mathbb{R}^n$, we can interpret any smooth function $f : B^n_1 \to \mathbb{R}^n$ as a vector field on B^n_1. Taking a bump function φ with support contained in B^n_1, which is identically one on a slightly smaller ball $B^n_{1-\epsilon}$ we can extend φf to a globally defined vector field on $\mathbb{R}^n$ with support in B^n_1. The flow of this vector filed defines a diffeomorphism on $\mathbb{R}^n$ which is the identity outside of B^n_1. Via local charts we can transport such diffeomorphisms to any smooth manifold of dimension n or combine different diffeomorphisms of that type on different balls and so on. This already implies that, the diffeomorphism group of any manifold has infinite dimension (i.e. cannot be finite dimensional).

Let us discuss some more explicit examples. Consider the unit circle S^1 and let $\partial_t \in \mathfrak{X}(S^1)$ be the derivative of an arc length parametrization of S^1. Choose a finite collection of points $x_i \in S^1$ for $i = 1, \dots, N$ which are ordered in an obvious sense. For each $i < N$, choose y_i between x_i and x_{i+1} and choose y_N between x_N and x_1 and take any function $\varphi : \{1, \dots, N\} \to \{\pm 1\}$. Then we claim that there is a diffeomorphism $f : S^1 \to S^1$ such that $f(x_i) = x_i$ for all i while $f(y_i)$ is moved towards x_{i+1} if $\varphi(i) = 1$ and towards x_i if $\varphi(i) = -1$. Indeed, we can find a connected open neighborhood U_i of y_i, which does not contain any x_j and a bump function $h_i : S^1 \to [0, \infty)$ with support contained in U_i such that $h_i(y_i) = 1$. Then we define $\xi := \sum_{i=1}^N \varphi(i) h_i \partial_t \in \mathfrak{X}(S^1)$. By construction, this vector field satisfies $\xi(x_i) = 0$ for all i while $\xi(y_i)$ is a positive multiple of ∂_t if $\varphi(i) = 1$ and a negative multiple of ∂_t if $\varphi(i) = -1$. This shows that for any $s > 0$, $\mathrm{Fl}^\xi_s \in \mathrm{Diff}(S^1)$ has the required property.

Another typical example of such a construction works on any connected manifold M of dimension $n \geq 3$. Choose any finite number N and points $x_i, y_i \in M$ for each $i = 1, \dots, N$ such that $x_i \neq x_j$ and $y_i \neq y_j$ for $i \neq j$. Then there is a diffeomorphism

$f : M \to M$ such that $f(x_i) = y_i$ for all i. The idea how to obtain this is as follows. For each i, one can find a smooth curve $c_i : (-\epsilon, 1+\epsilon) \to M$ such that $c_i(0) = x_i$ and $c_i(1) = y_i$. Since $\dim(M) > 3$ things can be arranged in such a way that the curves c_i have disjoint images. With a bit of differential topology (tubular neighborhoods) one can find disjoint open neighborhoods U_i of the images of the c_i with compact closure and bump functions $h_i : M \to \mathbb{R}$ with support contained in U_i, which are identically 1 along the image of c_i. Then one extends c_i' to a vector field $\tilde{\xi}_i$ defined on U_i and then $h_i\tilde{\xi}_i$ can be extended by zero to a vector field ξ_i on M, which by construction has compact support. Then also $\xi := \sum_i \xi_i$ has compact support and thus defines a global flow $f := \mathrm{Fl}_1^\xi \in \mathrm{Diff}(M)$. But by construction, each c_i is an integral curve for ξ_i and hence for ξ, and thus $f(x_i) = c_i(1) = y_i$ for each $i = 1, \dots, N$.

1.2. Riemannian metrics. By definition, a Riemannian metric on a smooth manifold M is given by choosing a positive definite inner product g_x on the tangent space T_xM for each point $x \in M$. This has to depend smoothly on x, thus defining a $\binom{0}{2}$-tensor field g on M. One then calls (M, g) a Riemannian manifold. The diffeomorphisms $f \in \mathrm{Diff}(M)$ that are compatible with g in the sense that $f^*g = g$ are called *isometries* of (M, g). Of course, they form a subgroup $\mathrm{Isom}(M, g) \subset \mathrm{Diff}(M)$. The main point we want to illustrate here is that compatibility with g is an extremely restrictive condition. This shows up in several different ways, which initially look rather unrelated. Exploring their relations in a general setting will be a major aim of this course.

A first way to observe this is to look at a model example. By linear algebra, there is only one positive definite inner product on an n-dimensional vector space up to isomorphism. Thus we can focus on the example of the standard inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$. Now if we consider $\mathbb{R}^n$ (or rather the affine space underlying $\mathbb{R}^n$) as a smooth manifold, we get a canonical trivialization of the tangent bundle as $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$. The inverse of this trivialization maps (x, v) to the tangent vector $\frac{d}{dt}|_{t=0}(x + tv)$ or, in the picture of derivations, to the directional derivative at x in direction v. Putting the standard inner product on $\mathbb{R}^n$ thus defines a Riemannian metric on $\mathbb{R}^n$, the resulting Riemannian manifold is usually called the *Euclidean space* $\mathbb{E}^n$. One would expect this to be the simplest and most symmetric Riemannian manifold of dimension n, which turns out to be the case. Still, its isometry group is quite small.

PROPOSITION 1.2. *Let $f : \mathbb{E}^n \to \mathbb{E}^n$ be an isometry. Then there is an orthogonal matrix $A \in O(n)$ and a vector $b \in \mathbb{R}^n$ such that $f(x) = Ax + b$ for all $x \in \mathbb{R}^n$. So the isometries of $\mathbb{E}^n$ are exactly the Euclidean motions, which form a Lie group of dimension $\frac{n(n+1)}{2}$.*

PROOF. Let $f : \mathbb{E}^n \to \mathbb{E}^n$ be a diffeomorphism. Then via the trivialization $T\mathbb{E}^n \cong \mathbb{E}^n \times \mathbb{R}^n$ from above, the tangent map of f corresponds to $(x, v) \mapsto (f(x), Df(x)(v))$. Hence the condition that f is an isometry is equivalent to

$$\langle Df(x)(v), Df(x)(w)\rangle = \langle v, w\rangle \tag{1.1}$$

for all $x \in \mathbb{E}^n$ and $v, w \in \mathbb{R}^n$. Equivalently, $Df(x) \in O(n) \subset GL(n, \mathbb{R})$ for each $x \in \mathbb{E}^n$. Taking (1.1) for fixed v and w as a function of x and differentiating in direction $z \in \mathbb{R}^n$, we obtain

$$\langle D^2f(x)(z, v), Df(x)(w)\rangle + \langle Df(x)(v), D^2f(x)(z, w)\rangle = 0. \tag{1.2}$$

Fixing x, the map $(z, v, w) \mapsto \langle D^2f(x)(z, v), Df(x)(w)\rangle$ defines a trilinear map $\Phi : (\mathbb{R}^n)^3 \to \mathbb{R}$ and by symmetry of $\langle\ ,\ \rangle$, (1.2) exactly says that $\Phi(z, v, w) = -\Phi(z, w, v)$. On the other hand, symmetry of the second derivative shows that $\Phi(z, v, w) = \Phi(v, z, w)$.

But these two properties algebraically imply that $\Phi = 0$, since we can compute as follows.

$$\begin{aligned}\Phi(z,v,w) =& -\Phi(z,w,v) = -\Phi(w,z,v) = \Phi(w,v,z)\\ =&\Phi(v,w,z) = -\Phi(v,z,w) = -\Phi(z,v,w).\end{aligned}$$

Since $Df(x)$ is a linear isomorphism, we conclude that $D^2f(x) = 0$ for any $x \in \mathbb{E}^n$. Thus $Df(x) = A$ for some fixed matrix $A \in O(n)$, so the curve $c(t) = f(tx)$ has derivative $c'(t) = Ax$ for all t. Hence $c(1) - c(0) = \int_0^1 c'(t)dt = Ax$, and putting $b := f(0) \in \mathbb{R}^n$, we get $f(x) = Ax + b$ as claimed.

Conversely, any such function determines an isometry of $\mathbb{E}^n$. For $f(x) = Ax + b$ and $\tilde{f}(x) = \tilde{A}x + \tilde{b}$, we see that $(\tilde{f} \circ f)(x) = \tilde{A}Ax + (\tilde{A}b + \tilde{b})$. Hence we can identify $\mathrm{Isom}(\mathbb{E}^n)$ with $O(n) \times \mathbb{R}^n$, endowed with the multiplication $(\tilde{A}, \tilde{b}) \cdot (A, b) = (\tilde{A}A, Ab + \tilde{b})$, which is evidently smooth. Thus we obtain a Lie group and the claim on the dimension is obvious. □

Notice that this in particular implies that isometries of $\mathbb{E}^n$ preserve the structure as an affine space, i.e. they map affine lines to affine lines and so on. This is not clear in advance but follows from our discussion below.

The extension of these ideas to general Riemannian manifolds needs some fundamental concepts of Riemannian geometry, in particular existence and uniqueness of the Levi-Civita connection. We discuss this only very briefly here, much more will be done later. On a Riemannian manifold (M, g), one obtains a canonical bilinear operator $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$. This is written as $(\xi, \eta) \mapsto \nabla_\xi \eta$ and can be thought of as defining a "directional derivative" of η in direction ξ. In particular, on $\mathbb{E}^n$, the Levi-Civita connection is just the directional derivative of vector fields viewed as $\mathbb{R}^n$-valued functions.

The Levi-Civita connection in turn leads to geodesics, which form a family of canonical curves on M. Given a point $x \in M$ and a tangent vector $X \in T_xM$, there is a unique curve $c : I \to M$ in that family such that $c(0) = x$ and $c'(0) = X$. Part of the uniqueness here is that I is taken to be a maximal interval containing zero. On $\mathbb{E}^n$, this simply gives the affine lines (parametrized with constant speed).

One way to phrase uniqueness of the Levi-Civita connection is that for any isometry $f : M \to M$ of (M, g) and vector fields $\xi, \eta \in \mathfrak{X}(M)$, one obtains $f^*(\nabla_\xi \eta) = \nabla_{f^*\xi} f^*\eta$. This in turn implies that for any geodesic c for g, also $f \circ c$ is a geodesic. This shows that an isometry of $\mathbb{E}^n$ has to map straight lines to straight lines and it leads to a general result on isometries of Riemannian manifolds:

Theorem 1.2. *Let (M, g) be a connected Riemannian manifold. Then any isometry of M is uniquely determined by its value and its tangent map in a single point $x \in M$. So if $f, \tilde{f} \in \mathrm{Isom}(M)$ have the property that $f(x) = \tilde{f}(x) =: y$ and $T_xf = T_x\tilde{f} : T_xM \to T_yM$, then $f = \tilde{f}$.*

Proof. Suppose that $f, \tilde{f} \in \mathrm{Isom}(M)$ are arbitrary, and consider the subset $N \subset M$ defined by $N := \{x \in M : f(x) = \tilde{f}(x), T_xf = T_x\tilde{f}\}$. Since f and Tf are continuous, N is evidently closed in M. On the other hand, suppose that $x \in N$. Then there is an open neighborhood U of x in M such that for each point $y \in U$ there is a geodesic $c : I \to M$ such that $c(0) = x$ and $c(1) = y$. But then $f \circ c$ is a geodesic for g with initial point $f(x)$ and initial direction $T_xf(c'(0))$. But since $x \in N$, we obtain $\tilde{f}(x) = f(x)$ and $T_x\tilde{f}(c'(0)) = T_xf(c'(0))$, $\tilde{f} \circ c$ also is a geodesic with the same initial point and initial direction as $f \circ c$. Hence $\tilde{f} \circ c = f \circ c$ and in particular $f(y) = (f \circ c)(1) = (\tilde{f} \circ c)(1) = \tilde{f}(y)$.

Thus we conclude that $f|_U = \tilde{f}|_U$ and since U is open, this implies that $T_y f = T_y \tilde{f}$ for all $y \in U$. Hence $U \subset N$ and thus N is open so if it is non-empty, it has to equal M by connectedness. □

This says that for isometries of M there are at most $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ "degrees of freedom", which gives us a bound on the "dimension" (in a naive sense) of the isometry group. There actually is a general result that says that the isometry group of a Riemannian manifold (M, g) always is a Lie group that acts smoothly on M and then Theorem 1.2 says that the dimension of this Lie group is at most $\frac{n(n+1)}{2}$. The Lie algebra of the isometry group is formed by all complete Killing vector fields on M. Here a Killing vector field ξ is a vector field such that that $\binom{0}{2}$-tensor field defined by $(\eta, \zeta) \mapsto g(\nabla_\eta \xi, \zeta)$ is symmetric. This property is equivalent to the fact that any local flow of ξ is an isometry. Again, we will say more about that (in a much more general setting) later.

The next crucial ingredient in the study of Riemannian manifolds is also obtained via the Levi-Civita connection, namely the curvature. The defining properties of the connection imply that the expression

$$R(\xi, \eta)(\zeta) := \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi,\eta]} \zeta,$$

is linear over smooth functions in all three variables. Thus this defines a $\binom{1}{3}$-tensor field R called the *Riemann curvature tensor*. This evidently defines a *local invariant* of a Riemannian manifold (M, g) since the value of R in $x \in M$ is determined by the restriction of g to any open neighborhood U of x in M. Explicit coordinate formulae for R show that indeed only only needs to know the values and the first and second partial derivatives in x of the components of g in some coordinate system to determine the value of R in x.

This readily shows that local coordinates cannot be adapted too well to a Riemannian metric, so one cannot find local coordinates in which a Riemannian metric looks like some model. The best adaption of this type is provided by so-called (Riemannian) normal coordinates centered at a point $x \in M$. But these are well adapted only along geodesics through x.

The curvature also imposes additional restrictions on isometries. From the compatibility of isometries with the Levi-Civita connection observed above it readily follows that $f^*R = R$ for any isometry $f \in \mathrm{Isom}(M, g)$. Most easily, this shows that for points $x, y \in M$ such that $R(x) = 0$ and $R(y) \neq 0$, there cannot be any isometry f with $f(x) = y$. But of course a much finer analysis is possible. If an isometry f with $f(x) = y$ exists, the trilinear maps $R(x) : (T_x M)^3 \to T_x M$ and $R(y) : (T_y M)^3 \to T_y M$ have to be related to each other via the orthogonal linear isomorphism $T_x f : T_x M \to T_y M$. This is a problem that can be studied using linear algebra and there are obstructions coming from concepts like rank. Things become a bit complicated since the Riemann curvature has relatively complicated symmetries, simplifications are obtained by decomposing R into parts (scalar curvature, Ricci curvature and Weyl curvature) and analyzing these parts separately. Of course, one can then go beyond a point-wise study to see that "complicated curvature" obstructs the existence of many isometries.

It can actually be shown that the maximal possible dimension of the isometry group $\mathrm{Isom}(M)$ is only attained if (M, g) has constant sectional curvature, which actually means that locally (M, g) is isometric to either $\mathbb{E}^n$, S^n or hyperbolic space of dimension n. Moreover, generic Riemannian metrics in dimension at least 3 do not admit any local isometries.

1.3. Almost symplectic structures. These structures are a skew-symmetric analog of Riemannian metrics, so instead of inner products we consider skew-symmetric bilinear forms on tangent spaces. It is well known from linear algebra that a skew symmetric matrix has to have even rank. This easily implies that on a vector space of odd dimension, any skew-symmetric bilinear form is degenerate. In turn, one proves that one any real vector space of even dimension, there is a unique non-degenerate, skew-symmetric bilinear form up to isomorphism. Since skew symmetric $\binom{0}{2}$-tensor fields are just two-forms, we arrive at the following definition.

DEFINITION 1.3. Let M be a smooth manifold of even dimension. Then an *almost symplectic form* on M is a two-form $\omega \in \Omega^2(M)$ such that for each $x \in M$, the bilinear map $\omega(x) : T_xM \times T_xM \to \mathbb{R}$ is non-degenerate. A *symplectic form* on M is an almost symplectic form ω such that $d\omega = 0$. One calls (M, ω) an *(almost) symplectic manifold*. A *symplectomorphism* of such a structure then is a diffeomorphism $f : M \to M$ such that $f^*\omega = \omega$.

Observe that by non-degeneracy, for an almost symplectic form ω, $\omega(x)$ can be viewed as defining a linear isomorphism $T_xM \to T_x^*M$. The point-wise inverses fit together to define a $\binom{2}{0}$-tensor field ω^{-1}, compare with Section 3.8 of [**AnaMF**]. There is an induced isomorphism $\mathfrak{X}(M) \to \Omega^1(M)$ that sends $\xi \in \mathfrak{X}(M)$ to $i_\xi\omega \in \Omega^1(M)$, with inverse similarly described via ω^{-1}. Given an almost symplectic form, one may thus, as in Riemannian geometry, identify $\binom{\ell}{k}$-tensor fields with $\binom{\ell'}{k'}$-tensor fields provided that $k + \ell = k' + \ell'$. Still, such structures show completely different behavior than Riemannian metrics.

Let us start by looking at the model example that is similar to Euclidean space. So we fix a skew symmetric bilinear map $b : V \times V \to \mathbb{R}$, where $V := \mathbb{R}^{2n}$ and view this as a two-form ω on the manifold V via the isomorphism $TV \cong V \times V$. (Similarly to Euclidean space, one could use affine space of dimension $2n$ here.) For a diffeomorphism $f : V \to V$, the condition to be a symplectomorphism of course is that for all $x, v, w \in V$, we get $b(Df(x)(v), Df(x)(w)) = b(v, w)$ and differentiating this in x in direction z, we get

$$0 = b(D^2f(x)(z, v), Df(x)(w)) + b(Df(x)(v), D^2f(x)(z, w)). \tag{1.3}$$

But because of skew symmetry of b, this now tells us that the trilinear map $\Phi : V^3 \to \mathbb{R}$ defined by $\Phi(z, v, w) := b(D^2f(x)(z, v), Df(x)(w))$ is symmetric both in the pair (z, v) and in the pair (v, w). But this just means that Φ has to be totally symmetric, so it leaves a lot of freedom for possible values for $D^2f(x)$ given a fixed value of $Df(x)$. Indeed one can differentiate equation (1.3) further to obtain restrictions on higher derivatives of f. But it turns out that even if we fix D^if for $i \leq k$, there always is some freedom for the choice of $D^{k+1}f$. This indicates that the situation is very different from the Riemannian case.

Indeed, we can show that symplectomorphisms can occur in infinite dimensional families and get some additional information by passing to a general construction which also provides a a nice interpretation of our model example. Let N be any smooth manifold of dimension n and put $M := T^*N$ the cotangent bundle of N, which has dimension $2n$. Then there is a smooth projection $\pi : M \to N$ which is a submersion. Now a point $y \in M$ is a linear map $\lambda : T_{\pi(y)}N \to \mathbb{R}$. Hence given $\xi \in T_yM$, we can define $\alpha(y)(\xi) := \lambda(T_y\pi \cdot \xi)$ and this defines the so-called *tautological one-form* $\alpha \in \Omega^1(M)$. Now choose local coordinates on $U \subset N$ which are traditionally denoted by q^i, and consider the associated coordinate one-forms dq^i. Mapping an element in $\pi^{-1}(U)$ to its coefficients with respect to the dq^i defines smooth functions $p_i : \pi^{-1}(U) \to \mathbb{R}$. Now it is also traditional to suppress the pullback along π and just view the q^i as functions defined on $\pi^{-1}(U)$. Together with the p_i, they then are the coordinate functions of a local chart for T^*N defined on $\pi^{-1}(U)$. In the same way, one suppresses the pullback and considers the dq^i as one-forms on $\pi^{-1}(U)$. Then by construction, the coordinate expression of α is simply given by $\alpha = \sum_i p_i dq^i$ and thus $\omega := -d\alpha = \sum_i dq^i \wedge dp_i$. Now one immediately verifies that this has non-degenerate values, while $d\omega = 0$ is obvious from the construction, so ω defines a symplectic form on $M = T^*N$.

In particular, doing this for $N := \mathbb{R}^n$, we get $M = T^*N = \mathbb{R}^n \times \mathbb{R}^{n*} \cong \mathbb{R}^{2n}$ and then $TM = M \times \mathbb{R}^n \times \mathbb{R}^{n*}$. Starting from the global coordinates q^i on $\mathbb{R}^n$, one easily verifies that ω actually comes from the constant skew symmetric bilinear form b on $\mathbb{R}^n \times \mathbb{R}^{n*}$ defined by $b((v,\lambda),(w,\mu)) := \lambda(w) - \mu(v)$. Hence our model example from above is the simplest example of the canonical symplectic form on a cotangent bundle.

The "universal" construction of the symplectic structure on cotangent bundles has an important consequence. Recall that for a local diffeomorphism $f : N \to P$, there is an induced map $T^*f : T^*N \to T^*P$. For a point $x \in N$ and linear map $\lambda : T_xN \to \mathbb{R}$ one defines $T^*f(\lambda) := \lambda \circ (T_xf)^{-1} : T_{f(x)}P \to \mathbb{R}$. This readily implies that $\pi \circ T^*f = f \circ \pi$, and smoothness of the tangent maps of local inverses to f easily implies that $T^*f : T^*N \to T^*P$ is smooth. Indeed, applying the same construction to local inverses of f, one obtains local inverses to T^*f, so T^*f is a local diffeomorphism. Now let $\alpha \in \Omega^1(T^*P)$ be the tautological one form and take $x \in N$ and $\lambda \in T^*N$ and $X_\lambda \in T_\lambda T^*N$. Then by definition we get

$$((T^*f)^*\alpha)(\lambda)(X_\lambda) = \alpha(T^*f(\lambda))(T_\lambda(T^*f) \cdot X_\lambda) = T^*f(\lambda)(T(\pi \circ T_\lambda(T^*f)) \cdot X_\lambda).$$

But then $\pi \circ T^*f = f \circ \pi$ implies that $T\pi \circ T(T^*f) = Tf \circ \pi$ and hence this equals $T^*f(\lambda)(T_xf(T_\lambda\pi \cdot X_\lambda)) = \lambda(T_\lambda\pi \cdot X_\lambda)$. Thus we see that, for any local diffeomorphism $f : N \to P$, T^*f pulls back the tautological one-form on T^*P to the tautological one-form on T^*N. Hence we obtain the following results

Proposition 1.3. *(1) For any smooth manifold N and any diffeomorphism $f : N \to N$ the induced map $T^*f : T^*N \to T^*N$ is a symplectomorphism. Hence groups of symplectomorphisms can be infinite dimensional.*

(2) For any smooth manifold N the symplectic manifold T^*N is locally isomorphic to the model example $T^*\mathbb{R}^n$ of symplectic manifolds with $n = \dim(N)$.

Proof. (1) For a diffeomorphism f, T^*f^{-1} by construction is inverse to T^*f, so T^*f is a diffeomorphism. From above, we know that for the tautological one-form α on T^*N, we get $(T^*f)^*\alpha = \alpha$. Since $\omega = -d\alpha$, $(T^*f)^*\omega = \omega$ readily follows from compatibility of the exterior derivative with pullbacks.

(2) Given a manifold N and $x \in N$, there of course is a local chart (U,u) for N with $x \in U$ and $u(U) = \mathbb{R}^n$. Then the open subset $\pi^{-1}(U) \subset T^*N$ can be naturally identified with T^*U and from above we know that $T^*(u^{-1})$ induces an isomorphism from the model space $T^*\mathbb{R}^n$ to T^*U. □

There actually is an improvement to part (2) that is well known under the name *Darboux theorem*, see Theorem 22.13 in [**Lee**]: Given any symplectic manifold (M,ω) and a point $x \in M$, there is a local chart U for M that contains x with coordinate functions q^i and p_i such that $\omega|_U = \sum_i dq^i \wedge dp_i$. Hence locally any symplectic structure on a smooth manifold of dimension $2n$ is isomorphic to the model structure on $T^*\mathbb{R}^n$. So in contrast to Riemannian metrics, there is a nice way to adapt local coordinates to a symplectic structure and in particular, symplectic structures cannot have any local invariants like curvature.

The situation is different for almost symplectic structures, since for an almost symplectic form ω on M, the exterior derivative $d\omega \in \Omega^3(M)$ defines a local invariant. So in a way the vanishing of the fundamental local invariant of an almost symplectic form is part of the definition of a symplectic form. In that sense, one might be tempted to compare symplectic structures to *flat* Riemannian metrics, i.e. metrics with vanishing Riemann curvature tensor. While those do admit canonical coordinates (defined by local isometries to $\mathbb{E}^n$), the situation is still fundamentally different in view of the sizes of the automorphism groups for the two examples.

Structures on a vector space and matrix groups

In both the examples discussed in Sections 1.2 and 1.3 we have defined some structure on a manifold by endowing each tangent space with some kind of additional data. This is exactly the general idea of a G-structure. We will next discuss what kind of additional data are appropriate for this as well as a large number of examples. Basically this is an issue of linear algebra, but as we shall see soon, we are naturally led to matrix groups here.

1.4. Examples. (1) An example of an additional structure on a manifold known from analysis on manifolds is an orientation. This comes from the concept of an orientation on a vector space V of dimension $n \geq 1$. Recall that to two ordered bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of V one canonically associates a matrix $A \in GL(n, \mathbb{R})$ that collects the coefficients needed to write the elements of $\mathcal{B}_2$ as linear combinations of the elements of $\mathcal{B}_1$. Calling the two ordered bases equivalent if $\det(A) > 0$, the set of all bases of V gets split into two equivalence classes, and an orientation on V is given by selecting one of these classes. So an orientation comes automatically with a (large) class of distinguished ordered bases for V.

(2) There are several further examples which admit an immediate description in terms of a class of distinguished ordered bases. For example, on an n-dimensional space V, we can consider, for some fixed $0 < k < n$, a distinguished k-dimensional subspace $W \subset V$. Associated to this is the class of all ordered bases for V for which the first k elements form a basis for W. This generalizes to so-called *flags*, i.e. sequences of $W_1 \subset W_2 \subset \cdots \subset W_j \subset V$ with $\dim(W_i) = k_i$ for fixed numbers j and $0 < k_1 < \cdots < k_j < n$.

Instead of just fixing one subspace $W \subset V$, we can also use a decomposition $V = W_1 \oplus W_2$ with $\dim(W_1) = k$ for some fixed k. The corresponding ordered bases are exactly those which are the union of a basis for W_1 and a basis for W_2. Of course, also this generalizes to more than two summands.

(3) From analysis on manifolds you also know the concept of a volume form on a manifold M of dimension n as a form $\nu \in \Omega^n(M)$ such that $\nu(x) \neq 0$ for all $x \in M$. The corresponding concept for a vector space V of dimension n simply is a nonzero element of $\Lambda^n V^*$, i.e. a non-zero alternating n-linear map $\alpha : V^n \to \mathbb{R}$. Similarly to an inner product or a non-degenerate skew symmetric bilinear map, such a map is unique up to isomorphism. This follows readily from the facts that $\dim(\Lambda^n V^*) = 1$ and that for $f \in GL(V)$ one gets $\alpha \circ f^n = \det(f)\alpha$, where $\det(f)$ can be obtained from any matrix representation of f.

Observe also that a non-zero element $\alpha \in \Lambda^n V^*$ determines an orientation on V. For an ordered basis $v_1, \dots, v_n$ of V, one has $\alpha(v_1, \dots, v_n) \neq 0$ and one defines the basis to be positively oriented if $\alpha(v_1, \dots, v_n) > 0$ and negatively oriented otherwise.

(4) A nice example of a structure on a vector space is "considering an even dimensional real vector space as a complex vector space". So let V be a real vector space of dimension $2n$. Then we define a *complex structure* on V as a linear map $J : V \to V$ such that $J \circ J = -\operatorname{id}$. Observe that the latter condition implies that $\det(J)^2 = (-1)^{\dim(V)}$, so such maps can only exist in even dimensions. Given J, one immediately verifies that $(a+ib)v := av+bJ(v)$ defines a map $\mathbb{C}\times V \to V$ which together with the given addition makes V into a complex vector space. This of course implies that V admits a complex basis making it isomorphic to $\mathbb{C}^n$. This in turn shows that for two such maps J and $\tilde{J}$ on V, there is $f \in GL(V)$ such that $\tilde{J} = f \circ J \circ f^{-1}$, so again there is a unique complex structure up to isomorphism.

In fact, this example is similar to (3) and to an inner product on V. Indeed, we can view J as an element of $L(V, V) \cong V^* \otimes V$ and hence as a tensor of some fixed "type" on V.

(5) There are several generalizations of the ideas in (2) and (4). For example, we can try to "view" a vector space of dimension mn as a tensor product of an n-dimensional and an m-dimensional space. Again, this can be easily described in terms of distinguished bases. For $V_1 \otimes V_2$, these are just the bases consisting of the tensor products of the elements of bases of the two spaces V_1 and V_2. Similarly, one can work with $L(V_1, V_2) \cong V_1^* \otimes V_2$, which leads to bases consisting of maps of rank one. Similarly one can "view" spaces of appropriate dimensions as symmetric or exterior powers of some smaller dimensional space, which again can be described in terms of an obvious class of bases.

(6) The examples from Sections 1.2 and 1.3 of course correspond to fixing an inner product on a vector space respectively a non-degenerate skew symmetric bilinear form on an even dimensional vector space. The significant differences between these two examples indicate that a general bilinear form on a vector space should not fall into the "structures on vector spaces" that we use. The problem here is that such bilinear forms fall into different classes, i.e. there are invariants that allow us to distinguish bilinear forms of different type. Indeed, if $b : V \times V \to \mathbb{R}$ is a bilinear form, we can decompose it as $b = b_s + b_a$, where b_s is symmetric and b_a is skew symmetric. Then both b_s and b_a have a well defined rank and b_s in addition has a well defined signature, and these certainly are invariants of b. These just give finitely many possible types, but in general the situation gets much worse. Assume for example that b_s is an inner product on V. Then there is a unique linear map $f : V \to V$ such that $b_a(v, w) = b_s(f(v), w)$ for all $v, w \in V$ and by construction f is skew symmetric with respect to b_s. This means that, passing to the complexification, f will be diagonalizable and the eigenvalues that occur are continuously varying invariants of b. So there are infinitely many different types of bilinear forms.

Observe that bilinear forms on V are in bijective correspondence with linear maps $V \to V^*$ (or $V^* \to V$) and non-degenerate bilinear forms correspond to linear isomorphisms here. So also prescribing a linear isomorphism $V \to V^*$ will not define a structure on V in our sense, since there are different types of such isomorphisms.

1.5. Structures and matrix groups. To arrive at the general definition of what we mean by a structure on a vector space, it is easier to first think about the "space of all structures of fixed type". A basic requirement here is that we can "transport" structures via linear isomorphisms between vector spaces. In particular, the set of all structures of some fixed type on a space V should be endowed with an action of the group $GL(V)$ of linear isomorphisms $V \to V$. Assuming that the structure is "unique up to isomorphism" then exactly means that this action is transitive. Fixing one of the structures, one obtains an isotropy group $G \subset GL(V)$ and acting on that fixed structure defines a bijection from the homogeneous space $GL(V)/G$ onto our "set of structures".

The group $GL(V)$ is an open subset in the vector space $L(V, V)$ and thus inherits a natural topology, which induces a quotient topology on $GL(V)/G$. This will be important, since we will want to talk about structures on tangent spaces depending continuously (or even smoothly) on the base point. Now for $GL(V)/G$ to be Hausdorff, we obviously need $G \subset GL(V)$ to be a closed subgroup. By [**LieGrp**][Theorem 1.11], this implies that G is a Lie subgroup of the Lie group $GL(V)$ and thus a Lie group. Identifying V with $\mathbb{R}^n$ with $n = \dim(V)$, $GL(V)$ gets identified with the group $GL(n, \mathbb{R})$ of invertible $n \times n$-matrices. Thus closed subgroups of $GL(V)$ are commonly called *matrix groups*. Moreover, [**LieGrp**][Theorem 1.16] shows that $GL(V)/G$ is canonically a smooth manifold, and the natural left action of $GL(V)$ on $GL(V)/G$ is smooth.

Thus we arrive at the point of view that a type of structures on a vector space V is described by a homogeneous space of $GL(V)$ or, after choosing one of the structures and thus a base-point in the homogeneous space, by a closed subgroup of $GL(V)$. The dependence on the base point is easy to understand using general facts about group actions: For two points in an orbit of a group action, it is well known that the corresponding isotropy groups are conjugate, so a different choice of base point replaces G by a conjugate subgroup of $GL(V)$.

EXAMPLE 1.5. It is easy to recast the examples we have discussed before in this language.

(0) The natural action of $GL(V)$ on bilinear forms on V is given by

$$(f \cdot b)(v, w) := b(f^{-1}(v), f^{-1}(w))$$

for a bilinear form $b : V \times V \to \mathbb{R}$ and $v, w \in V$. Linear algebra shows that this action is transitive on the subset of symmetric bilinear forms that are positive definite and on the subset of skew symmetric bilinear forms which are non-degenerate. In the first case, the stabilizer of b is just the orthogonal group $O(b) \subset GL(V)$, which is clearly closed. In particular, for the model example $(\mathbb{R}^n, \langle\ ,\ \rangle)$, we simply get $O(n) \subset GL(n, \mathbb{R})$ and the resulting interpretation of $GL(n, \mathbb{R})/O(n)$ as the space of all inner products on $\mathbb{R}^n$.

If b is non-degenerate and skew symmetric (and hence $\dim(V)$ is even) the stabilizer, which again is obviously closed, is commonly called the *symplectic group* determined by b and denoted by $Sp(b) \subset GL(V)$. For the model case of $\mathbb{R}^{2n}$ one obtains a group called $Sp(2n, \mathbb{R}) \subset GL(2n, \mathbb{R})$ and the corresponding homogeneous space of all non-degenerate, skew symmetric bilinear forms on $\mathbb{R}^{2n}$.

(1) The action of $f \in GL(V)$ on orientations maps the orientation represented by an ordered basis $v_1, \dots, v_n$ to the orientation determined by $f(v_1), \dots, f(v_n)$. This is easily seen to be well defined and the stabilizer of either of the two possible orientations is the subgroup $GL^+(V) \subset GL(V)$ consisting of all f such that $\det(f) > 0$. The homogeneous space $GL(V)/GL^+(V)$ can be identifies with the two-point sect $\{\pm 1\}$ with f acting by multiplication with the sign of $\det(f)$.

(2) For a vector space V and $k < n := \dim(V)$ let $Gr(k, V)$ be the set of k-dimensional subspaces of V. To obtain an action of $GL(V)$ on $Gr(k, V)$, we can either proceed via distinguished bases similarly as in (1). More easily, we just put $f \cdot W := f(W)$, the image of W under f, and then linear algebra shows that this action is transitive. For the model example $\mathbb{R}^k \subset \mathbb{R}^n$ the stabilizer is formed by the invertible matrices with block form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ with blocks of sizes k and $n-k$. This clearly is closed

and the diagonal blocks A and C have to be invertible. This realizes $Gr(k, \mathbb{R}^n)$ as a homogeneous space and thus as a smooth manifold. For more general flags as discussed in Example (2) in Section 1.4, on obtains an analogous description via matrices that are block-upper-triangular for finer block decompositions. This leads to realizations of the so called *flag manifolds*, i.e. the spaces of flags of some fixed type in $\mathbb{R}^n$, as homogeneous spaces of $GL(n, \mathbb{R})$.

(3) For multilinear alternating forms, the action of $f \in GL(V)$ is again given by $(f \cdot \alpha)(v_1, \dots, v_n) := \alpha(f^{-1}(v_1), \dots, f^{-1}(v_n))$. Linear Algebra then shows that $f \cdot \alpha = \det(f^{-1})\alpha$, so the stabilizer of any non-zero α is the subgroup $SL(V) \subset GL(V)$. In particular, the homogeneous space $GL(V)/SL(V)$ can be identified with the space of non-zero n-linear alternating maps $V^n \to \mathbb{R}$. The fact that a choice of any non-zero map α determines an orientation on V as observed in Section 1.4 is reflected in this picture by the fact that $SL(V) \subset GL^+(V)$.

(4) The natural action of $f \in GL(V)$ on $L(V, V)$ is given by $f \cdot h := f \circ h \circ f^{-1}$, which shows that the stabilizer of h simply is formed by those $f \in GL(V)$ which satisfy $f \circ h = h \circ f$. Of course, this action has many orbits (determined essentially by the real version of Jordan normal forms), but as discussed in Section 1.4 above, the maps J such that $J^2 = -\operatorname{id}_V$ (which exist only if $\dim(V)$ is even) form a single orbit. To obtain a model example, we take $\mathbb{C}^n$ viewed as the real vector space $\mathbb{R}^{2n}$ with $J(z) = iz$. Then $A \in GL(2n, \mathbb{R})$ commutes with J if and only if A is complex linear and thus lies in $GL(n, \mathbb{C})$. In matrices, this means that decomposing the real $2n \times 2n$-matrix A into blocks of size 2×2, any such block is of the form $\begin{pmatrix} u & -v \\ v & u \end{pmatrix}$. Hence $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$ is a closed subgroup and the corresponding homogeneous space can be identified with the space of all complex structures on $\mathbb{R}^{2n}$.

(5) In the situations discussed in Example (5) of Section 1.4 the easiest way is to directly describe the corresponding subgroup. Let us first do this in the situation of linear maps, and thus consider the space $L(V, W)$ for finite dimensional vector spaces V and W. To $(f, g) \in GL(V) \times GL(W)$ one associates a linear map $L(V, W) \to L(V, W)$ by $h \mapsto g \circ h \circ f^{-1}$. Clearly (f^{-1}, g^{-1}) induces an inverse to this map, so we actually obtain a homomorphism $GL(V) \times GL(W) \to GL(L(V, W))$. It turns out that image of this homomorphism is a closed subgroup $G \subset GL(L(V, W))$, so it can be viewed as an isotropy subgroup in the above sense:

The key to this is an alternative characterization of the image. This is based on the subset $\mathcal{C} \subset L(V, W)$ consisting of all linear maps of rank 1. Linear algebra immediately implies that for $(f, g) \in G$ and $h \in \mathcal{C}$ we get $g \circ h \circ f^{-1} \in \mathcal{C}$. Conversely, one shows that a map $F \in GL(L(V, W))$ that has the property that $F(h) \in \mathcal{C}$ for any $h \in \mathcal{C}$ must actually be contained in G. Passing to matrices, rank one can be characterized by vanishing of the determinants of all 2×2-minors of a matrix, so $\mathcal{C} \subset L(V, W)$ is a closed subset. But then continuity of the evaluation map $GL(L(V, W)) \times L(V, W) \to L(V, W)$ that sends (F, h) to $F(h)$ shows that for fixed $h \in \mathcal{C}$ the set $\{F \in GL(L(V, W)) : F(h) \in \mathcal{C}\}$ is closed, too. Since G is the intersection of these sets over all $h \in \mathcal{C}$, it is a closed subset of $GL(L(V, W))$.

Similar arguments apply to $V \otimes W$, where to $(f, g) \in GL(V) \times GL(W)$, one associates the map $f \otimes g$ defined by $(f \otimes g)(v \otimes w) := f(v) \otimes g(w)$. Again, this defines a homomorphism $GL(V) \times GL(W) \to GL(V \otimes W)$ whose image G is the relevant subgroup in this case. The proof that G is closed is similar as before, here it is based on the subset $\mathcal{C} := \{v \otimes w : v \in V, w \in W\} \subset V \otimes W$ of decomposable tensors.

1.6. G-structures on vector spaces. We can take these ideas even further and *define* a type of structure on vector spaces (of a fixed dimension n) by fixing a subgroup $G_0 \subset GL(V_0)$ for a vector space V_0 with $\dim(V_0) = n$. The idea is that we interpret this a having a "model structure" on V_0 which is preserved by a linear isomorphism $f : V_0 \to V_0$ if and only if $f \in G_0$. The general principles from Section 1.5 then imply that we should be able to realize any structure of that type on an n-dimensional vector space V by a linear isomorphism $\varphi : V_0 \to V$. Of course, this linear isomorphism should not be unique, for example, for any $f \in G_0$, $\varphi \circ f : V_0 \to V$ should induce the same structure on V. This however is the only freedom we should allow: If $\psi : V_0 \to V$ is another linear isomorphism, then $f := \varphi^{-1} \circ \psi : V_0 \to V_0$ is a linear isomorphism and if ψ induces the same structure on V as φ, the this linear isomorphism should preserve the model structure and hence lie in G_0. But this shows that $\psi = \varphi \circ f$ for some $f \in G_0$. This motivates the following definition:

Definition 1.6. Let $G_0 \subset GL(V_0)$ be a closed subgroup and put $n := \dim(V_0)$. Then a G_0*–structure* on an n-dimensional vector space V is given by a set $\mathcal{F} \subset L(V_0, V)$ of linear isomorphisms $\varphi : V_0 \to V$ such that for one (or equivalently any) $\varphi_0 \in \mathcal{F}$ the map $G_0 \to \mathcal{F}$ defined by $f \mapsto \varphi_0 \circ f$ is bijective.

The class $\mathcal{F}$ in this definition can be identified with G_0 after the choice of a base-point. Observe, however, that $\mathcal{F}$ does not carry a natural group structure. This should be compared to the notion of an affine space modelled on a vector space. The most natural point of view here is that we have got a right action $\mathcal{F} \times G_0 \to \mathcal{F}$, which is transitive and *free*, i.e. each point has trivial isotropy group. Such an object is sometimes called a *principal homogeneous space* of G_0. The connection of this point of view to what we did in Section 1.5 is easy: Given a G_0–structure $\mathcal{F}$ on V, we can take $\varphi \in \mathcal{F}$. Then $f \mapsto \varphi \circ f \circ \varphi^{-1}$ defines an isomorphism $\varphi_* : GL(V_0) \to GL(V)$ of Lie groups and hence $G := \varphi_*(G_0)$ is a closed subgroup of $GL(V)$. Evidently, replacing φ by $\varphi \circ g$ for $g \in G_0$ does not change the resulting subgroup G, so it depends only on $\mathcal{F}$.

The concept of an "underlying structure" or a "weaker structure" has a nice interpretation in this approach. Recall from Examples 1.4 and 1.5 that the choice of a non-zero element $\alpha \in \Lambda^n V^*$ for an n-dimensional vector space V also defines an orientation on V and that this is reflected in the isotropy groups via $SL(V) \subset GL^+(V)$. Now suppose in general that we have closed subgroups $G_0 \subset \tilde{G}_0 \subset GL(V_0)$ with $\dim(V_0) = n$ and that $\mathcal{F}$ is a G_0-structure on an n-dimensional vector space V. Then we define a set $\tilde{\mathcal{F}}$ of linear automorphisms $V_0 \to V$ as $\tilde{\mathcal{F}} := \{\varphi \circ \tilde{f} : \varphi \in \mathcal{F}, \tilde{f} \in \tilde{G}_0\}$. Using that $G_0 \subset \tilde{G}_0$, one immediately verifies that this indeed defines a $\tilde{G}_0$-structure on V, which we can view as the $\tilde{G}_0$-structure underlying the G_0-structure $\mathcal{F}$.

This occurs rather frequently, say in the fact that viewing a complex linear automorphism of $\mathbb{C}^n$ as a real linear automorphism of $\mathbb{R}^{2n}$, it always has positive determinant. Alternatively, $GL(n, \mathbb{C})$ is connected, so it has to be contained in $GL^+(2n, \mathbb{R}) \subset GL(2n, \mathbb{R})$. Hence also a complex structure on a real vector space V of dimension $2n$ defines an orientation on V. Similarly, this applies to a symplectic structure, and so on.

Our definition of a G_0–structure nicely connects to the point of view of distinguished (ordered) bases that we have met already. To have this for G_0–structures with $G_0 \subset GL(V_0)$, we only need one fixed basis $\mathcal{B}$ for V_0 that is compatible with our model structure. This will usually be implemented in such a way that V_0 is a space that has a "standard basis" (for example $\mathbb{R}^n$ or $\mathbb{R}^k \otimes \mathbb{R}^\ell$) and the group G_0 is chosen in such a way that this standard basis is compatible with the model structure. For a linear isomorphism $\varphi : V_0 \to V$ the images of our basis vectors form a basis $f(\mathcal{B})$ of V. Hence for a G_0–structure $\mathcal{F}$ on V, the corresponding set of distinguished basis is then the set of all $\varphi(\mathcal{B})$ for $\varphi \in \mathcal{F}$.

The different points of view introduced here fit together very nicely. For example, an inner product b on an n-dimensional vector space V defines an $O(n)$-structure by

$$\mathcal{F} := \{\varphi \in L(\mathbb{R}^n, V) : \forall x, y \in \mathbb{R}^n : b(\varphi(x), \varphi(y)) = \langle x, y\rangle\}.$$

Observe that is suffices to check the condition in the definition of $\mathcal{F}$ for all pairs of elements of a basis of $\mathbb{R}^n$. Hence $\varphi \in \mathcal{F}$ if and only if φ maps the standard basis of $\mathbb{R}^n$ to a basis which is orthonormal for b, and the distinguished bases obtained from $\mathcal{F}$ are exactly the orthonormal bases. Conversely, given an $O(n)$-structure $\mathcal{F}$ on V, we can take $\varphi \in \mathcal{F}$ and define an inner product on V by $b(v, w) := \langle \varphi^{-1}(v), \varphi^{-1}(w)\rangle$. Since any other element of $\mathcal{F}$ is of the form $\varphi \circ f$ for $f \in O(n)$, they all lead to the same inner product on V.

CHAPTER 2

Structures on manifolds

Having an appropriate notion of a structure on a vector space, we can now follow our program of defining a corresponding structure on a manifold via putting a structure on each tangent space. This should be done in such a way that the structure depends smoothly on the base point. It will be relatively easy to set this up via an alternative description of the tangent bundle via the so-called linear frame bundle. To do this, we first have to introduce the language of fiber bundles, which is fundamental in many areas of differential geometry.

Bundles

2.1. Fiber bundles. Recall that the tangent bundle TM of an n-dimensional manifold locally "looks like" a product of the manifold with the vector space $\mathbb{R}^n$. Generalizing this idea leads to the general concept of a fiber bundle.

Definition 2.1. (1) Let M and S be smooth manifolds. A *fiber bundle over* M with *standard fiber* S is given by a smooth manifold E (the *total space* of the bundle) and a smooth map $p : E \to M$ (the *bundle projection*) such that for each $x \in M$ there is an open subset $U \subset M$ with $x \in U$ and a diffeomorphism $\varphi : p^{-1}(U) \to U \times S$ such that $\mathrm{pr}_1 \circ \varphi = p$. For a point $x \in M$, the *fiber of* E *over* x is the subset $E_x := p^{-1}(\{x\}) \subset E$.
Such a diffeomorphism φ is called a *fiber bundle chart* for E.

(2) A *section* of a fiber bundle $p : E \to M$ is a smooth map $s : M \to E$ such that $p \circ s = \mathrm{id}_M$. A *local section* of $p : E \to M$ defined on an open subset $U \subset M$ is a smooth map $s : U \to E$ such that $p \circ s = \mathrm{id}_U$.

(3) A *morphism* between two fiber bundles $p : E \to M$ and $\tilde{p} : \tilde{E} \to \tilde{M}$ is a smooth map $F : E \to \tilde{E}$, which maps fibers to fibers. This means that there is a map $f : M \to \tilde{M}$ (the *base map* of F) such that $\tilde{p} \circ F = f \circ p$, so $F(E_x) \subset \tilde{E}_{f(x)}$. A morphism $F : E \to \tilde{E}$ is called an *isomorphism of fiber bundles* if there is a morphism $G : \tilde{E} \to E$ such that $G \circ F = \mathrm{id}_E$ and $F \circ G = \mathrm{id}_{\tilde{E}}$.

Observe that any fiber bundle $p : E \to M$ does admit local smooth sections. Indeed for a fiber bundle chart $\varphi : p^{-1}(U) \to U \times S$ and any smooth function $f : U \to S$, $s(x) := \varphi^{-1}(x, f(x))$ is a local smooth section defined on U. It is even true that for any $x \in M$ and any $y \in E_x$, there is a local section s such that $s(x) = y$. Taking this section and differentiating the equation $p \circ s = \mathrm{id}$ in x, we conclude that $T_y p : T_y E \to T_x M$ is surjective, so p is a surjective submersion. This in turn implies that each of the fibers $E_x \subset E$ is a smooth submanifold of E that is diffeomorphic to S via the restriction of any fiber bundle chart. Moreover, for a morphism $F : E \to \tilde{E}$ with base map $f : M \to \tilde{M}$, $\tilde{p} \circ F = f \circ p$ shows that $f \circ p$ is smooth which implies smoothness of f. In particular, both an isomorphism of fiber bundles and its base map are diffeomorphisms.

Observe that at the current stage there is no need to impose a compatibility condition between fiber bundle charts, this is already being taken care of by the requirement that any fiber bundle chart is a diffeomorphism. There is an obvious analog of the a chart change, however, which is usually referred to as *transition functions*: Suppose that $\varphi_\alpha : p^{-1}(U_\alpha) : U_\alpha \times S$ and $\varphi_\beta : p^{-1}(U_\beta) \to U_\beta \times S$ are fiber bundle charts for $p : E \to M$ such that $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$. Then both φ_α and φ_β restrict to diffeomorphism $p^{-1}(U_{\alpha\beta}) \to U_{\alpha\beta} \times S$ and $\varphi_\alpha \circ \varphi_\beta^{-1} : U_{\alpha\beta} \times S \to U_{\alpha\beta} \times S$ must be of the form $(x, z) \mapsto (x, \varphi_{\alpha\beta}(x, z))$ for a smooth function $\varphi_{\alpha\beta} : U_{\alpha\beta} \times S \to S$ which has the property that for each $x \in U_{\alpha\beta}$, the map $z \mapsto \varphi_{\alpha\beta}(x, z)$ is a diffeomorphism $S \to S$. We will soon require these transition functions to have a special form to define special classes of fiber bundles.

Example 2.1. (1) For arbitrary manifolds M and S, $\mathrm{pr}_1 : M \times S \to M$ is a fiber bundle with standard fiber S. A fiber bundle $p : E \to M$ is called *trivial* if it is isomorphic to such a product bundle. By definition, any fiber bundle therefore is *locally trivial* and therefore fiber bundle charts are also called *local trivializations*.

(2) For a smooth manifold M of dimension n, the tangent bundle $p : TM \to M$ is a fiber bundle with fiber $\mathbb{R}^n$. Indeed, for a local chart (U, u) on M, $Tu : p^{-1}(U) \to u(U) \times \mathbb{R}^n$ is a diffeomorphism with $\mathrm{pr}_1 \circ Tu = u \circ p$. Thus $(u^{-1} \times \mathrm{id}_{\mathbb{R}^n}) \circ Tu : p^{-1}(U) \to U \times \mathbb{R}^n$ is a vector bundle chart. Observe that the transition functions between two such fiber bundle charts have the form $(x, v) \mapsto (x, (T_x u_\alpha \circ (T_x u_\beta)^{-1})(v))$, so these are linear in the second variable.

If G is a Lie group, the usual left trivialization $TG \to G \times \mathfrak{g}$ shows that the fiber bundle TG is trivial. However, the hairy ball theorem shows that TS^2 cannot be isomorphic to $S^2 \times \mathbb{R}^2$, so tangent bundles are non-trivial in general.

(3) Let G be a Lie group, $H \subset G$ a closed subgroup and G/H the corresponding homogeneous space. Then we claim that the canonical projection $p : G \to G/H$ is a smooth fiber bundle with fiber H. Indeed, in the standard proof that G/H is a smooth manifold (see e.g. Theorem 1.16 in [**LieGrp**]), one actually constructs an open neighborhood U of eH in G/H and a smooth map $\sigma : U \to G$ such that $(x, h) \mapsto \sigma(x)h$ is a diffeomorphism $U \times H \to p^{-1}(U)$ and $p(\sigma(x)h) = x$. Hence its inverse is a fiber bundle chart around eH. For $g \in G$, let $\ell_g : G/H \to G/H$ be the diffeomorphism defined by $\ell_g(\tilde{g}H) = g\tilde{g}H$. Then one puts $U_g := \ell_g(U)$ and defines $\sigma_g : U_g \to G$ by $\sigma_g(y) = g\sigma(\ell_{g^{-1}}(y))$, which in the same way leads to a fiber bundle chart around gH. Observe that for $y \in U_g \cap U_{\tilde{g}}$, we obtain $\sigma_g(y)H = \sigma_{\tilde{g}}(y)H$ and thus $\psi(y) := \sigma_{\tilde{g}}(y)^{-1}\sigma_g(y) \in H$. Of course, this defines a smooth function $\psi : U_g \cap U_{\tilde{g}} \to H$ such that $\sigma_g(y)h = \sigma_{\tilde{g}}(y)(\psi(y)h)$. Hence the transition function between the two corresponding fiber bundle chart has the form $(y, h) \mapsto (y, \psi(y)h)$ for a smooth function ψ with values in H.

(4) There are various constructions with fiber bundles. For example if $p_i : E_i \to M_i$ is a fiber bundle with typical fiber S_i for $i = 1, 2$, then we can consider $p_1 \times p_2 : E_1 \times E_2 \to M_1 \times M_2$. One immediately verifies that fiber bundle charts of the factors can be combined to fiber bundle charts on the product, so this is a fiber bundle with typical fiber $S_1 \times S_2$. For bundles over the same base, there is a simple variation of this construction. Given $p_i : E_i \to M$, we define the *fibered product* $E_1 \times_M E_2 \subset E_1 \times E_2$ as the set $\{(u_1, u_2) : p_1(u_1) = p_2(u_2)\}$. Putting $p(u_1, u_2) := p_1(u_1) = p_2(u_2)$ we obtain a map $p : E_1 \times_M E_2 \to M$. Now we can take local fiber bundle charts for the factors defined over the same open subset $U \subset M$, i.e. $\varphi_1 : p_1^{-1}(U) \to U \times S_1$ and $\varphi_2 : p_2^{-1}(U) \to U \times S_2$ and the product chart which maps $(p_1 \times p_2)^{-1}(U)$ to $U \times U \times S_1 \times S_2$. Now by construction $(p_1 \times p_2)^{-1}(U) \cap (E_1 \times_M E_2) = p^{-1}(U)$ and $\varphi_1 \times \varphi_2$ restricts to a bijection from $p^{-1}(U)$ onto the subset of $U \times U \times S_1 \times S_2$ consisting of all points of the form (x, x, z_1, z_2) with $x \in U$ and $z_i \in S_i$. This can be used to construct submanifold charts, so $E_1 \times_M E_2$ is a smooth submanifold in $E_1 \times E_2$ and the restriction of $\varphi_1 \times \varphi_2$ to this can be viewed as

a fiber bundle chart (forgetting the redundant component). Thus $p : E_1 \times_M E_2 \to M$ is a smooth fiber bundle with typical fiber $S_1 \times S_2$.

2.2. Constructing fiber bundles via atlases. Defining manifolds, one usually starts with a topological space and then defines charts to be homeomorphisms. Alternatively, it is also possible to also define the topology via charts, see Lemma 1.6 of [**AnaMF**]. A similar approach works for fiber bundles:

LEMMA 2.2. *For smooth manifolds M and S, let E be a set and $p : E \to M$ a set map. Suppose that there is an open covering $\{U_\alpha : \alpha \in I\}$ of M together with bijective maps $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times S$ such that $\mathrm{pr}_1 \circ \varphi_\alpha = p|_{p^{-1}(U_\alpha)}$. Suppose further that for each $\alpha, \beta \in I$ such that $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ the map $(\varphi_\alpha \circ \varphi_\beta^{-1})|_{U_{\alpha\beta}\times S} : U_{\alpha\beta} \times S \to U_{\alpha\beta} \times S$ is a diffeomorphism.*

Then E can be uniquely made into a smooth manifold in such a way that $\{(U_\alpha, \varphi_\alpha)\}$ is a fiber bundle atlas.

SKETCH OF PROOF. Starting from a fixed atlas $\mathcal{A}$ for M, we can consider the intersections of the domains of the charts in $\mathcal{A}$ with the sets U_α and then pass to a countable subcovering of M. Since fiber bundle charts can clearly be restricted to open subsets of their domains, we may assume that we start from a countable atlas $\{(V_i, v_i)\}$ for M and from fiber bundle charts $\varphi_i : p^{-1}(V_i) \to V_i \times S$ such that each V_i is contained in some U_α and φ_i is the restriction of φ_α to $p^{-1}(V_i) \subset p^{-1}(U_\alpha)$. Observe that by construction this implies that also $\varphi_{i_1} \circ (\varphi_{i_2})^{-1}$ is a diffeomorphism from $(V_{i_1} \cap V_{i_2}) \times S$ to itself.

Next, let $\{(W_j, w_j)\}$ be a countable atlas for S and define $U_{(i,j)} := \varphi_i^{-1}(V_i \times W_j) \subset E$ and $u_{(i,j)} := (v_i \times w_j) \circ \varphi_i|_{U_{(i,j)}}$. By construction each $u_{(i,j)}$ is a bijection from $U_{(i,j)}$ onto the open subset $v_i(V_i) \times w_j(W_j)$ of $\mathbb{R}^N$, where $N = \dim(M) + \dim(S)$. Assuming that we have indices such that $U_{(i,j)} \cap U_{(i',j')} \neq \emptyset$, we must have $V_{ii'} = V_i \cap V_{i'} \neq \emptyset$ and the intersection is contained in $p^{-1}(V_{ii'})$. Indeed, in there it coincides with $\varphi_i^{-1}(V_{ii'} \times W_j) \cap \varphi_{i'}^{-1}(V_{ii'} \times W_{j'})$. But φ_i maps this intersection bijectively onto $(V_{ii'} \times W_j) \cap (\varphi_i \circ (\varphi_{i'})^{-1})(V_{ii'} \cap W_{j'})$. This is certainly open in $V_{ii'} \times S$, so we conclude that $u_{(i,j)}(U_{(i,j)} \cap U_{(i',j')})$ is open in $\mathbb{R}^N$. But then it follows from the construction that $u_{(i',j')} \circ (u_{(i,j)})^{-1}$ defines a smooth map from this open subset to $\mathbb{R}^N$.

Now we can proceed as in the proof of Lemma 1.6 of [**AnaMF**] to obtain a topology on E by declaring each $u_{(i,j)}$ to be a homeomorphism. Since we start for a countable family of $u_{(i,j)}$, this topology is second countable, and one easily verifies directly that it is Hausdorff. Then E can be uniquely made into a smooth manifold such that $\{(U_{(i,j)}, u_{(i,j)})\}$ is a smooth atlas for E. Having verified this, we immediately conclude that $p : E \to M$ is smooth and that each φ_i is a diffeomorphism. But since the φ_i are restrictions of the initial maps φ_α, we conclude that also each φ_α is a diffeomorphism, which completes the proof. □

2.3. Vector bundles. We can now impose restrictions on transition functions to define special classes of fiber bundles. In principle, one does this by first defining atlases, then equivalence of atlases and use equivalence classes of atlases in the definition. However, for the examples of primary importance for us (vector bundles and principal fiber bundles) there is a nice notion of morphism, which automatically takes care of equivalence.

DEFINITION 2.3. (1) Let $p : E \to M$ be a fiber bundle whose standard fiber is a finite dimensional $\mathbb{K}$-vector space V, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. A *vector bundle atlas* for E is a family $(U_\alpha, \varphi_\alpha)$ of fiber bundle charts for E such that $M = \cup_\alpha U_\alpha$ and for each pair of indices α, β such that $U_{\alpha\beta} \neq \emptyset$, the chart change $\varphi_\alpha \circ (\varphi_\beta)^{-1} : U_{\alpha\beta} \times V \to U_{\alpha\beta} \times V$ has the form $(x,v)\mapsto(x,\varphi_{\alpha\beta}(x,v))$, where $\varphi_{\alpha\beta}:U_{\alpha\beta}\times V\to V$ is $\mathbb{K}$-linear in the second variable.

(2) A *vector bundle* (for $\mathbb{K}=\mathbb{R}$) respectively a *complex vector bundle* (for $\mathbb{K}=\mathbb{C}$) is a fiber bundle $p:E\to M$ as in (1) endowed with an equivalence class of vector bundle atlases (in the obvious sense).

Let $p:E\to M$ be a vector bundle with standard fiber V. Then for each $x\in M$ we can consider the fiber $E_x:=p^{-1}(\{x\})\subset E$. Then for any chart $(U_\alpha,\varphi_\alpha)$ with $x\in U_\alpha$ the second component of the restriction $\varphi_\alpha|_{E_x}$ defines a bijection $E_x\to V$. We can make E_x into a vector space isomorphic to V by declaring this to be a linear isomorphism and the result is independent of the choice of the chart in our atlas by definition. Observe that for an open subset $U\subset M$ and a vector bundle $p:E\to M$ with fiber V, we can consider $E|_U:=p^{-1}(U)\to U$. Starting form a vector bundle atlas $\{(U_\alpha,\varphi_\alpha)\}$ for E, we form $(U\cap U_\alpha,\varphi_\alpha|_{p^{-1}(U\cap U_\alpha)})$ and one immediately verifies that this canonically makes $E|_U$ into a vector bundle over U.

The vector space structure on the fibers of a vector bundle $p:E\to M$ also allows us to add sections and multiply them by smooth functions, with both operations being defined point-wise. Locally, any vector bundle has many sections, since in the domain of a chart $(U_\alpha,\varphi_\alpha)$ there is an isomorphism $\Gamma(E|_{U_\alpha})\cong C^\infty(U_\alpha,V)$ which sends a section $\sigma:U_\alpha\to p^{-1}(U_\alpha)$ to the second component of $\varphi_\alpha\circ\sigma$. Multiplying a local section $\sigma:U\to E$ by a bump function f with support contained in U, we can extend $f\sigma$ by 0 to a section defined on all of M. More generally, we can glue local sections using partitions of unity, so any vector bundle has many global sections.

For two vector bundles $p:E\to M$ and $\tilde p:\tilde E\to\tilde M$, consider a morphism $F:E\to\tilde E$ of fiber bundles with base map $f:M\to\tilde M$. Then for each $x\in M$, F restricts to a map $E_x\to\tilde E_{f(x)}$ and we call F a *vector bundle homomorphism* if all these restrictions are linear maps. An *isomorphism* of vector bundles is a homomorphism of vector bundles that admits an inverse homomorphism. Evidently, the base map of such an isomorphism is a diffeomorphism.

The simplest vector bundles with fiber V are products, i.e. $\mathrm{pr}_1:M\times V\to M$ for any manifold M, with the (M,id) as a vector bundle atlas. A vector bundle $p:E\to M$ is called trivial if it is isomorphic to $M\times V$. A *vector bundle chart* for $p:E\to M$ is then a pair (U,φ), where $U\subset M$ is open and φ is an isomorphism $E|_U\to U\times V$ is an isomorphism of vector bundles.

The following result is typical for how one works with vector bundles.

PROPOSITION 2.3. *Let $p:E\to M$ and $\tilde p:\tilde E\to\tilde M$ be vector bundles and let $F:E\to\tilde E$ be a homomorphism of vector bundles with base map f. Suppose that f is a diffeomorphism and that for each $x\in M$, the restriction $F|_{E_x}:E_x\to\tilde E_{f(x)}$ is a linear isomorphism. Then F is an isomorphism of vector bundles.*

PROOF. Take any $w\in\tilde E$ and put $y:=\tilde p(w)$. Then by assumption, there is a unique element $v\in E_{f^{-1}(y)}$ such that $F(v)=w$ and this is the only element of E that gets mapped to w by F. Putting $G(w):=v$ defines a map $G:\tilde E\to E$ such that $p\circ G=f^{-1}\circ\tilde p$ and such that $G|_{\tilde E_y}$ is inverse to $F|_{E_{f^{-1}(y)}}$ and thus is linear. Hence it only remains to show that G is smooth in order to complete the proof. This can be done locally, so let us fix $y\in\tilde M$ and vector bundle charts (U,φ) for E with $f^{-1}(y)\in U$ and $(\tilde U,\tilde\varphi)$ for $\tilde E$ with $y\in\tilde U$. Passing to appropriate open subsets, we may assume that f restricts to a diffeomorphism $U\to\tilde U$.

Since both bundles have the same fiber dimension, we can fix linear isomorphisms between their standard fibers and $\mathbb{R}^n$ and assume that they both have standard fiber $\mathbb{R}^n$.

Then $\tilde{\varphi} \circ F \circ \varphi^{-1} : U \times \mathbb{R}^n \to \tilde{U} \times \mathbb{R}^n$ must be of the form $(x, v) \mapsto (f(x), \Phi(x, v))$, for a smooth map $\Phi : U \times \mathbb{R}^n \to \mathbb{R}^n$ which is linear in the second variable. Defining $\Psi : U \to M_n(\mathbb{R})$ by $\Psi(x)v := \Phi(x, v)$, we obtain a smooth function Ψ which by assumption has values in $GL(n, \mathbb{R})$. But then also $(y, w) \mapsto (f^{-1}(y), \Psi(f^{-1}(y))^{-1}w)$ defines a smooth map $\tilde{U} \times \mathbb{R}^n \to U \times \mathbb{R}^n$ and by construction, this has to equal $\varphi \circ G \circ \tilde{\varphi}^{-1}$. This shows that G is smooth on a neighborhood of y and this completes the proof. $\square$

We have actually verified in Example 2.1 (2) that for any smooth manifold M, the tangent bundle TM is a vector bundle. Similarly, the cotangent bundle and all tensor bundles are vector bundles over M. For a smooth map $f : M \to N$, the tangent map $Tf : TM \to TN$ is a vector bundle homomorphism with base map f and similarly for induced maps on tensor bundles.

Observe that for vector bundles $E \to M$ and $\tilde{E} \to M$ with standard fibers V and $\tilde{V}$ over the same manifold M, the fibered product $E \times_M \tilde{E}$ by construction is again a vector bundle with standard fiber $V \times \tilde{V}$. Since this product is commonly written as $V \oplus \tilde{V}$, one writes $E \oplus \tilde{E}$ for this fibered product and calls it the *Whitney sum* of the vector bundles E and $\tilde{E}$. Of course, this extends to more than two factors without problems.

There is a large number of similar constructions for vector bundles over a fixed base manifold. Loosely speaking, any functorial construction for finite dimensional vector spaces extends to vector bundles. We only discuss this briefly here, since we will only need it in a specific situation for which a simpler description of these constructions is available. For example, given a vector bundle $p : E \to M$ we consider the disjoint union $E^* := \sqcup_{x \in M}(E_x)^*$ of dual spaces to the fibers of E which comes with a projection $q : E^* \to M$. A vector bundle chart $\varphi : p^{-1}(U) \to U \times \mathbb{R}^n$ for E then gives rise to a bijection $\psi : q^{-1}(U) \to U \times \mathbb{R}^{n*}$ characterized by $\psi^{-1}(x, \lambda)(w) = \lambda(v)$, where $w \in E_x$ and $\varphi(w) = (x, v)$. Using Lemma 2.2, one easily shows that $q : E^* \to M$ is a vector bundle, called the *dual vector bundle* to E. Similarly we can work with tensor powers of the fibers for some $k \in \mathbb{N}$ and put $\otimes^k E := \sqcup_{x \in M} \otimes^k E_x$. Again, vector bundle charts for E can be used to construct bijections to $U \times \otimes^k \mathbb{R}^n$ and using Lemma 2.2 one shows that $\otimes^k E \to M$ is a vector bundle. This is the kth tensor power of E and the symmetric power $S^k E$ and the alternating power $\Lambda^k E$ are obtained similarly.

For two vector bundles E and F over M, one defines $E \otimes F := \sqcup_{x \in M} E_x \otimes F_x$. In a similar way as above, one then uses vector bundle charts for E and F to construct bijections which can be used to apply Lemma 2.2 to make $E \otimes F$ into a vector bundle over M. Analogously, this works for $L(E, F) := \sqcup_{x \in M} L(E_x, F_x)$, and sections of this bundle are easily seen to be in bijective correspondence with vector bundle homomorphism $E \to F$ with base map id_M in an evident way. Finally, natural isomorphism between different constructions for vector spaces carry over to vector bundles. For example, $\otimes^k(E^*) \cong (\otimes^k E)^*$, $(E \otimes F)^* \cong E^* \otimes F^*$, $L(E, F) \cong E^* \otimes F \cong L(F^*, E^*)$, and so on.

2.4. Principal fiber bundles. The second class of special fiber bundles that we will look at is less intuitive than vector bundles, but turns out to be very flexible. Although these bundles have standard fibers that are Lie groups, it is not the group structure that matters but rather the structure of a principal homogeneous space that we have met in Section 1.6.

Definition 2.4. (1) Let $p : P \to M$ be a fiber bundle, whose standard fiber is a Lie group G. A *principal bundle atlas* for P is a collection $(U_\alpha, \varphi_\alpha)$ of fiber bundle charts for P such that $M = \cup_\alpha U_\alpha$ and for each pair of indices α, β such that $U_{\alpha\beta} \neq \emptyset$, the chart change $\varphi_\alpha \circ (\varphi_\beta)^{-1} : U_{\alpha\beta} \times G \to U_{\alpha\beta} \times G$ has the form $(x, g) \mapsto (x, \varphi_{\alpha\beta}(x) \cdot g)$ for a smooth function $\varphi_{\alpha\beta} : U_{\alpha\beta} \to G$ with the dot denoting multiplication in G.

(2) A *principal fiber bundle with structure group G* (or a *principal G-bundle*) is a fiber bundle $p : P \to M$ as in (1) endowed with an equivalence class of vector bundle atlases (in the obvious sense).

Let us observe right away that the fiber P_x of a principal G-bundle is diffeomorphic to G but cannot be made into a Lie group in a canonical way. This is because left translations in a Lie group are not group homomorphism, so multiplication in a chart does not define a chart independent operation. However, one can make the fibers into principal homogeneous spaces for G in a nice way:

LEMMA 2.4. *Let $p : P \to M$ be a principal G-bundle and let $\mathfrak{g}$ be the Lie algebra of G. Then we have*

(1) There is a unique smooth right action $r : P \times G \to P$ of G on P which is induced by multiplication from the right in principal bundle charts. This action is free and its orbits are the fibers P_x of P.

(2) There exists a smooth map $\tau : P \times_M P \to G$ which is characterized by the fact that for $u, \tilde{u} \in P$ with $p(u) = p(\tilde{u})$, we get $\tilde{u} = r(u, \tau(u, \tilde{u}))$.

(3) For any $X \in \mathfrak{g}$, $\zeta_X(u) := \frac{d}{dt}|_{t=0} r(u, \exp(tX)) \in T_uP$ defines a smooth vector field $\zeta_X \in \mathfrak{X}(P)$. For each $u \in P$, the map $X \mapsto \zeta_X(u)$ defines a linear isomorphism from $\mathfrak{g}$ onto the vertical subspace $V_uP := \ker(T_up) \subset T_uP$.

PROOF. (1) Given $u \in P_x$ and $g \in G$, choose a principal bundle chart $(U_\alpha, \varphi_\alpha)$ with $x \in U_\alpha$. Define $r(u, g) := \varphi_\alpha^{-1}(x, hg)$, where $\varphi_\alpha(u) = (x, h) \in U_\alpha \times G$. This evidently defines a smooth map $p^{-1}(U_\alpha) \times G \to p^{-1}(U_\alpha)$. Since left and right translations in a Lie group commute, we conclude that the definition does not depend on the choice of the principal bundle chart containing x. In particular, the locally defined maps fit together to define a smooth map $r : P \times G \to P$, and one immediately verifies that this is a right action.

From the construction it is also obvious that the orbit $u \cdot G := \{r(u, g) : g \in G\}$ is exactly the fiber P_x. To prove freeness of the action, we have to show that if $r(u, g) = u$ for one $u \in P$, then $g = e$, but again this is obvious from the construction.

(2) We have just seen that for u and $\tilde{u}$ with $p(u) = p(\tilde{u})$, there is a unique element $g \in G$ such that $\tilde{u} = r(u, g)$. Thus there is a unique map τ as claimed and we only have to prove that this is smooth. This is a local question, so given $x \in M$ choose a principal bundle chart (U, φ) for P with $x \in U$. From the construction in Example (4) of Section 2.1, we see that the induced bundle chart on $P \times_M P$ is defined on the set of pairs $(u, \tilde{u})$ such that $x := p(u) = p(\tilde{u}) \in U$ and such a pair gets mapped to $(x, g, \tilde{g})$ where $\varphi(u) = (x, g)$ and $\varphi(\tilde{u}) = (x, \tilde{g})$. But then by construction $\tau(u, \tilde{u}) = g^{-1}\tilde{g}$ and since multiplication and inversion in G are smooth, also τ is smooth.

(3) Smoothness of the map r from (1) implies smoothness of its tangent map $Tr : TP \times TG \to TP$. Now both the zero vector field on P and the constant map $P \to \mathfrak{g} = T_eG$ that sends each $u \in P$ to X are smooth. Hence we obtain a smooth map $P \to TP$ by sending u to $T_{(u,e)}r(0_u, X)$, and since this lies in $T_{r(u,e)}P = T_uP$, it actually defines a vector field. But since the curve $c(t) := (u, \exp(tX))$ satisfies $c(0) = (u, e)$ and $c'(0) = (0, X)$, we see that $T_{(u,e)}r(0_u, X) = \zeta_X(u)$.

From the construction in (1) it is clear that $g \mapsto r(u, g)$ defines a diffeomorphism from G onto the fiber P_x, where $x = p(u)$. By construction the tangent map of this diffeomorphism is $X \mapsto \zeta_X(u)$, so this defines an injection $\mathfrak{g} \to T_uP$. By construction, the image is contained in V_uP and since $\dim(P) = \dim(M) + \dim(G)$ we conclude that it has to equal V_uP. □

The action r from part (1) is called the *principal right action* of G on P. If there is no risk of confusion, we will sometimes denote the principal right action simply by a dot, i.e. write $u \cdot g$ for $r(u, g)$. The vector field ζ_X from part (2) is called the *fundamental vector field generated by* X.

Example 2.4. (1) For a smooth manifold M and a Lie group G there is the trivial principal bundle $\mathrm{pr}_1 : M \times G \to M$, for which the principal right action is just multiplication from the right in the second component. This also shows that for the fundamental vector field ζ_X generated by $X \in \mathfrak{g}$, we get $\zeta_X(x, g) = (0, L_X(g))$ in the identification $T(M \times G) \cong TM \times TG$, where L_X is the left-invariant vector field generated by X.

(2) Let G be a Lie Group, $H \subset G$ a closed subgroup, and $p : G \to G/H$ the fiber bundle from Example (3) in Section 2.1. Then the atlas constructed there evidently is a principal bundle atlas making $p : G \to G/H$ into a principal H-bundle. Of course, the principal action $G \times H \to G$ is just the restriction of the multiplication map. As in (1), this also shows that the fundamental vector fields coincides with the left-invariant vector fields generated by elements of $\mathfrak{h} \subset \mathfrak{g}$.

2.5. Morphisms of principal bundles. Similarly to the case of vector bundles, the principal right action allows us to define a notion of morphism of principal fiber bundles without having to refer to charts. The basis for this is that on a group G, the maps commuting with multiplications from the right are exactly multiplications from the left. Indeed, associativity says the multiplications from the left commute with multiplications from the right. Conversely, if $f : G \to G$ has the property that $f(gh) = f(g)h$ for all g, h, then of course we get $f(g) = f(e)g$, so f coincides with left multiplication by $f(e)$.

Hence the basic requirement on a morphism of principal bundles is equivariancy with respect to the principal right action. This makes sense on several levels of generality, the most general version with an arbitrary homomorphism is used only rarely, but several special cases are of interest.

Definition 2.5. (1) Let $\alpha : G \to H$ be a homomorphism of Lie groups, $p : P \to M$ a principal G-bundle and $\tilde{p} : \tilde{P} \to \tilde{M}$ a principal H-bundle. Then a *morphism of principal bundles over* α is a fiber bundle morphism $F : P \to \tilde{P}$ such that for each $u \in P$ and $g \in G$, we get $F(u \cdot g) = F(u) \cdot \alpha(g)$.

In case that $G = H$, a *principal bundle morphism* is a principal bundle morphism over id_G.

(2) Let G be a Lie group $H \subset G$ a closed subgroup and $p : P \to M$ a principal G-bundle. Then a *reduction of P to the structure group H* is given by a principal H-bundle $\tilde{p} : \tilde{P} \to M$ together with a principal bundle morphism $F : \tilde{P} \to P$ over the inclusion $i : H \to G$ with base map id_M. In this case, P is also called an *extension to the structure group G of $\tilde{P}$*.

(3) Let $p : P \to M$ be a principal G-bundle. Then a *gauge transformation* of P is a principal fiber bundle morphism $F : P \to P$ with base map id_M.

Lemma 2.5. *(1) Let $F : P \to \tilde{P}$ be a principal bundle morphism between two principal G-bundles whose base map f is a diffeomorphism. Then F is an isomorphism of principal fiber bundles, i.e. F is a diffeomorphism and F^{-1} is a morphism of principal bundles, too. In particular, any gauge transformation of a principal bundle is an isomorphism.*

(2) Let $F : \tilde{P} \to P$ be a reduction of structure group to H of a principal G-bundle. Then $F(\tilde{P}) \subset P$ is a closed submanifold and F is an embedding.

(3) A principal bundle $p : P \to M$ admits a smooth section defined on an open subset $U \subset M$ if and only if it is trivial over U i.e. $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is isomorphic to $U \times G$ as a principal bundle.

(4) Let $p : P \to M$ and $\tilde{p} : \tilde{P} \to \tilde{M}$ be principal bundles with structure groups G and $\tilde{G}$ and fix a homomorphism $\alpha : G \to \tilde{G}$. Let $F : P \to \tilde{P}$ be a set map which is equivariant over α, i.e. such that $F(r(u, g)) = r(F(u), \alpha(g))$, and such that $\tilde{p} \circ F = f \circ p$ for some smooth map $f : M \to \tilde{M}$. Suppose that for each $x \in M$, there is an open neighborhood U of x in M and a local smooth section $\sigma : U \to P$ such that $F \circ \sigma : U \to \tilde{P}$ is smooth. Then F is a principal bundle homomorphism over α

Proof. (1) By definition, we have $F(u \cdot g) = F(u) \cdot g$. Now we know that $g \mapsto u \cdot g$ defines a bijection from G onto the fiber P_x where $x = p(u)$ and similarly for $\tilde{P}_{f(x)}$. Thus we see that the restriction of F to P_x is a bijection onto $\tilde{P}_{f(x)}$. Together with bijectivity of f this implies that F is bijective, so there is and inverse function $F^{-1} : \tilde{P} \to P$. Now $\tilde{p} \circ F = f \circ p$ readily implies $p \circ F^{-1} = f^{-1} \circ \tilde{p}$ and equivariancy of F implies equivariancy of F^{-1}. Thus it suffices to show that F^{-1} is smooth in order to complete the proof of (1). But this is a local problem, so we may take principal bundle charts (U, φ) for P and $(\tilde{U}, \tilde{\varphi})$ for $\tilde{P}$ and assume that f restricts to a diffeomorphism $U \to \tilde{U}$. Then $\Phi := \tilde{\varphi} \circ F \circ \varphi^{-1} : U \times G \to \tilde{U} \times G$ is smooth and defining $\alpha(x)$ to be the second component of $\Phi(x, e)$, we obtain a smooth map $\alpha : U \to G$. By construction and equivariancy, we see that $\Phi(x, g) = (f(x), \alpha(x)g)$. But this readily shows that $\Phi^{-1}(y, h) = (f^{-1}(y), \nu(\alpha(f^{-1}(y)))h)$, where ν is the inversion in G. So Φ^{-1} is evidently smooth, and $F^{-1}|_{p^{-1}(f(U))} = \varphi^{-1} \circ \Phi^{-1} \circ \tilde{\varphi}$, so this is smooth, too.

(2) Here equivariancy of F evidently implies that the restriction of F to each fiber $\tilde{P}_x$ is injective and hence F is injective. Locally around a point $x \in \tilde{M}$, we can take principal bundle charts $(U, \tilde{\varphi})$ for $\tilde{P}$ and (U, φ) for P defined on the same open set $U \subset M$ with $x \in U$. Then $\Phi := \varphi \circ F \circ \tilde{\varphi}^{-1} : U \times H \to U \times G$ has the form $\Phi(x, h) = (x, \alpha(x)h)$ for a smooth map $\alpha : U \to G$ as in (1). This is obviously an immersion, so also F is an immersion and it suffices to verify that $F(\tilde{P}) \subset P$ is a submanifold to conclude the proof. But the map $(x, g) \mapsto \varphi^{-1}(x, \nu(\alpha(x))g)$ clearly defines a diffeomorphism $U \times G \to p^{-1}(U)$ which restricts to a diffeomorphism $U \times H \to F(\tilde{p}^{-1}(U))$ by construction. Since H is a submanifold of G, we can use this to construct a submanifold chart for $F(\tilde{P}) \subset P$ around each point in $F(\tilde{P}_x)$.

(3) Observe first that by definition $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is a principal G-bundle. Given a section $\sigma : U \to p^{-1}(U)$, we define $F : U \times G \to p^{-1}(U)$ by $F(x, g) := r(\sigma(x), g)$. By construction this is a fiber bundle morphism with base map id_U and clearly, it is G-equivariant, so it is an isomorphism by (1). (Alternatively, F^{-1} is given by $u \mapsto (p(u), \tau(\sigma(p(u)), u))$, where τ is the map from Lemma 2.4.) Conversely, given an isomorphism $F : U \times G \to p^{-1}(U)$, the map $x \mapsto F(x, e)$ clearly defines a smooth section of $p^{-1}(U) \to U$.

(4) We only have to prove that F is smooth and since this is a local question, it suffices to prove smoothness on $p^{-1}(U)$ for an open subset $U \subset M$ for which there is a section σ such that $F \circ \sigma$ is smooth. Taking the isomorphism $\varphi : U \times G \to p^{-1}(U)$ from (3), i.e. $\varphi(x, g) = r(\sigma(x), g)$, we conclude that $(F \circ \varphi)(x, g) = r((F \circ \sigma)(x), \alpha(g))$ and the right hand side is smooth by assumption. Since φ is a diffeomorphism, smoothness of $F \circ \varphi$ implies smoothness of $F|_{p^{-1}(U)}$. □

The linear frame bundle and structures on manifolds

Having the necessary background at hand, we can discuss the linear frame bundle of a manifold and this quickly leads to the concept of a G-structure.

2.6. The linear frame bundle. Let M be a smooth manifold of dimension n. Then we will construct a canonical principal bundle $p : \mathcal{P}M \to M$ with structure group $GL(n, \mathbb{R})$. This can actually be viewed as a variation of the construction of the tangent bundle. Given a point $x \in M$, we define $\mathcal{P}_xM$ to be the set of linear isomorphisms $v = v_x : \mathbb{R}^n \to T_xM$. *As a set*, we then define $\mathcal{P}M$ to be the disjoint union of the sets $\mathcal{P}_xM$, which immediately gives as a canonical map $p : \mathcal{P}M \to M$ that sends each $\mathcal{P}_xM$ to x. Now take a chart (U_α, u_α) for M and define a map $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times GL(n, \mathbb{R})$ by sending $v_x : \mathbb{R}^n \to T_xM$ to $(x, T_xu \circ v_x)$, where we use the standard identification $T_xu(U) \cong \mathbb{R}^n$. Since T_xu is a linear isomorphism for each $x \in U$, the construction readily implies that φ_α is bijective and of course $\mathrm{pr}_1 \circ \varphi_\alpha = p$.

For a second chart (U_β, u_β) and $x \in U_{\alpha\beta}$ consider $\varphi_\alpha \circ (\varphi_\beta)^{-1}$, which maps $U_{\alpha\beta} \times GL(n, \mathbb{R})$ to itself. By construction, this maps (x, A) to $\varphi_\alpha((T_xu_\beta)^{-1} \circ A) = T_xu_\alpha \circ (T_xu_\beta)^{-1} \circ A$. But now $T_xu_\alpha \circ (T_xu_\beta)^{-1} = Du_{\alpha\beta}(x)$, where $u_{\alpha\beta} : u_\beta(U_{\alpha\beta}) \to u_\alpha(U_{\alpha\beta})$ is the chart-change diffeomorphism for the two charts. This show that the chart change has the form $(x, A) \mapsto (x, Du_{\alpha\beta}(x)A)$ so smoothness of $Du_{\alpha\beta} : U_{\alpha\beta} \to GL(n, \mathbb{R})$ implies that $\varphi_\alpha \circ (\varphi_\beta)^{-1}$ is smooth. Starting from an atlas for M, we can invoke Lemma 2.2 to conclude that $p : \mathcal{P}M \to M$ is a smooth manifolds and the $(U_\alpha, \varphi_\alpha)$ form a fiber bundle atlas. But then the form of the chart changes says that this is a principal bundle atlas so $p : \mathcal{P}M \to M$ is a principal $GL(n, \mathbb{R})$ bundle as claimed. Clearly an equivalent atlas for M leads to an equivalent principal bundle atlas, so the structure on $\mathcal{P}M$ is canonical.

Note that the principal right action of $GL(n, \mathbb{R})$ on $\mathcal{P}M$ admits a very simple description: If $u \in \mathcal{P}_xM$ and $\varphi_\alpha(u) = (x, B)$ for one of the charts constructed above, then $u = (T_xu_\alpha)^{-1} \circ B : \mathbb{R}^n \to T_xM$. Now by definition for $A \in GL(n, \mathbb{R})$, we get $r(u, A) = (\varphi_\alpha)^{-1}(x, BA) = u \circ A$, so the principal right action is just composition from the right. Observe that we could have started from any n-dimensional vector space V (using charts for M with values in V) to get $\mathcal{P}M$ as a principal $GL(V)$-bundle with an analogous description of the principal right action.

The name "frame bundle" is derived from a natural interpretation of local sections of $p : \mathcal{P}M \to M$. Let $U \subset M$ be open and let $\sigma : U \to \mathcal{P}M$ be a smooth section. Then for each $x \in U$, $\sigma(x)$ is a linear isomorphism $\mathbb{R}^n \to T_xM$ so taking the standard basis $\{e_1, \dots, e_n\}$ for $\mathbb{R}^n$, we obtain a map $\xi_i : U \to TM$ by putting $\xi_i(x) = \sigma(x)(e_i)$ for $i = 1, \dots n$. By construction, for each $x \in M$ the vectors $\xi_1(x), \dots, \xi_n(x)$ form a basis of T_xM. We claim that the ξ_i are smooth and hence form a local frame for TM defined on U. This is a local question, so we can work in the principal bundle chart $(U_\alpha, \varphi_\alpha)$ derived from a chart (U_α, u_α) as above and assume that $U_\alpha \subset U$. Then the second component of $\varphi_\alpha \circ \sigma$ is a smooth map $U_\alpha \to GL(n, \mathbb{R})$, which we write as $x \mapsto A(x) = (a_{ij}(x))$. (We know that this matrix is invertible in each point, but this is not important here.) But then $\sigma(x) = T_xu_\alpha^{-1} \circ A(x)$ which readily shows that on U_α, we can write ξ_j as $\sum_i a_{ij} \frac{\partial}{\partial u_\alpha^i}$, which is evidently smooth. The same line of argument shows that conversely, a local frame for TM over U determines a local section $\sigma : U \to \mathcal{P}M$, so local sections of $\mathcal{P}M \to M$ are equivalent to local frames for TM.

The construction of the frame bundle easily leads to a the existence of a canonical differential form with values in a vector space, called the *soldering form*. Here a V-valued k-form on M simply associates to each $x \in M$ a k-linear alternating map $(T_xM)^k \to$

V. This has to be smooth in the usual sense that plugging in k-vector fields, one obtains a smooth V-valued function. The space of V-valued k-forms on M is denoted by $\Omega^k(M,V)$. Observe that such forms can be pulled back along smooth maps in the same way as ordinary forms. Using this, we now formulate

PROPOSITION 2.6. *Let M be a smooth manifold of dimension n and $p:\mathcal{P}M\to M$ its linear frame bundle. For $A\in GL(n,\mathbb{R})$ let $r^A:\mathcal{P}M\to\mathcal{P}M$ be the principal right action by A. Then there is a canonical $\mathbb{R}^n$-valued one-form $\theta\in\Omega^1(\mathcal{P}M,\mathbb{R}^n)$ characterized by $T_up\cdot\xi=u(\theta(u)(\xi))$ for $u\in\mathcal{P}M$ and $\xi\in T_u\mathcal{P}M$. This form has the following properties*

- *For any $u\in\mathcal{P}M$, $\ker(\theta(u))=V_u\mathcal{P}M\subset T_u\mathcal{P}M$ ("θ is strictly horizontal")*
- *For any $A\in GL(n,\mathbb{R})$ and $u\in\mathcal{P}M$ we get $((r^A)^*\theta)(u)=A^{-1}\circ\theta(u):T_u\mathcal{P}M\to\mathbb{R}^n$ ("θ is $GL(n,\mathbb{R})$-equivariant").*

PROOF. By definition, a point $u\in\mathcal{P}M$ is a linear isomorphism $u:\mathbb{R}^n\to T_{p(u)}M$ and for $\xi\in T_u\mathcal{P}M$ we get $T_up\cdot\xi\in T_{p(u)}M$. Thus we can define $\theta(u)(\xi):=u^{-1}(T_up\cdot\xi)$. This satisfies $T_up\cdot\xi=u(\theta(u)(\xi))$ and is evidently characterized by this property. Since u^{-1} is a linear isomorphism, we see that $\ker(\theta(u))=\ker(T_up)$. Finally, since $p\circ r^A=p$ we get $T_{r(u,A)}p\circ T_ur^A=T_up$ and since $r(u,A)=(u\circ A)$ we conclude that $\theta(r(u,A))(T_ur^A\cdot\xi)=A^{-1}\circ u^{-1}(T_up\cdot\xi)$. Thus all claimed properties are satisfied and we only have to prove that this defines a smooth $\mathbb{R}^n$-valued form on $\mathcal{P}M$.

To verify smoothness, it suffices to show that one gets smooth functions after inserting the coordinate vector fields for some local charts. We can do this for charts obtained via local fiber bundle charts $\varphi_\alpha:p^{-1}(U_\alpha)\to U_\alpha\times GL(n,\mathbb{R})$ from product charts for the image. The coordinate vector fields for the second factor simply produce the zero function after insertion into θ. More specifically, we can use the principal bundle charts obtained from charts (U_α,u_α) for M. Then applying Tp to the corresponding coordinate fields on $\mathcal{P}M$, one simply obtains the coordinate vector fields $\frac{\partial}{\partial u^i_\alpha}$ on U_α. But for these charts, $u\in\mathcal{P}M$ corresponds to $T_uu_\alpha\circ u\in GL(n,\mathbb{R})$. This easily implies that u^{-1} maps $\frac{\partial}{\partial u^i_\alpha}$ to the ith column of the inverse of the matrix that forms the second component of $\varphi_\alpha(u)$. Since this evidently depends smoothly on u, this completes the proof. $\square$

Suppose now that we view M as modelled on an n-dimensional vector space V and hence $\mathcal{P}M$ as a principal bundle with structure group $GL(V)$. Then the construction in the proposition of course gives rise to a soldering form $\theta\in\Omega^1(\mathcal{P}M,V)$ which is strictly horizontal and $GL(V)$-equivariant in an obvious sense.

2.7. G-structures on manifolds. Having the picture of the frame bundle at hand, it is now clear how to carry over the notion of G-structures on a vector space as developed in Section 1.6 to the individual tangent spaces of a manifold. This would lead to a definition as a subset of $\mathcal{P}M$, but there are several ways to equivalently rephrase this, which are less closely tied to the frame bundle. On has to choose one of these equivalent possibilities as a definition, and we choose a version that is a special instance of a general concept for principal bundles.

DEFINITION 2.7. Let V be a vector space, put $n=\dim(V)$ and let $G\subset GL(V)$ be a closed subgroup. For a smooth manifold M of dimension n, let us view M as being modelled on V and hence $\mathcal{P}M$ as a principal $GL(V)$-bundle. Then a *G-structure* on M is a reduction $P\to\mathcal{P}M$ of the linear frame bundle to the structure group G.

To get to the equivalent descriptions mentioned above, we observe that on the one hand that by Lemma 2.5 a reduction of structure group is an embedding, so we can view P as a submanifold of $\mathcal{P}M$. This leads to the picture of putting a G-structure in the sense of Section 1.6 on each tangent space of M in a way depending smoothly on the point. On the other hand, we can pull back the soldering form on $\mathcal{P}M$ to a form on P which has analogous properties. It then turns out that prescribing such a form is equivalent to the principal bundle map defining a reduction. This leads to the picture that a G-structure on M is an abstract principal bundle together with an analog of the soldering form.

Theorem 2.7. *A G-structure $F : P \to \mathcal{P}M$ corresponding to $G \subset GL(V)$ can be equivalently described in either of the two following ways.*

(1) As a subset $Q \subset \mathcal{P}M$ such that for each $x \in M$ the following conditions are satisfied

- *$Q_x := Q \cap \mathcal{P}_xM \neq \emptyset$ and for one or equivalently any $u \in Q_x$ the map $G \to Q_x$ defined by $A \mapsto r(u, A)$ is bijective.*
- *There is an open neighborhood U of $x \in M$ and a smooth section $\sigma : U \to \mathcal{P}M$ such that σ has values in Q.*

(2) An abstract principal G-bundle $\pi : P \to M$ endowed with a form $\tilde{\theta} \in \Omega^1(P, V)$ which is strictly horizontal and G-equivariant in the sense of Proposition 2.6.

Proof. (1) Having given $F : P \to \mathcal{P}M$, we define $Q := F(P) \subset \mathcal{P}M$, which immediately implies that the required property of Q_x is satisfied for each x. Moreover since for a local section τ of P, $F \circ \tau$ is a local smooth section of $\mathcal{P}M$, the second property is satisfied, too.

Conversely, suppose we have given $Q \subset \mathcal{P}M$ with the two listed properties. Given U and $\sigma : U \to \mathcal{P}M$ with values in Q, we know from the proof of Lemma 2.5 that $(x, A) \mapsto r(\sigma(x), A)$ is the inverse of a fiber bundle chart for $\mathcal{P}M$. But by the first property, this restricts to a bijection $U \times G \to Q \cap p^{-1}(U)$. This can be used to construct submanifold charts for Q and to obtain principal bundle charts for $p|_Q : Q \to M$, so this is a principal G-bundle. But then the inclusion $Q \hookrightarrow \mathcal{P}M$ evidently defines a reduction of structure group and hence a G-structure on M.

(2) Having given $F : P \to \mathcal{P}M$, we define $\tilde{\theta} := F^*\theta \in \Omega^1(P, V)$. For $\tilde{u} \in P$ and $\xi \in T_{\tilde{u}}\xi$, we get $T_{F(\tilde{u})}p \circ T_{\tilde{u}}F = T_{\tilde{u}}\pi$. This immediately shows that $\ker(\tilde{\theta}(\tilde{u})) = \ker(T_{\tilde{u}}\pi)$, so $\tilde{\theta}$ is strictly horizontal. On the other hand, for $A \in G$, we get $F \circ r^A = r^A \circ F$ and thus $(r^A)^*F^*\theta = F^*(r^A)^*\theta$. By equivariancy of θ, this sends $\xi \in T_{\tilde{u}}P$ to

$$A^{-1}(\theta(F(\tilde{u}))(T_{\tilde{u}}F \cdot \xi)) = A^{-1}(\tilde{\theta}(\tilde{u})(\xi)),$$

so $\tilde{\theta}$ is equivariant, too.

Conversely, given a strictly horizontal, equivariant one form $\tilde{\theta} \in \Omega^1(P, V)$, we claim that there is a unique homomorphism $F : P \to \mathcal{P}M$ of principal bundles such that $\tilde{\theta} = F^*\theta$. Indeed, for a point $\tilde{u} \in P_x$ the map $\tilde{\theta}(\tilde{u}) : T_{\tilde{u}}P \to V$ by assumption has kernel $\ker(T_{\tilde{u}}\pi)$. Hence it is surjective and there is a unique linear isomorphism $F(\tilde{u}) : V \to T_xM$ such that $\tilde{\theta}(\tilde{u}) = F(\tilde{u})^{-1} \circ T_{\tilde{u}}\pi$. Thus we have defined a map $F : P \to \mathcal{P}M$ such that $p \circ F = \pi$. Moreover, $(r^A)^*\tilde{\theta}(\tilde{u}) = A^{-1} \circ \tilde{\theta}(\tilde{u})$ together with $\pi \circ r^A = \pi$ easily implies that $F(r(\tilde{u}, A)) = F(\tilde{u}) \circ A$, so F satisfies the assumptions of part (4) of Lemma 2.5.

Now take a local smooth section σ of P defined on $U \subset M$ and consider $F \circ \sigma$, which is a section of $\mathcal{P}M$. On the other hand, $\sigma^*\tilde{\theta} \in \Omega^1(U, V)$ by construction has the property that its value in each point $y \in U$ is a linear isomorphism $T_yM \to V$. Pulling back the constant functions determined by a basis of V, we obtain a smooth frame of TM defined over U. This in turn gives rise to a smooth local section of $\mathcal{P}M$ defined over U. Going through the construction, one easily concludes that the resulting section has the same value as $F \circ \sigma$ in each point $y \in U$, so F is smooth by Lemma 2.5. Since $Tp \circ TF = T\pi$ we also conclude that $F^*\theta = \tilde{\theta}$ by construction. □

Remark 2.7. The description in part (2) of Theorem 2.7 allows for a generalization of the notion of a G-structure that is important in several applications. An extreme version of this would be to call any principal fiber bundle endowed with a strictly horizontal, equivariant $\mathbb{R}^n$-valued one-form over an n-dimensional manifold a G-structure. This would be rather misleading however, as we shall see later on. An important class of examples arises, however, if one allows G to be a covering of a closed subgroup of $GL(V)$. Otherwise put, one requires that there is a homomorphism $\varphi : G \to GL(V)$ such that the derivative φ' is injective and such that $\varphi(G) \subset GL(V)$ is closed. The most important example of this are so-called *spin structures*. The point here is that the group $SO(n)$ is not simply connected and the universal covering is the spin group $Spin(n)$. This comes with a surjective homomorphism $Spin(n) \to SO(n)$ whose kernel is isomorphic to $\mathbb{Z}_2$. A spin structure can then be defined via a principal $Spin(n)$-bundle $P \to M$ endowed with a strictly horizontal, equivariant $\mathbb{R}^n$-valued form. As in the proof of part (2) of Theorem 2.7 this defines a homomorphism $P \to \mathcal{P}M$ whose image satisfies the conditions of part (1) of Theorem 2.7 for the group $SO(n) \subset GL(n, \mathbb{R})$. Hence a spin-structure has an underlying $SO(n)$-structure which, as we shall see soon, corresponds to a Riemannian metric and an orientation.

We will only mention this more general concept from time to time and not study it systematically in the course. With appropriate small changes, most of the theory extends rather easily to this more general setting.

2.8. Morphisms. To discuss morphisms of G-structures, we first look at the case of the linear frame bundle. Suppose that M and $\tilde{M}$ are manifolds of dimension n and that $f : M \to \tilde{M}$ is a local diffeomorphism. Then there is an obvious idea how to lift f to a map $\mathcal{P}f : \mathcal{P}M \to \mathcal{P}\tilde{M}$ by sending a linear isomorphism $u : \mathbb{R}^n \to T_xM$ to $T_xf \circ u : \mathbb{R}^n \to T_{f(x)}\tilde{M}$. This has nice properties:

Proposition 2.8. *The map $\mathcal{P}f$ defined above is a principal bundle homomorphism with base map f such that $(\mathcal{P}f)^*\theta = \theta$, where we denote the soldering forms on both bundles by the same letter. It is uniquely characterized by these properties.*

Proof. By definition, $\mathcal{P}f$ is equivariant for the principal right actions and maps $\mathcal{P}_xM$ to $\mathcal{P}_{f(x)}\tilde{M}$. Now let $U \subset M$ be open such that $f(U) \subset \tilde{M}$ is open, $f|_U : U \to f(U)$ is a diffeomorphism and there is a local section $\sigma : U \to P$. This corresponds to a local frame $\{\xi_1, \dots, \xi_n\}$ for TM defined on U. For each i, $\tilde{\xi}_i := Tf \circ \xi_i \circ (f|_U)^{-1} : f(U) \to T\tilde{M}$ is a smooth vector field on $\tilde{M}$, and these together form a smooth local frame for $T\tilde{M}$ defined on $f(U)$. By construction, for the corresponding local section $\tilde{\sigma} : f(U) \to \mathcal{P}\tilde{M}$, we obtain $\tilde{\sigma} \circ f = \mathcal{P}f \circ \sigma$ on U. Thus $\mathcal{P}f \circ \sigma$ is smooth, so smoothness of $\mathcal{P}f$ follows from Lemma 2.5 and we have verified all properties of a principal bundle homomorphism.

Let $F : \mathcal{P}M \to \mathcal{P}\tilde{M}$ be a principal bundle homomorphism with base map f. Then

$$(F^*\theta)(u)(\xi) = \theta(F(u))(T_uF \cdot \xi) = F(u)^{-1}((T_xf \circ T_up)(\xi)). \tag{2.1}$$

This shows that $(F^*\theta)(u) = \theta(u)$ if and only if $F(u)^{-1} \circ T_xf = u^{-1}$ which clearly is equivalent to $F(u) = T_xf \circ u$. This shows that $(\mathcal{P}f)^*\theta = \theta$ as well as uniqueness. □

This has several nice consequences. On the one hand, it directly leads to the definition of a morphism of G-structures. In the picture of part (1) of Theorem 2.7, we have $Q \subset \mathcal{P}M$ and $\tilde{Q} \subset \mathcal{P}\tilde{M}$ and a local diffeomorphism $f : M \to \tilde{M}$ is a *morphism of*

G-structures if and only if $\mathcal{P}f(Q)\subset\tilde{Q}$. Observe that this definition also makes sense if f is only defined on a neighborhood of x, so there is a concept of local morphisms.

It is easy to see that in the picture of reductions, this is equivalent to a homomorphism $F:P\to\tilde{P}$ of principal bundles with base map f such that the diagram

$$\begin{array}{ccc} P & \longrightarrow & \mathcal{P}M \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{P}f} \\ \tilde{P} & \longrightarrow & \mathcal{P}\tilde{M} \end{array}$$

commutes. Evidently, there is at most one homomorphism F with that property and f is a morphism if F exists. It is also easy to rephrase this in the picture of part (3) of Theorem 2.7. Here f is a morphism if and only if there is a principal bundle homomorphism $F:P\to\tilde{P}$ with base map f, which is compatible with the given V-valued one-forms on the bundles. An obvious analog of formula (2.1) shows that there is at most one such F. In this form, the concept of a morphism also extends to the more general setting of a covering of a closed subgroup of $GL(V)$ as discussed in Remark 2.7. However, in this case a morphism F of G-structures is not necessarily determined uniquely by its base map f. Hover, the relation to the principal bundle homomorphisms $\mathcal{P}f$ on the linear frame bundles still persists.

Second, we obtain a concept of pullback of G-structures along local diffeomorphisms. Again, this is most easily described in the picture of part (1) of Theorem 2.7. Given $\tilde{Q}\subset\mathcal{P}\tilde{M}$ and a local diffeomorphism $f:M\to\tilde{M}$, then for a point $x\in M$, we put $Q_x:=\{u:\mathbb{R}^n\to T_xM : T_xf\circ u\in\tilde{Q}_{f(x)}\}$. Since T_xf is a linear isomorphism, the resulting subset $Q\subset\mathcal{P}M$ evidently satisfies the first condition in part (1) of Theorem 2.7. For the second condition, we can choose an open neighborhood U of x in M such that $f(U)\subset\tilde{M}$ is open, $f|_U:U\to f(U)$ is a diffeomorphism and there is local smooth section $\tilde{\sigma}:f(U)\to\mathcal{P}\tilde{M}$ that has values in $\tilde{Q}$. Then we obtain an appropriate smooth section $\sigma:U\to\mathcal{P}M$ with values in Q via $\mathcal{P}(f|_U)^{-1}\circ\tilde{\sigma}\circ f$. Hence Q defines a G-structure on M, which we also denote by $f^*\tilde{Q}$. Clearly, the construction can be rephrased by saying that Q is the unique structure for which f becomes a morphism of G-structures.

To describe this construction in the picture of part (2) of Theorem 2.7, we need yet another general construction of bundle theory. Suppose that $p:E\to N$ is any fiber bundle and $f:M\to N$ is any smooth map. Then one defines $f^*E:=\{(x,y)\in M\times E : f(x)=p(y)\}$ and denotes by $f^*p:f^*E\to M$ and $p^*f:f^*E\to E$ the restriction of the projections on the product $M\times E$. Then one easily proves that f^*E is a submanifold of $M\times E$ and hence the maps f^*p and p^*f are smooth. Having this at hand, consider a fiber bundle chart $\varphi:p^{-1}(U)\to U\times S$ for E and let $\varphi_2:p^{-1}(U)\to S$ be its second component. Then $V:=f^{-1}(U)\subset M$ is open and $(p^*f)^{-1}(V)=(f^*p)^{-1}(p^{-1}(U))$ and one readily concludes that mapping $(x,y)\in(p^*f)^{-1}(V)$ to $(x,\varphi_2(y))$ is a fiber bundle chart, too. Hence $f^*E\to M$ is a fiber bundle with the same standard fiber S as E. Looking at the chart changes, one readily concludes that any pullback of a vector bundle is again a vector bundle and any pullback of a principal G-bundle is a principal G-bundle.

Now in the setting of part (2) of Theorem 2.7, assume that we have given a principal G-bundle $\tilde{p}:\tilde{P}\to\tilde{M}$ and a strictly horizontal, equivariant one-form $\tilde{\theta}\in\Omega^1(\tilde{P},\mathbb{R}^n)$. Then for a local diffeomorphism $f:M\to\tilde{M}$, we consider the principal G-bundle $P:=f^*\tilde{P}\to M$, whose projection we denote by p and the map $F:=p^*f:P\to\tilde{P}$. By construction, this is a principal bundle homomorphism with base map f, and we can consider $F^*\tilde{\theta} \in \Omega^1(P, \mathbb{R}^n)$. One easily verifies that this is again strictly horizontal and G-equivariant (which needs that f is a local diffeomorphism), and hence $(p : P \to M, F^*\tilde{\theta})$ is a G-structure.

Example 2.8. Let us quickly convince ourselves that the concepts we have developed here lead to the expected results in some of the examples we have discussed so far.

(1) Take $G = O(n) \subset GL(n, \mathbb{R})$ and a smooth manifold M of dimension n. Given an $O(n)$-structure $Q \subset \mathcal{P}M$ and a point $x \in M$, we can take any $u \in Q_x$ to define an inner product g_x on T_xM by $g_x(X, Y) := \langle u^{-1}(X), u^{-1}(Y)\rangle$. Any other element of Q_x is of the form $u \circ A$ for $A \in O(n)$ and hence leads to the same inner product g_x. Taking these, we obtain a function g that associates to each x and element $g_x \in \otimes^2 T_x^*M$ which is symmetric. By definition there is an open neighborhood U of x in M and a local smooth section $\sigma : U \to \mathcal{P}M$ with values in U. From the construction in Section 2.6, we see that the corresponding local frame $\{\xi_i\}$ for TM has the property that $\{\xi_1(y), \dots, \xi_n(y)\}$ is orthonormal for each $y \in U$. But given $\xi, \eta \in \mathfrak{X}(M)$, we find smooth functions f_i and g_i such that $\xi|_U = \sum f_i\xi_i$ and $\eta|_U = \sum g_j\xi_j$ and hence $g(\xi, \eta)|_U = \sum_i f_ig_i$, so g defines a Riemannian metric on M.

Conversely, given a Riemannian metric g on M, one defines $Q \subset \mathcal{P}M$ by putting Q_x the subset of those $u : \mathbb{R}^n \to T_xM$ which are orthogonal with respect to $\langle\ ,\ \rangle$ and g_x. From Theorem 2.7, it easily follows that this indeed is an $O(n)$-structure. From this description it is also obvious that given (M, g), $(\tilde{M}, \tilde{g})$ and a local diffeomorphism $f : M \to \tilde{M}$, the condition that $\mathcal{P}f(Q_x) \subset Q_{f(x)}$ is equivalent to the fact that $T_xf : T_xM \to T_{f(x)}\tilde{M}$ is orthogonal with respect to the inner products g_x and $\tilde{g}_{f(x)}$. Thus morphisms of $O(n)$-structures are exactly isometries in the sense of Riemannian geometry. The construction also readily shows that the pullback of $O(n)$-structures corresponds to the usual pullback of Riemannian metrics.

In the same way, one verifies that for $G = Sp(2m, \mathbb{R}) \subset GL(2m, \mathbb{R})$, G-structures are exactly almost symplectic structures and morphisms are exactly local diffeomorphisms that are compatible with the two-forms defining the structure.

(2) Let $G \subset GL(n, \mathbb{R})$ be the stabilizer of the subspace $\mathbb{R}^k \subset \mathbb{R}^n$ as in Example (2) of Section 1.5. Given a G-structure $Q \subset \mathcal{P}M$, a point $x \in M$ and $u \in Q_x$, one defines a k-dimensional linear subspace $E_x := u(\mathbb{R}^k) \subset T_xM$. This is immediately seen to be independent of u. Moreover, local smooth sections of $\mathcal{P}M$ with values in Q give rise to local frames for TM for which the values of the first k elements in any point y form a basis of E_y. This shows that the spaces E_x fit together to define a smooth distribution $E \subset TM$ of rank k.

Conversely, given a smooth distribution $E \subset TM$ of rank k, one defines $Q \subset \mathcal{P}M$ by defining Q_x to be the set of those linear isomorphisms $u : \mathbb{R}^n \to T_xM$ which map $\mathbb{R}^k$ to E_x. This immediately implies that $Q \subset \mathcal{P}M$ satisfies the first condition in part (1) of Theorem 2.7, so it remains to construct appropriate smooth local frames to show that the distribution E determines a G-structure. But it is easy to see that a local frame for E can be extended (possibly on a smaller subset) to a local frame of TM, which provides exactly the kind of frames we need. A morphism between the G-structures corresponding to $E \subset TM$ and $\tilde{E} \subset T\tilde{M}$ by construction exactly is a local diffeomorphism $f : M \to \tilde{M}$ such that for each $x \in M$, we get $T_xf(E_x) = \tilde{E}_x$. Finally, the pullback of the G-structure corresponding to $\tilde{E} \subset T\tilde{M}$ along a local diffeomorphism $f : M \to \tilde{M}$ is determined by the distribution E characterized by $E_x = \{X \in T_xM : T_xf(X) \in \tilde{E}_{f(x)}\}$.

(3) Put $G := GL(m,\mathbb{C}) \subset GL(2m,\mathbb{R})$ as in Example (4) of Section 1.5. Then for a G-structure $Q \subset \mathcal{P}M$, a point $x \in M$ and $u \in Q_x$, we define $J_x : T_xM \to T_xM$ by $J_x(X) = u(iu^{-1}(X))$. Any other element of Q_x is of the form $u \circ A$ for a complex linear map A, so this is independent of the choice of u and by construction $J_x \circ J_x = -\operatorname{id}_{T_xM}$. Putting these together, we have defined a map J that associates to each $x \in M$ an element of $L(T_xM, T_xM) \cong T_x^*M \otimes T_xM$. The explicit description of G as a matrix group in Example (4) of Section 1.5 is based on a basis for $\mathbb{R}^{2m} \cong \mathbb{C}^m$ of the form $\{v_1, iv_1, \dots, v_m, iv_m\}$ for a complex basis $\{v_j\}$ of $\mathbb{C}^m$. Hence for a local smooth section σ of $\mathcal{P}M$ with values in Q the values of the corresponding local frame for TM have the form $\{\xi_1(y), J_y(\xi_1(y)), \dots, \xi_m(y), J_y(\xi_m(y))\}$ for some vector fields $\xi_1, \dots, \xi_m$ on M. In this frame, J is given by a simple constant matrix, which shows that J is a smooth $\binom{1}{1}$-tensor field on M.

Conversely, consider an *almost complex structure* on M, i.e. a tensor field $J \in \mathcal{T}^1_1(M)$ such that for each $x \in M$ and viewing J_x as a linear map $T_xM \to T_xM$, we get $J_x \circ J_x = -\operatorname{id}_{T_xM}$. Then as observed Example (4) of Section 1.5, this makes T_xM into a complex vector space and we define Q_x as the set of complex linear isomorphisms $\mathbb{C}^m \to T_xM$, which sits inside the fiber $\mathcal{P}_xM$ of all real linear isomorphisms between these two spaces. One immediately verifies that this satisfies the first condition in Theorem 2.7, so we have to construct appropriate local frames in order to see that we have defined a G-structure. Given $x \in M$, consider the complex vector space (T_xM, J_x), choose a complex basis $\{X_1, \dots, X_m\}$ for this space and extend these tangent vectors to local vector fields $\xi_1, \dots, \xi_m$ defined on some open neighborhood U of x in M. Then also $J(\xi_1), \dots, J(\xi_m)$ are smooth local vector fields defined on the same neighborhood. Since the values at x of $\xi_1, J(\xi_1), \dots, \xi_m, J(\xi_m)$ are linearly independent these fields form a local frame on a (possibly smaller) neighborhood of x in M, which provides a local frame of the required form. Hence we see that $GL(m,\mathbb{C})$-structures on manifolds of dimension $2m$ are equivalent to almost complex structures.

The construction also shows that morphisms between the structures corresponding to (M, J) and $(\tilde{M}, \tilde{J})$ are exactly those local diffeomorphism $f : M \to \tilde{M}$ whose derivatives are all complex linear in the sense that for each $x \in M$ we get $T_xf \circ J_x = \tilde{J}_x \circ T_xM$. The pullback of the structure corresponding to $(\tilde{M}, \tilde{J})$ by a local diffeomorphism $f : M \to \tilde{M}$ is readily seen to correspond to $J_x := (T_xf)^{-1} \circ \tilde{J}_{f(x)} \circ T_xf$, which exactly says that $J = f^*\tilde{J}$ for the pullback of $\binom{1}{1}$-tensor fields.

2.9. Associated bundles. The construction of the linear frame bundle suggests a simple possibility to recover the tangent bundle. Indeed, there is a natural map $q : \mathcal{P}M \times \mathbb{R}^n \to TM$ which sends an isomorphism $u : \mathbb{R}^n \to T_xM$ and a vector $v \in \mathbb{R}^n$ to $u(v) \in T_xM$. One easily verifies that q is smooth and by construction it is compatible with the obvious projections to M on both sides and surjective. It is also easy to characterize when two pairs (u, v) and $(\tilde{u}, \tilde{v})$ have the same image in TM. This clearly is possible only if $p(u) = p(\tilde{u})$, and then there is a unique element $A \in GL(n,\mathbb{R})$ such that $\tilde{u} = r(u, A) = u \circ A$. Then $u(v) = \tilde{u}(\tilde{v}) = u(A\tilde{v})$ if and only if $\tilde{v} = A^{-1}v$. Expressed in a more fancy way, $(u, v) \cdot A := (r(u, A), A^{-1}v)$ is immediately seen to define a smooth right action of $GL(n,\mathbb{R})$ on $\mathcal{P}M \times \mathbb{R}^n$ and two elements in this space have the same image in TM if and only if they lie in one orbit of this action.

The congenial fact about this is that one can work in a similar way with vector spaces that are obtained from $\mathbb{R}^n$ and T_xM by some functorial construction. A linear isomorphism $u : \mathbb{R}^n \to T_xM$ induces a linear isomorphism $\mathbb{R}^{n*} \to (T_xM)^*$ that sends $\lambda : \mathbb{R}^n \to \mathbb{R}$ to $\lambda \circ u^{-1} : T_xM \to \mathbb{R}$. This induces a similar surjection $\mathcal{P}M \times \mathbb{R}^{n*} \to T^*M$

and (u, λ) and $(\tilde{u} = u \circ A, \tilde{\lambda})$ have the same image under this if $\tilde{\lambda} = \lambda \circ A = A^{-1} \cdot \lambda$ for the natural action of $GL(n, \mathbb{R})$ on $\mathbb{R}^{n*}$. This extends analogously to tensor bundles and symmetric and exterior powers. All these are special cases of the construction of associated bundles that we describe in general next.

Let $p : P \to M$ be a principal G bundle and let S be a smooth manifold that is endowed with a left G-action, which we will denote by a dot. Then we can form the product $P \times S$ which is a fiber bundle over M with fiber $G \times S$. Then $(u, y) \cdot g := (r(u, g), g^{-1} \cdot y)$ defines a smooth right action of G on $P \times S$. Now we consider the set of orbits of this action, i.e. the set of equivalence classes of the equivalence relation $\sim$ defined by $(u, y) \sim (\tilde{u}, \tilde{y})$ iff there is a $g \in G$ such that $(\tilde{u}, \tilde{y}) = (u, y) \cdot g$. We denote this by $P \times_G S$ or by $P[S]$ and we write $q : P \times S \to P \times_G S$ for the obvious map that sends each pair to its equivalence class. Note that $q(u, v) = q(\tilde{u}, \tilde{v})$ implies that $\tilde{u} = r(u, g)$ for some $g \in G$ and hence $p(u) = p(\tilde{u})$. Thus we obtain a natural map $\pi : P \times_G S \to M$ characterized by $\pi \circ q = p \circ \mathrm{pr}_1$. An important special case is an action of G on a finite dimensional vector space V the comes from a representation of G on V, i.e. from a smooth homomorphism $G \to GL(V)$.

Lemma 2.9. *In the setting described above, the set $P \times_G S$ can be canonically made into a smooth manifold and we get:*

(1) $\pi : P \times_G S \to M$ is a smooth fiber bundle with typical fiber S. In the case of a representation of G on V, $\pi : P \times_G V \to M$ is a vector bundle with typical fiber V.

(2) $q : P \times S \to P \times_G S$ is a principal G-bundle.

(3) There is a smooth map $\tau_S : P \times_M (P \times_G S) \to S$ which is uniquely characterized by the fact that for $u \in P$ and $z \in P \times_G S$ with $p(u) = \pi(z)$ we get $z = q(u, \tau_S(u, z))$.

Proof. (1) Take a principal bundle chart $(U_\alpha, \varphi_\alpha)$ for P, so $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times G$. Take $u \in p^{-1}(U_\alpha)$, write $\varphi_\alpha(u) = (x, g)$ and consider the map $p^{-1}(U_\alpha) \times S \to U_\alpha \times S$ that sends (u, y) to $(x, g \cdot y)$. Then $\varphi_\alpha(r(u, h)) = (x, gh)$, which shows that $(r(u, h), h^{-1} \cdot y)$ is also sent to $(x, g \cdot y)$. Hence this induces well defined map $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times S$ such that $\psi_\alpha(q(u, y)) = (x, g \cdot y)$, which shows that ψ_α is surjective. On the other hand a pair $(\tilde{u}, \tilde{y})$ can be mapped to $(x, g \cdot y)$ only if $x = p(\tilde{u}) = p(u)$ and hence there is an element $h \in G$ such that $\tilde{u} = r(u, h)$ and thus $\varphi_\alpha(\tilde{u}) = (x, gh)$. But then $gh \cdot \tilde{y} = g \cdot y$ implies $\tilde{y} = h^{-1} \cdot y$ and hence $(\tilde{u}, \tilde{y}) = (u, y) \cdot h$. Hence $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times S$ is bijective.

Taking a compatible principal bundle chart (U_β, φ_β) such that $U_{\alpha\beta} \neq \emptyset$, we obtain a smooth function $\varphi_{\alpha\beta} : U_{\alpha\beta} \to G$ such that $(\varphi_\alpha \circ \varphi_\beta^{-1})(x, g) = (x, \varphi_{\alpha\beta}(x)g)$. This readily implies that $(\psi_\alpha \circ \psi_\beta^{-1})(x, y) = (x, \varphi_{\alpha\beta}(x) \cdot y)$, where we use the left action of G on S in the second component. This shows that starting from a principal atlas for P, we obtain a family (U_α, ψ_α) that satisfies the assumptions of Lemma 2.2. Thus we can make $P \times_G S$ canonically into a manifold in such a way that (U_α, ψ_α) is a fiber bundle atlas. In case that we start from a representation of G on V, the chart changes of this atlas by construction are linear in each point, so we obtain a vector bundle atlas.

(2) The construction in (1) says that On $p^{-1}(U_\alpha) \times S$, we get $\psi_\alpha \circ q = (\mathrm{id}_{U_\alpha} \times \ell) \circ (\varphi_\alpha \times \mathrm{id}_S)$, where $\ell : G \times S \to S$ is the left action, so q is smooth. Moreover $p^{-1}(U_\alpha) \times S = q^{-1}(\pi^{-1}(U_\alpha))$ and we define $\tau_\alpha(u, y) := (q(u, y), g)$ where $\varphi_\alpha(u) = (x, g)$. One immediately verifies that this defines a diffeomorphism $\tau_\alpha : p^{-1}(U_\alpha) \times S \to \pi^{-1}(U_\alpha) \times G$ and that these fit together to define a principal bundle atlas for $q : P \times S \to P \times_G S$.

(3) The construction in (1) shows that for $u \in P$ with $p(u) = x$ the map $y \mapsto q(u, y)$ defines a bijection from S onto the fiber of $P \times_G S$ over x. This shows that τ_S is uniquely defined by the required property, and we only have to verify smoothness. But

now for $u, \tilde{u} \in P_x$ put $\varphi_\alpha(u) = (x, g)$ and $\varphi_\alpha(\tilde{u}) = (x, \tilde{g})$. Then for $y \in S$ we get $\psi_\alpha(q(\tilde{u}, y)) = (x, \tilde{g} \cdot y)$. But now $\tilde{u} = r(u, g^{-1}\tilde{g})$ and hence $q(\tilde{u}, y) = q(u, g^{-1}\tilde{g} \cdot y)$. The induced chart on $P \times_M (P \times_G S)$ sends $(u, q(\tilde{u}, y))$ to $(x, g, \tilde{g} \cdot y)$. Hence in this chart, τ_S is given by $(x, h, z) \mapsto h^{-1} \cdot z$ and this is obviously smooth. □

In the setting of vector bundles that are associated to a fixed principal bundle, the constructions discussed in the end of Section 2.3 become much easier, since they can be obtained from constructions with the inducing representations. Suppose that $E = P \times_G V$ for a principal G-bundle $p : P \to M$ and a representation V of G. Then we have the dual representation V^* to V which is characterized by $g \cdot \lambda(v) := \lambda(g^{-1} \cdot v)$ and it is easy to see that $P \times_G V^*$ can be naturally identified with the dual bundle E^* as described in Section 2.3. Similarly, there are representations $\otimes^k V$, $S^k V$, and $\Lambda^k V$ which induce the bundles $\otimes^k E$, $S^k E$, and $\Lambda^k E$, respectively.

Given a second associated bundle $F = P \times_G W$ for a representation of G on W, we get representations on $V \otimes W$ characterized by $g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$ and on $L(V, W)$ given by $(g \cdot f)(v) := g \cdot (f(g^{-1} \cdot v))$. Passing to associated bundles, these exactly give the bundles $E \otimes F$ and $L(E, F)$ as discussed in Section 2.3.

2.10. Sections of associated bundles. The construction of an associated bundle directly leads to a description of sections of such a bundle in terms of functions that satisfy an equivariancy condition.

Proposition 2.10. *Let $p : P \to M$ be a principal G-bundle, S a smooth manifold endowed with a left G-action and $\pi : P \times_G S \to M$ the corresponding associated bundle. Then there is a natural isomorphism between the space $\Gamma(P \times_G S)$ of smooth sections and the space $C^\infty(P, S)^G$ of smooth functions $f : P \to S$ that are equivariant in the sense that for any $u \in P$ and $g \in G$, we get $f(r(u, g)) = g^{-1} \cdot f(u)$. Explicitly, this correspondence is characterized by $s(x) = q(u, f(u))$ for any $u \in \mathcal{P}_x$.*

In that case that the action of G comes from a finite dimensional representation of G on V, this isomorphism is linear for the obvious (point-wise) operations on both spaces.

Proof. Given a smooth section $s \in \Gamma(P \times_G S)$, $u \mapsto (u, s(p(u)))$ defines a smooth map $P \to P \times (P \times_G S)$ which by construction has values in $P \times_M (P \times_G S)$. Hence $f(u) := \tau_S(u, s(p(u)))$, where τ_S is the function from part (3) of Lemma 2.9 defines a smooth map $f : P \to S$. The characterizing property of τ_S then shows that $s(p(u)) = q(u, f(u))$ for each $u \in P$. By construction, this means that $q(u, f(u)) = q(r(u, g), f(r(u, g))) = q(u, g \cdot f(r(u, g)))$ and this implies $g \cdot f(r(u, g)) = f(u)$ and hence the required equivariancy condition. It is also clear from the construction that in the case of an associated vector bundle, the map $s \mapsto f$ is linear for the point-wise structure on $C^\infty(P, V)$.

Conversely, given $f \in C^\infty(P, S)^G$, we define $\tilde{s}(u) := q(u, f(u))$ to obtain a smooth map $\tilde{s} : P \to P \times_G S$ and by construction $\pi \circ \tilde{s} = p$. As above, equivariancy of f implies $\tilde{s}(r(u, g)) = \tilde{s}(u)$ for any $g \in G$. Hence $\tilde{s}(u)$ only depends on $p(u) \in M$, so there is a map $s : M \to P \times_G S$ such that $\tilde{s} = s \circ p$. Since p is a surjective submersion, s is smooth and by construction $\pi \circ s \circ p = p$ and hence $\pi \circ s = \mathrm{id}_M$ and hence s is a smooth section of $P \times_G S$. This construction evidently is inverse to the one above, so this completes the proof. □

As a first application of these ideas (in an unexpected setting), we can describe G-structures on a fixed manifold M as sections of an associated bundle.

Corollary 2.10. *Fix a closed subgroup $G \subset GL(V)$ for a vector space V of dimension n and consider the corresponding homogeneous space $GL(V)/G$. Then for any* *smooth manifold M of dimension n, G-structures on M are in bijective correspondence with smooth sections of the fiber bundle $\pi : E := \mathcal{P}M \times_{GL(V)} (GL(V)/G) \to M$.*

Proof. Let $o := eG$ be the base point of the homogeneous space $GL(V)/G$, and consider the map $q : \mathcal{P}M \times (GL(V)/G) \to E$. Given a G-structure $Q \subset \mathcal{P}M$ and a point $x \in M$, take an open neighborhood U of x in M such that there exists a smooth section $\sigma : U \to \mathcal{P}M$ with values in Q. Putting $s(y) := q(\sigma(y), o)$ defines a smooth map $s : U \to E$ such that $\pi \circ s = \mathrm{id}_U$, so we obtain a local smooth section of E. Suppose that $\hat{\sigma} : U \to \mathcal{P}M$ is another local section. Then for each $y \in U$, there is an element $A \in G$ such that $\hat{\sigma}(y) = r(\sigma(y), A)$, which shows that $q(\hat{\sigma}(y), o) = q(\sigma(y), A \cdot o) = s(y)$. Thus all local sections of Q defined on U give rise to the same local section of E. But this also shows that for a covering $\{U_i : i \in I\}$ of M by such subsets the resulting local sections of E agree on the intersection of their domains and hence they piece together to a global smooth section s of E.

Conversely, given a smooth section $s : M \to E$, Proposition 2.10 gives us a smooth function $f : \mathcal{P}M \to GL(V)/G$ such that $f(r(u, A)) = A^{-1} \cdot f(u)$ for all $u \in \mathcal{P}M$ and $A \in GL(V)$. Now we put $Q := f^{-1}(o) \subset \mathcal{P}M$ and verify that this satisfies the conditions of part (1) of Theorem 2.7. Given $x \in M$ and $u \in \mathcal{P}_xM$ there is an element $A \in GL(V)$ such that $f(u) = A \cdot o$ and then $r(u, A) \in Q_x$ by equivariancy of F so $Q_x \neq \emptyset$. Equivariancy of f also readily implies that for $u \in Q_x$ we get $r(u, A) \in Q_x$ if and only if $A \in G$, so the first condition is satisfied. To verify the second condition, take $u \in Q_x$ and a local section $\tilde{\sigma}$ of $\mathcal{P}M$ defined on a neighborhood of x such that $\tilde{\sigma}(x) = u$. Then $f \circ \tilde{\sigma}$ is a smooth map to $GL(V)/G$ that maps x to o. Since the projection $GL(V) \to GL(V)/G$ admits local smooth sections, we find an open neighborhood U of x in M and a smooth function $A : U \to GL(V)$ such that $f(\tilde{\sigma}(y)) = A(y) \cdot o$ for any $y \in U$. But then $\sigma(y) := r(\tilde{\sigma}(y), A(y))$ defines a smooth section $U \to \mathcal{P}M$ with values in Q and this completes the proof. □

Remark 2.10. Any fiber bundle admits local smooth sections but global smooth sections do not exist in general (see part (3) of Lemma 2.5). Hence we conclude that locally G-structures exist on any manifold (of the "right" dimension) but globally, the question of existence of sections is interesting. Indeed, Corollary 2.10 offers a systematic approach to the study of existence of G-structures. There is a part of algebraic topology called *obstruction theory*, which studies obstructions to existence of global sections of fiber bundles in terms of cohomology.

The actual behavior may be quite different, even for closely related structures. For example, it is well known that any manifold admits a Riemannian metric. From the topological point of view, this is due to the fact that $GL(n, \mathbb{R})/O(n)$ can be identified with the space of positive definite $(n \times n)$-matrices, which is convex and thus contractible. In contrast, if one looks for Lorentzian metrics, i.e. pseudo-Riemannian metrics of signature $(n-1, 1)$, it is known that such a metric exists on a smooth manifold M of dimension n if and only if there is a nowhere vanishing vector field on M. So for example such metrics do not exist on spheres of even dimension. Indeed, it turns out that existence of a nowhere vanishing vector field on M is equivalent to the fact that the Euler characteristic of M is zero.

To extend the classification of G-structures to the more general setting of a covering $\tilde{G} \to G \subset GL(V)$ discussed in Remark 2.7, an additional step is needed. Starting from a principal G-bundle $P \to M$ one can study the questions of existence and classification of *extensions* of P to the structure group $\tilde{G}$. Here an extension is a principal $\tilde{G}$-bundle $\tilde{P} \to M$ endowed with a homomorphism $\tilde{P} \to P$ over the given homomorphism $\tilde{G} \to G$

with base map id_M. Again, algebraic topology offers tools to study this questions, for the situation for spin structures is well understood in terms of so-called Stiefel-Whitney classes.

2.11. Associated bundles and morphisms. It is not surprising, that associated bundles have good functorial properties in both arguments. Suppose that $F : P \to \tilde{P}$ is a principal bundle morphism over a homomorphism $\varphi : G \to H$ with base map f. Then for a space S with a left action of H, we can define a left action of G via $g \cdot y := \varphi(g) \cdot y$. Then $F \times \mathrm{id}_S : P \times S \to \tilde{P} \times S$ has the property that $F((u, y) \cdot g) = F(u, y) \cdot \varphi(g)$, so there is a map $F[S] : P \times_G S \to \tilde{P} \times_H S$ such that $F[S] \circ q = q \circ (F \times \mathrm{id}_S)$. Since q is a surjective submersion, $F[S]$ is smooth and thus a morphism of fiber bundles with base map f. In case that we start from a representation of H on V, the above procedure leads to a representation of G, and $F[V]$ will be a homomorphism of vector bundles. Moreover, by construction $F[V]$ restricts to a linear isomorphism in each fiber of $P \times_G V$. In particular, if F is a reduction of structure group (so $f = \mathrm{id}_M$), we just obtain the restriction of the representation of H on V to the subgroup G and by Proposition 2.3, we see that $P \times_G V \cong \tilde{P} \times_H V$. In this situation, we will identify the associated bundles without further mentioning.

Similarly, assume that we have given a principal G-bundle P, two manifolds S and $\tilde{S}$ endowed with left actions of G and a G-equivariant smooth map $\alpha : S \to \tilde{S}$, i.e. $\alpha(g \cdot y) = g \cdot \alpha(y)$ for all $g \in G$ and $y \in S$. Then also $\mathrm{id}_P \times \alpha : P \times S \to P \times \tilde{S}$ is G-equivariant, so there is a map $P[\alpha] : P \times_G S \to P \times_G \tilde{S}$ such that $P[\alpha] \circ q = q \circ (\mathrm{id}_P \times \alpha)$. As above, $P[\alpha]$ is smooth and thus a morphism of fiber bundles with base-map id_M. Hence there is an induced map $\Gamma(P \times_G S) \to \Gamma(P \times_G \tilde{S})$, which can be described very easily in the language of Proposition 2.10. For $s \in \Gamma(P \times_G S)$ corresponding to an equivariant function $f : P \to S$, the induced section $P[\alpha] \circ s$ simply corresponds to $\alpha \circ f : P \to \tilde{S}$. Of course we can apply this in particular to a homomorphism $\alpha : V \to \tilde{V}$ between two representations of V and obtain a homomorphism $P[\alpha] : P \times_G V \to P \times_G \tilde{V}$ of vector bundles.

Observe that in the setting of associated vector bundles, these observations provide a simple proof for the fact that natural isomorphisms between constructions for vector spaces extend to the corresponding constructions for vector bundles. For example, given any representation of a Lie group G on V, the representations $\otimes^k(V^*)$ and $(\otimes^k V)^*$ are naturally isomorphic. Now this isomorphism induces an isomorphism between associated bundles, so $\otimes^k(E^*) \cong (\otimes^k E)^*$ where $E = P \times_G V$. Likewise, for a second representation of G on W, we get $(V \otimes W)^* \cong V^* \otimes W^*$ and $L(V, W) \cong V^* \otimes W \cong L(W^*, V^*)$ of representations. Putting $F := P \times_G W$, the induced isomorphisms between associated bundles show that $(E \otimes F)^* \cong E^* \otimes F^*$ and $L(E, F) \cong E^* \otimes F \cong L(F^*, E^*)$.

These simple observations have tremendous consequences for our perspective on G-structures. First, we get a relation between representation theory of the group G and the geometry of G-structures. Suppose that $F : P \to \mathcal{P}M$ is the reduction of structure group defining a G-structure on M. Then any representation of G on a vector space V gives rise to an associated vector bundle $P \times_G V$, and we call these the *natural vector bundles* for the G-structure. The inclusion of G into $GL(n, \mathbb{R})$ gives rise to a representation on $\mathbb{R}^n$, which in turn gives rise to representations on $\mathbb{R}^{n*}$, all tensor products $\otimes^k \mathbb{R}^{n*} \otimes \otimes^\ell \mathbb{R}^n$ and the exterior powers $\Lambda^k \mathbb{R}^{n*}$ that are again restrictions of representations of $GL(n, \mathbb{R})$. Hence we conclude that we can realize TM, T^*M, all tensor bundles and all bundles of differential forms as such natural bundles.

But it may happen, that there are more homomorphisms between representations of G than between the underlying representations of $GL(n,\mathbb{R})$. As an example, for $G := O(n)$, mapping $v \in \mathbb{R}^n$ to $\langle v, \_\rangle$ defines an isomorphism $\mathbb{R}^n \to \mathbb{R}^{n*}$ of representations of $O(n)$. Hence this induces an isomorphism between the corresponding associated bundles, which are TM and T^*M, respectively. So a choice of $O(n)$-structure gives rise to an isomorphism $TM \to T^*M$, which is certainly not available without a choice. To describe this explicitly, take $x \in M$ and $u \in Q_x$. Then $X \in T_xM$ corresponds to $u^{-1}(X) \in \mathbb{R}^n$, which then should be mapped to the functional $\langle u^{-1}(X), \_\rangle$ on $\mathbb{R}^n$. The induced functional on T_xM, then maps Y to $\langle u^{-1}(X), u^{-1}(Y)\rangle$, which equals $g_x(X,Y)$ since $u \in Q_x$. This also shows that the result is independent of the choice of u, so we have recovered the isomorphism $TM \to T^*M$ induced by a Riemannian metric. This of course works similarly for all types of tensor bundles.

A slightly different situation occurs for the representation $S^2\mathbb{R}^{n*}$ that induces the bundle of symmetric $\binom{0}{2}$-tensor fields. This is irreducible as a representation of $GL(n,\mathbb{R})$ but as a representation of $O(n)$ it splits as $\mathbb{R}\oplus S^2_0\mathbb{R}^{n*}$ into trace part and trace-free part. Correspondingly, there are homomorphisms $\mathbb{R} \to S^2\mathbb{R}^{n*}$, $t \mapsto t\langle\ ,\ \rangle$ and $S^2\mathbb{R}^{n*} \to \mathbb{R}$ given by $b \mapsto \sum_i b(v_i, v_i)$ where $\{v_i\}$ is any orthonormal basis of $\mathbb{R}^n$. Now the associated bundle corresponding to $S^2_0\mathbb{R}^{n*}$ comes with an inclusion into S^2T^*M so it defines a *subbundle* that is commonly denoted by $S^2_0T^*M$, but one has to note that this depends on a choice of $O(n)$-structure respectively a Riemannian metric. Then the bundle S^2T^*M can be identified with the Whitney sum of $M \times \mathbb{R}$ and $S^2_0T^*M$. So here one obtains a new natural bundle, which is not available without the G-structure.

The occurrence of additional natural bundles also happens in the generalization of the notion of G-structures discussed in Remark 2.7. For a covering $\tilde{G} \to G$ of a closed subgroup of $GL(V)$, the group $\tilde{G}$ may have more representations than the group G, which lead to natural bundles. Indeed, a major reason for the interest in spin structures as discussed in Remark 2.7 is that there are representations of the Lie algebra $\mathfrak{so}(n)$ that integrate to the simply connected group $Spin(n)$ with Lie algebra $\mathfrak{so}(n)$ but not to the group $SO(n)$. Given a spin structure, one may use these so-called *spin representations* to form associated bundles that are known as *spinor bundles*. The *Dirac operator* that is a cornerstone of spin geometry actually acts on sections of appropriate spinor bundles.

Second, functoriality in the other variable shows that morphisms of G-structures nicely act on all natural bundles as defined above. As we have seen in Section 2.8 a morphism f always comes as the base maps of principal bundle homomorphism. Taking a natural bundle corresponding to a left action of G on S one thus obtains a homomorphism of fiber bundles with base map f. In the case of natural vector bundles, these always are vector bundle homomorphisms and hence we get an induced action of the morphism on sections of the induced vector bundle.

2.12. The standard flat G-structure on $\mathbb{R}^n$. In the discussion of Riemannian metrics in Section 1.2, we started from Euclidean space $\mathbb{E}^n$. The corresponding $O(n)$-structure on $\mathbb{R}^n$ (or on an affine space of dimension n) has an analog for any closed subgroup $G \subset GL(n,\mathbb{R})$. Indeed, the trivialization of $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ gives rise to a trivialization of $\mathcal{P}\mathbb{R}^n \cong \mathbb{R}^n \times GL(n,\mathbb{R})$. The inverse of this trivialization maps (x,A) to the linear isomorphism $A : \mathbb{R}^n \to \mathbb{R}^n \cong T_x\mathbb{R}^n$. Of course, $\mathbb{R}^n \times G \subset \mathbb{R}^n \times GL(n,\mathbb{R})$ defines a G-structure on $\mathbb{R}^n$, which is called the *standard (flat) G-structure*. Similarly to the case of Euclidean motions discussed in Proposition 1.2, we see that that maps of the form $f(x) = Ax + b$ with $A \in G$ and $b \in \mathbb{R}^n$ form a Lie group, which is diffeomorphic to $G \times \mathbb{R}^n$. In group theory terms, this is a semi-direct product, so we write it as $G \ltimes \mathbb{R}^n$.

Note that for $G = GL(n, \mathbb{R})$ one exactly obtains the group of affine motions of $\mathbb{R}^n$ in that way, so we will refer to the elements of $G \ltimes \mathbb{R}^n$ as *G-motions*.

Given a G-motion of the form $f(x) = Ax + b$, we of course have $Df(x) = A$ for all $x \in \mathbb{R}^n$. Consequently, defining $F : \mathbb{R}^n \times G \to \mathbb{R}^n \times G$ as $F(x, B) := (f(x), AB)$, we see that F coincides with the restriction of $\mathcal{P}f$ to our G-structure. Hence any G-motion is an automorphism of the standard flat G-structure on $\mathbb{R}^n$. The proof of Proposition 1.2 actually shows that in the case that $G = O(n)$, these are the only isometries and hence the only automorphisms of the G-structure in question. Now we can rephrase the computation from that proof in Lie theoretic terms and then it can be applied to any closed subgroup $G \subset GL(n, \mathbb{R})$. Recall that for a Lie group G and a smooth function $\varphi : M \to G$ from some manifold M to G, there is the *left logarithmic derivative* $\delta\varphi \in \Omega^1(M, \mathfrak{g})$, see Section 2.8 of [**LieGrp**]. By definition $\delta\varphi(x)(X) = T_{\varphi(x)}\lambda_{\varphi(x)^{-1}}(T_x\varphi(X))$, so one just transports the value of the tangent map to the origin by the appropriate left translation. In particular, for $M = \mathbb{R}^n$, we can interpret $\delta\varphi$ as a smooth map $\delta\varphi : M \to L(\mathbb{R}^n, \mathfrak{g})$ and since $\mathfrak{g} \subset L(\mathbb{R}^n, \mathbb{R}^n)$, we can interpret the target space as a subspace of the space $\otimes^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$ of bilinear maps $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Now we first meet an algebraic object that will be crucial for the further development.

Definition 2.12. Let $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ be a Lie subalgebra. Then the *first prolongation* $\mathfrak{g}^{(1)}$ of $\mathfrak{g}$ is the intersection of $L(\mathbb{R}^n, \mathfrak{g})$ with the subspace of symmetric bilinear maps $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. So explicitly, a linear map $\alpha : \mathbb{R}^n \to \mathfrak{g} \subset L(\mathbb{R}^n, \mathbb{R}^n)$ lies in $\mathfrak{g}^{(1)}$ if and only if $\alpha(v)(w) = \alpha(w)(v)$ for any $v, w \in \mathbb{R}^n$.

Theorem 2.12. *Let $G \subset GL(n, \mathbb{R})$ be a closed subgroup and consider the standard flat G-structure on $\mathbb{R}^n$. Then for any automorphism F of this structure with base map f, the derivative Df can be viewed as a smooth function $\mathbb{R}^n \to G$ and then $\delta(Df) : \mathbb{R}^n \to L(\mathbb{R}^n, \mathfrak{g})$ has values in the subspace $\mathfrak{g}^{(1)}$. In particular, if $\mathfrak{g}^{(1)} = \{0\}$, then the automorphism of the standard flat g-structure on $\mathbb{R}^n$ are exactly the G-motions.*

Proof. By definition, an automorphism $F : \mathbb{R}^n \times G \to \mathbb{R}^n \times G$ must be of the form $F(x, A) = (f(x), Df(x)A)$ so in particular we must have $Df(x) \in G$ for any $x \in \mathbb{R}^n$. Putting $\varphi = Df : \mathbb{R}^n \to G$, we by definition get $T_x\varphi(v) = T_e\lambda_{\varphi(x)}(\delta\varphi(x)(v))$. Since we are working in a matrix group, we can view $\delta\varphi(x)(v)$ as an $n \times n$-matrix and left translations are linear, so $T_x\varphi(v)$ is just the matrix product $\varphi(x) \cdot (\delta\varphi(x)(v))$. Now the second derivative $D^2f(x)(v, w)$ is simply given by evaluating $T_x\varphi(v)$ on $w \in \mathbb{R}^n$, so putting things together, we get

$$D^2f(x)(v, w) = Df(x)((\delta Df)(x)(v)(w)) \tag{2.2}$$

Now $Df(x)$ is a linear isomorphism, so symmetry of the second derivative readily implies that $\delta Df(x) \in \mathfrak{g}^{(1)} \subset L(\mathbb{R}^n, \mathfrak{g})$.

Assuming that $\mathfrak{g}^{(1)} = \{0\}$ we conclude that $D^2f(x) = 0$ for any $x \in \mathbb{R}^n$ and hence $Df(x) = A$ for all $x \in \mathbb{R}^n$ and some fixed $A \in G$. Putting $b = f(0) \in \mathbb{R}^n$ one concludes that $f(x) = Ax + b$ as in the proof of Proposition 1.2. □

In the case that $\mathfrak{g} = \mathfrak{o}(n)$, we can easily see that the computation in the proof of Proposition 1.2 actually shows that $\mathfrak{o}(n)^{(1)} = \{0\}$. Indeed, given a linear map $\alpha : \mathbb{R}^n \to \mathfrak{gl}(n, \mathbb{R})$ consider the trilinear map $\Phi : (\mathbb{R}^n)^3 \to \mathbb{R}$ defined by $\Phi(u, v, w) := \langle \alpha(u)(v), w \rangle$. Then α has values in $\mathfrak{o}(n)$ iff $\alpha(u)$ is skew symmetric for any u, i.e. iff $\Phi(u, v, w) = -\Phi(u, w, v)$. But then α lies in $\mathfrak{o}(n)^{(1)}$ iff in addition $\Phi(u, v, w) = \Phi(v, u, w)$. In the proof of Proposition 1.2 we have shown that these symmetries imply that $\Phi = 0$ and hence $\alpha = 0$. Observe that this argument also shows that $\mathfrak{g}^{(1)} = \{0\}$ for any Lie subalgebra $\mathfrak{g} \subset \mathfrak{o}(n)$.

Similarly, the computation in Section 1.3 shows that for the Lie algebra $\mathfrak{sp}(n,\mathbb{R}) \subset \mathfrak{gl}(n,\mathbb{R})$ determined by a non-degenerate, skew symmetric bilinear form b (with n even), the first prolongation $\mathfrak{sp}(n,\mathbb{R})^{(1)}$ is non-zero. In fact one easily verifies that sending $A \in \mathfrak{sp}(n,\mathbb{R})$ to the bilinear map $(v,w) \mapsto b(Av,w)$ defines a linear isomorphism to $S^2\mathbb{R}^{n*}$ and then the first prolongation is isomorphic to $S^3\mathbb{R}^{n*} \subset \mathbb{R}^{n*} \otimes S^2\mathbb{R}^{n*}$.

What we are doing here admits a nice interpretation in terms of PDEs. Indeed, we can consider $Df(x) \in G \subset GL(n,\mathbb{R})$ as a (non-linear) first order PDE on a smooth function $f : U \to \mathbb{R}^n$ for any open subset $U \subset \mathbb{R}^n$. Theorem 2.12 then identifies fundamental differential consequences of this equation that lead to restrictions on possible solutions. If $\mathfrak{g}^{(1)} \neq \{0\}$ then one can go ahead and study restrictions on higher derivatives in a similar fashion and this leads to notions of higher prolongations of a Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{R})$. Indeed, an analysis of this type is available for rather large classes of PDE replacing Lie subalgebras of $\mathfrak{gl}(n,\mathbb{R})$ by so called tableaux, which are subspaces of $L(\mathbb{R}^n,\mathbb{R}^m)$ for general n and m. An introduction to these ideas can be found in the book [**Ivey-Landsberg**].

CHAPTER 3

Connections

As discussed in Section 1.2, a fundamental ingredient for Riemannian geometry is the Levi-Civita connection. This defines a notion of directional derivative for vector fields which is compatible with the Riemannian metric in an appropriate sense and uniquely determined by this property and torsion-freeness. It turns out that the Levi-Civita connection and similar objects can be obtained from data on a principal bundle. This will lead to the fundamental result that any G-structure on a manifold admits connections that are compatible in an appropriate sense. The question of whether one may select one of these connections by some natural condition turns out to be of algebraic nature and is closely related to the first prolongation of the Lie algebra $\mathfrak{g}$ of G discussed in Section 2.12.

Principal and induced connections

There are different versions of the concept of a connection. The notion that relates to Riemannian geometry and is easier to understand intuitively is a linear connection on a vector bundle. On the other hand, there is the concept of principal connections, which is less intuitive but turns out to be more versatile and technically useful. This then leads to a linear connection on any associated bundle and indeed, the concept is suggested by the description of sections of a vector bundle associated to a principal bundle in Proposition 2.10. Since we will only be interested in linear connections obtained in that way, we are rather sketchy on how to obtain concepts and results directly in the language of linear connections.

3.1. Linear connections. Let $p : E \to M$ be a vector bundle on a smooth manifold M. Then one defines a *linear connection* on E as a bilinear operator $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ written as $(\xi, s) \mapsto \nabla_\xi s$ such that for any $f \in C^\infty(M, \mathbb{R})$, we get $\nabla_{f\xi} s = f\nabla_\xi s$ and $\nabla_\xi(fs) = \xi(f)s + f\nabla_\xi s$. As usual, linearity over smooth functions in the first entry implies that for a point $x \in M$, the value $\nabla_\xi s(x) \in E_x$ depends only on $\xi(x)$. The Leibniz rule in the second variable easily implies that, in a local trivialization, $\nabla_\xi s(x) \in E_x$ depends on the values and first derivatives of the functions describing s in the point x. So this is exactly what one would expect from an abstract version of a directional derivative.

It is easy to see that locally there are linear connections on any vector bundle E. Given a local frame $\{s_i\}$ defined on $U \subset M$, any local smooth section of E over U can be written as $\sum_i f_i s_i$ for smooth functions $f_i : U \to \mathbb{R}$. Then one can simply define $\nabla_\xi s := \sum_i \xi(f_i)s_i$. Using a vector bundle atlas for E and a subordinate partition of unity, one can glue such local connections to a global connection on E, compare with the second proof of Theorem 1.9 in [**Riem**]. Moreover, for two linear connections ∇ and $\hat{\nabla}$ on E, define $A : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ by $A(\xi, s) := \hat{\nabla}_\xi s - \nabla_\xi s$. Then by definition, this expression is linear over smooth functions in both arguments. If $E = TM$, this shows that A is a $\binom{1}{2}$-tensor field. In general the same proof as for Lemma 3.3 in [**AnaMF**] shows that $A(\xi, s)(x)$ only depends on $\xi(x)$ and $s(x)$. Hence A defines a vector bundle homomorphism $T^*M \otimes E \to E$ with base map id_M, or, equivalently, a section of the vector bundle $T^*M \otimes L(E,E)$. Hence the structure of the space of all linear connections on E is rather simple.

Parallel to the Riemann curvature tensor, one defines the *curvature* of a general linear connection. Given a linear connection ∇ on a vector bundle E, one considers the trilinear map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ defined by

$$R(\xi,\eta)(s) := \nabla_\xi \nabla_\eta s - \nabla_\eta \nabla_\xi s - \nabla_{[\xi,\eta]} s. \tag{3.1}$$

By definition, this is skew symmetric in ξ and η. Moreover, a direct computation shows that R is linear over smooth functions in all arguments, so as above, we conclude that it defines a section of $\Lambda^2 T^*M \otimes L(E,E)$. In the case that $E = TM$ this means that R is a $\binom{1}{3}$-tensor field which is skew symmetric in the first two entries.

In the case of linear connections on the tangent bundle TM of a smooth manifold M, there is an additional invariant which will be of crucial importance in what follows. We define the *torsion* of a linear connection ∇ on TM as a bilinear operation $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by

$$T(\xi,\eta) := \nabla_\xi \eta - \nabla_\eta \xi - [\xi,\eta]. \tag{3.2}$$

Again, this is skew symmetric and by definition and standard properties of the Lie bracket it is bilinear over smooth functions. Hence T defines a skew-symmetric $\binom{1}{2}$-tensor field, i.e. a section of the vector bundle $\Lambda^2 T^*M \otimes TM$.

To formulate analogs of compatibility of the Levi-Civita connection with the Riemannian metric, one has to extend linear connections to bundles obtained by constructions. This is based on the same naturality principles as the extension of the Levi-Civita connection to tensor fields, see Section 2.7 of [**Riem**]. Let $E \to M$ be a vector bundle, ∇^E a linear connection on E and $E^* \to M$ the dual bundle to E. Given sections $\varphi \in \Gamma(E^*)$ and $s \in \Gamma(E)$, we can form $\varphi(s) \in C^\infty(M,\mathbb{R})$ via the point-wise dual pairing. For a vector field $\xi \in \mathfrak{X}(M)$ one then defines

$$(\nabla^{E^*}_\xi \varphi)(s) := \xi \cdot \varphi(s) - \varphi(\nabla^E_\xi s). \tag{3.3}$$

More precisely, one first fixes ξ and φ and views the right hand side as an operator $\Gamma(E) \to C^\infty(M,\mathbb{R})$. By definition, this is linear over $C^\infty(M,\mathbb{R})$ (in s) and hence defines a section of E^* that one denotes by $\nabla^{E^*}_\xi \varphi$. Since the right hand side is clearly linear over smooth functions in ξ, we see that $\nabla^{E^*}_{f\xi}\varphi = f\nabla^{E^*}_\xi \varphi$. On the other hand, the definition readily implies that $\nabla^{E^*}_\xi (f\varphi) = \xi(f)\varphi + f\nabla^{E^*}_\xi \varphi$. Hence we have indeed defined a linear connection on E^*.

Next for two vector bundles E and F over M with linear connections ∇^E and ∇^F consider the tensor product $E \otimes F$. Given sections $s_1 \in \Gamma(E)$ and $s_2 \in \Gamma(F)$, one can form the point-wise tensor product $s_1 \otimes s_2 \in \Gamma(E \otimes F)$. Then one easily proves that there is a unique linear connection $\nabla^{E\otimes F}$ on $E \otimes F$, such that for each $\xi \in \mathfrak{X}(M)$ and s_1, s_2 as above, one gets

$$\nabla^{E\otimes F}_\xi (s_1 \otimes s_2) = (\nabla^E_\xi s_1) \otimes s_2 + s_1 \otimes \nabla^F_\xi s_2. \tag{3.4}$$

Of course, this then extends to tensor products of more than two factors. In particular, for a vector bundle E with linear connection ∇^E, one obtains connections on all the bundles $\otimes^k E \otimes \otimes^\ell E^*$. It is easy to see that these preserve symmetry properties of tensors, so this also works with symmetric and alternating powers. It is then usual to simply denote all these connections by ∇. In particular, a linear connection on TM in this way gives rise to linear connections on all tensor bundles, so one can for example form the covariant derivative of the torsion of the connection. To form a covariant derivative of the curvature of a linear connection ∇ on a vector bundle E, one however needs additional input, namely an auxiliary linear connection on TM.

3.2. Towards principal connections. As mentioned already, the motivation for the concept of a principal connection comes from the description of sections of an associated vector bundle in Proposition 2.10. For a principal G-bundle $p : P \to M$, a representation of G on V and the corresponding associated vector bundle $E := P \times_G V$ this shows that $\Gamma(E) \cong C^\infty(P, V)^G$. Now smooth functions $P \to V$ can be differentiated in the direction of vector fields on P without problems. It is also easy to make sure that such a derivative is again equivariant. Given $g \in G$, let us denote by $r^g : P \to P$ the principal right action by g. A vector field $\tilde{\xi} \in \mathfrak{X}(P)$, is then called *$G$-invariant* if $(r^g)^*\tilde{\xi} = \tilde{\xi}$ for any $g \in G$. Evidently, G-invariant vector fields form a linear subspace of $\mathfrak{X}(P)$ which we denote by $\mathfrak{X}(P)^G$.

For $\tilde{\xi} \in \mathfrak{X}(P)^G$ we get $\tilde{\xi} \sim_{r^g} \tilde{\xi}$ and hence $\tilde{\xi}(f \circ r^g) = \tilde{\xi}(f) \circ r^g$ for any $f \in C^\infty(P, V)$ and any $g \in G$, see Proposition 2.2 of [**AnaMF**]. For $f \in C^\infty(P, V)^G$, we get $f \circ r^g = \ell_{g^{-1}} \circ f$, where $\ell_{g^{-1}} : V \to V$ denotes the action of g^{-1} coming from the given representation. Since this is a linear map, we conclude that $\tilde{\xi}(\ell_{g^{-1}} \circ f) = \ell_{g^{-1}} \circ \tilde{\xi}(f)$, so we conclude that $\tilde{\xi}(f) : P \to V$, is G-equivariant, too. Thus the derivative of a G-equivariant function in direction of a G-invariant vector field is G-equivariant. Observe further that for $\tilde{\xi} \in \mathfrak{X}(P)^G$ and a point $u \in P$, we get $\tilde{\xi}(r^g(u)) = T_u r^g(\tilde{\xi}(u))$. Now $p \circ r^g = p$ implies $T_{r^g(u)}p \circ T_u r^g = T_u p$ and hence $T_{r^g(u)}p \cdot \tilde{\xi}(r^g(u)) = T_u p \cdot \tilde{\xi}(u)$. Since the points $r^g(u)$ exhaust the fiber of P through u, this means that $\tilde{\xi}$ is *projectable.* Hence there is a unique vector field $\xi \in \mathfrak{X}(M)$ such that $\tilde{\xi} \sim_p \xi$ and thus $\xi(p(u)) = T_u p(\tilde{\xi}(u))$ for all $u \in P$.

Now recall that for $u \in P$, we have the vertical subspace $V_u P \subset T_u P$ and elements in there can be uniquely written as values of fundamental vector fields $\zeta_X(u)$ for $X \in \mathfrak{g}$, see Lemma 2.4. By definition, $\zeta_X(u) = \frac{d}{dt}|_{t=0} r^{\exp(tX)}(u)$, so for $f \in C^\infty(M, V)^G$ we obtain

$$(\zeta_X \cdot f)(u) = \tfrac{d}{dt}|_{t=0} f(r^{\exp(tX)}(u)) = \tfrac{d}{dt}|_{t=0} \exp(-tX) \cdot (f(u)) = -X \cdot (f(u)), \tag{3.5}$$

where in the right hand side we use the infinitesimal action of $X \in \mathfrak{g}$ on $f(u) \in V$. Hence the derivatives of equivariant functions in vertical directions are completely determined by equivariancy. Observe finally, that for any linear subspace $W \subset T_u P$ which is complementary to $V_u P$, the linear map $T_u p$ restricts to a linear isomorphism $W \to T_{p(u)} M$.

These considerations now motivate the definition of a principal connection as a distribution on P with specific properties:

Definition 3.2. Let $p : P \to M$ be a principal G-bundle.

(1) A *principal connection* on P is a smooth distribution $H \subset TP$ of rank $n := \dim(M)$ which is complementary to the vertical subbundle and G-invariant, i.e. $H_u \cap V_u P = \{0\}$ and $H_{r^g(u)} = T_u r^g(H_u)$ for any $u \in P$ and $g \in G$.

(2) One refers to H as a the *horizontal distribution* of the principal connection. A tangent vector $X \in T_u P$ is called *horizontal* (with respect to the given principal connection) if $X \in H_u$. Similarly, a vector field $\tilde{\xi} \in \mathfrak{X}(P)$ is called *horizontal* if $\tilde{\xi}(u) \in H_u$ for all $u \in P$.

Having these preparations at hand, it is rather easy to see that a principal connection gives rise to a linear connection on any associated vector bundle.

THEOREM 3.2. *Let $p : P \to M$ be a principal G-bundle and let $H \subset TP$ be a principal connection on P. Then we have*

(1) Take a point $x \in M$ and a tangent vector $X \in T_xM$. Then for each point $u \in P$ with $p(u) = x$ there is a unique tangent vector $X^h \in H_u \subset T_uP$ such that $T_up \cdot X^h = X$.

(2) For a smooth vector field $\xi \in \mathfrak{X}(M)$ the construction in (1) gives rise to a vector field $\xi^h \in \mathfrak{X}(P)$ which is horizontal and G-invariant.

(3) Let V be a representation of G and $E := P \times_G V \to M$ the corresponding associated vector bundle. Given a vector field $\xi \in \mathfrak{X}(M)$, consider $\xi^h \in \mathfrak{X}(P)$ as in (2). For a section $s \in \Gamma(E)$ corresponding to $f \in C^\infty(P, V)^G$, we have $\xi^h(f) \in C^\infty(P, V)^G$ and denoting the corresponding section by $\nabla_\xi s$ defines a linear connection ∇ on E.

PROOF. By definition $\dim(H_u) = \dim(T_xM)$ and since $H_u \cap V_uP = \{0\}$, the restriction of T_up to H_u is injective. Thus it has to be a linear isomorphism and (1) follows.

(2) By (1), $\xi^h(u) \in H_u$ is uniquely determined by $T_up(\xi^h(u)) = \xi(p(u))$. Moreover, equivariancy of H implies that $T_ur^g(\xi^h(u)) \in H_{r^g(u)} \subset T_{r^g(u)}P$ and of course $T_{r^g(u)}p$ maps this tangent vector to $\xi(p(u))$. Hence we conclude that $\xi^h(r^g(u)) = T_ur^g(\xi^h(u))$ and since ξ^h is horizontal by construction, we only have to show that ξ^h is smooth in order to complete the proof of (2).

This is a local question, so we may restrict to an open subset $W \subset P$. Since H is a smooth distribution, we may choose W in such a way that there is a local frame $\{\tilde\eta_i : i = 1 \dots n\}$ for H defined on W. In addition, we may assume that there is a principal bundle chart (U, φ) for P such that $W \subset p^{-1}(U)$. This means that $\varphi : p^{-1}(U) \to U \times G$ is a diffeomorphism with first component p, so $T\varphi : Tp^{-1}(U) \to TU \times TG$ is a diffeomorphism with first component Tp. Now we define $\tilde\xi \in \mathfrak{X}(p^{-1}(U))$ as $\tilde\xi(y) := (T_y\varphi)^{-1}(\xi(p(y)), 0)$. Clearly this is a local vector field on $p^{-1}(U)$ such that $Tp \circ \tilde\xi = \xi \circ p$, so it is a lift of ξ.

Now choosing a basis $\{X_\alpha\}$ for $\mathfrak{g}$, the vector fields ζ_{X_α} form a global frame for VP, so together with the $\tilde\eta_i$, their restrictions to W form a local frame for TP defined on W. Hence there are smooth functions $a_i, b_\alpha : W \to \mathbb{R}$ such that

$$\tilde\xi|_W = \textstyle\sum_i a_i\tilde\eta_i + \sum_\alpha b_\alpha \zeta_{X_\alpha}|_W. \tag{3.6}$$

Hence also $\sum_i a_i\tilde\eta_i$ is a smooth vector field on W, which by construction is horizontal. Since the second sum in (3.6) has values in the kernel of Tp, we conclude that $T_yp(\sum_i a_i\tilde\eta_i) = T_yp(\tilde\xi(y)) = \xi(p(y))$. But this shows that we must have $\sum_i a_i\tilde\eta_i = \xi^h|_W$, which completes the argument.

(3) In (2) we have seen that $\xi^h \in \mathfrak{X}(P)^G$, so we already know that for $f \in C^\infty(P, V)^G$, we also have $\xi^h(f) \in C^\infty(P, V)^G$ and hence the operator ∇ is well defined. Now let $a \in C^\infty(M, \mathbb{R})$ be a smooth function. Then for $\xi \in \mathfrak{X}(M)$ of course $(a \circ p)\xi^h \in \mathfrak{X}(P)$ has values in H and is a lift of $a\xi$, so we must have $(a\xi)^h = (a \circ p)\xi^h$. Using this to differentiate $f \in C^\infty(P, V)^G$, we obtain $(a \circ p)\xi^h(f)$, which clearly corresponds to $a\nabla_\xi s$, where f corresponds to s. Thus we conclude that $\nabla_{a\xi}s = a\nabla_\xi s$. Likewise, as corresponds to $(a \circ p)f : P \to V$ and applying ξ^h to this, we obtain

$$(\xi^h(a \circ p))f + (a \circ p)\xi^h(f).$$

Since ξ^h is a lift of ξ, the first summand coincides with $(\xi(a) \circ p)f$, so our expression corresponds to $\xi(a)s + a\nabla_\xi s$, which is exactly what we need to complete the proof. □

The tangent vector X^h in (1) is called the *horizontal lift* of X and the vector field ξ^h in (2) is called the *horizontal lift* of ξ. The linear connection ∇ on $P \times_G V$ obtained in (3) is referred to as the *induced connection* obtained from the principal connection H.

3.3. Connection forms. While the description of a principal connections in terms of a horizontal distribution gives a very satisfactory explanation for the existence of induced linear connections, it is not so easy to handle technically. There is a simpler equivalent description of a principal connection, which is easier to handle technically and is often used as the definition of a principal connection in the literature. The route towards this description is visible to some extent from the proof of Theorem 3.2.

In linear algebra terms, a linear subspace $H_u \subset T_uP$ such that $T_uP = H_u \oplus V_uP$ can be equivalently encoded into the projection $\Pi_u : T_uP \to V_uP$ onto the second component. By definition $X \in T_uP$ can be uniquely written as $X_1 + X_2$ with $X_1 \in H_u$ and $X_2 \in T_uP$, and $\Pi_u(X) := X_2$. Of course, the linear map Π_u satisfies $\Pi_u \circ \Pi_u = \Pi_u$ and $\mathrm{im}(\Pi_u) = V_uP$ and it is characterized by these two properties. Moreover, $\ker(\Pi_u) = H_u$ and given a projection onto V_uP, one obtains a complementary subspace H_u via this definition. The projections Π_u fit together to define $\Pi : TP \to TP$ (or $TP \to VP$). As in the proof of Theorem 3.2, one readily shows that for $\tilde{\xi} \in \mathfrak{X}(P)$ we get $\Pi(\tilde{\xi}) \in \mathfrak{X}(P)$, so Π is a vector bundle homomorphism and can also be interpreted as a $\binom{1}{1}$-tensor field.

As a second step, we can now use the fact that the vertical subbundle VP is trivialized via fundamental vector fields. This means that for a tangent vector $X \in T_uP$ and a vertical projection Π as above, there is a unique element $A \in \mathfrak{g}$ such that $\Pi_u(X) = \zeta_A(u)$. Sending X to A defines a linear map $T_uP \to \mathfrak{g}$ which by construction has the property that it maps $\zeta_A(u)$ to A for any $A \in \mathfrak{g}$. Hence we can interpret this construction as associating to each $u \in P$ a linear map $\gamma(u) : T_uP \to \mathfrak{g}$, so if this is smooth, then it defines a $\mathfrak{g}$-valued one-form $\gamma \in \Omega^1(M, \mathfrak{g})$; see Section 2.7 of [**LieGrp**] for information on differential forms with values in a vector space. Now we need a few more notions to proceed.

Definition 3.3. Let $p : P \to M$ be a principal G bundle and let W be a finite dimensional vector space.

(1) A W-valued differential form $\varphi \in \Omega^k(M, W)$ is called *horizontal* if for any point $u \in P$ and tangent vectors $X_1, \dots, X_k \in T_uP$, we get $\omega(u)(X_1, \dots, X_k) = 0$ provided that one of the X_i lies in $V_uP \subset T_uP$.

(2) If W is a representation of G, then $\varphi \in \Omega^k(M, W)$ is called *G-equivariant* if for any $g \in G$, $u \in P$ and $X_1, \dots, X_k \in T_uP$, we get

$$((r^g)^*\varphi)(u)(X_1, \dots, X_k) = g^{-1} \cdot (\varphi(u)(X_1, \dots, X_k)) \tag{3.7}$$

(3) A *principal connection form* on P is a $\mathfrak{g}$-valued one-form $\gamma \in \Omega^1(P, \mathfrak{g})$ which is G-equivariant (for the adjoint representation of G on $\mathfrak{g}$) and has the property that $\gamma(u)(\zeta_A(u)) = A$ for any $u \in P$ and $A \in \mathfrak{g}$.

Observe that in this terminology, the soldering form $\theta \in \Omega^1(\mathcal{P}M, \mathbb{R}^n)$ from Proposition 2.6 is horizontal and equivariant for the standard representation of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$. The same holds for the forms $\tilde{\theta}$ from Theorem 2.7 for the standard representation of a closed subgroup of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$.

Theorem 3.3. *Let $p : P \to M$ be a principal G-bundle.*

(1) There is a principal connection form γ on P.

(2) There is a bijective correspondence between principal connection forms $\gamma \in \Omega^1(P, \mathfrak{g})$ and principal connections $H \subset TP$, which is characterized by $H_u = \ker(\gamma(u))$.

(3) If γ is a principal connection form on P and $\psi \in \Omega^1(P, \mathfrak{g})$ is horizontal and G-equivariant, then $\gamma + \psi$ is a principal connection form, too. Conversely, if γ and $\hat{\gamma}$ are principal connection forms than $\hat{\gamma} - \gamma \in \Omega^1(P, \mathfrak{g})$ is horizontal and G-equivariant.

PROOF. (1) Let $\varphi : p^{-1}(U) \to U \times G$ be a principal bundle chart for P. Let ω be the left Maurer-Cartan form on G, so $\omega(g)(X) = T_g\lambda_{g^{-1}}(X) \in T_eG = \mathfrak{g}$ an consider $\gamma := (\mathrm{pr}_2 \circ \varphi)^*\omega \in \Omega^1(p^{-1}(U), \mathfrak{g})$. By definition, $(\mathrm{pr}_2 \circ\varphi) \circ r^g = \rho^g \circ (\mathrm{pr}_2 \circ\varphi)$, where ρ^g denotes right translation by g in G. Hence $(r^g)^*\gamma = (\mathrm{pr}_2 \circ\varphi)^*(\rho^g)^*\omega$ and the definition of ω easily implies that $(\rho^g)^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega$, see Proposition 2.7 in [**LieGrp**]. This shows that $(r^g)^*\gamma = \mathrm{Ad}(g^{-1}) \circ \gamma$, so γ is equivariant. For $g = \exp(tA)$, we can differentiating the equation $(\mathrm{pr}_2 \circ\varphi) \circ r^{\exp(tA)} = \rho^{\exp(tA)} \circ (\mathrm{pr}_2 \circ\varphi)$ with respect to t at $t = 0$ to see that $T_u(\mathrm{pr}_2 \circ\varphi)(\zeta_A(u)) = L_A(\mathrm{pr}_2 \circ\varphi(u))$, the left invariant vector field generated by A. This is mapped to A by ω, so $\gamma(\zeta_A(u)) = A$ follows and hence γ is a connection form on $p^{-1}(U)$.

Now we take a principal bundle atlas $(U_\alpha, \varphi_\alpha)$ for P and a partition $\{f_i : i \in \mathbb{N}\}$ of unity subordinate to the open covering $\{U_\alpha\}$ of M. For each $i \in \mathbb{N}$ choose an index $\alpha(i)$ such that $\mathrm{supp}(f_i) \subset U_{\alpha(i)}$ and a principal connection form γ_i on $p^{-1}(U_{\alpha(i)})$. Then $(f_i \circ p)\gamma_i$ can be extended by 0 to an element of $\Omega^1(P, \mathfrak{g})$ and we put $\gamma := \sum_i (f_i \circ p)\gamma_i$. This is a well defined $\mathfrak{g}$-valued one-form on P. Since each $f_i \circ p$ is constant along the fibers of P, equivariancy of the forms γ_i implies equivariancy of γ. Moreover $\gamma(u)(\zeta_A(u)) = \sum_i f_i(p(u))A = A$ for each $u \in P$ and each $A \in \mathfrak{g}$.

(2) Given a principal connection form γ, we know that each $\gamma(u) : T_uP \to \mathfrak{g}$ is surjective, so $H_u := \ker(\gamma(u))$ has dimension $\dim(P) - \dim(G) = \dim(M)$. Moreover, one immediately verifies that for a smooth function $\psi : P \to \mathfrak{g}$, $u \mapsto \zeta_{\psi(u)}(u)$ is smooth and hence defines a vector field on P. This implies that for any vector field $\tilde{\xi} \in \mathfrak{X}(P)$, $u \mapsto \tilde{\xi}(u) - \zeta_{\gamma(\tilde{\xi})(u)}(u) \in H_u$ defines a vector field on P. Starting from a basis of H_u this leads to a local frame for H in a neighborhood of u, which shows that H_u is a smooth distribution. The fact that $T_uP = H_u \oplus V_uP$ is obvious from the construction. Finally, for $X \in H_u$ we get $((r^g)^*\gamma)(u)(X) = \gamma(r^g(u))(T_ur^g(X))$ and by equivariancy, the left hand side vanishes. This shows that $T_ur^g(X) \in H_{r^g(u)}$ and hence $T_ur^g(H_u) \subset H_{u\cdot g}$ and since both spaces have the same dimension, they have to agree.

Conversely, if $H \subset TP$ is a principal connection, we denote by Π the corresponding vertical projection and define $\gamma(u) : T_uP \to \mathfrak{g}$ by $\Pi_u(X) = \zeta_{\gamma(u)(X)}(u)$ for $X \in T_uP$. Hence $\ker(\gamma(u)) = \ker(\Pi_u) = H_u$ and $\gamma(u)$ is characterized by this property together with $\gamma(u)(\zeta_A(u)) = A$ for all $A \in \mathfrak{g}$. We have already observed that Π is smooth, which easily implies that γ is smooth and thus a $\mathfrak{g}$-valued one-form on P. Hence it remains to verify equivariancy, i.e. that $\gamma(r^g(u))(T_ur^g(X)) = \mathrm{Ad}(g^{-1})(\gamma(u)(X))$ for any $u \in P$, $X \in T_uP$ and $g \in G$. Since both sides of the equation are linear in X, it suffices to insert elements of some basis of T_uP and we can take this to consist of elements of H_u and of V_uP. But for $X \in H_u$, we have $T_ur^g(X) \in H_{r^g(u)}$ and hence both sides of the equation vanish. On the other hand, take $A \in \mathfrak{g}$ and consider $X = \zeta_A(u)$. Now recall that $\exp(t\,\mathrm{Ad}(g^{-1})(A)) = g^{-1}\exp(tA)g$ and acting with this on $r^g(u)$ and differentiating at $t = 0$, we get $\zeta_{\mathrm{Ad}(g^{-1})(A)}(r^g(u))$. But by the property of a right action, the curve can be written as $r^g(r^{\exp(tA)}(u))$, so differentiating we get $T_ur^g(\zeta_A(u))$. But this exactly says that $((r^g)^*\gamma)(u)(\zeta_A(u)) = \mathrm{Ad}(g^{-1})(A)$ which is the required property, so the proof of (2) is complete.

(3) Of course the sum and difference of equivariant forms is equivariant. Since $\gamma+\psi$ acts on fundamental fields in the same way as γ, we see that this is again a connection

form. Conversely, γ and $\hat{\gamma}$ act in the same way on fundamental vector fields and hence on vertical tangent vectors, their difference vanishes on vertical tangent vectors. □

Observe that it is not yet clear whether there are non-zero horizontal equivariant $\mathfrak{g}$-valued one-forms on a principal G-bundle P and how many such forms there are. This will be clarified in Lemma 3.4 below.

3.4. Curvature of principal connections. The idea for the definition of the curvature of a principal connection is rather simple. As we shall see, the curvature can be viewed as the obstruction to integrability of the horizontal distribution. There is a slick way to formulate this in terms of the principal connection form as follows.

DEFINITION 3.4. Let $p : P \to M$ be a principal G-bundle and let $\gamma \in \Omega^1(P, \mathfrak{g})$ be the connection form of a principal connection $H \subset TP$. Then the *curvature form* $\Omega \in \Omega^2(P, \mathfrak{g})$ is defined by

$$\Omega(u)(X,Y) := d\gamma(u)(X,Y) + [\gamma(u)(X), \gamma(u)(Y)] \tag{3.8}$$

for $u \in P$ and $X, Y \in T_uP$. Here the bracket in the right hand side is the Lie bracket of $\mathfrak{g}$.

PROPOSITION 3.4. *(1) The curvature form $\Omega \in \Omega^2(P, \mathfrak{g})$ is horizontal and G-equivariant.*

(2) For two sections $\tilde{\xi}$ and $\tilde{\eta}$ of $H \subset TP$ and any point $u \in P$, the vertical component of $[\tilde{\xi}, \tilde{\eta}](u)$ equals $\zeta_A(u)$ where $A = -\Omega(u)(\tilde{\xi}, \tilde{\eta})$. In particular, Ω vanishes identically if and only if the horizontal distribution H is involutive.

PROOF. (1) The fact that d commutes with pullbacks extends without problems to forms with values in a vector space. Hence $(r^g)^*d\gamma = d((r^g)^*\gamma) = d(\mathrm{Ad}(g^{-1}) \circ \gamma)$. Since $\mathrm{Ad}(g^{-1})$ is a linear map, the definition of d readily implies that this equals $\mathrm{Ad}(g^{-1}) \circ d\gamma$. For the second term, we observe that pulling back along r^g gives

$$\begin{aligned}[((r^g)^*\gamma)(u)(X), ((r^g)^*\gamma)(u)(Y)] &= [\mathrm{Ad}(g^{-1})(\gamma(u)(X)), \mathrm{Ad}(g^{-1})(\gamma(u)(Y))] \\ &= \mathrm{Ad}(g^{-1})([\gamma(u)(X), \gamma(u)(Y)]),\end{aligned}$$

so equivariancy follows.

To prove horizontality, we have to show that $i_{\zeta_A}\Omega \in \Omega^1(P, \mathfrak{g})$ vanishes identically for any $A \in \mathfrak{g}$. Since any tangent vector can be extended to a G-invariant vector field, it suffices to prove that $\Omega(\zeta_A, \tilde{\xi}) = 0$ for any $\tilde{\xi} \in \mathfrak{X}(P)^G$. Now since the flow of ζ_A is $r^{\exp(tA)}$ we conclude that $(\mathrm{Fl}_t^{\zeta_A})^*\tilde{\xi} = \tilde{\xi}$ and hence $0 = \mathcal{L}_{\zeta_A}(\tilde{\xi}) = [\zeta_A, \tilde{\xi}]$. Since $\gamma(\zeta_A) = A$ is constant, this shows that $d\gamma(\zeta_A, \tilde{\xi}) = \zeta_A \cdot \gamma(\tilde{\xi})$. But invariance of $\tilde{\xi}$ implies that $\tilde{\xi}(r^g(u)) = T_ur^g(\tilde{\xi}(u))$ and applying $\gamma(r^g(u))$ to this, we get $((r^g)^*\gamma)(\tilde{\xi}(u))$. Hence equivariancy of γ implies that the smooth function $\gamma(\tilde{\xi}) : P \to \mathfrak{g}$ satisfies $\gamma(\tilde{\xi})(r^g(u)) = \mathrm{Ad}(g^{-1})(\gamma(\tilde{\xi})(u))$. Applying this to $g = \exp(tA)$ and differentiating at $t = 0$ we conclude that

$$d\gamma(\zeta_A, \tilde{\xi}) = -[A, \gamma(\tilde{\xi})] = -[\gamma(\zeta_A), \gamma(\tilde{\xi})],$$

and horizontality follows.

(2) Our assumptions imply that the functions $\gamma(\tilde{\xi})$ and $\gamma(\tilde{\eta})$ vanish identically. So using the global formula for the exterior derivative, we conclude that $\Omega(\tilde{\xi}, \tilde{\eta}) = -\gamma([\tilde{\xi}, \tilde{\eta}])$ and the claim follows. □

To relate this to the curvature of induced linear connections, we need a generalization of Proposition 2.10 to differential forms. For a vector bundle $E \to M$ we define an E-valued k-form as a smooth section of the bundle $\Lambda^k T^*M \otimes E$. Such a form thus associates to each $x \in M$ a k-linear, alternating map $(T_xM)^k \to E_x$ that depends smoothly on x in an obvious sense. The space of all such forms will be denoted by $\Omega^k(M, E)$. This generalizes forms with values in a finite dimensional vector space V, which arise via the trivial vector bundle $M \times V \to M$. Note however, that for bundle valued forms, there a priori is neither a well defined pullback nor a wedge product nor an exterior derivative.

LEMMA 3.4. *Let $p : P \to M$ be a principal G-bundle, V a representation of G, denote by $\pi : P \times_G V \to M$ the corresponding associated bundle. Then there is a natural bijective correspondence between the space of horizontal, equivariant V-valued k-forms on P and the space $\Omega^k(M, P \times_G V)$.*

Denoting by $q : P \times V \to P \times_G V$ the canonical map, the correspondence between $\varphi \in \Omega^k(P, V)$ and $\alpha \in \Omega^k(M, P \times_G V)$ is characterized by

$$\alpha(p(u))(T_up(X_1), \dots, T_up(X_k)) = q(u, \varphi(u)(X_1, \dots, X_k)) \tag{3.9}$$

for $u \in P$ and $X_1, \dots, X_k \in T_uP$.

PROOF. Having given $\alpha \in \Omega^k(M, P \times_G V)$ and a point $u \in P$, we can define a map $\varphi(u) : (T_uP)^k \to V$ via (3.9). In the language of Lemma 2.9, this means that

$$\varphi(u)(X_1, \dots, X_k) = \tau_V\left(u, \alpha(p(u))(T_up(X_1), \dots, T_up(X_k))\right). \tag{3.10}$$

Obviously, this vanishes if one of the X_i is vertical, and replacing u by $r^g(u)$, we readily see that $\varphi(r^g(u))(T_ur^g(X_1), \dots, T_ur^g(X_k)) = g^{-1} \cdot \varphi(u)(X_1, \dots, X_k)$.

Conversely, given a horizontal, equivariant form φ, we observe that for $X \in T_uP$ and $Y \in T_{r^g(u)}P$ with $T_up(X) = T_{r^g(u)}p(Y)$ the difference $Y - T_ur^g(X)$ is vertical. This implies that Y and $T_ur^g(X)$ lead to the same result when inserted into φ and using this, we conclude that we can use (3.9) to define $\alpha(p(u)) : (T_xM)^k \to E_x$.

Hence the correspondence works out point-wise and it remains to show that smoothness has the same meaning in both pictures. To prove this, we can choose some principal connection on P and use the resulting horizontal lift for vector fields. Starting from a smooth form φ, take vector fields $\xi_1, \dots, \xi_k \in \mathfrak{X}(M)$. Then (3.9) shows that the smooth function $P \to V$ that maps u to $q(u, \varphi(u)(\xi_1^h(u), \dots, \xi_k^h(u)))$ descends to the function $\alpha(\xi_1, \dots, \xi_k)$ on M, which thus is smooth. Conversely, (3.10) shows that starting with a smooth form α, also $\varphi(\xi_1^h, \dots, \xi_k^h)$ is smooth for arbitrary $\xi_i \in \mathfrak{X}(M)$. But locally, there are smooth frames for TP which consist of such horizontal lifts and of fundamental vector fields. Since φ vanishes upon insertion of one of the latter, this completes the proof. □

At this point, we need just one more observation. For a representation $\beta : G \to GL(V)$ the derivative $\beta' : \mathfrak{g} \to L(V, V)$ is not only a homomorphism of Lie algebras but also a homomorphism for the natural representations of G. Indeed, for $g \in G$, $A \in \mathfrak{g}$ and $v \in V$, we can compute $\operatorname{Ad}(g)(A) \cdot v = \frac{d}{dt}|_{t=0} \exp(t \operatorname{Ad}(g)(A)) \cdot v = \frac{d}{dt}|_{t=0} g \exp(tA) g^{-1} \cdot v$. Since the action of g is by a linear map, this just gives $g \cdot A \cdot g^{-1} \cdot v$, which is exactly what we claimed. For the associated bundle $E := P \times_G V$ to a principal G-bundle P, we of course get $L(E, E) = P \times_G L(V, V)$ and hence β' induces a vector bundle homomorphism $P \times_G \mathfrak{g} \to L(E, E)$. It is common to suppress this from the notation and just say that a section of $P \times_G \mathfrak{g}$ gives rise to a section of $L(E, E)$. Of course this applies analogously to differential forms with values in these bundles.

THEOREM 3.4. *Let $p : P \to M$ be a principal G-bundle, V a representation of G and $E := P \times_G V$ the corresponding associated bundle. Consider a principal connection form γ on P with curvature form Ω. Then the form in $\Omega^2(M, L(E,E))$ induced by the horizontal equivariant form $\Omega \in \Omega^2(P, \mathfrak{g})$ via Lemma 3.4 and the infinitesimal representation $\mathfrak{g} \to L(V,V)$ coincides with the curvature R of the induced linear connection on E.*

PROOF. This is just a direct computation. Take the definition of $R(\xi,\eta)(s)$ in formula (3.1) for $\xi, \eta \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$. Denoting by $f : P \to V$ the equivariant function corresponding to s, we can expand this as

$$\xi^h(\eta^h(f)) - \eta^h(\xi^h(f)) - [\xi,\eta]^h(f) = ([\xi^h,\eta^h] - [\xi,\eta]^h)(f). \tag{3.11}$$

Since ξ^h is a lift of ξ, we get $\xi^h \sim_p \xi$ and similarly for η. This implies that $[\xi^h,\eta^h] \sim_p [\xi,\eta]$ so $[\xi^h,\eta^h]$ is a lift of $[\xi,\eta]$. But this means that $([\xi^h,\eta^h]-[\xi,\eta]^h)(u) \in V_uP$ for any $u \in P$. To compute this, we can apply γ, which of course kills $[\xi,\eta]^h$ and since $\gamma(\xi^h)$ and $\gamma(\eta^h)$ vanish identically, we see that

$$\gamma([\xi^h,\eta^h]) = -d\gamma(\xi^h,\eta^h) = -\Omega(\xi^h,\eta^h).$$

This means that the right hand side of (3.11) can be written as $\zeta_{-\Omega(\xi^h,\eta^h)}(f)$. By formula (3.5) this sends $u \in P$ to $\Omega(\xi^h,\eta^h)(u) \cdot (f(u))$ where the dot now indicates the action of $\mathfrak{g}$ on V. But by Lemma 3.4, this implies our claim. □

3.5. Torsion. To close the circle, we consider the case of a principal G-bundle $p : P \to M$ such that the tangent bundle TM is associated to P. This means that there is a representation of G on $\mathbb{R}^n$ with $n = \dim(M)$ such that $P \times_G \mathbb{R}^n \cong TM$. With the tools we have at hand, we can get a nice characterization of this situation which generalizes the ideas from Section 2.7. Given a representation of G on $\mathbb{R}^n$ an isomorphism $TM \to E := P \times_G \mathbb{R}^n$ defines a one-form $\alpha \in \Omega^1(M,E)$ such that $\alpha(x) : T_xM \to E_x$ is a linear isomorphism for each $x \in M$. By Proposition 2.3 a form α with this property conversely gives rise to an isomorphism $TM \to E$. Lemma 3.4 shows that $\Omega^1(M,E)$ is isomorphic to the space of horizontal, equivariant, $\mathbb{R}^n$-valued one-forms on P. For such a form θ, the condition that $\alpha(p(u))$ is injective is equivalent to the fact that $\ker(\theta(u)) = V_uP$, and, as in 2.7, we say that θ is *strictly horizontal* in this case. Thus we can equivalently phrase the fact that TM is associated to P as the fact that we have given a representation of G on $\mathbb{R}^n$ and a strictly horizontal, G-equivariant one-form $\theta \in \Omega^1(P,\mathbb{R}^n)$.

For a linear connection ∇ on TM, the torsion is a section of $\Lambda^2T^*M \otimes TM$, so we can view it as $T \in \Omega^2(M,TM)$. Hence this corresponds to a form $\tau \in \Omega^2(P,\mathbb{R}^n)$ which is horizontal and G-equivariant. In case that ∇ is induced by a principal connection on P, we can easily describe τ explicitly in terms of the connection form γ and the form θ. It is common to refer to T and τ also as the *torsion of* γ and to call γ *torsion-free* if this torsion vanishes identically.

PROPOSITION 3.5. *Let $p : P \to M$ be a principal G-bundle endowed with a principal connection form γ and fix a representation of G on $\mathbb{R}^n$ such that there is a strictly horizontal, G-equivariant one-form $\theta \in \Omega^1(P,\mathbb{R}^n)$. Then the form τ describing the torsion of the induced linear connection on $TM \cong P \times_G \mathbb{R}^n$ is explicitly given by*

$$\tau(u)(X,Y) = d\theta(u)(X,Y) + \gamma(u)(X) \cdot \theta(u)(Y) - \gamma(u)(Y) \cdot \theta(u)(X) \tag{3.12}$$

for $u \in P$ and $X, Y \in T_uP$. Here in the right hand side the dot indicates the infinitesimal representation of $\mathfrak{g}$ on $\mathbb{R}^n$.

Proof. By construction, for $u \in P$ and $X \in T_uP$ we get $\alpha(p(u))(T_up \cdot X) = q(u, \theta(u)(X))$. As we have noted already, this means that the equivariant function $P \to \mathbb{R}^n$ corresponding to $\xi \in \mathfrak{X}(M)$ can be written as $\theta(\tilde{\xi})$ where $\tilde{\xi}$ is any lift of ξ. In particular, we may use the horizontal lift ξ^h of ξ with respect to the principal connection determined by γ. Hence the equivariant function corresponding to $\nabla_\xi \eta$ can be written as $\xi^h \cdot \theta(\eta^h)$ and similarly for $\nabla_\eta \xi$. Since $[\xi^h, \eta^h]$ is a lift of $[\xi, \eta]$ the equivariant function corresponding to $[\xi, \eta]$ equals $\theta([\xi^h, \eta^h])$. Overall, we see that $T(\xi, \eta)$ corresponds to $d\theta(\xi^h, \eta^h)$ and thus $\tau(u)(X, Y) = d\theta(u)(X^h, Y^h)$ for any $u \in P$ and $X, Y \in T_uP$.

Now we can write $X^h = X - \zeta_{\gamma(u)(X)}(u)$ and similarly for Y and use bilinearity of $d\gamma(u)$. But as in the proof of Proposition 3.4, one shows that equivariancy of θ implies that $d\theta(u)(\zeta_A(u), Y) = -A \cdot (\theta(u)(Y))$ and using this (3.12) follows immediately. □

Observe that equation (3.12) in particular shows that for fixed P and θ, $\tau(u)$ depends only on $\gamma(u)$ and not on derivatives. This is in sharp contrast to the curvature of γ, which involves $d\gamma$ and will be absolutely crucial for the further development.

For later use, we show that the construction of gluing local principal connections to a global connection used in the proof of Theorem 3.1 is nicely compatible with torsion.

Corollary 3.5. *Let $p : P \to M$ be a principal G-bundle as in Proposition 3.5. Let $\{U_k : k \in \mathbb{N}\}$ be a countable covering of M, $\{f_k : k \in \mathbb{N}\}$ a subordinate partition of unity. For each k, let $\gamma_k \in \Omega^1(p^{-1}(U_k), \mathfrak{g})$ be a principal connection form with torsion T_k and consider the connection $\gamma = \sum_k (f_k \circ p)\gamma_k$ as in the proof of Theorem 3.1. Then the torsion T of γ is given by $\sum_k f_k T_k$.*

Proof. Of course, if T_k corresponds to $\tau_k \in \Omega^2(p^{-1}(U_k), \mathbb{R}^n)$ then $\sum_k f_k T_k$ corresponds to $\sum (f_k \circ p)\tau_k$. Now by equation (3.12) we get

$$\tau_k(u)(X, Y) = d\theta(u)(X, Y) + \gamma_k(u)(X) \cdot \theta(u)(Y) - \gamma_k(u)(Y) \cdot \theta(u)(X)$$

for each $u \in p^{-1}(U_k)$ and $X, Y \in T_uP$. Multiplying both sides with $f_k(p(u))$, the equation then trivially extends to all $u \in P$ and all $X, Y \in T_uP$. Observe that in this equation the last term can be written as $-((f_k \circ p)\gamma_k)(u)(Y) \cdot \theta(u)(X)$ and similarly for the last but one term. Summing over all k, the first term in the right hand side simply reproduces $d\theta(u)(X, Y)$ and the claim follows from Proposition 3.5. □

Connections on G-structures

3.6. Definitions and an example. Since a G-structure on a smooth manifold M in particular comes with a principal G-bundle $p : P \to M$, we can consider principal connections on this principal bundle. These are usually just referred to as *connections on a G-structure* or as *connections compatible with a G-structure*. In particular, part (1) of Theorem 3.3 implies that there is a principal connection on any G-structure and the space of all such connections is described by part (3) of that theorem. Moreover, the tangent bundle TM is just the associated bundle $P \times_G \mathbb{R}^n$ corresponding to the standard representation of $G \subset GL(n, \mathbb{R})$, so by Theorem 3.2 any connection on a G-structure induces a linear connection on TM. The terminology of being compatible with the G-structure is extended to the resulting linear connections. Let us discuss what this means in an example.

Example 3.6. Let (M, g) be a Riemannian manifold, so the corresponding $O(n)$-structure is given by the orthonormal frame bundle $p : P \to M$ for g. We claim that principal connections on P are in bijective correspondence with linear connections on

TM which are *metric for* g. This can be either phrased as $\nabla g = 0$ for the induced connection on S^2T^*M or as the fact that for $\xi, \eta_1, \eta_2 \in \mathfrak{X}(M)$ one always gets

$$\xi(g(\eta_1, \eta_2)) = g(\nabla_\xi \eta_1, \eta_2) + g(\eta_1, \nabla_\xi \eta_2). \tag{3.13}$$

Let us start from principal connection form $\gamma \in \Omega^1(P, \mathfrak{o}(n))$ on P. Given $\eta_1, \eta_2 \in \mathfrak{X}(M)$ consider the associated functions $f_1, f_2 : P \to \mathbb{R}^n$. These are characterized by $u(f_i(u)) = \eta_i(p(u))$ for $i = 1, 2$. But this means that

$$g(\eta_1, \eta_2)(x) = g_x(u(f_1(u)), u(f_2(u))) = \langle f_1(u), f_2(u) \rangle$$

for any $u \in P_x$ and the standard inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$. Now take another vector field $\xi \in \mathfrak{X}(M)$ and its horizontal lift $\xi^h \in \mathfrak{X}(P)$ with respect to γ.

Then from Theorem 3.2, we know that the induced linear connection ∇ on TM has the property that $\nabla_\xi \eta_i$ corresponds to the function $\xi^h \cdot f_i$ for $i = 1, 2$. On the other hand, since ξ^h lifts ξ, we see that

$$\xi \cdot g(\eta_1, \eta_2) = \xi^h \cdot (g(\eta_1, \eta_2) \circ p) = \xi^h \cdot \langle f_1, f_2 \rangle = \langle \xi^h \cdot f_1, f_2 \rangle + \langle f_1, \xi^h \cdot f_2 \rangle.$$

Here for the last equality, we have used bilinearity of the standard inner product. But from above, we conclude that this exactly gives (3.13). In fact, it is even easier to directly verify $\nabla g = 0$. We know that $S^2T^*M \cong P \times_G S^2\mathbb{R}^{n*}$ and hence g corresponds to an equivariant function $P \to S^2\mathbb{R}^{n*}$ and our above observations exactly mean that this function is constant and sends each point $u \in P$ to the standard inner product. But this immediately shows that $\nabla_\xi g = 0$ for the induced connection on S^2T^*M.

So let us conversely assume that ∇ is a linear connection on TM which is metric for g. Choose any principal connection $\hat{\gamma}$ on P and consider the induced linear connection $\hat{\nabla}$ on TM. Then $(\xi, \eta) \mapsto \hat{\nabla}_\xi \eta - \nabla_\xi \eta$ is bilinear over smooth functions, so it equals $A(\xi, \eta)$ for a tensor field $A \in \mathcal{T}^1_2(M)$. Moreover, the fact that both ∇ and $\hat{\nabla}$ are metric for g readily implies that

$$0 = g(A(\xi, \eta_1), \eta_2) + g(\eta_1, A(\xi, \eta_2)).$$

Now observe that $L(TM, TM) = P \times_G L(\mathbb{R}^n, \mathbb{R}^n)$ and $\mathfrak{o}(n) \subset L(\mathbb{R}^n, \mathbb{R}^n)$ is an $O(n)$-invariant subspace. Hence this gives rise to a natural subbundle $\mathfrak{o}(TM)$ that consist of maps that are skew symmetric with respect to g_x in each point x. So our observation just says that we can consider A as a one-form on M with values in the bundle $\mathfrak{o}(TM)$. But now by Lemma 3.4, A corresponds to a form $\varphi \in \Omega^1(P, \mathfrak{o}(n))$, which is horizontal and G-equivariant. Moreover, by Theorem 3.3, $\gamma := \hat{\gamma} - \varphi$ is a principal connection form on P and we claim that this induces ∇.

Given $\xi \in \mathfrak{X}(M)$ let us denote by $\hat{\xi}^h$ and ξ^h the horizontal lifts with respect to $\hat{\gamma}$ and γ, respectively. Then $\gamma(\hat{\xi}^h)(u) = -\varphi(\hat{\xi}^h)(u)$ and hence $\xi^h(u) = \hat{\xi}^h(u) + \zeta_{\varphi(\hat{\xi}^h)(u)}(u)$. Now for an equivariant function $f : P \to \mathbb{R}^n$, we get $\zeta_{\varphi(\hat{\xi}^h)(u)}(u) \cdot f = -\varphi(\hat{\xi}^h)(u)(f(u))$ where in the right hand side we apply a linear map lying in $\mathfrak{o}(n)$ to $f(u) \in \mathbb{R}^n$. Viewing this as a function of u, this is G-equivariant and by construction corresponds to $-A(\xi, \eta)$, where η denotes the vector field corresponding to f. But this exactly implies that the linear connection induced by γ acts on η as $\hat{\nabla}_\xi \eta - A(\xi, \eta) = \nabla_\xi \eta$, which completes the proof of our claim.

What one should have in mind here is that a connection on a G-structure is compatible with all the natural tensor fields and bundle maps induced by the G-structure. We will discuss this in several examples below. Here we just observe that Theorem 3.3 gives us a precise description of the space of all connections on a G-structure $p : P \to M$ on M. By part (1) of that theorem, there is at least one such connection, say it is given by $\gamma \in \Omega^1(P, \mathfrak{g})$ and all such connections are of the form $\hat{\gamma} = \gamma + \varphi$ for a G-equivariant, horizontal one-form $\varphi \in \Omega^1(P, \mathfrak{g})$. From Lemma 3.4 we know in turn that these forms are in bijective correspondence with all forms in $\Omega^1(M, P \times_G \mathfrak{g})$. But the latter associated bundle can be easily made explicit. By construction $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ and it is invariant under the adjoint action of G. Hence we can view it as a G-invariant linear subspace in $L(\mathbb{R}^n, \mathbb{R}^n)$ and hence $P \times_G \mathfrak{g}$ naturally is a subbundle of $P \times_G L(\mathbb{R}^n, \mathbb{R}^n) \cong L(TM, TM)$. As in the discussion in Example 3.6 above, it is easy to describe this subbundle explicitly in specific cases.

3.7. Intrinsic torsion. We can use the considerations about connections on G-structures obtain a fundamental invariant for G-structures. Whether this invariant can be non-trivial depends on the type of G-structure, so we can observe different behavior here. For a Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ consider the linear map

$$\partial : \mathbb{R}^{n*} \otimes \mathfrak{g} \to \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n \qquad \partial\varphi(v, w) := \varphi(v)(w) - \varphi(w)(v). \tag{3.14}$$

Here we view the left hand side as $L(\mathbb{R}^n, \mathfrak{g})$ and the right hand side as skew-symmetric bilinear maps $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Alternatively, since $\mathfrak{g} \subset \mathbb{R}^{n*} \otimes \mathbb{R}^n$, we can view ∂ as the restriction of the alternation $\mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^n \to \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$ to the subspace $\mathbb{R}^{n*} \otimes \mathfrak{g}$. Observe that by definition, we get $\ker(\partial) = \mathfrak{g}^{(1)}$, the first prolongation of $\mathfrak{g}$ from Section 2.12, and this will be important later on.

But now we start by looking at the cokernel of ∂, i.e. the quotient

$$\mathcal{I} := (\Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n)/\operatorname{im}(\partial). \tag{3.15}$$

Observe that G naturally acts on both the domain and the source of ∂. Explicitly, the actions are given by $(A\cdot\varphi)(v) = \mathrm{Ad}(A)(\varphi(A^{-1}v)) = A\varphi(A^{-1}v)A^{-1}$ and by $(A\cdot\psi)(v, w) = A\psi(A^{-1}v, A^{-1}w)$, respectively. But this immediately shows that ∂ is G-equivariant and consequently $\ker(\partial) \subset \mathbb{R}^{n*} \otimes \mathfrak{g}$ and $\operatorname{im}(\partial) \subset \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$ are G-invariant subspaces. In particular, $\mathcal{I}$ naturally carries a representation of G and hence for any G-structure $p : P \to M$, we can form the associated bundle $\mathcal{I}M := P \times_G \mathcal{I}$. In an obvious sense, this is a quotient bundle of $\Lambda^2T^*M \otimes TM$, so there is is natural bundle map Π from that bundle to $\mathcal{I}M$. In particular, any form $\psi \in \Omega^2(M, TM)$ can be naturally projected to a section $\Pi(\psi) \in \Gamma(\mathcal{I}M)$.

Theorem 3.7. *Fix a closed subgroup $G \subset GL(n, \mathbb{R})$ with Lie algebra $\mathfrak{g}$ and let $\mathcal{I}$ be the representation of G defined in* (3.15). *Let $p : P \to M$ be a G-structure on M and let $\gamma \in \Omega^1(P, \mathfrak{g})$ be a principal connection on P with torsion $T \in \Omega^2(M, TM)$. Then*

(1) The the section $T_i := \Pi(T) \in \Gamma(\mathcal{I}M)$ is independent of the choice of γ.

(2) T_i is an invariant of the G-structure in the sense that for any morphism $F : \tilde{P} \to P$ of G-structures with base map f, we get (with obvious notation) $f^*T_i = \tilde{T}_i$.

(3) T_i vanishes identically if and only if the G-structure admits a compatible torsion-free connection.

Proof. (1) Given γ, any other principal connection on P is of the form $\hat{\gamma} = \gamma + \varphi$ where $\varphi \in \Omega^1(P, \mathfrak{g})$ is horizontal and G-equivariant. Using formula (3.12) from Theorem 3.5, we conclude that the torsion $\hat{\tau}$ of $\hat{\gamma}$ is given by

$$\hat{\tau}(u)(X, Y) = \tau(u)(X, Y) + \varphi(u)(X) \cdot \theta(u)(Y) - \varphi(u)(Y) \cdot \theta(u)(X). \tag{3.16}$$

By Lemma 3.4, we can view φ as a one-form on M with values in the bundle $P \times_G \mathfrak{g} \subset L(TM, TM)$ and hence as a section of $P \times_G (\mathbb{R}^{n*} \otimes \mathfrak{g})$. This in turn is represented by an equivariant function $h : P \to \mathbb{R}^{n*} \otimes \mathfrak{g}$ and evidently the last two terms in the right hand side of (3.16) correspond to $\partial \circ h$. But this exactly means that the functions corresponding to the torsions $\hat{T}$ and T differ by some element in the image of ∂ and hence $\Pi(\hat{T}) = \Pi(T)$.

(2) Let $F : \tilde{P} \to P$ be a morphism of G-structures, which implies that $F^*\theta = \tilde{\theta}$. Then it follows readily from the definitions $\tilde{\gamma} := F^*\gamma \in \Omega^1(\tilde{P}, \mathfrak{g})$ is a principal connection form, so this defines a connection on the G-structure $\tilde{P}$. Using compatibility of pullbacks with the exterior derivative, formula (3.12) then shows the torsion $\tilde{\tau}$ of $\tilde{\gamma}$ is simply given by $F^*\tau$. In the language of forms on M, this says that $\tilde{T} = f^*T$, which implies the claim by part (1).

(3) Existence of a torsion-free compatible connection evidently implies vanishing of the intrinsic torsion. Conversely, take a G-structure with vanishing intrinsic torsion. Then it suffices to construct a compatible torsion-free connection locally, since by Corollary 3.5 local torsion-free connections can be glued to a global torsion-free connection.

Assume that γ is a connection on a G-structure for which the intrinsic torsion T_i vanishes and let $h : P \to \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$ be the equivariant function corresponding to its torsion T. By assumption, h has values in the image of ∂. Of course, we can choose a linear map $s : \mathrm{im}(\partial) \to \mathbb{R}^{n*} \otimes \mathfrak{g}$ such that $\partial \circ s = \mathrm{id}$, but s is not G-equivariant in general. However, let $U \subset M$ be open such that there is a local section $\sigma : U \to P$. Then $-s \circ h \circ \sigma : U \to \mathbb{R}^{n*} \otimes \mathfrak{g}$ is smooth, and of course there exists a unique G-equivariant smooth function $\psi : p^{-1}(U) \to \mathbb{R}^{n*} \otimes \mathfrak{g}$ such that $\psi \circ \sigma = -s \circ h \circ \sigma$ and hence $\partial \circ \psi \circ \sigma = -h \circ \sigma$. But since $\partial \circ \psi$ and $-h$ both are G-equivariant, this implies that $\partial \circ \psi = -h$ on all of $p^{-1}(U)$.

Now ψ corresponds to an element of $\Omega^1(U, P \times_G \mathfrak{g})$ and hence to a form $\varphi \in \Omega^1(p^{-1}(U), \mathfrak{g})$, which is horizontal and G-equivariant. Then by Theorem 3.3, $\hat{\gamma} := \gamma|_{p^{-1}(U)} + \varphi$ is a principal connection on $p^{-1}(U)$ and then the computation in the beginning of this proof shows that $\hat{\gamma}$ is torsion-free. □

Definition 3.7. The invariant T_i constructed in the theorem is called the *intrinsic torsion* of a G-structure.

3.8. Normalizing torsion. By Theorem 3.7, we can restrict our interest to compatible torsion-free connections in the case of G-structures with vanishing intrinsic torsion. To move towards the question of existence of canonical connections on G-structures, the next step is trying to find a distinguished value for the torsion in the general case. Again by Theorem 3.7, we know that the torsions of all connections on a G-structure have the same image in the quotient by $\mathrm{im}(\partial)$. The natural way to pin down one of these values is to require that it lies in a specified complement to $\mathrm{im}(\partial)$ in $\Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$. However, since we are dealing with equivariant functions and want to get a condition that has geometric meaning, we have to restrict our attention to G-invariant complements.

Definition 3.8. Fix a matrix group $G \subset GL(n, \mathbb{R})$, let $\mathfrak{g}$ be its Lie algebra and let ∂ be the map from (3.14). A *normalization condition* for the torsion of G-structures is a G-invariant linear subspace $\mathcal{N} \subset \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$, which is complementary to the subspace $\mathrm{im}(\partial)$.

In general, invariant subspaces do not admit invariant complements, so such normalization conditions need not exist for general groups G. Under certain assumptions, for example if G is compact or semi-simple, the existence of invariant complements follows from general results on complete reducibility of representations. If a normalization conditions exists, then it need not be unique in general. If $\mathcal{N}$ and $\mathcal{N}'$ are normalization conditions, then for each $v \in \mathcal{N}$, there is a unique element $f(v) \in \mathrm{im}(\partial)$ such that

$v + f(v) \in \mathcal{N}'$ and this defines a linear map $f : \mathcal{N} \to \operatorname{im}(\partial)$. Equivariancy of $\mathcal{N}'$ then implies that for $g \in G$, we get $g \cdot v + g \cdot f(v) \in \mathcal{N}'$ and uniqueness shows that $g \cdot f(v) = f(g \cdot v)$, so f is G-equivariant. Conversely, given $\mathcal{N}$ and a G-equivariant linear map $f : \mathcal{N} \to \operatorname{im}(\partial)$, one can put $\mathcal{N}' := \{v + f(v) : v \in \mathcal{N}\}$ to obtain another normalization condition. So one can prove that a normalization condition is unique by showing that there are no G-equivariant maps $\mathcal{N} \to \operatorname{im}(\partial)$. In any case, non-uniqueness of normalization conditions is not a big problem, since one can fix a choice of $\mathcal{N}$ as a part of the ingredients for the study of G-structures for fixed G.

By definition, a normalization condition $\mathcal{N}$ is a representation of G. Hence we can consider the associated bundle $P \times_G \mathcal{N}$ and this is a subbundle of $\Lambda^2 T^*M \otimes TM$. Now we call a principal connection on P *normal* (with respect to $\mathcal{N}$) iff its torsion has values in $P \times_G \mathcal{N} \subset \Lambda^2 T^*M \otimes TM$. On the other hand, the restriction of the projection from $\Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$ to its quotient by $\operatorname{im}(\partial)$ restricts to an isomorphism of representations on $\mathcal{N}$. Consequently, $P \times_G \mathcal{N}$ is isomorphic to $\mathcal{I}M = P \times_G \mathcal{I}$, so once we have fixed $\mathcal{N}$, we can view the intrinsic torsion of a G-structure as a section of $P \times_G \mathcal{N}$.

Theorem 3.8. *Let G be a matrix group such that there is a normalization condition $\mathcal{N}$ for the torsion of G-structures and let us fix $\mathcal{N}$.*

Then any G-structure $p : P \to M$ admits a normal connection. Moreover, for any normal connection γ on P, the torsion of T of γ coincides with the intrinsic torsion T_i, viewed as a section of $P \times_G \mathcal{N}$.

Proof. If we have a normal connection, then its torsion has to lie in $P \times_G \mathcal{N}$ and we know that it has to project to the intrinsic torsion T_i in $P \times_G \mathcal{I}$, so the last statement follows. Moreover, Corollary 3.5 shows that gluing local connections with torsion in $P \times_G \mathcal{N}$ we get a global connection with torsion in $P \times_G \mathcal{N}$. Hence we conclude that it suffices to construct normal connections locally, so we take $U \subset M$ open for which there exists a local section $\sigma : U \to P$ of P. Let γ be a connection form on $p^{-1}(U)$ and let $h : P \to \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$ be the equivariant function corresponding to the torsion T of γ. By assumption, the target space equals $\mathcal{N} \oplus \operatorname{im}(\partial)$ and correspondingly we can write $h = h_1 + h_2$ for equivariant functions h_1 with values in $\mathcal{N}$ and h_2 with values in $\operatorname{im}(\partial)$. Now as in the proof of Theorem 3.7, we can construct an equivariant function $\psi : p^{-1}(U) \to \mathbb{R}^{n*} \otimes \mathfrak{g}$ such that $\partial \circ \psi = -h_2$. As there, this corresponds to a form $\varphi \in \Omega^1(p^{-1}(U), \mathfrak{g})$ which is horizontal and G-equivariant and the principal connection $\gamma + \varphi$ has torsion represented by h_1 and hence is normal. □

3.9. Canonical connections. We are now ready to exhibit a class of G-structures which, similar to Riemannian metrics, admit canonical connections. As we shall see, this has strong consequences.

Theorem 3.9. *Let $G \subset GL(n, \mathbb{R})$ be a matrix group such that there is a normalization condition $\mathcal{N}$ for the torsion of G-structures and let us fix $\mathcal{N}$. Assume in addition that the first prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$ of G is trivial.*

Then for any G-structure $p : P \to M$, there is a unique normal connection $\gamma_{\mathcal{N}}$ on P. This connection is an invariant of the G structure, in the sense that for any isomorphism $F : \tilde{P} \to P$ of G-structures we obtain (in obvious notation) $F^*\gamma_{\mathcal{N}} = \tilde{\gamma}_{\mathcal{N}}$. Hence also the curvature of $\gamma_{\mathcal{N}}$ is an invariant of the G-structure.

Proof. We know from Theorem 3.8 that there is a normal connection γ on P. If $\hat{\gamma}$ is another normal connection on P, then again by Theorem 3.8, γ and $\hat{\gamma}$ have the same torsion. Now from Theorem 3.3, we know that $\hat{\gamma} = \gamma + \varphi$ for a horizontal equivariant form $\varphi \in \Omega^1(P, \mathfrak{g})$. But from the proof of Theorem 3.7, we know that $\hat{\tau} = \tau$ implies that $\varphi(u)(X)(Y) = \varphi(u)(Y)(X)$ for all $u \in P$ and $X, Y \in T_uP$. But this exactly means that the equivariant function $P \to \mathbb{R}^{n*} \otimes \mathfrak{g}$ corresponding to φ actually has values in $\mathfrak{g}^{(1)} = \{0\}$ and uniqueness follows.

If $F : \tilde{P} \to P$ is an isomorphism of G-structures, then $F^*\gamma_{\mathcal{N}}$ is a principal connection on $\tilde{P}$. As we have noted in the proof of Theorem 3.7, the torsion $\tilde{T}$ of $F^*\gamma_{\mathcal{N}}$ is the pullback f^*T, where f is the base map of F and T is the torsion of $\gamma_{\mathcal{N}}$. In terms of equivariant functions with values in $\Lambda^2\mathbb{R}^{n*}\otimes\mathbb{R}^n$ pulling back just means composing with F, so the torsion of $F^*\gamma_{\mathcal{N}}$ has values in $\mathcal{N}$, so this connection is normal. By uniqueness, it has to coincide with $\tilde{\gamma}_{\mathcal{N}}$. The definition of curvature easily implies that the curvature of $F^*\gamma_{\mathcal{N}}$ is $F^*\Omega$, where Ω is the curvature of $\gamma_{\mathcal{N}}$, so the curvature is an invariant. □

Let us revisit the standard flat G-structure on $\mathbb{R}^n$ from Section 2.12 from our current perspective. By definition, this is given by the trivial principal bundle $P := \mathbb{R}^n \times G$ viewed as a subbundle of the (trivial) linear frame bundle $\mathbb{R}^n \times GL(n,\mathbb{R}) \to \mathbb{R}^n$. Hence we can identify $TP \cong T\mathbb{R}^n \times TG$ and thus view tangent vectors at (x, g) as pairs (v, X) with $v \in \mathbb{R}^n$ and $X \in T_gG$. In this language, it follows readily from the definitions that the canonical $\mathbb{R}^n$-valued form is given by $\theta(x, g)(v, X) = g^{-1}(v)$. On the other hand, we can define $\gamma \in \Omega^1(P, \mathfrak{g})$ by $\gamma(x, g)(v, X) := T_g\lambda_{g^{-1}}(X)$, where $\lambda_g : G \to G$ denotes left translation by g. The principal right action on P is just given by $r^h(x, g) = (x, gh)$. Hence

$$((r^h)^*\gamma)(x, g)(v, X) = \gamma(x, gh)(v, T_g\rho^h(X)) = \mathrm{Ad}(h^{-1})(\gamma(x, g)(v, X)),$$

so γ is G-equivariant. Moreover, the fundamental vector fields on P are given by $\zeta_Y(x, g) = (0, L_Y(g))$, where L_Y is the left invariant vector field generated by $Y \in \mathfrak{g}$ and hence $\gamma(\zeta_Y) = Y$. Thus γ defines a principal connection on P for which the horizontal subspace consists of the vectors of the form $(v, 0)$.

Now we can easily compute that both the torsion and the curvature of γ vanish identically. We can either go through the constructions and verify that the induced linear connection on $T\mathbb{R}^n$ is simply given by taking directional derivatives of vector fields viewed as $\mathbb{R}^n$-valued functions. Alternatively, we can observe that the forms Ω from Definition 3.4 and τ from Proposition 3.5 that describe the torsion and the curvature are horizontal. Thus it suffices to verify that they vanish upon insertion of two horizontal lifts $\partial_i^h = (\partial_i, 0)$ of coordinate vector fields $\partial_i = \frac{\partial}{\partial x_i}$ on $\mathbb{R}^n$. Now $\theta(x, g)(\partial_i^h) = g^{-1}(e_i)$ and $\gamma(\partial_i^h) = 0$ and $[\partial_i^h, \partial_j^h] = 0$ and plugging these into the formulae, it follows readily that $\Omega(\partial_i^h, \partial_j^h) = 0$ and $\tau(\partial_i^h, \partial_j^h) = 0$.

In particular, the intrinsic torsion of the standard flat G-structure vanishes identically. If we in addition assume that we are in our current setting that $\mathfrak{g}^{(1)} = \{0\}$, we see that independent of the choice of normalization condition we have $0 \in \mathcal{N}$ and hence $\gamma = \gamma_{\mathcal{N}}$. Hence in this case, also the second invariant we have obtained, namely the curvature of $\gamma_{\mathcal{N}}$ vanishes identically. We can next prove in general that this locally characterizes the standard flat G-structure on $\mathbb{R}^n$.

Proposition 3.9. *Let $G \subset GL(n,\mathbb{R})$ be a matrix group such that there is a normalization condition $\mathcal{N}$ for the torsion of G-structures and such that the first prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$ of G is trivial.*

Then for a G-structure $p : P \to M$ both the intrinsic torsion T_i and the curvature of $\gamma_{\mathcal{N}}$ vanish identically if and only if for each point $x \in M$ has an open neighborhood $U \subset M$ such that $p^{-1}(U) \to U$ is isomorphic (as a G-structure) to the standard flat G-structure on an open subset of $\mathbb{R}^n$.

Proof. This is based on another interpretation of the standard flat G-structure. Consider the group $\tilde{G} := G \rtimes \mathbb{R}^n$ of G-motions and $G \subset \tilde{G}$ as the isotropy group of $0 \in \mathbb{R}^n$. Then we can consider the projection $\pi : \tilde{G} \to \mathbb{R}^n$ defined by $\pi(f) := f(0)$ as the trivial principal G-bundle with the principal right action given by multiplication by elements of the subgroup G from the right. So this is just the standard principal G-bundle $\pi : \tilde{G} \to \tilde{G}/G$. Now of course $T_e\tilde{G} \cong \mathfrak{g} \oplus \mathbb{R}^n$ as a vector space, so we can combine the canonical $\mathbb{R}^n$-valued form and the connection form to a form with values in $\tilde{\mathfrak{g}}$, and we claim that this is just the left Maurer-Cartan form ω of $\tilde{G}$. This is easily verified by realizing $\tilde{G}$ as the closed subgroup of $GL(n+1,\mathbb{R})$ consisting of all matrices of the form $\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$ with $A \in G$ and $b \in \mathbb{R}^n$. In this picture, the action as G-motions simply comes from acting on vectors of the form $\binom{v}{1}$ with $v \in \mathbb{R}^n$.

Now clearly the tangent space $T_{(A,b)}\tilde{G}$ consists of all matrices of the form $\begin{pmatrix} X & v \\ 0 & 0 \end{pmatrix}$ with $X \in T_AG$ and $v \in \mathbb{R}^n$. Moreover, in a matrix group left translations are linear, and hence coincide with their tangent maps. Since evidently $(A,b)^{-1} = (A^{-1}, -A^{-1}b)$, one immediately computes that $\omega(A,b)(X,v) = (A^{-1}X, A^{-1}v)$, which coincides with what we obtained above. Then interpretation as a matrix group also readily shows that the Lie bracket on $\tilde{\mathfrak{g}}$ is given by $[(X,v),(Y,w)] = ([X,Y], Xw - Yv)$. This then shows that vanishing of the torsion and the curvature of our connection is equivalent to the Maurer-Cartan equation $0 = d\omega(\xi,\eta) + [\omega(\xi),\omega(\eta)]$ for the left Maurer-Cartan form.

Now let us assume that we have given a G-structure $p : P \to M$ with canonical form θ and normal connection $\gamma_{\mathcal{N}}$ such that both the torsion and the curvature of $\gamma_{\mathcal{N}}$ vanish identically. Then we put $\psi := \gamma_{\mathcal{N}} \oplus \theta \in \Omega^1(P,\tilde{\mathfrak{g}})$ and, as above, vanishing of the torsion and curvature of $\gamma_{\mathcal{N}}$ exactly say that ψ satisfies the Maurer-Cartan equation. Now fix a point $x \in M$ and a local section σ of P defined on a neighborhood of $x \in M$ and put $u := \sigma(x)$. By Theorem 2.9 of [**LieGrp**] there exists an open neighborhood V of u in P and a unique smooth function $\tilde{F} : V \to \tilde{G}$ with $\tilde{F}(u) = e$ such that $\psi = \tilde{F}^*\omega$. By construction, both ψ and ω restrict to linear isomorphisms on each tangent space, so each tangent map of $\tilde{F}$ has to be a linear isomorphism. Hence we may assume that $\tilde{F}$ is a diffeomorphism from V onto an open neighborhood of e in $\tilde{G}$.

Since the fundamental vector fields on our two bundles are characterized by their values under ω and ψ, respectively, it follows that $\tilde{F}^*\zeta_A = \zeta_A$ for any $A \in \mathfrak{g}$, where we denote fundamental fields on both bundles by the same symbol. Hence the flows of the fields are $\tilde{F}$-related, and hence $\tilde{F} \circ r^{\exp(tX)} = r^{\exp(tX)} \circ \tilde{F}$ for any $X \in \mathfrak{g}$ and sufficiently small t. Shrinking V we can assume that $V \subset p^{-1}(U)$ where σ is defined on U and $\sigma(U) \subset V$. Possibly shrinking further, we can assume that $y \mapsto \pi(\tilde{F}(\sigma(y)))$ defines a diffeomorphism from U onto an open neighborhood W of 0 in $\mathbb{R}^n = \tilde{G}/G$. But then mapping $\pi(\tilde{F}(\sigma(y)))$ to $\tilde{F}(\sigma(y))$ defines a local smooth section of $\tilde{G} \to \tilde{G}/G$ on W. Hence we can define a G-equivariant diffeomorphism $F : p^{-1}(U) \to \pi^{-1}(W)$ by $F(r^A(\sigma(y))) := \tilde{F}(\sigma(y))A$ for any $y \in U$ and $A \in G$. By construction, this is G-equivariant and from above we see that it coincides with $\tilde{F}$ on the neighborhood V of $\sigma(U)$.

The last observation in particular implies that $F^*\omega|_V = \tilde{F}^*\omega|_V = \psi|_V$. To conclude the proof, it suffices to show that this holds on all of $p^{-1}(U)$ since then F is a morphism of G-structures. But the representation of $\tilde{G}$ as a matrix group readily implies that for $A \in G$, we get $\mathrm{Ad}(A)(X,v) = (AXA^{-1}, Av)$. This shows that $(r^A)^*\psi = \mathrm{Ad}(A^{-1}) \circ \psi$, so ψ is G-equivariant as a form with values in $\tilde{\mathfrak{g}}$, and we know that also ω has this property. Now for $A \in G$ consider the open set $r^A(V)$, observe that $r^{A^{-1}} : r^A(V) \to V$ to conclude that $F^*\omega|_V = \psi|_V$ implies that on $r^A(V)$ we obtain $(r^{A^{-1}})^*F^*\omega = (r^{A^{-1}})^*\psi = \mathrm{Ad}(A) \circ \psi$. But then $F \circ r^{A^{-1}} = r^{A^{-1}} \circ F$ shows that the left hand side can be rewritten as $F^*((r^{A^{-1}})^*\omega) = F^*(\mathrm{Ad}(A) \circ \omega) = \mathrm{Ad}(A) \circ F^*\omega$. Hence $F^*\omega|_{r^A(V)} = \psi_{r^A(V)}$ and since A is arbitrary, this completes the proof. □

3.10. Consequences for morphisms. To finalize the general discussion of G-structures admitting a canonical connection, we discuss morphisms and automorphisms in this setting. In particular, we show that there is an analog of the dimension bound for isometry groups of Riemannian manifolds as well as a general proof for the fact that automorphisms form a Lie group that we sketch. All this is based on joining the canonical form θ and the normal connection $\gamma_{\mathcal{N}}$ to a one-form ψ with values in the Lie algebra $\tilde{\mathfrak{g}}$ of the group $\tilde{G} = G \rtimes \mathbb{R}^n$ of G-motions that we used in the proof of Proposition 3.9. We have already observed there that ψ is G-equivariant and noted that $\psi(u) : T_uP \to \tilde{\mathfrak{g}}$ is a linear isomorphism for each $u \in P$. Observe that by Proposition 2.3 this implies that we can view ψ as defining an isomorphism $TP \to M \times \tilde{\mathfrak{g}}$ of vector bundles, which strengthens the analogy to the Maurer-Cartan form used in Section 3.9. The properties observed here are the defining properties of a *Cartan connection* with model the homogeneous space $\tilde{G}/G$. From Section 2.8 and Theorem 3.9, we know that under our current assumptions any morphism of G-structures preserves both θ and $\gamma_{\mathcal{N}}$ and hence also ψ.

To prove that automorphisms form a Lie group, we need some results from the theory of Lie transformation groups, see [**Palais**] or Theorem 3.1 in [**Kobayashi**] or Proposition 1.5.11 in [**Cap-Slovak**]. Suppose that N is a smooth manifold and $H \subset \mathrm{Diff}(N)$ is a group of diffeomorphisms. Then one considers the space E of all vector fields $\xi \in \mathfrak{X}(N)$ which are complete and have the property that $\mathrm{Fl}^{\xi}_t \in H$ for all $t \in \mathbb{R}$. The key condition then is that the Lie subalgebra of $\mathfrak{X}(N)$ generated by E is finite dimensional. In practice, one proves this by showing that there is a finite dimensional Lie subalgebra of $\mathfrak{X}(N)$ that contains E. Then it turns out that E itself is already a Lie subalgebra of $\mathfrak{X}(N)$ and one can uniquely make H into a Lie group with Lie algebra $\mathfrak{h} = E$. Moreover, the inclusion of H into $\mathrm{Diff}(N)$ defines a smooth left action of the Lie group H on N.

THEOREM 3.10. *Let $G \subset GL(n, \mathbb{R})$ be a matrix group such that there is a normalization condition $\mathcal{N}$ for the torsion of G-structures and such that the first prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$ of G is trivial.*

(1) Let $p : P \to M$ and $\tilde{p} : \tilde{P} \to \tilde{M}$ be G-structures such that M is connected. Then any morphism $F : P \to \tilde{P}$ of G-structures is uniquely determined by its value in one point $u \in P$.

(2) For any G-structure $p : P \to M$ with M connected, the group of automorphisms is a Lie group of dimension $\leq \dim(M) + \dim(G)$. For a fixed normalization condition $\mathcal{N}$, the Lie algebra of this group can be identified with the space of all G-invariant vector fields $\tilde{\xi}$ on P which have the property that $\mathcal{L}_{\xi}\theta = 0$ and $\mathcal{L}_{\xi}\gamma_{\mathcal{N}} = 0$. Moreover any such vector field is projectable to M and uniquely determined by its projection.

PROOF. Fix a normalization condition $\mathcal{N}$ and put $\psi = \gamma_{\mathcal{N}} \oplus \theta$ and similarly for $\tilde{\psi}$.

(1) As we have noted above any morphism F satisfies $F^*\tilde{\psi} = \psi$ and since already $F^*\tilde{\theta} = \tilde{\theta}$ implies that F is a morphism of G-structures, this is an equivalent characterization. Now given $Y \in \tilde{\mathfrak{g}}$, there is a unique element $Y_P \in \mathfrak{X}(P)$ such that $\psi(Y_P) \equiv Y$ and similar we get $Y_{\tilde{P}} \in \mathfrak{X}(\tilde{P})$. But then $F^*\tilde{\psi} = \psi$ readily shows that $\psi(F^*Y_{\tilde{P}}) \equiv Y$ and hence $F^*Y_{\tilde{P}} = Y_P$. Hence we see that $F \circ \mathrm{Fl}^{Y_P}_t = \mathrm{Fl}^{Y_{\tilde{P}}}_t \circ F$ for all $Y \in \tilde{\mathfrak{g}}$ and all t

such that both flows are defined. Given a point $u \in P$, we conclude that there is an open neighborhood V of u in P such that $F|_V$ is determined by $F(u)$. By equivariancy, the value in any point of P determines the values of F on the whole fiber of P through this point. Hence $F(u)$ even determines the values of F on $p^{-1}(\underline{V})$ for some open neighborhood $\underline{V}$ of $p(u)$ in M. This shows that for two morphisms F_1 and F_2 the set $A \subset M$ defined by $\{x \in M : F_1|_{P_x} = F_2|_{P_x}\}$ is open and closed in M, and the claim follows from connectedness of M.

(2) The set of principal bundle automorphisms $F : P \to P$ such that $F^*\psi = \psi$ evidently forms a subgroup of $\mathrm{Diff}(P)$. Now let us take vector field $\xi \in \mathfrak{X}(P)$ and study what it means that $\mathrm{Fl}^\xi_t \circ r^A = r^A \circ \mathrm{Fl}^\xi_t$ for all $A \in G$ and $(\mathrm{Fl}^\xi_t)^*\psi = \psi$. Differentiating these equations with respect to t at $t = 0$, we readily get $\xi(r^A(u)) = T_u r^A(\xi(u))$ and hence $(r^A)^*\xi = \xi$ and $\mathcal{L}_\xi\psi = 0$, respectively. Denoting by $\mathfrak{a}$ the space of all vector fields that satisfy these two conditions, naturality of the Lie bracket and the fact that $\mathcal{L}_{[\xi,\eta]} = \mathcal{L}_\xi \circ \mathcal{L}_\eta - \mathcal{L}_\eta \circ \mathcal{L}_\xi$ imply that $\mathfrak{a}$ is a subalgebra of $\mathfrak{X}(P)$. Observe that $\mathcal{L}_\xi\psi = 0$ is equivalent to $\mathcal{L}_\xi\theta = 0$ and and $\mathcal{L}_\xi\gamma_{\mathcal{N}} = 0$.

On the other hand, if $\xi \in \mathfrak{a}$ is complete, then one easily verifies that the two conditions on the flows are satisfied. Given any $\xi, \eta \in \mathfrak{X}(P)$, we can compute $\mathcal{L}_\xi\psi(\eta) = \xi \cdot \psi(\eta) - \psi([\xi, \eta])$. For $\eta = Y_P$, the first summand vanishes, and we conclude that $\xi \in \mathfrak{a}$ implies that $[\xi, Y_P] = 0$ for any $Y \in \tilde{g}$. This implies that $(\mathrm{Fl}^{Y_P}_t)^*\xi = 0$ wherever the flow is defined, so $\xi(\mathrm{Fl}^{Y_P}_t(u))$ is determined by $\xi(u)$ in this case. Together with invariance, this shows that ξ is determined by $\xi(u)$ on $p^{-1}(V)$ for an open neighborhood V of $p(u)$ in M. Connectedness of M implies that any $\xi \in \mathfrak{a}$ is uniquely determined by its value in a single point $u \in P$. Hence $\mathfrak{a}$ has dimension at most $\dim(P)$ and Palais' characterization of Lie transformation groups discussed above implies that automorphisms form a Lie group as well as the description of the Lie algebra.

For $\xi \in \mathfrak{a}$, $(r^A)^*\xi = \xi$ implies that ξ is projectable, so the Lie algebra of the automorphism group consists of projectable vector fields. To complete the proof, it suffices to show that an element $\xi \in \mathfrak{a}$ that projects to zero has to vanish identically. If ξ projects to zero, then by definition we get $i_\xi\theta = 0$ and since $\xi \in \mathfrak{a}$, we get $\mathcal{L}_\xi\theta = 0$ and hence $i_\xi d\theta = 0$. But by construction, ξ is a section of the vertical subbundle in TP. Hence choosing a basis $\{A_i\}$ of $\mathfrak{g}$, there are smooth functions $a_i : P \to \mathbb{R}$ such that $\xi = \sum_i a_i\zeta_{A_i}$. But then $i_\xi d\theta = \sum_i a_i(i_{\zeta_{A_i}} d\theta)$. As we have noted in the proof of Theorem 3.5, $d\theta(\zeta_{A_i}, \eta) = -A_i(\theta(\eta))$. Putting $\eta = v_P$ for $v \in \mathbb{R}^n \subset \tilde{\mathfrak{g}}$ we finally conclude that $\sum_i a_i(u)A_i v = 0$ for all $v \in \mathbb{R}^n$. Hence $0 = \sum_i a_i(u)A_i \in \mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ for any u and since the A_i form a basis, we conclude that $\xi = 0$. □

Recall that in the case of Riemannian metrics we used arguments based on geodesics to prove similar results. Indeed, the arguments in the proof above can be phrased in a similar fashion. In particular, the geodesics for the linear connection on TM induced by a connection on a G-structure can be described as the projections of the integral curves of the vector fields v_P, with $v \in \mathbb{R}^n$, i.e. those that are mapped to zero by γ and to a fixed vector by θ.

CHAPTER 4

Examples

We will now discuss several fundamental examples of G-structures and study their properties via the methods exhibited so far. We will sometimes use results from representation theory without providing complete details.

4.1. (Pseudo-)Riemannian metrics. Let us start with the example $G = O(n) \subset GL(n, \mathbb{R})$ which corresponds to Riemannian geometry, see Example 2.8 (1) for the equivalence between a Riemannian metric g on M and an $O(n)$-structure $p : P \to M$. We have seen in Example 3.6 that connections on this G-structure exactly correspond to those linear connections on TM that are metric for g in the usual sense. Now in this case, the analysis of torsion is very easy: We have already observed in Section 2.12 that for $\mathfrak{g} = \mathfrak{o}(n)$, we get $\mathfrak{g}^{(1)} = \{0\}$, so the map $\partial : \mathbb{R}^{n*} \otimes \mathfrak{o}(n) \to \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$ defined in (3.14) is injective. However, $\dim(\mathfrak{o}(n)) = \frac{n(n-1)}{2} = \dim(\Lambda^2\mathbb{R}^{n*})$ readily shows that ∂ maps between two spaces of the same dimension, so it has to be a linear isomorphism.

Hence there is no intrinsic torsion for $O(n)$-structures and $\mathcal{N} = \{0\}$ is the only possible complement to $\operatorname{im}(\partial)$. Hence Theorem 3.9 shows that any $O(n)$-structure admits a unique torsion-free connection, which exactly is existence and uniqueness of the Levi-Civita connection. This also shows that the Riemann curvature tensor is a basic invariant of an $O(n)$-structure and Corollary 3.9 says that vanishing of the Riemann curvature tensor is equivalent to local isometry to the flat metric on $\mathbb{R}^n$. Finally, Theorem 3.10 shows that the isometry group of any Riemannian manifold is a Lie group (Myers-Steenrod theorem) of dimension at most $\frac{n(n+1)}{2}$. It is not difficult to verify that the Lie algebra of this group is formed by the complete Killing vector fields on M.

Replacing Riemannian metrics in dimension n by pseudo-Riemannian metrics of some fixed signature (p, q) with $p + q = n$, we simply have to replace $O(n)$ by $O(p, q)$. But the arguments from the proof of Proposition 1.2 and from Section 2.12 extend without any change if one replaces the standard inner product on $\mathbb{R}^n$ by any non-degenerate, symmetric bilinear form. Hence the first prolongation also vanishes for $\mathfrak{o}(p, q)$ and since $\dim(\mathfrak{o}(p, q)) = \dim(\mathfrak{o}(n))$ also all the remaining arguments extend to the pseudo-Riemannian case without changes.

Observe also, that things clearly extend to subgroups of $O(p, q)$ which also have Lie algebra $\mathfrak{o}(p, q)$. In the definite case, this only applies to the subgroup $SO(n) \subset O(n)$ which is the connected component of the identity. This corresponds to oriented Riemannian manifolds, so the additional orientation does not lead to additional torsion or curvature invariants. If one goes deeper into the study of invariants, the fact that one gets a volume form rather than a volume density does cause differences, though. In the indefinite case, the situation is slightly more complicated since for $p, q \neq 0$, the group $O(p, q)$ has four connected components and there are more subgroups with the same Lie algebra. In particular, one always has $SO(p, q) \subset O(p, q)$ which corresponds to oriented pseudo-Riemannian manifolds and the connected component $SO_0(p, q)$ which corresponds to a stronger notion of orientation (space- and time-orientation).

Finally, let us also remark that the proof of existence of a unique torsion free connections extends to the cases of groups that cover $O(p,q)$ or its subgroups with the same Lie algebra. So in particular, for the two-fold covering $Spin(n) \to SO(n)$, once existence of a spin-structure is known, it follows that one the principal $Spin(n)$-bundle, there is a unique principal connection with vanishing torsion. This is often called the spin-connection.

4.2. Riemannian G-structures. Let us next consider the case that G is a closed subgroup of $O(n)$ (which of course implies that G is also closed in $GL(n,\mathbb{R})$). Then the standard inner product on $\mathbb{R}^n$ is invariant under G, which shows that any G-structure $p : P \to M$ in particular gives rise to a Riemannian metric on M. Hence such structures can be viewed as refinements of Riemannian metrics and they are often called *Riemannian G-structures*. Two obvious examples of such groups are $SU(m) \subset U(m) \subset SO(n)$ for $n = 2m$, but one can also use examples like distinguished subspaces or flags and combine them with an inner product. Then by construction the Lie algebra $\mathfrak{g}$ is a subalgebra of $\mathfrak{o}(n)$ and since we have discussed the case that $\mathfrak{g} = \mathfrak{o}(n)$ already, we assume that $\mathfrak{g} \subset \mathfrak{o}(n)$ is a proper subalgebra. Since the resulting map ∂ for $\mathfrak{g}$ is just the restriction of the corresponding map for $\mathfrak{o}(n)$, we conclude that $\partial : \mathbb{R}^{n*} \otimes \mathfrak{g} \to \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$ is injective (and hence $\mathfrak{g}^{(1)} = \{0\}$).

We also conclude that the codimension of $\mathrm{im}(\partial)$ in $\Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$ equals $n(\dim(\mathfrak{o}(n)) - \dim(\mathfrak{g}))$. Of course, this equals the dimension of the space $\mathcal{I}$ from formula (3.15), so there is room for intrinsic torsion here. As a closed subgroup of $O(n)$, the group G is compact, so by general results any invariant subspace in a representation of G admits an invariant complement. This implies that for $G \subset O(n)$, there always is a normalization condition for the torsion of G-structures. Hence we are in the setting of Theorem 3.9, so choosing $\mathcal{N}$, we conclude that any Riemannian G-structure admits a unique connection $\gamma_{\mathcal{N}}$ with torsion in $\mathcal{N}$ and the torsion of this connection coincides with the intrinsic torsion of our G-structure by Theorem 3.8. Of course, also Theorems 3.9 and 3.10 apply. Similarly as in Section 3.6 one immediately shows that $\gamma_{\mathcal{N}}$ is metric for the underlying Riemannian metric g, however it differs from the Levi-Civita connection of g unless the intrinsic torsion T_i of the G-structure vanishes identically.

We can easily make things much more explicit: For $O(n)$ the adjoint representation on $\mathfrak{o}(n)$ is simply given by conjugation and one immediately verifies that $(X,Y) \mapsto -\operatorname{tr}(XY) = \operatorname{tr}(XY^t)$ defines an $O(n)$-invariant inner product on $\mathfrak{o}(n)$. Now we can restrict the adjoint representation of $O(n)$ to G to obtain a representation of G on $\mathfrak{o}(n)$ for which this inner product is invariant, too. The subspace $\mathfrak{g} \subset \mathfrak{o}(n)$ of course is G-invariant, so it follows that the orthocomplement $\mathfrak{g}^\perp$ of $\mathfrak{g}$ in $\mathfrak{o}(n)$ defines a G-invariant complement to $\mathfrak{g}$. Since $\partial : \mathbb{R}^{n*} \otimes \mathfrak{o}(n) \to \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$ is a linear isomorphism and by construction G-equivariant, we conclude that $\mathcal{N} := \partial(\mathbb{R}^{n*} \otimes \mathfrak{g}^\perp)$ defines a G-invariant complement to $\partial(\mathbb{R}^{n*} \otimes \mathfrak{g})$ in $\Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$. Hence we have obtained a normalization condition for the torsion of G-structure, and we know that as a representation of G, the space $\mathcal{I}$ from formula (3.15) is isomorphic to $\mathbb{R}^{n*} \otimes \mathfrak{g}^\perp$.

For this choice of normalization condition, we can directly characterize the normal connection $\gamma_{\mathcal{N}}$ in terms of the tensor describing the difference to the Levi-Civita connection. To formulate this in terms of linear connections, we first recall that for an $O(n)$-structure $p : P \to M$, the associated bundle $P \times_{O(n)} \mathfrak{o}(n)$ is the subbundle of $L(TM, TM)$ consisting of those endomorphisms of TM, whose value in each point $x \in M$ is skew symmetric for the inner product $g(x)$. It is natural to denote this bundle by $\mathfrak{o}(TM)$. Starting with a G-structure $p : P \to M$, we of course get

$P \times_G \mathfrak{o}(n) \cong \mathfrak{o}(TM)$. Thus the decomposition $\mathfrak{o}(n) = \mathfrak{g} \oplus \mathfrak{g}^\perp$ defines a decomposition $\mathfrak{o}(TM) = E_1 \oplus E_2$ as a Whitney sum. Now if we denote by g the Riemannian metric underlying the G-structure and by ∇^g its Levi-Civita connection, any linear connection on TM can be written as $\nabla_\xi \eta = \nabla^g_\xi \eta + A(\xi, \eta)$ for a $\binom{1}{2}$-tensor field A on M, see Section 3.1. From Section 3.6, we know that ∇ is metric for g if and only if $A_x(\xi, \ ) \in \mathfrak{o}(T_xM)$ for each $x \in M$, and of course this has to be satisfied for any connection compatible with the G-structure. But then the torsion of ∇ is given by $T(\xi, \eta) = A(\xi, \eta) - A(\eta, \xi)$ so by construction we conclude that this has values in $\mathcal{N}$ if and only if $A(\xi, \ )$ has values in $E_2 \subset \mathfrak{o}(TM)$. So we know from our general results that there is a unique tensor A such that the resulting connection ∇ is compatible with the G structures and such that $A(\xi, \ )$ has values in E_2 and then the resulting connection ∇ is induced by the canonical connection $\gamma_{\mathcal{N}}$.

This can be converted into a natural interpretation of what vanishing of the intrinsic torsion means for a Riemannian G-structure and it leads to a general approach for analyzing the intrinsic torsion of a Riemannian G-structure. We discuss this in a specific example next.

4.3. Example: Almost Hermitian structures. Consider $G = U(m) \subset SO(2m)$ in dimension $n = 2m$. Starting with $\mathbb{C}^m \cong \mathbb{R}^{2m}$, the standard inner product on $\mathbb{R}^{2m}$ simply is the real part of the standard Hermitian inner product on $\mathbb{C}^m$. In linear algebra one shows that an orthogonal linear map f lies in $U(m) \subset SO(2m)$ if and only if f is complex linear. Otherwise put, $U(m)$ is the joint stabilizer of the standard inner product $\langle\ ,\ \rangle$ and the linear map $J(z) := iz$ on $\mathbb{R}^{2m} \cong \mathbb{C}^m$. This readily translates to the fact that a $U(m)$-structure on a manifold M of dimension $n = 2m$ is equivalent to a pair (g, J), where $g \in T^0_2(M)$ is a Riemannian metric on M and $J \in \mathcal{T}^1_1(M)$ is an almost complex structure as discussed in Example 2.8 (3) such that for each $x \in M$ and $X, Y \in T_xM$ we get $g_x(J_x(X), J_x(Y)) = g_x(X, Y)$. This means that g is Hermitian with respect to J and hence $U(m)$-structures are commonly called *almost Hermitian structures*.

Similarly as in Section 3.6 one concludes that for any connection γ on a $U(m)$-structure with induced linear connection(s) ∇, one obtains $\nabla g = 0$ and $\nabla J = 0$, so g and J are parallel as tensor fields. As there, our characterization of $U(m)$ also implies that conversely, any linear connection with these two properties is induced by a connection on the $U(m)$-structure. Having observed this, we claim that a $U(m)$-structure $p : P \to M$ has vanishing intrinsic torsion if and only if the Levi-Civita connection ∇^g of g satisfies $\nabla^g J = 0$. If this is the case, then ∇^g is induced by a connection on P, which therefore admits a torsion-free connection and hence has vanishing intrinsic torsion. Conversely, if P has vanishing intrinsic torsion, then the canonical connection $\gamma_{\mathcal{N}}$ is torsion free and since the induced connection on TM is metric, it has to coincide with the Levi-Civita connection so $\nabla^g J = 0$. Such structures are called *Kähler structures* and they are of fundamental importance.

The general version of this statement says that a Riemannian G-structure has vanishing intrinsic torsion if and only if all the structures that one has to add to the Riemannian metric to obtain the G-structure are preserved by the Levi-Civita connection. Such G-structures are often called *integrable*. They are important since a Riemannian metric underlies an integrable G-structure if and only if the so-called *holonomy group* of g is a subgroup of G. Since there are classifications of possible holonomy groups of Riemannian metrics this is an important ingredient for Riemannian geometry.

Understanding the intrinsic torsion of almost Hermitian structures in detail is a more complicated issue. It is possible to approach this via the almost complex structure J (see the discussion in Section 4.5 below), but there is a more popular alternative approach. This is known as the *Gray-Hervella classification*, see [**GH76**], and we briefly discuss this from our perspective. The starting point for this approach is that for an almost Hermitian structure (g, J) on a smooth manifold M of dimension $2m$, on obtains the so-called *fundamental two-form* $\omega \in \Omega^2(M)$ via $\omega(x)(X, Y) := g(x)(X, J(x)(Y))$ for $x \in M$ and $X, Y \in T_xM$. Under a linear isomorphism $T_xM \to \mathbb{C}^m$ that sends $g(x)$ to the real part of the Hermitian inner product and $J(x)$ to multiplication by i, $\omega(x)$ exactly corresponds to the imaginary part of the Hermitian form. By construction $\omega(x)$ is non-degenerate and Hermitian in the sense that $\omega(x)(J(x)(X), J(x)(Y)) = \omega(x)(X, Y)$. Hence in particular, ω defines an almost symplectic structure on M.

Since ω is constructed from g and J, it follows readily that for the canonical connection ∇ of the almost-Hermitian structure, we get $\nabla\omega = 0$. Denoting by ∇^g the Levi-Civita connection of g, one easily verifies that $(\nabla^g_\xi\omega)(\eta, \zeta) = \omega(\eta, (\nabla^g_\xi J)(\zeta))$. Thus from above we conclude that vanishing of the intrinsic torsion of (g, J) can be equivalently characterized as $\nabla^g\omega = 0$. But the situation is much better than that, since it turns out that the $\binom{0}{3}$-tensor field $\nabla^g\omega$ provides an equivalent encoding of the intrinsic torsion T_i, and the Gray-Hervella approach is to study T_i via the properties of $\nabla^g\omega$.

Let us start with $\mathfrak{o}(2m)$ viewed as a the space of real linear maps $f : \mathbb{C}^m \to \mathbb{C}^m$ which are skew symmetric with respect to the real part $\langle\ ,\ \rangle$ of the standard Hermitian inner product. Of course, f lies in the subalgebra $\mathfrak{u}(m)$ if and only if f is $\mathbb{C}$-linear, i.e. $f(iz) = if(z)$. Now one can decompose any real linear map on $\mathbb{C}^m$ into a complex linear and a conjugate linear part, given by $z \mapsto \frac{1}{2}(f(z) \mp if(iz))$. Since $\langle\ ,\ \rangle$ is Hermitian, it follows readily that for $f \in \mathfrak{o}(2m)$ also $z \mapsto if(iz)$ lies in $\mathfrak{o}(2m)$. But this shows that $V := \{f \in \mathfrak{o}(2m) : f(iz) = -if(z)\}$ is a linear subspace in $\mathfrak{o}(2m)$ such that $\mathfrak{o}(2m) = \mathfrak{u}(m) \oplus V$. Moreover for $X \in \mathfrak{u}(m)$ and $Y \in V$, the composition XY is conjugate linear and thus has vanishing real trace, so $V = \mathfrak{u}(m)^\perp$ for the inner product on $\mathfrak{o}(2m)$ discussed above. In particular, we conclude that the representation $\mathcal{I}$ of $U(m)$ from Section 3.7 is isomorphic to the space $\mathbb{C}^{m*} \otimes V$ of real linear maps $\mathbb{C}^m \to V$.

Elements of the latter space can be equivalently described as as real-bilinear maps $\Phi : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^m$, which are conjugate linear and skew symmetric with respect to $\langle\ ,\ \rangle$ in the second variable. Defining $\Psi(X, Y, Z) := \langle Y, \Phi(X, Z)\rangle$, we obtain a trilinear map $\Psi : (\mathbb{C}^m)^3 \to \mathbb{R}$ which is skew symmetric in Y and Z and satisfies $\Psi(X, iY, iZ) = -\Psi(X, Y, Z)$. Denoting by $\mathcal{W}$ the space of all maps with these two properties, one easily verifies that $\dim(\mathcal{W}) = 2m\dim(V)$, so we conclude that $\mathcal{W} \cong \mathcal{I}$ as a representation of $U(m)$. Gray and Hervella then show that for $m \geq 3$, $\mathcal{W}$ decomposes as a direct sum of 4 irreducible representations $\mathcal{W}_i$ of $U(m)$ and characterize these components by equations on the maps Ψ contained in them. Correspondingly, the intrinsic torsion of an almost Hermitian structure in dimensions $2m \geq 6$ can be written as a sum of 4 components, and the 16 classes referred to in the title of [**GH76**] correspond to the 16 possibilities of which of these components are non-zero. In low dimensions, the decomposition into irreducibles becomes simpler, so there are fewer possible types.

We can also make the characterization of the canonical connection from above explicit here. Of course the decomposition $\mathfrak{o}(TM) = E_1 \oplus E_2$ of the bundle of endomorphisms of TM which are skew symmetric with respect to g exactly corresponds to the fact that the endomorphisms in addition are complex linear respectively conjugate linear with respect to J in each point. In particular writing the canonical connection ∇ as $\nabla_\xi\eta = \nabla^g_\xi\eta + A(\xi, \eta)$, it is uniquely characterized by the fact that $A(\xi,\ )$ is skew symmetric with respect to g and conjugate linear with respect to J. But this provides a relation to $\nabla^g\omega$ as follows. By definition, we get

$$0 = (\nabla_\xi\omega)(\eta,\zeta) = \xi\cdot\omega(\eta,\zeta) - \omega(\nabla_\xi\eta,\zeta) - \omega(\eta,\nabla_\xi\zeta).$$

Inserting $\nabla_\xi\eta = \nabla^g_\xi\eta + A(\xi,\eta)$ and similarly for $\nabla_\xi\zeta$, we conclude that

$$0 = (\nabla^g_\xi\omega)(\eta,\zeta) - \omega(A(\xi,\eta),\zeta) - \omega(\eta, A(\xi,\zeta)).$$

Inserting the definition of ω and using skew-symmetry of $A(\xi,\ )$ in the first term and then conjugate linearity, we conclude that

$$(\nabla^g_\xi\omega)(\eta,\zeta) = g(\eta, -A(\xi, J(\zeta)) + JA(\xi,\zeta)) = -2g(\eta, A(\xi, J(\zeta))).$$

But this shows that $(\nabla^g_\xi\omega)(\eta,\zeta)$ simply is an equivalent encoding of the trilinear map Ψ associated to A and hence of the intrinsic torsion of the almost Hermitian structure (g,J). The 16 classes defined by Gray and Hervella (including nearly Kähler, almost Kähler and quasi-Kähler structures) are then defined by requiring specific properties of $\nabla^g\omega$, e.g. being symmetric in the first two arguments or totally skew symmetric and so on.

4.4. Almost symplectic structures. Here we put $n = 2m$ and consider $G = Sp(2m,\mathbb{R}) \subset GL(n,\mathbb{R})$, the group of maps that preserve a fixed, non-degenerate, skew-symmetric bilinear form b on $\mathbb{R}^{2m}$. The resulting G-structure clearly is equivalent to an almost symplectic structure as defined in Section 1.3. Similarly as in Example 3.6 one shows that a linear connection ∇ on TM is compatible with the G-structure inducing an almost symplectic structure $\omega\in\Omega^2(M)$ if and only if $\nabla\omega = 0$.

The Lie algebra $\mathfrak{g} = \mathfrak{sp}(2m,\mathbb{R})$ of G consist of all linear maps $f:\mathbb{R}^n\to\mathbb{R}^n$ which are skew symmetric with respect to b. Via b, linear maps $f:\mathbb{R}^n\to\mathbb{R}^n$ can be identified with bilinear forms on $\mathbb{R}^n$ by looking at $(x,y)\mapsto b(f(x),y)$, and a map f lies in $\mathfrak{g}$ if and only if the corresponding bilinear form is symmetric. Hence we obtain an isomorphism $\mathfrak{g}\cong\mathbb{S}^2\mathbb{R}^{n*}$ of representations of G. This readily implies that the kernel $\mathfrak{g}^{(1)}$ of the map $\partial:\mathbb{R}^{n*}\otimes\mathfrak{g}\to\Lambda^2\mathbb{R}^{n*}\otimes\mathbb{R}^n$ is isomorphic to $S^3\mathbb{R}^{n*}\subset\mathbb{R}^{n*}\otimes S^2\mathbb{R}^{n*}$. The complete symmetrization defines a projection onto this subspace, so we see that $\mathbb{R}^{n*}\otimes S^2\mathbb{R}^{n*}\cong S^3\mathbb{R}^{n*}\oplus W$, where W is the kernel of the symmetrization. It is well know that this is the decomposition into irreducible representations for $GL(n,\mathbb{R})$. To avoid low dimensional special phenomena, let us assume $n\geq 6$ from now on.

Now using b, we can also identify $\Lambda^2\mathbb{R}^{n*}\otimes\mathbb{R}^n$ with $\Lambda^2\mathbb{R}^{n*}\otimes\mathbb{R}^{n*}$ as a representation of G. It is well known that, as a representation of $GL(n,\mathbb{R})$, the latter decomposes as $\tilde{W}\oplus\Lambda^3\mathbb{R}^{n*}$, where $\tilde{W}$ is the kernel of the complete alternation. Again, the two summands are irreducible as representations of $GL(n,\mathbb{R})$ and it is well known that $\tilde{W}\cong W$. As a map $\mathbb{R}^{n*}\otimes S^2\mathbb{R}^{n*}\to\Lambda^2\mathbb{R}^{n*}\otimes\mathbb{R}^{n*}$ the alternation in the first two entries (which corresponds to ∂) is $GL(n,\mathbb{R})$-equivariant, so it has to map W isomorphically onto $\tilde{W}$. Hence we conclude that here $\operatorname{im}(\partial)$ coincides with the kernel of the complete alternation. In particular, the representation $\mathcal{I}$ from Section 3.7 is isomorphic to $\Lambda^3\mathbb{R}^{n*}$, so the intrinsic torsion of an almost symplectic structure can be viewed as a 3-form. Likewise, viewing $\Lambda^3\mathbb{R}^{n*}$ as a subspace of $\Lambda^2\mathbb{R}^{n*}\otimes\mathbb{R}^{n*}$ it defines a normalization condition $\mathcal{N}$ for the torsion of almost symplectic structures. However, since $\mathfrak{g}^{(1)}\cong S^3\mathbb{R}^{n*}$, normal connections are always far from being unique, they form an affine space modelled on completely symmetric $\binom{0}{3}$-tensor fields on M.

Taking a compatible connection with induced linear connection ∇ on TM with torsion $T\in\Omega^2(M,TM)$, this leads to the following description of the intrinsic torsion of (M,ω): We first convert the torsion T to a $\binom{0}{3}$-tensor field via $(\xi,\eta,\zeta)\mapsto\omega(T(\xi,\eta),\zeta)$

and then the intrinsic torsion T_i is given by the complete alternation of this expression. Since T is skew symmetric, this can be computed (up to a non-zero factor) as the sum of the expression of over all cyclic permutations of the three arguments. This allows a very nice interpretation as follows.

PROPOSITION 4.4. *Consider an almost symplectic structure (M, ω) on a smooth manifold M of dimension $n = 2m \geq 6$. Then the intrinsic torsion of the corresponding G-structure is equivalently encoded as $d\omega \in \Omega^3(M)$. In particular, an almost symplectic structure has vanishing intrinsic torsion and thus admits a compatible torsion-free connection if an only if it is symplectic.*

PROOF. Let ∇ be a linear connection on TM that is compatible with the almost symplectic structure ω. Then

$$0 = (\nabla_\xi \omega)(\eta, \zeta) = \xi(\omega(\eta, \zeta)) - \omega(\nabla_\xi \eta, \zeta) - \omega(\eta, \nabla_\xi \zeta). \tag{4.1}$$

Now we can sum this equation over all cyclic permutations of the arguments, which adds the expressions

$$\begin{gathered}\zeta(\omega(\xi, \eta)) - \omega(\nabla_\zeta \xi, \eta) - \omega(\xi, \nabla_\zeta \eta) \\ \eta(\omega(\zeta, \xi)) - \omega(\nabla_\eta \zeta, \xi) - \omega(\zeta, \nabla_\eta \xi).\end{gathered}$$

Now we can for example collect the second term in the right hand side of (4.1) with the last term in the bottom line to get

$$-\omega(\nabla_\xi \eta - \nabla_\eta \xi, \zeta) = -\omega([\xi, \eta], \zeta) - \omega(T(\xi, \eta), \zeta)$$

by the definition of torsion. Collecting the Lie bracket terms with the terms in which the values of ω are differentiated, we obtain $d\omega(\xi, \eta, \zeta)$. Hence we conclude that $d\omega(\xi, \eta, \zeta)$ coincides with the sum of $\omega(T(\xi, \eta), \zeta)$ over all cyclic permutations of the arguments. As we have noted above, this sum provides an equivalent encoding of the intrinsic torsion of the G-structure determined by ω, so the result follows. □

Torsion-free connections preserving a symplectic form are called *Fedosov connections*, so such connections always exist but they are far from being unique.

4.5. Almost complex structures. Here we consider $\mathbb{C}^m = \mathbb{R}^{2m}$ and the closed subgroup $G := GL(m, \mathbb{C}) \subset GL(2m, \mathbb{R})$ and hence $\mathfrak{g} := \mathfrak{gl}(m, \mathbb{C}) \subset \mathfrak{gl}(2m, \mathbb{R})$. In this case, we can analyze the map ∂ completely and we assume $m > 1$ throughout the discussion. First, the space $\mathbb{R}^{2m*} \otimes \mathfrak{g}$ has real dimension $2m(2m^2) = 4m^3$ and we can identify it with the space of $\mathbb{R}$-bilinear maps $f : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^m$ which are complex linear in the second variable. In this interpretation $\partial f(z, w) = f(z, w) - f(w, z)$, so $\partial f = 0$ if and only if f is symmetric. But then of course f is symmetric and complex bilinear, and conversely, any symmetric complex bilinear map defines an element of $\ker(\partial)$. Thus we see that $\mathfrak{g}^{(1)}$ coincides with the space of complex bilinear symmetric maps $\mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^m$. This is a complex vector space of complex dimension $m(\frac{1}{2}m(m+1))$ and hence for the real dimension we get $\dim(\mathfrak{g}^{(1)}) = m^3 + m^2$. This implies that $\operatorname{im}(\partial)$ has real dimension $3m^3 - m^2$.

Now the image space $W := \Lambda^2 \mathbb{R}^{2m*} \otimes \mathbb{R}^{2m}$ of ∂ is the space of skew symmetric, real bilinear maps $\mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^m$ and we have to understand this as a representation of $G = GL(m, \mathbb{C})$. This can be done via complex linearity properties as follows. Since the target space $\mathbb{C}^m$ is complex, we can consider skew symmetric maps that are either complex linear or conjugate linear in both arguments. Let us denote by $\Lambda^{2,0}$ the space of complex bilinear maps and by $\Lambda^{0,2}$ the space of maps that are conjugate linear in both arguments. Evidently, both spaces are G-invariant complex subspaces of W whose dimension can be easily determined. One can specify a map φ in either of the two spaces by fixing the values $\varphi(e_j, e_k) \in \mathbb{C}^m$ for $j \neq k$ and the standard basis $\{e_1, \dots, e_m\}$ for $\mathbb{C}^m$. This shows that the complex dimension of either of the two spaces equals $m(\frac{1}{2}m(m-1))$ and hence they have real dimension $m^3 - m^2$.

Now if φ is either $\mathbb{C}$-bilinear or conjugate linear in both arguments, then $\varphi(iz, iw) = -\varphi(z, w)$. Thus we are led to the idea to also consider the subspace $\Lambda^{1,1} \subset W$ consisting of all maps φ such that $\varphi(iz, iw) = \varphi(z, w)$, so these are Hermitian skew symmetric bilinear forms. Of course, this also is a G-invariant complex subspace of W. Now it is easy to see that $W = \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}$, for example by decomposing a general element $\varphi \in W$ as $\varphi = \varphi_{2,0} + \varphi_{1,1} + \varphi_{0,2}$ with

$$\begin{aligned}
\varphi_{2,0}(z, w) :&= \tfrac{1}{4} \left(\varphi(z, w) - \varphi(iz, iw) - i(\varphi(iz, w) + \varphi(z, iw))\right) \\
\varphi_{1,1}(z, w) :&= \tfrac{1}{2} \left(\varphi(z, w) + \varphi(iz, iw)\right) \\
\varphi_{0,2}(z, w) :&= \tfrac{1}{4} \left(\varphi(z, w) - \varphi(iz, iw) + i(\varphi(iz, w) + \varphi(z, iw))\right)
\end{aligned}$$

and verifying that the components lie in the indicated subspaces. This in particular implies that $\dim(\Lambda^{1,1}) = 2m^3$. But now one verifies immediately that for $f \in \mathbb{R}^{2m*} \otimes \mathfrak{g}$, one gets $(\partial f)_{0,2} = 0$, since f is complex linear in the second variable. Thus we conclude that $\operatorname{im}(\partial) \subset \Lambda^{2,0} \oplus \Lambda^{1,1}$ and checking dimensions, we see that we must have equality. This also shows that $\mathcal{N} := \Lambda^{0,2}$ is a normalization condition for the torsion of G-structures. The resulting condition on the torsion can be immediately made explicit in the language of almost complex structures. Since the torsion is an element $T \in \Omega^2(M, TM)$ its value in a point $x \in M$ is a skew-symmetric real bilinear map $\Lambda^2 T_xM \to T_xM$. Now the value $J(x)$ of the almost complex structure J on M makes T_xM into a complex vector space and so it makes sense to require that $T(x)$ is conjugate linear in both arguments with respect to $J(x)$, and this is exactly the normalization condition we obtain.

The projection $\varphi \mapsto \varphi_{0,2}$ can be applied point-wise on M and our above considerations also tell us that the intrinsic torsion of the G-structure corresponding to an almost complex structure J on M can be equivalently encoded in $T_{0,2}$ where T is the torsion of any compatible connection. In terms of the induced linear connection ∇ on TM, the compatibility condition implies that for $J \in \mathcal{T}^1_1(M)$, we get $\nabla J = 0$. Conversely, suppose we have given two connections ∇ and $\hat{\nabla}$ such that $\nabla J = 0$ and $\hat{\nabla} J = 0$. This means that $0 = (\nabla_\xi J)(\eta) = \nabla_\xi J\eta - J\nabla_\xi \eta$ and similarly for $\hat{\nabla}$ so writing $A(\xi, \eta) = \hat{\nabla}_\xi \eta - \nabla_\xi \eta$ we conclude that $A(\xi, J(\eta)) = J(A(\xi, \eta))$. Using this, one verifies as in Example 3.6 that a linear connection ∇ on TM is compatible with the G-structure if and only if $\nabla J = 0$.

Theorem 4.5. *Let J be an almost complex structure on a smooth manifold M of even dimension $2m \geq 4$. Then there is a $\binom{1}{2}$-tensor field $N = N_J$ on M such that for $\xi, \eta \in \mathfrak{X}(M)$ one gets*

$$N_J(\xi, \eta) = [\xi, \eta] - [J\xi, J\eta] + J([J\xi, \eta] + [\xi, J\eta]) \tag{4.2}$$

and the intrinsic torsion of the G-structure determined by J is given by $T_i = -\frac{1}{4}N$. In particular, an almost complex structure admits a compatible torsion-free connection if and only if N vanishes identically.

Proof. While existence of N follows from the following considerations, it is a good exercise to verify directly that the right hand side of (4.2) is linear over $C^\infty(M, \mathbb{R})$ in both arguments. Indeed, replacing η by $f\eta$ and taking into account that J is a tensor field, we see that the additional terms to $fN(\xi,\eta)$ in the right hand side of (4.2) are

$$\xi(f)\eta - (J\xi)(f)J\eta + J((J\xi)(f)\eta + \xi(f)J\eta) = 0.$$

Since N is visibly skew symmetric, this completes the argument.

Now we know from above that we can take any linear connection ∇ on TM such that $\nabla J = 0$ and then compute the intrinsic torsion as $T_i = T_{0,2}$, where T is the torsion of ∇. Now take the definition of T, i.e.

$$T(\xi,\eta) = \nabla_\xi\eta - \nabla_\eta\xi - [\xi,\eta],$$

and we can obtain $T_{0,2}(\xi,\eta)$ as

$$\tfrac{1}{4}\left(T(\xi,\eta) - T(J\xi,J\eta) + J(T(J\xi,\eta) + T(\xi,J\eta))\right).$$

Inserting the definition of T, we can use the fact that $\nabla_\xi J\eta = J\nabla_\xi\eta$ and similar expression to conclude that the terms involving ∇ do not contribute to $T_{0,2}$ at all. So the only contributions comes from taking the appropriate combinations of the negative Lie bracket, which shows that $T_{0,2} = -\frac{1}{4}N_J$ as claimed. □

The tensor N defined in (4.2) is called the *Nijenhuis-tensor* of J. There is a beautiful interpretation of vanishing of N_J via the so-called *Newlander-Nirenberg theorem*, see [**NN57**]. Consider a complex manifold M of complex dimension m, i.e. a manifold endowed with an atlas that has charts with values in open subsets of $\mathbb{C}^m$ such that the chart changes are holomorphic. This exactly means that the derivatives of the chart changes are complex linear, and hence each tangent space of M canonically inherits the structure of a complex vector space. Since multiplication by i is a constant map in such charts, this indeed defines an almost complex structure J on M. Looking at complex coordinate vector fields, one immediately deduces that for such a structure the Nijenhuis tensor automatically vanishes. Observe in particular, that this implies that J is real analytic. The Newlander-Nirenberg theorem states that conversely any almost complex structure which is least C^2 and whose Nijenhuis tensor vanishes is obtained from a holomorphic atlas and hence is automatically real analytic. Indeed, proving real analyticity is the first step in the proof and is rather hard. Assuming real analyticity of J from the outset, the proof becomes much easier.

Let us briefly mention what happens for $m = 1$. Multiplications by complex numbers of modulus one on $\mathbb{C}$ are exactly rotations, which corresponds to $U(1) \cong SO(2)$. Thus we see that $GL(1,\mathbb{C}) \subset GL(2,\mathbb{R})$ is generated by $SO(2)$ and by multiples of the identity. Thus it coincides with those linear automorphisms of $\mathbb{R}^2$ which are orientation preserving and conformal, i.e. they preserve the standard inner product up to multiples. This is the basis for the relation between complex structures and conformal structures on real surfaces which is fundamental for the theory of Riemann surfaces. On the level of Lie algebras, this means that $\mathfrak{gl}(1,\mathbb{C}) \supset \mathfrak{o}(2)$, so from Section 4.1 we conclude that $\partial : \mathbb{R}^{2*} \otimes \mathfrak{gl}(1,\mathbb{C}) \to \Lambda^2\mathbb{R}^{2*} \otimes \mathbb{R}^2$ is surjective with 2-dimensional kernel. Hence there is no intrinsic torsion for almost complex structures in real dimension two. Thus any such structure admits a compatible torsion-free connection, but this is not unique. Indeed, it turns out that in dimension two any almost complex structure comes from a complex atlas.

4.6. Distributions. Here we assume that $n \geq 3$ and $2 \leq k < n$ and let G be the stabilizer of $\mathbb{R}^k \subset \mathbb{R}^n$, so G-structures correspond to smooth distributions of rank k of smooth manifolds of dimension n. As in 4.5, we can completely analyze the map $\partial : \mathbb{R}^{n*} \otimes \mathfrak{g} \to \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$ in this case. From the description of G it follows readily that $\dim(\mathfrak{g}) = kn + (n-k)^2$. We can identify $\mathbb{R}^{n*} \otimes \mathfrak{g}$ with the space of bilinear maps

$f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ which have the property that $f(v,w) \in \mathbb{R}^k \subset \mathbb{R}^n$ provided that $w \in \mathbb{R}^k \subset \mathbb{R}^n$. In this picture ∂f is just the alternation of f, so if $\partial f = 0$, then f is a symmetric bilinear map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ which has values in $\mathbb{R}^k$ provided that one of its entries lies in $\mathbb{R}^k$. Conversely, any such form evidently lies in $\mathbb{R}^{n*} \otimes \mathfrak{g}$, so we have found a complete description of $\mathfrak{g}^{(1)}$.

To determine the dimension of $\mathfrak{g}^{(1)}$, we observe that $\dim(S^2\mathbb{R}^{n*}) = \frac{1}{2}n(n+1)$. Among the elements of the basis of this space induced by the standard basis of $\mathbb{R}^n$, there are $\frac{1}{2}k(k+1) + k(n-k) = \frac{1}{2}(k(2n-k+1))$ elements which only are non-zero if one of their entries lies in $\mathbb{R}^k$ while the other $\frac{1}{2}(n-k)(n-k+1)$ elements vanish on this subspace. Hence the above description of $\mathfrak{g}^{(1)}$ shows that

$$\dim(\mathfrak{g}^{(1)}) = \tfrac{1}{2}(k^2(2n-k+1) + n(n-k)(n-k+1)).$$

Subtracting this from $kn^2 + n(n-k)^2$, we conclude that

$$\dim(\mathrm{im}(\partial)) = \tfrac{1}{2}(n(n-k)(n+k-1) + k^2(k-1)). \tag{4.3}$$

On the side of the torsion, we observe that for a bilinear map f as above, we certainly get $\partial f(v,w) \in \mathbb{R}^k$ if both v and w lie in the subspace $\mathbb{R}^k$. There is a simple way to form a quotient of $\Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$ by restricting to entries from $\mathbb{R}^k$ and then projecting the result to $\mathbb{R}^n/\mathbb{R}^k$. Now observe that the group G has canonical representations on $\mathbb{R}^k$ and on $\mathbb{R}^n/\mathbb{R}^k$ that are induced from the two diagonal blocks an a block decomposition. Using these representations, the map $\pi : \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n \to \Lambda^2\mathbb{R}^{k*} \otimes (\mathbb{R}^n/\mathbb{R}^k)$ we have described is G-equivariant. By construction $\mathrm{im}(\partial) \subset \ker(\pi)$. But we can easily compute the dimension of $\ker(\pi)$ as $\frac{1}{2}(n^2(n-1) - k(k-1)(n-k))$ and one immediately verifies that this coincides with the right hand side of (4.3). Hence we conclude that $\mathrm{im}(\partial) = \ker(\pi)$ and hence the representation $\mathcal{I}$ from Section 3.7 can be identified with $\Lambda^2\mathbb{R}^{k*} \otimes (\mathbb{R}^n/\mathbb{R}^k)$.

Translating this setup to geometry is rather easy. As we know from Section 2.8, a G-structure $p : P \to M$ on a smooth manifold of dimension n is equivalent to a smooth distribution $E \subset TM$ of rank k. From the construction, it is clear that $E = P \times_G \mathbb{R}^k$, and as usual $TM = P \times_G \mathbb{R}^n$. We can also form the bundle $P \times_G (\mathbb{R}^n/\mathbb{R}^k)$, and the G-equivariant quotient projection $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{R}^k$ induces a surjective bundle map from TM to this bundle. For each point $x \in M$ the kernel of the corresponding map on T_xM is E_x, so it is natural to denote $P \times_G (\mathbb{R}^n/\mathbb{R}^k)$ by TM/E. Indeed this is an instance of the general notion of the quotient of a vector bundle by a smooth subbundle. Let us denote by $q : TM \to TM/E$ the natural bundle projection.

THEOREM 4.6. *Let $E \subset TM$ be a smooth distribution of rank $k \geq 2$ on a smooth manifold M of dimension $n > k$. Then there is a skew symmetric bilinear bundle map $\mathcal{L} : E \times E \to TM/E$ such that for $\xi, \eta \in \Gamma(E)$ we get $\mathcal{L}_x(\xi(x), \eta(x)) = q([\xi,\eta](x))$. The intrinsic torsion of the G-structure corresponding to $E \subset TM$ is given by $T_i = -\mathcal{L} \in \Gamma(\Lambda^2E^* \otimes (TM/E))$. In particular, the G-structure admits a compatible torsion-free connection if and only if the distribution E is involutive and hence integrable.*

PROOF. As before, it is not really necessary to to this, but one can easily verify that $\mathcal{L}$ indeed defines a bundle map. We just observe that $[\xi, f\eta] = f[\xi,\eta] + \xi(f)\eta$, but the second summand has values in E and hence its image under q vanishes. As an operator $\Gamma(E) \times \Gamma(E) \to \Gamma(TM/E)$, $\mathcal{L}$ therefore is bilinear over smooth functions and hence defines a bundle map. Alternatively, it can be interpreted as a smooth section of the bundle $\Lambda^2E^* \otimes (TM/E)$.

Consider a G-structure $p : P \to M$. In terms of equivariant functions $f : P \to M$ sections of the corresponding distribution $E \subset TM$ correspond to functions with values in $\mathbb{R}^k \subset \mathbb{R}^n$. Differentiating such a function with an equivariant vector field on P, one again obtains a function with values in $\mathbb{R}^k$. Thus for any principal connection on P, the induced linear connection ∇ on TM has the property that for $\eta \in \Gamma(E) \subset \mathfrak{X}(M)$ and $\xi \in \mathfrak{X}(M)$, one has $\nabla_\xi \eta \in \Gamma(E)$. Similarly as in Section 3.6, one shows that this characterizes the linear connections induced from principal connections on P.

Hence our above discussion implies that we can obtain the intrinsic torsion T_i of the G-structure by projecting the torsion T of any linear connection with this property to the bundle $\Lambda^2 E^* \otimes (TM/E)$. So this means that we have to evaluate T on two sections $\xi, \eta \in \Gamma(E)$ and then project the result to TM/E using q. But in the standard formula $T(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta]$ the first two summands by definition have values in E and hence do not contribute to the image under q. Hence we readily conclude that $T_i = -\mathcal{L}$ and since $\mathcal{L}(\xi, \eta) = 0$ if and only if $[\xi, \eta] \in \Gamma(E)$, the last statement follows. $\square$

Let us briefly discuss the case $k = 1$. Calculating dimensions in the same way as for $k \geq 2$, one easily verifies that (4.3) continues to hold for $k = 1$. This shows that $\dim(\mathrm{im}(\partial)) = \frac{1}{2}(n^2(n-1))$, and hence ∂ is surjective for $k = 1$. This means that there is no intrinsic torsion for distributions of rank 1 (which also are automatically involutive) and hence for each such distribution, there is a compatible torsion-free connection by Theorem 3.7. But such connections are far from being unique. Indeed, also the formula for $\dim(\mathfrak{g}^{(1)})$ from above remains correct if one naively inserts $k = 1$, and this gives $n + \frac{1}{2}n^2(n-1)$.

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