Geometry of homogeneous spaces

rough lecture notes

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1. Introduction

1.1. Basic questions. Recall the concept of Lie groups. For a Lie group G, any subgroup $H \subset G$, which is closed in the natural topology of G is a Lie subgroup. In particular, it is a smooth submanifold of G. Moreover, the space G/H of left cosets of H in G can be naturally made into a smooth manifold in such a way that the canonical map $p : G \to G/H$ defined by $p(g) := gH$ is a surjective submersion. In particular, a mapping f from G/H to any smooth manifold M is smooth if and only if $f \circ p : G \to M$ is smooth.

The manifold G/H carries a natural smooth action of G, defined by $g \cdot (\tilde{g}H) := (g\tilde{g})H$. This action is transitive, so G/H "looks the same" around each point, whence it is called a *homogeneous space* of G. Consider a smooth action $G \times M \to M$ of G on a smooth manifold M. For a point $x \in M$, the *isotropy group* $G_x := \{g \in G : g \cdot x = x\}$ is a closed subgroup of G and acting on x defined a smooth bijection from G/G_x onto the *orbit* $G \cdot x := \{g \cdot x : g \in G\}$. Thus homogeneous spaces of G are the models for orbits of smooth actions of G.

The basic question to be answered in this course concerns *G–invariant geometric structures* on a homogeneous space G/H. As a typical example, consider the question of existence of a Riemannian metric on G/H for which each element of G acts as an isometry. A priori, this sounds like a very difficult question, since the space of Riemannian metrics on G/H certainly is infinite dimensional. However, it turns out that such questions can be reduced to finite dimensional representation theory and hence in many cases to questions in linear algebra, which can be solved effectively.

This may sound like a rather restricted setting, but indeed there are general results that make sure that automorphism groups of certain geometric structures are Lie groups. A classical example of such a result is the so-called Myers–Steenrod theorem that says that the isometries of a Riemannian manifold always form a Lie group which acts smoothly on M. Now suppose that (M, g) is a Riemannian manifold which is homogeneous in the sense that for any two points $x, y \in M$, there is an isometry $f : M \to M$ such that $f(x) = y$. But this just says that the obvious action of the Lie group $G := \mathrm{Isom}(M)$ of isometries of M is transitive, so fixing a point $x \in M$, its stabilizer $H := \{f \in G : f(x) = x\}$ is a closed subgroup in G and $f \mapsto f(x)$ induces a smooth bijection $G/H \to M$, which can be shown to be a diffeomorphism. Hence the above considerations actually apply to all homogeneous Riemannian manifolds.

1.2. An introduction to the Erlangen program. Apart from its intrinsic interest, this topic fits into a much wider perspective, since it provides a connection between classical geometry and differential geometry. This is due to the approach to classical geometry known as the "Erlangen program" by F. Klein. To outline this, consider the example of affine and Euclidean geometry. Affine geometry can be phrase on an abstract affine space, which basically means that one takes a vector space, called the *modeling vector space* of the affine space and forgets its origin. This leads to two basic operations. On the one hand, given to points in the affine space A, there is a vector $\overrightarrow{pq}$ in the modeling vector space V connecting the two points. On the other hand, given $p \in A$ and $v \in V$, one obtains a point $p + v$ by "attaching v to p". In this way, one can axiomatically define affine spaces similarly to vector spaces. Less formally, one can use the fact that for any $p \in A$, the maps $q \mapsto \overrightarrow{pq}$ and $v \mapsto v + p$ are inverse bijections to identify A with V, but keep in mind that none of the resulting identifications is preferred. In any case, there is an obvious notion of an *affine subspace* obtained by adding to some point $p \in A$ all elements of a linear subspace $W \subset V$.

To do Euclidean geometry, one has to consider in addition a positive definite inner product $\langle\ ,\ \rangle$ on the modeling vector space V. This allows on to measure distances of points and angles between intersecting lines, etc. A crucial ingredient in any form of geometry is the concept of a motion. The difference between affine and Euclidean motions is also the simplest way to explain the difference between affine and Euclidean geometry. Initially, on may define *affine motions* as set functions $\Phi : A \to A$, such that for any affine line $\ell \subset A$ also $\Phi(\ell) \subset A$ is an affine line. It turns out that identifying A with V, affine motions are exactly the functions of the form $\Phi(v) = f(v) + v_0$ for a fixed vector $v_0 \in V$ and a linear map $f : V \to V$.

Likewise, *Euclidean motions* can be defined as set maps which preserve the Euclidean distance of points. In this case, it is even less obvious that there is a relation to the linear (or affine) structure. However, it turns out that, identifying A with V, Euclidean motions are exactly the maps of the form $v \mapsto f(v)+v_0$ for a fixed vector $v_0 \in V$ and an orthogonal linear map $f : V \to V$. Another interpretation is that the inner product on V can be viewed as defining a Riemannian metric on Euclidean space, and Euclidean motions are exactly the isometries of this metric (in the sense of Riemannian geometry). The equivalence of these three pictures is proved as Proposition 1.1 in my lecture notes on Riemannian geometry.

The basic role of affine respectively Euclidean motions is that any result of affine respectively Euclidean geometry should be (in an appropriate sense) compatible with the corresponding concept of motions. The basic idea of the Erlangen program is to take this as the fundamental definition of (a general version of) "geometry". By definition both affine an Euclidean motions form Lie groups and they act transitively on affine space. The general version of geometry advocated by the Erlangen program thus is the study of properties of subsets of a homogeneous space G/H, which are invariant under the action of G. This of course raises the question what kind of "geometric objects" could be available on a homogeneous space. To formally define this, we will need the concept of various types of bundles which will be developed in Section 2.

1.3. The groups of affine and Euclidean motions. To describe the groups $\mathrm{Aff}(n)$ of affine motions and $\mathrm{Euc}(n)$ of Euclidean motions in dimension n, one starts from a fixed affine hyperplane in $\mathbb{R}^{n+1}$. Let us take the hyperplane of all points $(x_1, \dots, x_{n+1})$ for which $x_{n+1} = 1$, and briefly write such a point as $\binom{x}{1}$. An invertible matrix in $GL(n+1, \mathbb{R})$ evidently maps this hyperplane to itself if and only if its last row has the form $(0, \dots, 0, 1)$. Otherwise put, we can write the matrix in block form as $\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}$, and invertibility means that $A \in GL(n, \mathbb{R})$ while $v \in \mathbb{R}^n$ is arbitrary. Evidently, these matrices form a closed subgroup of $GL(n+1, \mathbb{R})$. Such a matrix clearly maps $\binom{x}{1}$ to $\binom{Ax+b}{1}$, so we see that this is the affine group $\mathrm{Aff}(n)$. The stabilizer of the obvious base–point $\binom{0}{1}$ is formed by all block matrices with $v = 0$, so this is a closed subgroup isomorphic to $GL(n, \mathbb{R})$. Hence we get a realization of affine n–space as $\mathrm{Aff}(n)/GL(n, \mathbb{R})$.

To obtain Euclidean motions rather than affine motions, we can simply form the subgroup $\mathrm{Euc}(n) \subset \mathrm{Aff}(n)$ of all block matrices as above with $A \in O(n)$. Since $O(n)$ is a closed subgroup of $GL(n, \mathbb{R})$, this is a Lie subgroup. Similarly to above, we obtain a realization of Euclidean space as $\mathrm{Euc}(n)/O(n)$.

It is easy to see that these identifications lead to a nice perspective on geometry. To see this, let us look at the Lie algebras. The affine Lie algebra $\mathfrak{aff}(n)$, is obviously formed by all $(n + 1) \times (n + 1)$–matrices with last row consisting of zeros. Writing such a matrix as a pair (X, w) with $X \in M_n(\mathbb{R})$ and $w \in \mathbb{R}^n$, the Lie bracket is given by $[(X_1, w_1), (X_2, w_2)] = ([X_1, X_2], X_1w_2 - X_2w_1)$. (This is called a semidirect sum of $\mathfrak{gl}(n, \mathbb{R})$ and the abelian Lie algebra $\mathbb{R}^n$.) For the Euclidean Lie algebra, we obtain the same picture, but with $X \in \mathfrak{o}(n)$, i.e. $X^t = -X$. If we restrict to the subgroups $GL(n, \mathbb{R}) \subset \mathrm{Aff}(n)$ respectively $O(n) \subset \mathrm{Euc}(n)$, then the adjoint action is given by $\mathrm{Ad}(A)(X, w) = (AXA^{-1}, Aw)$ in both cases. Hence $\mathfrak{aff}(n) = \mathfrak{gl}(n, \mathbb{R}) \oplus \mathbb{R}^n$ as a representation of $GL(n, \mathbb{R}) \subset \mathrm{Aff}(n)$ and $\mathfrak{euc}(n) = \mathfrak{o}(n) \oplus \mathbb{R}^n$ as a representation of $O(n) \subset \mathrm{Euc}(n)$. Now we can for example take an element $w \in \mathbb{R}^n$, view it as an element of $\mathfrak{aff}(n)$ respectively $\mathfrak{euc}(n)$, and form the one–parameter subgroup e^{tw} in $\mathrm{Aff}(n)$ respectively $\mathrm{Euc}(n)$. Since the matrix corresponding to w has zero square, we see that $e^{tw} = \begin{pmatrix} \mathbb{I} & tw \\ 0 & 1 \end{pmatrix}$ in both cases. Acting with this on any point $\binom{x}{1}$, we get the affine line $\{x + tw : t \in \mathbb{R}\}$ through x in direction w.

1.4. The tangent bundle of a homogeneous space. To get a perspective on how to describe geometric objects on homogeneous spaces, let us give a description of the tangent bundle $T(G/H)$ of G/H. We already noted that $p : G \to G/H$ is a surjective submersion, so in particular $T_ep : T_eG \to T_{eH}(G/H)$ is surjective. Now T_eG is the Lie algebra $\mathfrak{g}$ of G, and since $p(g) = p(e)$ if and only if $g \in H$, we see that the kernel of T_ep is $\mathfrak{h} \subset \mathfrak{g}$. Hence $T_{eH}(G/H)$ is isomorphic to the quotient space $\mathfrak{g}/\mathfrak{h}$. Now the adjoint action of G can be restricted to the subgroup $H \subset G$. For each $h \in H$ the subspace $\mathfrak{h} \subset \mathfrak{g}$ of course is invariant under the adjoint action $\mathrm{Ad}(h)$, so there is an induced action $\underline{\mathrm{Ad}}(h) : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$. Observe that in the examples of affine and Euclidean motions discussed in 1.3 above, this gives rise to the standard action of $GL(n, \mathbb{R})$ respectively of $O(n)$ on $\mathbb{R}^n$. Using this, we now formulate:

PROPOSITION 1.4. *The tangent bundle $T(G/H)$ can be naturally identified with the space of equivalence classes of the equivalence relation on $G \times (\mathfrak{g}/\mathfrak{h})$ defined by $(g, X + \mathfrak{h}) \sim (g', X' + \mathfrak{h})$ if and only if there is an element $h \in H$ such that $g' = gh$ and $X' + \mathfrak{h} = \underline{\mathrm{Ad}}(h^{-1})(X + \mathfrak{h})$.*

PROOF. Consider the map $G \times \mathfrak{g} \to T(G/H)$ defined by $(g, X) \mapsto T_gp \cdot L_X(g)$, where L_X is the left invariant vector field generated by X. Since the left invariant vector fields span each tangent space of G and p is a surjective submersion, this map is surjective. Denoting by $\lambda_g : G \to G$ the left translation by g and by $\ell_g : G/H \to G/H$ the natural action by g, one by definition has $p \circ \lambda_g = \ell_g \circ p$. Differentiating this, we conclude that $T_gp \cdot T_e\lambda_g \cdot X = T_{eH}\ell_g \cdot T_ep \cdot X$. Since ℓ_g is a diffeomorphism, we conclude that $T_gp \cdot L_X(g) = 0$ if and only if $T_ep \cdot X = 0$ and hence $X \in \mathfrak{h}$.

Thus we see that $(g, X + \mathfrak{h}) \mapsto T_gp \cdot L_X(g)$ induces a well–defined surjection $G \times (\mathfrak{g}/\mathfrak{h}) \to T(G/H)$. Thus it remains to show that to elements have the same image under this map if and only if they are equivalent in the sense defined in the proposition. Now above we have already seen that for fixed $g \in G$, our map restricts to a bijection $\mathfrak{g}/\mathfrak{h} \to T_{gH}(G/H)$. Thus, given g, X and $h \in H$, it suffices to characterize the (unique) element $Y + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$ such that $T_{gh}p \cdot L_Y(gh) = T_gp \cdot L_X(g)$. But denoting by ρ^h the right translation by h, we can differentiate $p \circ \rho^h = p$ to get $T_{gh}p \circ T_g\rho^h = T_gp$. Hence, up to an element of $\mathfrak{h}$, we can compute Y as

$$T_{gh}\lambda_{(gh)^{-1}} \cdot T_g\rho^h \cdot T_e\lambda_g \cdot X = T_h\lambda_{h^{-1}}T_{gh}\lambda_{g^{-1}} \cdot T_h\lambda_g \cdot T_e\rho^h \cdot X = \mathrm{Ad}(h^{-1})(X).$$

In the computation, we have used that $\rho^h \circ \lambda_g = \lambda_g \circ \rho^h$ and that $\lambda_{g^{-1}}$ is inverse to λ_g. $\square$

2. Bundles

This section develops the general theory of various types of bundles (fiber bundles, vector bundles, principal bundles), which provides a basic tool for the geometric study of homogeneous spaces.

2.1. Fiber bundles. A smooth map $p : E \to M$ is a fiber bundles with *standard fiber* S if and only if for each $x \in M$ there is an open neighborhood U of x in M and a diffeomorphism $\varphi : p^{-1}(U) \to U \times S$. Then E is called the *total space* of the fiber bundle, while M is called its *base*. In this setting the diffeomorphism φ is called a *fiber bundle chart* and this naturally leads to the concept of a fiber bundle atlas.

Observe that at this stage there is no natural compatibility condition between fiber bundle charts, since they are already assumed to diffeomorphisms. We will impose such compatibility conditions later to define special classes of bundles.

A (smooth) *section* of E is a smooth map $\sigma : M \to E$ such that $p \circ \sigma = \mathrm{id}_M$. A *local section* is characterized by the same condition but only defined on some open subset of M. The spaces of sections and of local sections defined on U are denote by $\Gamma(E)$ or $\Gamma(E|_U)$, respectively. Evidently, there always are many local sections defined on the domain of a fiber bundle chart, since these are equivalent to smooth functions $U \to S$. Global sections of a fiber bundle do not exist in general.

From the existence of local smooth sections, it follows readily that the *bundle projection* p of any fiber bundle is a surjective submersion. Consequently, for any point $x \in M$, the *fiber of E over x*, $E_x := p^{-1}(\{x\})$ is a smooth submanifold of E, which is diffeomorphic to S.

A *morphism* between two fiber bundles $p : E \to M$ and $\tilde{p} : \tilde{E} \to \tilde{M}$ is a smooth map $F : E \to \tilde{E}$, which maps fibers to fibers. This means that there is a map $f : M \to \tilde{M}$ such that $\tilde{p} \circ F = f \circ p$, so $F(E_x) \subset \tilde{E}_{f(x)}$. Since p is a surjective submersion, it follows that f is automatically smooth. An *isomorphism* of fiber bundles is a morphism F of fiber bundles, which is a diffeomorphism. In this case, also F^{-1} is a morphism of fiber bundles.

A fiber bundle is called *trivial* if it is isomorphic to $pr_1 : M \times S \to S$. The existence of fiber bundle charts then exactly means that fiber bundles are *locally trivial*, and fiber bundle charts are also called *local trivializations*.

Example 2.1. (1) For arbitrary manifolds M and S, the first projection $M \times S \to M$ defines a (trivial) fiber bundle with standard fiber S.

(2) For any smooth manifold M, the tangent bundle $p : TM \to M$ is a fiber bundle with standard fiber $\mathbb{R}^n$, where $n = \dim(M)$.

(3) Let G be a Lie group G and $H \subset G$ a closed subgroup and consider the homogeneous space G/H and the canonical canonical map $p : G \to G/H$. In the standard proof that G/H is a smooth manifold (see e.g. §1.16 of [**LG**]), one chooses a complement $\mathfrak{k}$ to the $\mathfrak{h} \subset \mathfrak{g}$. Then one proves that for a sufficiently small neighborhood W of 0 in $\mathfrak{k}$, the map $\psi : W \times H \to G$ defined by $(X, h) \mapsto \exp(X)h$ defines a diffeomorphism onto an open neighborhood of H in G. Then one shows that $p(W)$ is an open neighborhood of eH in G/H and that $p \circ \exp : W \to p(W)$ is a diffeomorphism. But by construction $p^{-1}(p(W))$ is the image of ψ, so ψ defines a local trivialization around eH. This can be transported around using the left action of G, thus showing that $p : G \to G/H$ is a fiber bundle with standard fiber H.

Let us next analyze the analog of chart-changes for fiber bundle charts. Of course, these are only defined if the domains of the bundle charts have non–empty intersection.

For later use, we denote these domains by $U_\alpha, U_\beta \subset M$ and the charts by φ_α and φ_β and we put $U_{\alpha\beta} := U_\alpha \cap U_\beta$. Then the chart change is $(\varphi_\alpha \circ \varphi_\beta^{-1})|_{U_{\alpha\beta}\times S}$ and maps $U_{\alpha\beta} \times S$ to itself. By definition it is of the form $(\varphi_\alpha \circ \varphi_\beta^{-1})(x.y) = (x, \varphi_{\alpha\beta}(x, y))$ for a smooth function $\varphi_{\alpha\beta} : U_{\alpha\beta} \times S \to S$, which has the property that for each x, the maps $y \mapsto \varphi_{\alpha\beta}(x, y)$ is a diffeomorphism of S.

For a general fiber bundle, there is no restriction on the chart changes, these will later be used to define special types of bundles. The examples of $p : TM \to M$ (changes are linear in the second variable) and of $p : G \to G/H$ (chart chart changes are given by left multiplication in H) will be typical models.

There is a general way how to make a set into a fiber bundle from fiber bundle charts, provided that the chart changes are smooth. This will be frequently used to construct fiber bundles later on.

LEMMA 2.1. *Let E be a set, M and S smooth manifolds and $p : E \to M$ a set map. Suppose that there is an open covering $\{U_\alpha : \alpha \in I\}$ of M together with bijective maps $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times S$ such that $\mathrm{pr}_1 \circ \varphi_\alpha = p|_{U_\alpha}$. Suppose further that for each $\alpha, \beta \in I$ such that $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ the map $(\varphi_\alpha \circ \varphi_\beta^{-1})|_{U_{\alpha\beta}\times S} : U_{\alpha\beta} \times S \to U_{\alpha\beta} \times S$ is given by $(x, y) \mapsto (x, \varphi_{\alpha\beta}(x, y))$ for a smooth function $\varphi_{\alpha\beta} : U_{\alpha\beta} \times S \to S$.*

Then E can be uniquely made into a smooth manifold in such a way that $\{(U_\alpha, \varphi_\alpha)\}$ is a fiber bundle atlas.

SKETCH OF PROOF. We first fix an atlas for M, then consider intersections of the domains of the charts in this atlas with the U_α and then pass to a countable subcover (which exists since M is a Lindelöff space). Since fiber bundle charts can clearly be restricted to open subsets of their domain, we may assume that we start from a countable atlas $\{(V_i, v_i)\}$ of M and from fiber bundle charts $\varphi_i : p^{-1}(V_i) \to V_i \times S$ such that each φ_i is the restriction of some φ_α.

Now we consider the collection of the subsets $U \subset E$ such that for each i, $\varphi_i(U \cap p^{-1}(V_i))$ is open in $V_i \times S$, which clearly define a topology on E. If $V \subset M$ is open then $V \cap V_i$ is open in V_i for all i, so $p^{-1}(V)$ is open in this topology and hence p is continuous. The topology on E is Hausdorff, since points in different fibers can be separated by preimages of open subsets of M, while different points in one fiber can be separated by open subsets in S. Further, since M and S are second countable, it follows that E is second countable, which is sufficient for allowing it to be the underlying topological space of a smooth manifold.

For an open subset $W \in S$, we claim that $\varphi_i^{-1}(V_i \times W) \subset E$ is open for each i. Thus we have to take V_ℓ such that $V_{i\ell} \neq \emptyset$ and prove that $\varphi_\ell(p^{-1}(V_\ell) \cap \varphi_i^{-1}(V_i \times W))$ is open in $V_\ell \times S$. But now $V_\ell \cap V_i$ is open in V_i, so $V_{i\ell} \times W$ is open in $V_{i\ell} \times S$, and the subset we consider is the image of this under $\varphi_\ell \circ \varphi_i^{-1}$. But by assumption this composition is smooth as a map from $V_{i\ell} \cap S$ to itself. Since the same holds for $\varphi_i \circ \varphi_\ell^{-1}$ it even is a diffeomorphism and thus a homeomorphism, which implies the claim.

Now fixing a countable atlas $\{(W_j, w_j)\}$ for S and the sets $\varphi_i^{-1}(V_i \times W_j)$ form an open covering of E and for each i and j, $(v_i \times w_j) \circ \varphi_i$ is a homeomorphism onto $v_i(V_i) \times w_j(W_j)$, which is an open subset in $\mathbb{R}^N$ for appropriate N. One immediately verifies that the resulting chart changes are smooth, so we can use this as an atlas to define a smooth structure on E. By construction, the map p just corresponds to the first projection in these charts and thus is smooth. Continuity of p implies that each of the sets $p^{-1}(U_\alpha)$ is open in E, and one easily verifies directly that each φ_α is a diffeomorphism, which shows that $\{(U_\alpha, \varphi_\alpha)\}$ defines a fiber bundle atlas on E. □

2.2. Bundles with structure group. Fix a left action of a Lie group G on S. Then two fiber bundle charts φ_α and φ_β on $p : E \to M$ corresponding to open subsets U_α and U_β in M are called *G–compatible* if either $U_{\alpha\beta} = \emptyset$, or the chart change is given by $\varphi_{\alpha\beta}(x, y) = \psi_{\alpha\beta}(x) \cdot y$ for a smooth map $\psi_{\alpha\beta} : U_{\alpha\beta} \to G$. As in the case of manifolds, this leads to the concept of a *G–atlas* and a notion of *compatibility* of G–atlases. A fiber bundle together with an equivalence class of compatible G–atlases is then called a *fiber bundle with structure group G*. (Although the given action of G should be mentioned, this is not usual.)

In this language, the discussion in 2.1 shows that for any smooth manifold M of dimension n, the tangent bundle $p : TM \to M$ is a bundle with structure group $GL(n, \mathbb{R})$ (acting on $\mathbb{R}^n$ in the usual way). Likewise, $p : G \to G/H$ has structure group H (acting on itself by left translations).

2.3. Vector bundles. Let V be a real vector space. Then a *vector bundle* with typical fiber V is a fiber bundle with fiber V and structure group $GL(V)$. This means that the chart changes are linear in the second variable. The charts in a G-atlas are then referred to as *vector bundle charts*. For a *complex vector bundle* one requires that V is a complex vector space and that the structure group is the group of complex linear automorphisms of V.

The definition easily implies that each fiber E_x of a (complex) vector bundle inherits the structure of a (complex) vector space. Given $y_1, y_2 \in E_x$ and $t \in \mathbb{K}$, one chooses a vector bundle chart (U, φ) with $x \in U$. Then $\varphi(y_i) = (x, v_i)$ for $v_i \in V$, and one defines $y_1 + ty_2 := \varphi^{-1}(x, v_1 + tv_2)$. This is independent of the choice of chart by construction. Hence for a $\mathbb{K}$–vector bundle E ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), the spaces $\Gamma(E)$ and $\Gamma(E|_U)$ are vector spaces and modules over $C^\infty(M, \mathbb{K})$ and $C^\infty(U, \mathbb{K})$, respectively, (pointwise operations). This readily shows that any vector bundle has many smooth sections, since local smooth sections can be globalized by multiplying by bump functions and extending by zero. Partitions of unity can be used (as for vector fields or tensor fields) to construct sections with prescribed properties.

There is a general concept of morphisms of vector bundles. Given $p : E \to M$ and $q : F \to N$, a *vector bundle homomorphism* from E to F is a fiber bundle morphism $f : E \to F$ with underlying map $\underline{f} : M \to N$ such that for each $x \in M$, the restriction of Φ to E_x is a linear map $E_x \to F_{\underline{f}(x)}$. In the special case that $M = N$ and $\underline{f} = \mathrm{id}_M$, a vector bundle homomorphism $f : E \to F$ induces a linear map $f_* : \Gamma(E) \to \Gamma(F)$, defined by $f_*(\sigma) = f \circ \sigma$. These operators can be characterized similarly to the characterization of the action of tensor fields.

Proposition 2.3. *Let $E \to M$ and $\tilde{E} \to M$ be vector bundles. Then a linear map $\Phi : \Gamma(E) \to \Gamma(\tilde{E})$ comes from a vector bundle homomorphism if and only if it is linear over $C^\infty(M, \mathbb{R})$.*

Sketch of proof. The proof is analogous to the characterization of tensor fields as maps on vector fields and one-forms that are multilinear over smooth functions. Necessity of the condition is easy to verify directly.

For sufficiency, one has to show that linearity over $C^\infty(M, \mathbb{R})$ implies that $\Phi(\sigma(x)) \in \tilde{E}_x$ depends only on $\sigma(x) \in E_x$. Having done that, one defines $F : E \to \tilde{E}$ by $F(v) := \Phi(\sigma)(x)$for $v \in E_x$ and any smooth section σ of E such that $\sigma(x) = v$, which easily implies that $\Phi = F_*$. To see that this is well defined, it suffices to show that $\sigma(x) = 0$ implies $\Phi(\sigma)(x) = 0$ by linearity. This is proved in two steps. First if σ vanishes on an open set U containing x, then one chooses a bump function ψ with support contained in U such that $\psi(x) = 1$. Then $\psi\sigma = 0$ so $\Phi(\psi\sigma) = 0$ by linearity of Φ. But this shows that $0 = \psi(x)\Phi(\sigma)(x) = \Phi(\sigma)(x)$.

Again using linearity, this implies that, for any open subset $U \subset M$, $\Phi(\sigma)|_U$ depends only on $\sigma|_U$. Now for $x \in M$ and $\sigma(x) = 0$ we choose a vector bundle chart $\varphi : p^{-1}(U) \to U \times V$ with $x \in U$ and a basis $\{v_1, \dots, v_n\}$ for V. Defining $\sigma_i(y) := \varphi^{-1}(y, v_i)$ for $y \in U$, we get local sections $\sigma_1, \dots, \sigma_n$ for E whose values form a basis in each point. This implies that there are smooth functions $\psi_i : U \to \mathbb{R}$ for $i = 1, \dots, n$ such that $\sigma|_U = \sum \psi_i\sigma_i$, and $\sigma(x) = 0$ implies that $\psi_i(x) = 0$ for all i. But then $\Phi(\sigma)|_U = \sum \psi_i\Phi(\sigma_i)$ shows that $\Phi(\sigma)(x) = 0$. $\square$

Example 2.3. (1) For any smooth manifold M of dimension n the tangent bundle $p : TM \to M$ is a vector bundle with n–dimensional fibers. Indeed, a chart (U, u) of M induces a diffeomorphism $Tu : TU \to u(u) \times \mathbb{R}^n$ and composing $u^{-1} \times \mathrm{id}$ with that, we arrive at a diffeomorphism $TU \to U \times \mathbb{R}^n$. The changes between two such charts are well known to be linear in the second variable (they are the derivative of the chart–changes) so these charts form a vector bundle atlas.

For a smooth map $f : M \to N$, the tangent map $Tf : TM \to TN$ is a homomorphism of vector bundles with base map f.

(2) Suppose that $E \subset TM$ is a smooth distribution of rank k on a smooth manifold M of dimension n. This means that for each $x \in M$, there is a specified subspace $E_x \subset T_xM$ of dimension k and E is the union of these spaces. Hence there is a natural projection $p : E \to M$.

The condition of smoothness says that for each x there is an open neighborhood U of $x \in M$ and there are local smooth vector fields $\xi_1, \dots, \xi_k \in U$ such that for each $y \in U$, the vectors $\xi_1(y), \dots, \xi_k(y)$ span the subspace $E_y \subset T_yM$. But this exactly means that each $\xi \in E_y$ can be uniquely written as $a_1\xi_1(y) + \dots + a_k\xi_k(y)$ for coefficients $a_1, \dots, a_k \in \mathbb{R}$. Mapping ξ to $(y, (a_1, \dots, a_k)) \in U \times \mathbb{R}^k$ then defines a vector bundle chart $p^{-1}(U) \to U \times \mathbb{R}^k$. It is easy to see that the inclusion $E \to TM$ is a homomorphism of vector bundles.

One can apply the same construction to TM itself, showing that for local vector fields $\xi_1, \dots, \xi_n$ defined on U whose values in each $y \in U$ form a basis of T_yM, one obtains a vector bundle chart for TM defined on U. Such a collection is called a *local frame* for TM.

(3) Recall that the Grassmann–manifold $Gr(k, \mathbb{R}^n)$ is the set of all k–dimensional linear subspace of $\mathbb{R}^n$. This can be made into a smooth manifold by identifying it with a homogeneous space of $GL(n, \mathbb{R})$ (or of $O(n)$, which shows that it is compact). The simplest example is real projective space $\mathbb{R}P^{n-1}$, the space of one–dimensional linear subspaces of $\mathbb{R}^n$.

Each of the Grassmann manifolds carries a so–called *tautological bundle* defined as follows. Consider the trivial bundle $Gr(k, \mathbb{R}^n) \times \mathbb{R}^n$, and define a subset in there as $E := \{(V, v) : v \in V\}$. Hence one attaches to each subspace V the space V itself. We will see later, that E is indeed a locally trivial vector bundle (which is nicely related to $GL(n, \mathbb{R}) \to Gr(k, \mathbb{R}^n)$).

2.4. Principal fiber bundles.

Definition 2.4. Let G be a Lie group. A *principal fiber bundle* with structure G (or a *principal G–bundle*) is a fiber bundle $p : P \to M$ with fiber G structure group G acting on itself by multiplication from the left.

By definition this means that we have an atlas $\{(U_\alpha, \varphi_\alpha)\alpha \in I\}$ of fiber bundle charts $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times G$ such that for $U_{\alpha\beta} \neq \emptyset$, there is a smooth function $\varphi_{\alpha\beta} : U_{\alpha\beta} \to G$

such that $(\varphi_\alpha \circ \varphi_\beta^{-1})(x, g) = (x, \varphi_{\alpha\beta}(x) \cdot g)$. In particular, for any Lie group G and closed subgroup H, the natural map $p : G \to G/H$ is a principal H–bundle.

While principal bundle are a very versatile and effective tool, they are a bit hard to imagine and initially the definition may be slightly mysterious. On should first observe that the fibers of a principal G–bundle are diffeomorphic to the Lie group G, but they do *not* carry a natural group structure. This is simply because of the fact that left translations in a group are not group homomorphisms.

The structure on a principal bundle which comes closest to a group structure is the *principal right action*. For a principal G–bundle $p : P \to M$, this is a smooth right action $P \times G \to P$ of G on P. To define this action, take $u \in P$ and $g \in G$, put $x = p(u)$ and choose a principal bundle chart $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times G$. If $\varphi_\alpha(u) = (x, h)$, the define $u \cdot g := \varphi_\alpha^{-1}(x, hg)$. Since left and right translations commute, this is independent of the chart, and smoothness is obvious. From the definition it follows readily that the orbits of the principal right action are exactly the fibers of $p : P \to M$ and that the action is free, i.e. if for $g \in G$ there is one point $u \in P$ such that $u \cdot g = u$, then $g = e$, the neutral element of G.

The principal right action is also a crucial ingredient in the definition of morphism between principal bundles. The most general version of a morphism is defined as follows. Suppose that $\psi : G \to H$ is a homomorphism between two Lie groups, $p : P \to M$ is a principal G–bundle and $q : Q \to N$ is a principal H–bundle. Then a *morphism of principal bundles over* ψ is a morphism $F : P \to Q$ of fiber bundles, which is equivariant for the principal right actions over ψ, i.e. such that $F(u \cdot g) = F(u) \cdot \psi(g)$ for all $u \in P$ and $g \in G$.

An important special case with a non–trivial homomorphism is the following. Suppose that $H \subset G$ is a closed subgroup and $i : H \to G$ is the inclusion. Then for a principal G–bundle $p : P \to M$ a *reductions to the structure group* H is a principal H–bundle $q : Q \to M$ together with a morphism $F : Q \to P$ of principal bundles over i, which covers the identity on M, i.e. is such that $p \circ F = q$.

When considering morphisms of principal G–bundles on usually does not consider nontrivial homomorphism, but just equivariancy in the sense the $F(u \cdot g) = F(u) \cdot g$. In particular, one can consider one principal G–bundle $p : P \to M$ and morphisms $F : P \to P$ covering the identity on M. These are called *gauge transformations*, and the concept of gauge theories in physics is formulated on principal fiber bundles.

Example 2.4. (1) We have already noted the $p : G \to G/H$ is a principal H bundle. Of course, the principal right action $G \times H \to G$ in this case is just the restriction of the group multiplication.

(2) To understand the second basic example, let us first recall a bit of linear algebra. Consider a real vector space V of dimension n and the set of all linear isomorphism $u : \mathbb{R}^n \to V$. Given such an isomorphism u and $A \in GL(n, \mathbb{R})$ also $u \circ A$ is a linear isomorphism. If u and v are two isomorphisms, then $u^{-1} \circ v =: A$ is a linear isomorphism $\mathbb{R}^n \to \mathbb{R}^n$ and thus an element of $GL(n, \mathbb{R})$ and $v = u \circ A$. This shows that $GL(n, \mathbb{R})$ acts freely and transitively from the right on the set of linear isomorphisms $\mathbb{R}^n \to V$, which thus can be interpreted as a principal bundle over a point.

Equivalently, one may interpret a linear isomorphism $\mathbb{R}^n \to V$ as a choice of basis of V by looking at the image of the standard basis of $\mathbb{R}^n$.

(3) The example in (2) generalizes to the tangent bundle TM of a manifold M and indeed to arbitrary vector bundles. For $x \in M$, one defines $\mathcal{P}_x$ to be the set of a linear isomorphisms $\mathbb{R}^n \to T_xM$, where $n = \dim(M)$. Defining $\mathcal{P}$ to be the union of of the

spaces $\mathcal{P}_x$, there is an obvious projection $p : \mathcal{P} \to M$ sending $\mathcal{P}_x$ to x. For a local chart (U_α, u_α) of M on defines a map $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times GL(n, \mathbb{R})$ as follows. For $u \in \mathcal{P}_x$ with $x \in U_\alpha$, one defines $\varphi_\alpha(u) := (x, T_x u_\alpha \circ \varphi)$, observing that $T_x u_\alpha : T_x M \to \mathbb{R}^n$ is a linear isomorphism. One easily verifies that the resulting chart changes are of the form $(x, A) \mapsto (x, \varphi_{\alpha\beta}(x)A)$ for a smooth map $\varphi_{\alpha\beta} : U_{\alpha\beta} \to GL(n, \mathbb{R})$ (the derivative of the chart–change). Lemma 2.1 then implies that $p : \mathcal{P} \to M$ can be made into a fiber bundle in such a way that $(U_\alpha, \varphi_\alpha)$ becomes a fiber bundle atlas, thus making $\mathcal{P}$ into a principal $GL(n, \mathbb{R})$ bundle over M. This is called the *linear frame bundle* of M.

The reason for the name "frame bundle" becomes clear if one looks at local sections. If $\sigma : U \to \mathcal{P}M$ is a local smooth section, then for each $x \in U$, $\sigma(x)$ is a linear isomorphism $\mathbb{R}^n \to T_x M$. Denoting by $\{e_1, \dots, e_n\}$ the standard basis of $\mathbb{R}^n$, consider $\xi_i(x) := \sigma(x)(e_i)$. This is easily seen to define a vector field on U and clearly, these fields form a local frame on U. Conversely, a local frame determines an isomorphism $\mathbb{R}^n \to T_x M$ for each x in the domain of definition of the frame. It is easy to see that the resulting local section of $\mathcal{P}M$ is smooth. This also shows that $\mathcal{P}M$ does not admit global smooth sections in general, since this would lead to a global frame. Such frames do not exist in general, for example by the hairy ball theorem, even dimensional spheres do not admit any global vector field which is nowhere vanishing.

(4) In the same way, if $E \to M$ is a vector bundle with typical fiber V, one define a frame bundle for E, which is a principal bundle with structure group $GL(V)$. One sets P_x to be the set of all linear isomorphisms $V \to E_x$, defines P to be the union of the spaces P_x, and uses local vector bundle charts for E to construct principal bundle charts for P. Again local smooth sections of P are equivalent to local frames for E.

Lemma 2.4. *(1) If $F : P \to Q$ is a morphism of principal bundles and $u \in E$ is a point, then $F(u)$ determines the values of F on the fiber containing u.*
(2) Any morphism $F : P \to Q$ of principal G–bundles whose base map is a diffeomorphism is an isomorphism. In particular, any gauge transformation is an isomorphism.

Proof. (1) Equivariancy of F implies that $F(u)$ determines the values $F(u \cdot g)$ for each g in the structure group of P, and the set of these points coincides with the fiber containing U.

(2) Since $F(u \cdot g) = F(u) \cdot g$ for all $g \in G$, we see that the restriction of F to each fiber is bijective. Together with bijectivity of the base map f of F, this easily implies that F is bijective. Now it suffices to prove that the inverse map F^{-1} is smooth, since then it is automatically a morphism of fiber bundles and equivariancy of F readily implies equivariancy of F^{-1}. But this is a local problem, so we can use principal bundle charts $\varphi : p^{-1}(U) \to U \times G$ for P and $\psi : q^{-1}(V) \to V \times G$ for Q with $f(U) = V$. Evidently $\psi \circ F \circ \varphi^{-1}$ has the form $(x, g) \mapsto (f(x), \Phi(x)g)$ for some smooth function $\Phi : U \to G$ by equivariancy. But then also $y \mapsto \Phi(f^{-1}(y))^{-1}$ is smooth since f is a diffeomorphism and inversion in G is smooth. But then clearly $\varphi \circ F^{-1} \circ \psi^{-1}$ must be given by $(y, g) \mapsto (f^{-1}(y), \Phi(f^{-1}(y))^{-1} \cdot g)$ and this is smooth, too. □

Let us use this to develop some perspective on interpretations of reduction of structure group. Consider a smooth manifold M with linear frame bundle $\mathcal{P}M$. Then by definition, the fiber $\mathcal{P}_x M$ over $x \in M$ consists of all linear isomorphisms $u : \mathbb{R}^n \to T_x M$. Now suppose that we have given a Riemannian metric g on M. Then g_x is a (positive definite) inner product on $T_x M$. Since up to isomorphism there is just one such inner production, there are isomorphisms u as above, which are orthogonal for the standard inner product on $\mathbb{R}^n$ and the inner product g_x. Similarly as before, one verifies that v

is another such isomorphism if and only if $v = u \circ A$ for some $A \in O(n) \subset GL(n, \mathbb{R})$. Denote the resulting subsets by $\mathcal{O}_x M \subset \mathcal{P}_x M$ and by $\mathcal{O}M \subset \mathcal{P}M$.

Now the Gram–Schmidt procedure can be used to construct local orthonormal frames for TM. The vector bundle chart constructed from an orthonormal frame can then be used to construct a principal bundle chart on $\mathcal{O}M$, showing the $\mathcal{O}M$ is a principal bundle with structure group $O(n)$. Of course, the inclusion of $\mathcal{O}M$ into $\mathcal{P}M$ defines a reduction of structure group.

Conversely, suppose that $Q \to M$ is a principal $O(n)$–bundle, and that $F : Q \to M$ is a reduction of structure group. For $x \in M$ consider $F(Q_x) \subset \mathcal{P}_x M$ and choose an isomorphism $u : \mathbb{R}^n \to T_x M$ in this subset. Then $F(Q_x)$ consist exactly of the isomorphisms $u \circ A$ for $A \in O(n)$. Consequently, defining $g_x(\xi, \eta) := \langle u^{-1}(\xi), u^{-1}(\eta)\rangle$ we obtain an inner product on $T_x M$, which does not depend on the choice of u. It is easy to see that g_x depends smoothly on x, so we obtain an induced Riemannian metric on M. Hence Riemannian metrics on M are equivalent to reductions of structure group of $\mathcal{P}M$ to the subgroup $O(n) \subset GL(n, \mathbb{R})$.

This construction is extremely flexible. For example a reduction of $\mathcal{P}M$ to the structure group $GL^+(n, \mathbb{R}) := \{A \in GL(n, \mathbb{R}) : \det(A) > 0\}$ is equivalent to an orientation on M. For the stabilizer of $\mathbb{R}^k \subset \mathbb{R}^n$, a reduction of structure group corresponds to a distribution $E \subset TM$ of rank k. Identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, we see that $GL(n, \mathbb{C})$ is a closed subgroup of $GL(2n, \mathbb{R})$. Taking a real vector bundle $E \to M$ with fibers of dimension $2n$, we obtain a frame bundle with structure group $GL(2n, \mathbb{R})$. A reduction of structure group to $GL(n, \mathbb{C})$ is equivalent to making E into a complex vector bundle with n–dimensional fibers. A further reduction to $U(n)$ then is equivalent to a so–called Hermitian bundle metric, i.e. a choice of a Hermitian inner product on each fiber which depends smoothly on the base point.

2.5. Cocycles of transition functions. Suppose that $p : P \to M$ is a principal G–bundle, and that $\{(U_\alpha, \varphi_\alpha)\alpha \in I\}$ is a principal bundle atlas with transition functions $\varphi_{\alpha\beta} : U_{\alpha\beta} \to G$. Since the bundle is trivial over each of the sets U_α, one may expect that indeed the transition functions carry the main information about the bundle, and it turns out that this is indeed the case. From the definition, it readily follows that for $x \in U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$, one always gets

$$\varphi_{\alpha\beta}(x)\varphi_{\beta\gamma}(x) = \varphi_{\alpha\gamma}(x),$$

which is called the *cocycle* equation. For $\alpha = \beta = \gamma$, this implies $\varphi_{\alpha\alpha}(x)\varphi_{\alpha\alpha}(x) = \varphi_{\alpha\alpha}(x)$ and hence $\varphi_{\alpha\alpha}(x) = e$ for all x. Knowing this, and putting $\gamma = \alpha$, we conclude that $\varphi_{\beta\alpha}(x) = \varphi_{\alpha\beta}(x)^{-1}$.

Conversely, assume that for a smooth manifold M, we have given an open covering $\{U_\alpha : \alpha \in I\}$. Then a *cocycle of transition functions* is a family $\varphi_{\alpha\beta} : U_{\alpha\beta} \to G$ of smooth functions such for each $x \in U_{\alpha\beta\geq}$ the above equation is satisfied. Then one defines $\tilde{P} := \{(\alpha, x, g) : \alpha \in I, x \in U_\alpha, g \in G\}$. On this set define a relation by $(\alpha, x, g) \sim (\beta, x', g')$ if $x = x'$ (and hence lies in $U_{\alpha\beta}$) and $g = \varphi_{\alpha\beta}(x)g'$. This relation is evidently reflexive and symmetry easily follows from $\varphi_{\beta\alpha}(x) = \varphi_{\alpha\beta}(x)^{-1}$. On the other hand, the cocycle equation easily implies transitivity, so we have defined an equivalence relation. Putting $P := P/\sim$ and denoting the obvious projection by $p : P \to M$ one constructs (on the level of sets) a principal bundle atlas $\{(U_\alpha, \varphi_\alpha)\alpha \in I\}$ with transition functions $\varphi_{\alpha\beta}$. Hence by Lemma 2.1 we can make $p : P \to M$ into a smooth principal bundle, thus realizing the given cocycle of transition functions.

So it remains to understand when two cocycles lead to isomorphic principal bundles. By restricting charts, we may without loss of generality assume that we deal with principal bundle charts φ_α and ψ_α and hence with cocycles $\varphi_{\alpha\beta}$ and $\psi_{\alpha\beta}$ corresponding to the same covering $\{U_\alpha\}$. If F is an isomorphism between the two bundles (with base map the identity on M), then for each $\alpha \in U$ and $x \in U_\alpha$, we can write $\psi_\alpha(F(\varphi_\alpha^{-1}(x,g))) = (x, f_\alpha(x)g)$ for some element $f_\alpha(x) \in G$. This defines a smooth function $f_\alpha : U_\alpha \to G$ and we can analyze how the family $\{f_\alpha : \alpha \in I\}$ is compatible with the transition functions. For $x \in U_{\alpha\beta}$, we can recast the definition as $F(\varphi_\beta^{-1}(x,g) = \psi_\beta^{-1}(x, f_\beta(x)g)$ and applying ψ_α to this, we obtain $(x, \varphi_{\alpha\beta}(x)f_\beta(x)g)$. But the definition of transition functions also shows that $\varphi_\beta^{-1}(x,g) = \varphi_\alpha^{-1}(x, \varphi_{\alpha\beta}(x)g)$, and applying $\varphi_\alpha \circ F$ to this, we obtain $(x, f_\alpha(x)\varphi_{\alpha\beta}(x)g)$. Thus we conclude that $\psi_{\alpha\beta}(x)f_\beta(x) = f_\alpha(x)\varphi_{\alpha\beta}(x)$ has to hold for all $\alpha, \beta \in I$ and all $x \in U_{\alpha\beta}$. In this case, one calls the cocycle cohomologous. Conversely, one easily shows that one can use a family of functions f_α with these compatibility conditions to define an isomorphism in local charts.

This construction can be viewed as a non–commutative version of Čech–cohomology in degree 1. Apart from providing a nice way to describe an construct principal bundles, this leads at least in the case of commutative groups, to a relation to algebraic topology. This for example to classify principal bundles with structure group $\mathbb{R} \setminus \{0\}$ and $\mathbb{C} \setminus \{0\}$ and thus also real and complex vector bundles with one–dimensional fibers (so–called *line–bundles*) in terms of algebraic topology.

2.6. Pullbacks. Suppose that M and N are smooth manifolds, $p : E \to N$ is a fiber bundle and $f : M \to N$ is a smooth function. Then one defines $f^*E := \{(x,u) \in M \times E : f(x) = p(u)\}$, which evidently is a closed subspace of $M \times E$. Restricting projections of the product to this subspace one obtains $f^*p : f^*E \to M$ and $p^*f : f^*E \to E$. By definition, for each $x \in M$ the pre–image $(f^*p)^{-1}(\{x\})$ coincides with the fiber $E_{f(x)}$ of E over $f(x) \in N$.

Now suppose that $U \subset N$ is open and that $\varphi : p^{-1}(U) \to U \times S$ is a fiber bundle chart defined over U. Then $f^{-1}(U)$ is open in M and for $(x,u) \in (f^*p)^{-1}(f^{-1}(U))$ we have $f(x) \in U$ and $u \in E_{f(x)}$. Thus we obtain a well defined map $\tilde{\varphi} : (f^*p)^{-1}(f^{-1}(U)) \to f^{-1}(U) \times S$ by putting $\tilde{\varphi}(x,u) := (x, pr_2(\varphi(u)))$. For two such charts φ_α, φ_β such that $(\varphi_\alpha \circ \varphi_\beta^{-1})(x,y) = (x, \varphi_{\alpha\beta}(x,y))$ we readily see that $(\tilde{\varphi}_\alpha \circ \tilde{\varphi}_\beta^{-1})(x,y) = (x, \varphi_{\alpha\beta}(f(x),y))$. Hence we get smooth chart changes and so $f^*p : f^*E \to M$ is a smooth fiber bundle by Lemma 2.1. Then by definition $p^*f : f^*E \to E$ is a fiber bundle morphism covering $f : M \to N$. Moreover, the form of the chart changes readily shows that any pullback of a vector bundle is again a vector bundle, while the pullback of a principal bundle is a principal bundle, too.

Proposition 2.6. *Let $E \to M$ and $\tilde{E} \to \tilde{M}$ be vector bundles and let $f : M \to \tilde{M}$ be a smooth map. Then vector bundle homomorphisms $F : E \to \tilde{E}$ with base map f are in bijective correspondence with vector bundle homomorphisms $\hat{F} : E \to f^*\tilde{E}$ with base map the identity. Moreover $\hat{F}$ is an isomorphism of vector bundles if and only if for each $x \in M$, the restriction $F_x : E_x \to \tilde{E}_{f(x)}$ is a linear isomorphism.*

Proof. Given $\hat{F} : E \to f^*\tilde{E}$ with base map the identity, we define $F : E \to \tilde{E}$ as $p^*f \circ \hat{F}$. By construction, this is a vector bundle homomorphism with base map f. Conversely, suppose we are given $F : E \to \tilde{E}$ with base map f. For $u \in E$ with $x = p(u)$, we by definition have $F(u) \in \tilde{E}_{f(x)}$. But this readily implies that

$\hat{F}(u) := (p(u), F(u)) \in f^*\tilde{E}$ and it lies over x. Obviously, $\hat{F}$ preserves fibers, is linear in each fiber and has the identity as a base map.

Concerning the last claim, it is clear that a vector bundle isomorphisms induces linear isomorphisms in each fiber. To prove the converse direction, it suffices to show that if E and $\tilde{E}$ are vector bundles over M and $F : E \to \tilde{E}$ is a vector bundle homomorphisms with base map the identity which restricts to a linear isomorphism in each fiber, then F is an isomorphism of vector bundles. But under this assumption, we immediately conclude that F is bijective, and is suffices to show that the inverse is a smooth homomorphisms of vector bundles. But this can be done in local vector bundle charts, where F is described by a smooth function to $GL(n, \mathbb{R})$. But then also forming the pointwise inverse defines a smooth function, which implies the result. □

This has rather surprising applications showing how general bundles be realized as pullbacks. For example let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension k. Then for each $x \in M$, the tangent space T_xM is a k–dimensional subspace of $\mathbb{R}^n$, so we can view $x \mapsto T_xM$ as a smooth map f from M to the Grassmann manifold $Gr(k, \mathbb{R}^n)$ of k–dimensional subspaces of $\mathbb{R}^n$. This is an analog of the Gauß map in the theory of hypersurfaces. Recall that in Section 2.3 we have met the tautological subbundle $E \to Gr(k, \mathbb{R}^n)$. This was defined as the subspace of $Gr(k, \mathbb{R}^n) \times \mathbb{R}^n$ consisting of all (V, v) with $v \in V$. Hence we see that our map $f : M \to Gr(k, \mathbb{R}^n)$ naturally lifts to a map $TM \to E$ which is a vector bundle homomorphism inducing linear isomorphisms in each fiber. Hence $TM \cong f^*E$, any the tangent bundle of any submanifold can be realized as a pullback of a tautological bundle.

This generalizes rather easily: Suppose that $E \to M$ is a vector bundle and suppose that we can find a vector bundle homomorphism $E \to M \times \mathbb{R}^N$ for some large N. Then this maps each fiber of E to a linear subspace in $\mathbb{R}^N$, and we obtain a smooth map from M to a Grassmannian of $\mathbb{R}^N$. As above we can use this to show that E is a pullback of the tautological bundle on this Grassmannian.

There are further vast generalizations, which provide a generalization to algebraic topology. They are best formulated in the setting of topological principal bundles with structure group a topological group G. So one considers a continuous map $p : P \to X$ which admits local homeomorphisms $p^{-1}(U) \to U \times G$ such that the chart changes are are of the form $(x, g) \mapsto (x, \varphi_{\alpha\beta}(x)g)$ for a continuous function $\varphi_{\alpha\beta} : U_{\alpha\beta} \to G$. Such bundles can be pulled back along continuous maps. A first crucial result is that (assuming sufficiently nice base spaces, e.g. paracompact ones) that homotopic continuous maps lead isomorphic pull–backs. Moreover, for any topological group G, it turns out that there is a so called *universal* principal G–bundle $p : EG \to BG$, which has the property that the total space EG is contractible. Similarly as for the Grassmannians above, it turns out that any principal G–bundle over a sufficiently nice space X can be realized as f^*EG for some continuous map $f : X \to BG$. Moreover, the fact that EG is contractible implies that two maps $f, g : X \to BG$ are homotopic if and only if the bundles f^*EG and g^*EG are isomorphic. Hence isomorphism classes of principal G–bundles over X are in bijective correspondence with the set $[X, BG]$ of continuous maps from X to BG. Therefore, BG is called the *classifying space* for principal G–bundles.

2.7. Fibered products. The next construction we discuss is the fibered product of two fiber bundles. Formally, this is just the pullback of one bundle to the other bundle, but we give it a different interpretation.

DEFINITION 2.7. Let $p : E \to M$ and $\tilde{p} : \tilde{E} \to M$ be two fiber bundles over the same base with standard fibers S and $\tilde{S}$. Then we define the *fibered product* $E \times_M \tilde{E} \subset E \times \tilde{E}$ as the set $\{(u, v) \in E \times \tilde{E} : p(u) = \tilde{p}(v)\}$.

There is an obvious projection $q : E \times_M \tilde{E} \to M$ mapping (u, v) to $p(u) = \tilde{p}(v)$ and $q^{-1}(\{x\}) = E_x \times \tilde{E}_x$. Now consider fiber bundle charts (without loss of generality defined on the same open subset $U \subset M$) $\varphi : p^{-1}(U) \to U \times S$ and $\tilde{\varphi} : \tilde{p}^{-1}(U) \to U \times \tilde{S}$. Then we define $\psi : q^{-1}(U) \to U \times (S \times \tilde{S})$ by $\psi(u, v) = (x, (y, \tilde{y}))$, where $\varphi(u) = (x, y)$ and $\tilde{\varphi}(v) = (x, \tilde{y})$. Obviously, this leads to smooth chart changes, so by Lemma 2.1, $q : E \times_M \tilde{E} \to M$ is a fiber bundle with standard fiber $S \times \tilde{S}$.

Restricting the two projections to the fibered product defines morphisms $pr_1 : E \times_M \tilde{E} \to E$ and $pr_2 : E \times_M \tilde{E} \to E$ of fiber bundles with base map id_M. The fibered product has an obvious universal property with respect to these projections. Suppose that $F \to N$ is any fiber bundle and $\Phi : F \to E$ and $\tilde{\Phi} : F \to \tilde{E}$ are morphisms of fiber bundles over the same base map $f : N \to M$, then there is a unique morphism $(\Phi, \tilde{\Phi}) : F \to E \times_M \tilde{E}$ with base map f, such that $pr_1 \circ (\Phi, \tilde{\Phi}) = \Phi$ and $\mathrm{pr}_1 \circ (\Phi, \tilde{\Phi}) = \tilde{\Phi}$.

Similarly, one may associate to sections $\sigma \in \Gamma(E)$ and $\tilde{\sigma} \in \Gamma(\tilde{E})$ a unique section $(\sigma, \tilde{\sigma}) \in \Gamma(E \times_M \tilde{E})$. This construction clearly gives rise to an isomorphism $\Gamma(E) \times \Gamma(\tilde{E}) \cong \Gamma(E \times_M \tilde{E})$.

It is also clear how chart changes for $E \times_M \tilde{E}$ look in terms of chart changes on the two factors. In particular, if E is a principal G–bundle and $\tilde{E}$ is a principal $\tilde{G}$–bundle, then the fibered product is a principal bundle with structure group $G \times \tilde{G}$.

Likewise, if E and $\tilde{E}$ are vector bundles, then $E \times_M \tilde{E}$ is also a vector bundle. Since for vector spaces, the product equals the direct sum, this bundle is usually denoted by $E \oplus \tilde{E}$ and called the *direct sum* or the *Whitney sum* of the two vector bundles. Similarly to the case of vector spaces there is a natural homomorphism $i_1 : E \to E \oplus \tilde{E}$ defined by $i_1(u) = (u, 0)$ and likewise for the second factor.

Using fibered products, we can now construct a kind of an inverse to the principal right action on a principal bundle, which is technically very useful.

PROPOSITION 2.7. *Let $p : P \to M$ be a principal fiber bundle with structure group G. Then there is a smooth map $\tau : P \times_M P \to G$ such that for all $u, v \in P$ with $p(u) = p(v)$ one has $v = u \cdot \tau(u, v)$.*

PROOF. For $(u, v) \in P \times_M P$, we by definition have $p(u) = p(v)$, so there is a unique element $g \in G$ such that $v = u \cdot g$. Hence the map τ is well defined and it remains to show that it is smooth. But for a principal bundle chart $\varphi : p^{-1}(U) \to U \times G$ for P, the induced chart of $P \times_M P$ maps (u, v) to $(x, \mathrm{pr}_2(\varphi(u)), \mathrm{pr}_2(\varphi(v)))$. By definition $\tau(u, v) = \mathrm{pr}_2(\varphi(u))^{-1} \cdot \mathrm{pr}_2(\varphi(v))$, so smoothness follows. □

A simple consequence of this is that a principal fiber bundle $p : P \to M$ admits a global smooth section if and only if it is trivial. Since the trivial bundle $M \times G$ evidently admits smooth sections, it suffices to prove the converse. But ff $\sigma : M \to P$ is a global section that $(x, g) \mapsto \sigma(x) \cdot g$ and $u \mapsto (p(u), \tau(u, \sigma(p(u))))$ clearly define inverse isomorphisms between $M \times G$ and P.

2.8. Associated bundles. One of the key features of principal bundles is that a single principal bundle with structure group G can be used to construct a whole family of fiber bundles with that structure group. Consider a principal G–bundle $p : P \to M$ and a left action $G \times S \to S$ on some smooth manifold S. Then we can define a smooth right action $(P \times S) \times G \to P \times S$ by $(u, y) \cdot g := (u \cdot g, g^{-1} \cdot y)$. Let $P[S] := P \times_G S := (P \times S)/G$

be the set of orbits of this action and let us write $[u,y]$ for the orbit of $(u,y)\in P\times S$. On the one hand, we then get an obvious projection $q: P\times S\to P\times_G S$ defined by $q(u,y):=[u,y]$. On the other hand, if (u,y) and (v,z) lie in the same orbit, then $v=u\cdot g$ for some $g\in G$. Hence $p(u)=p(v)$, so there is a well defined map $\pi: P\times_G S\to M$ given by $\pi([u,y]):=p(u)$.

DEFINITION 2.8. We call $\pi: P\times_G S\to M$, the *associated bundle* to $p: P\to M$ corresponding to the given left action $G\times S\to S$.

PROPOSITION 2.8. *For a principal G–bundle $p: P\to M$ and a smooth left action $G\times S\to S$ we have:*

(1) The associated bundle $\pi: E\times_G S\to M$ is a smooth fiber bundle with typical fiber S and structure group G. In particular, if we start with a representation of G on a vector space V, $P\times_G V$ is a vector bundle.

(2) The projection $q: P\times S\to P\times_G S$ is a smooth principal bundle with structure group G.

(3) There is a smooth map $\tau_S: P\times_M(P\times_G S)\to S$ which is uniquely characterized by the property that for $z\in P\times_G S$ and $u\in P$ with $\pi(z)=p(u)$ we have $z=q(u,\tau_S(u,z))$. In particular, $\tau_S(u\cdot g,z)=g^{-1}\cdot\tau_S(u,z)$.

PROOF. (1) Take a principal bundle atlas $\{(U_\alpha,\varphi_\alpha):\alpha\in I\}$ for P. For each α, define $\psi_\alpha:\pi^{-1}(U_\alpha)\to U_\alpha\times S$ by $\psi_\alpha([u,y]):=(p(u),\mathrm{pr}_2(\varphi_\alpha(u))\cdot y)$. Observe first that $\pi([u,y])=p(u)\in U_\alpha$, so $\varphi_\alpha(u)$ makes sense. Moreover, if $[u,y]=[v,z]$, then there is an element $g\in G$ such that $v=u\cdot g$ and $z=g^{-1}\cdot y$. But then $\mathrm{pr}_2(\varphi_\alpha(v))=\mathrm{pr}_2(\varphi_\alpha(u))\cdot g$ and acting with this on z, we obtain $\mathrm{pr}_2(\varphi_\alpha(u))\cdot y$, so ψ_α is well defined.

Given $x\in U_\alpha$ and $y\in S$, we put $u=\varphi_\alpha^{-1}(x,e)$ and then obtain $\psi_\alpha([u,y])=(x,y)$, so ψ_α is surjective. On the other hand, if $\psi_\alpha([u,y])=\psi_\alpha([v,z])$, then $p(u)=p(v)$, so there is an element $g\in G$ such that $v=u\cdot g$. As above, $\mathrm{pr}_2(\varphi_\alpha(v))=\mathrm{pr}_2(\varphi_\alpha(u))\cdot g$, so we conclude $\mathrm{pr}_2(\varphi_\alpha(u))\cdot y=\mathrm{pr}_2(\varphi_\alpha(u))\cdot g\cdot z$ and hence $z=g^{-1}\cdot y$. This shows that $[u,y]=[v,z]$, so ψ_α is bijective.

Computing the chart changes, we see that for $x\in U_{\alpha\beta}$ and $y\in S$, we get $\psi_\beta^{-1}(x,y)=[\varphi_\beta^{-1}(x,e),y]$. Applying ψ_α to this, we get $(x,pr_2((\varphi_\alpha\circ\varphi_\beta^{-1})(x,e))\cdot y)$. But denoting by $\varphi_{\alpha\beta}:U_{\alpha\beta}\to G$ the transition function of P, this equals $(x,\varphi_{\alpha\beta}(x)\cdot y)$. Thus part (1) follows from Lemma 2.1.

(2) Take a principal bundle chart $(U_\alpha,\varphi_\alpha)$ for P and the bundle chart (U_α,ψ_α) for $P\times_G S$ as constructed in the proof of (1). Then $\psi_\alpha(q((\varphi_\alpha^{-1}(x,g),y)))=(x,g\cdot y)$, so q is smooth. Moreover, for the open subset $\pi^{-1}(U_\alpha)\subset P\times_G S$, we have $q^{-1}(\pi^{-1}(U_\alpha))=p^{-1}(U_\alpha)\times S$. Define a map τ_α from this subset to $\pi^{-1}(U_\alpha)\times G$ by $\tau_\alpha(u,y):=([u,y],\mathrm{pr}_2(\varphi_\alpha(u)))$. One easily verifies directly that $(\pi^{-1}(U_\alpha),\tau_\alpha)$ is a principal bundle atlas for $q: P\times S\to P\times_G S$.

(3) A typical point in $P\times_M(P\times_G S)$ has the form $(u,[v,y])$ with $p(u)=\pi([v,y])=p(v)$. But this means that $v=u\cdot g$ for some $g\in G$ and thus $[v,y]=[u\cdot g,y]=[u,g\cdot y]$. Of course $[u,y]=[u,z]$ implies $y=z$, so τ_S is well defined and we only have to verify that it is smooth. But for bundle charts $(U_\alpha,\varphi_\alpha)$ and (U_α,ψ_α) as above, the corresponding chart on $P\times_M(P\times_G S)$ is given by $(u,[v,y])\mapsto(p(u),\mathrm{pr}_2(\varphi_\alpha(u)),\mathrm{pr}_2(\varphi_\alpha(v))\cdot y)$. Since $g=\mathrm{pr}_2(\varphi_\alpha(u))^{-1}\,\mathrm{pr}_2(\varphi_\alpha(v))$, we see that in charts τ_S is given by $(x,h,z)\mapsto h^{-1}\cdot z$ and hence is smooth. □

We next discuss a description of sections of associated bundles which will be technically very useful. Let $p: P\to M$ be a G–principal bundle and $G\times S\to S$ a smooth left action. Then a smooth map $f: P\to S$ is said to be *G–equivariant* if and only if

$f(u \cdot g) = g^{-1} \cdot f(u)$ for all $u \in P$ and $g \in G$. The space of all such maps is denoted by $C^\infty(P, S)^G$.

Corollary 2.8. *Let $p : P \to M$ be a principal G–bundle and consider the associated bundle $P \times_G S$ with respect to a smooth left action of G on a manifold S. Then the space $\Gamma(P \times_G S)$ is naturally isomorphic to $C^\infty(P, S)^G$.*

Proof. For a smooth section σ of $P \times_G M$, the function $f : P \to S$ defined by $f(u) := \tau_S(u, \sigma(p(u)))$ is evidently smooth and it says that $\sigma(p(u)) = [u, f(u)]$. For $g \in G$, we get $p(u \cdot g) = p(u)$ and hence $[u, f(u)] = [u \cdot g, f(u \cdot g)]$, which readily implies that $f(u \cdot g) = g^{-1} \cdot f(u)$.

Conversely, suppose that $f : P \to S$ is an equivariant smooth function and take a point $x \in M$. Then the above computation shows that the element $[u, f(u)] \in P \times_G S$ is the same for all $u \in P_x$, and we define this to be $\sigma(x)$. Hence we have defined a map $\sigma : M \to P \times_G S$ such that $\pi \circ \sigma = \mathrm{id}_M$. This is smooth since locally around each point, we can choose a smooth local section τ of P and then write $\sigma(x)$ as $q(\tau(x), f(\tau(x)))$. Evidently, the two constructions are inverse to each other. □

Example 2.8. (1) Consider the case of linear isomorphisms $\mathbb{R}^n \to V$, which can be viewed as a principal bundle over a point with structure group $GL(n, \mathbb{R})$, see Example 2.4 (2). Let us form the associated bundle corresponding to the standard representation $\mathbb{R}^n$ of $GL(n, \mathbb{R})$. We have to take $P \times \mathbb{R}^n$ and consider the action $(\varphi, x) \cdot A = (\varphi \circ A, A^{-1}x)$. But then evidently mapping $(\varphi, x) \to \varphi(x)$ induces a bijection between $P \times_{GL(n,\mathbb{R})} \mathbb{R}^n$ and V.

The description of section in the corollary can also be understood in this picture. Let us interpret P as the space of all ordered bases in V. This means that $\varphi : \mathbb{R}^n \to V$ corresponds to the basis $\{\varphi(e_1), \dots, \varphi(e_n)\}$. Given a vector $v \in V$, we can compute its coordinates in each basis and view them as an element of $\mathbb{R}^n$. So we can view v as a function $f : P \to \mathbb{R}^n$. A moment of thought shows that f is explicitly given by $f(\varphi) = \varphi^{-1}(v) \in \mathbb{R}^n$. This immediately shows that $f(\varphi \circ A) = A^{-1} f(\varphi)$. Conversely, given $f = (f_1, \dots, f_n) : P \to \mathbb{R}^n$, we can consider $f(\varphi) := \sum_i f_i(\varphi_i) \cdot \varphi(e_i) \in V$. Equivariancy of f is exactly what is needed in order to obtain the same result for all choices of φ.

(2) This now works in a completely similar way for frame bundles of vector bundles. Let $p : E \to M$ be a vector bundle with n–dimensional fibers and let $P \to M$ be its frame bundle. Then P_x consists of all linear isomorphisms $u : \mathbb{R}^n \to E_x$ and the map $P \times \mathbb{R}^n \to E$ defined by $(u, y) \mapsto u(y)$ descends to an isomorphism $P \times_{GL(n,\mathbb{R})} \mathbb{R}^n \to E$.

(3) We can now easily show that the tautological bundle on a Grassmann manifold as introduced in Example 2.3 (3) is indeed a locally trivial vector bundle. We consider the Grassmannian $Gr(k, \mathbb{R}^n)$ as the homogeneous space G/H, where $G = GL(n, \mathbb{R})$ and H is the stabilizer of $\mathbb{R}^k \subset \mathbb{R}^n$. Then we can restrict the H–action to the invariant subspace $\mathbb{R}^k$, thus obtaining a representation of H on this space. Since we know that $p : G \to G/H$ is a principal H–bundle, Proposition 2.8 shows that $G \times_H \mathbb{R}^k$ is a locally trivial vector bundle over G/H. Now consider the map $G \times \mathbb{R}^k \to (G/H) \times \mathbb{R}^n$ defined by $(g, v) \mapsto (gH, gv)$. Evidently, this descends to a map $G \times_H \mathbb{R}^k$ which is then automatically smooth since q is a surjective submersion. But $gH \in G/H$ exactly corresponds to the subspace $g(\mathbb{R}^k)$ so by construction v lies in that subspace. This implies that the descended map is a bijection onto the subspace $\{(W, w) : w \in W\}$ which was our definition of the tautological bundle.

(4) We can also nicely illustrate the flexibility of the construction of associated bundles by discussing a description of reductions of structure group. Consider a principal G–bundle $p : P \to M$ and a closed subgroup $H \subset G$. Then we can consider the homogeneous space G/H with its canonical G–action and form the associated bundle $P \times_G (G/H)$. Now we can restrict the principal right action of G on P to the subgroup H, and we first claim that $P \times_G (G/H)$ can be identified with the orbit space P/H.

To see this, consider the map $\psi : P \to P \times_G (G/H)$ defined by $\psi(u) := q(u, eH)$, where $q : P \times (G/H) \to P \times_G (G/H)$ is the natural map. Since $q(u, gH) = g(u \cdot g, eH)$, this map is surjective. Moreover, $q(u, eH) = q(v, eH)$ if and only if there is an element $g \in G$ such that $v = u \cdot g$ and $g^{-1}H = eH$ and hence $g \in H$.

Now we claim that smooth sections of $P \times_G (G/H)$ are in bijective correspondence with reductions $Q \to M$ of P to the structure group H. Given a reduction $F : Q \to P$, consider a local section σ of Q over some open subset of U and form $\psi \circ F \circ \sigma$, which clearly is a local smooth section of $P \times_G (G/H)$. But any other local section of Q over U is of the form $\sigma(x) \cdot \varphi(x)$ for some smooth function $\varphi : U \to H$. But then H–equivariancy of F and H–invariance of ψ imply that this leads to the same section of $P \times_G (G/H)$. Covering M by the charts of a principal bundle atlas for Q, we get a family of local sections of $P \times_G (G/H)$, which agree on the intersections of their domains and thus fit together to define a global smooth section.

Conversely, give a smooth section of $P \times_G (G/H)$, we get an associated G–equivariant function $f : P \to G/H$ and we consider $Q := f^{-1}(eH) \subset G$. For a point $x \in M$ and an element $u \in P_x \cap Q$ we see that $f(u \cdot g) = g \cdot f(u)$ implies that $P_x \cap Q = \{u \cdot h : h \in H\}$. Moreover, equivariancy easily implies that f is a regular function, so locally Q is a submanifold in P and it is easy to verify that it defines a reduction of structure group.

This give a nice perspective on the description of reductions of structure group of the frame bundle $\mathcal{P}M \to M$ to the subgroup $O(n) \subset GL(n, \mathbb{R})$. The point here is that the homogeneous space $\mathcal{G}_n := GL(n, \mathbb{R})/O(n)$ can be naturally identified with the space of positive definite inner products on $\mathbb{R}^n$, see Section 1.17 of my lecture notes [**LG**] on Lie groups. This easily implies that $\mathcal{P}M \times_{GL(n,\mathbb{R})} \mathcal{G}_n$ is the bundle whose fiber in each point is the space of positive definite inner products on the fiber of $\mathcal{P}M \times_{GL(n,\mathbb{R})} \mathbb{R}^n \cong TM$. So sections of this bundle are exactly Riemannian metrics on M.

2.9. Generalized Frame bundles. We can now prove that under fairly week assumptions bundles with structure group G can always be realized as associated bundles. Recall that a left action $G \times S \to S$ is called *effective* if $g \cdot y = y$ for all $y \in S$ implies $y = e$. For a general action, it is easy to see that $\{g \in G : \forall y \in S g \cdot y = y\}$ is a closed normal subgroup of G and one obtains an induced effective action of the quotient group.

PROPOSITION 2.9. *Let $\pi : E \to M$ be a fiber bundle with fiber S and structure group G which acts effectively on S. Then there is a unique (up to isomorphism) G–principal bundle $p : P \to M$ such that $E \cong P \times_G S$ as a bundle with structure group G.*

PROOF. By definition, there is an atlas $\{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$ for E such that the chart changes have the form $\varphi_\alpha \circ \varphi_\beta^{-1}(x, y) = (x, \varphi_{\alpha\beta}(x) \cdot y)$ for smooth functions $\varphi_{\alpha\beta} : U_{\alpha\beta} \to G$. Now for $x \in U_{\alpha\beta\gamma}$ and each $y \in S$, we obtain $\varphi_{\alpha\beta}(x)\varphi_{\beta\gamma}(x) \cdot y = \varphi_{\alpha\gamma}(x) \cdot y$. By effectivity, this implies that $\varphi_{\alpha\beta}$ is a cocycle of transition functions corresponding to the covering $\{U_\alpha : \alpha \in I\}$ of M. From 2.5 we know that we can realize this as the transition function of a principal bundle atlas of a principal G–bundle $p : G \to M$. Taking the induced atlas $\{(U_\alpha, \psi_\alpha) : \alpha \in I\}$ for $P \times_G S$, its transition functions are also given by $\psi_\alpha \circ \psi_\beta^{-1}(x, y) = (x, \varphi_{\alpha\beta}(x) \cdot y)$. But this implies that the maps defined by

$F(\psi_\alpha^{-1}(x,y)) = \varphi_\alpha^{-1}(x,y)$ fit together to define a isomorphism $P \times_G S \to E$ of bundles with structure group G.

To prove uniqueness assume that $P \to M$ and $\tilde{P} \to M$ are principal G–bundles corresponding to cocycles $\varphi_{\alpha\beta}$ and $\tilde{\varphi}_{\alpha\beta}$, without loss of generality for the same covering $\{U_\alpha : \alpha \in I\}$. Suppose further that $F : P \times_G S \to \tilde{P} \times_G S$ is an isomorphism of bundles with structure group G. Then for each α, $\tilde{\varphi}_\alpha \circ F|_{p^{-1}(U_\alpha)} : p^{-1}(U_\alpha) \to U_\alpha \times Y$ must be G–compatible to the chart φ_α. Hence there is a smooth map $f_\alpha : U_\alpha \to G$ such that $\tilde{\varphi}_\alpha(F(\varphi_\alpha^{-1}(x,y))) = (x, f_\alpha(x) \cdot y)$. Changing charts in both atlases, we conclude that for $x \in U_{\alpha\beta}$ and $y \in S$, we obtain $f_\beta(x) \cdot y = \tilde{\varphi}_{\beta\alpha}(x) f_\alpha(x) \varphi_{\alpha\beta}(x) \cdot y$. By effectivity, this implies $\tilde{\varphi}_{\alpha\beta}(x) f_\beta(x) = f_\alpha(x) \varphi_{\alpha\beta}(x)$, and thus we see from Section 2.5 that $P \cong \tilde{P}$. $\square$

2.10. Functorial properties of associated bundles. We can next show that the construction of associated bundles is functorial in both arguments. Consider first a fixed principal bundle $p : P \to M$ with structure group G and two left actions $G \times S \to S$ and $G \times \tilde{S} \to \tilde{S}$ on smooth manifolds S and $\tilde{S}$. Then a smooth map $\varphi : S \to \tilde{S}$ is called *G–equivariant* if it satisfies $\varphi(g \cdot y) = g \cdot \varphi(y)$ for all $g \in G$ and $y \in S$. Then $\mathrm{id}_P \times \varphi : P \times S \to P \times \tilde{S}$ is G–equivariant, too, so there is a unique map $P[\varphi] : P \times_G S \to P \times_G \tilde{S}$ such that $P[\varphi]([u,y]) := [u, \varphi(y)]$ and this is smooth since $q : P \times S \to P \times_G S$ is a surjective submersion. Moreover, in the local associated bundle charts induced by a principal bundle chart ψ for U, the map $P[\varphi]$ has the form $(x,y) \mapsto [\psi^{-1}(x,e), y] \mapsto [\psi^{-1}(x,e), \varphi(y)] \mapsto (x, \varphi(y))$. Equivariancy of φ then readily implies that $P[\varphi]$ is a smooth morphism of fiber bundles with structure group G.

Since the base map of $P[\varphi]$ is the identity, there is an induced map $\Gamma(P \times_G S) \to \Gamma(P \times_G \tilde{S})$ on the spaces of sections. Under the isomorphism to spaces of equivariant functions from Corollary 2.9, this map corresponds to $f \mapsto \varphi \circ f$, which evidently defines a map $C^\infty(P,S)^G \to C^\infty(P,\tilde{S})^G$ by equivariancy. This readily follows from the above description in view of the equation $\sigma(x) = [u, f(u)]$ relating sections to equivariant functions.

In particular, we can apply this to an morphism $\varphi : V \to W$ between two representations of G, i.e. a linear map between the vector spaces which is equivariant for the actions of G. Then of course we obtain an induced homomorphism $P[\varphi] : P \times_G V \to P \times_G W$ of vector bundles.

On the other hand, consider a principal G–bundle $p : P \to M$, a principal $\tilde{G}$–bundle $\tilde{p} : \tilde{P} \to \tilde{M}$, a homomorphism $\tau : \tilde{G} \to G$ and a morphism $F : \tilde{P} \to P$ of principal bundles over τ with with base map $f : \tilde{M} \to M$. Given a smooth left action $G \times S \to S$, we define a smooth left action of $\tilde{G}$ on S by $\tilde{g} \cdot y := \tau(\tilde{g}) \cdot y$. Then we can form the associated bundles $P \times_G S$ and $\tilde{P} \times_{\tilde{G}} S$. The map $F \times \mathrm{id}_S : \tilde{P} \times S \to P \times S$ maps $(u,y) \cdot \tilde{g}$ to $F(u \cdot \tilde{g}, \tau(\tilde{g}^{-1}) \cdot s) = (F(u), y) \cdot \tau(\tilde{g})$. Hence there is an induced map $F[S] : \tilde{P} \times_{\tilde{G}} S \to P \times_G S$, which is smooth by the universal property of surjective submersions and has base map f. Evidently, this is a morphism of fiber bundles, and if we start from a representation of G on V, then both associated bundles are vector bundles, and we obtain a homomorphism of vector bundles with base map f.

2.11. Linear algebra on vector bundles. Large parts of linear algebra can be carried over to vector bundles using the connection to the representation theory of the Lie groups $GL(V)$. In particular, most functorial constructions with vector spaces can be extended to vector bundles.

Let us start with the concept of a vector–subbundle, which we have already met implicitly in the example of distributions in Section 2.4. For a vector bundle $p : E \to M$, a *vector–subbundle* is a subset $\tilde{E} \subset E$ such that for each $x \in M$, $\tilde{E}_x := \tilde{E} \cap E_x$ is a linear subspace and such that $p|_{\tilde{E}} : \tilde{E} \to M$ is itself a vector bundle. Observe that this implies that all the spaces $\tilde{E}_x$ have the same dimension (over each connected component of M).

We claim that this is equivalent to the fact that for each $x \in M$ that is a local vector bundle chart $\varphi : p^{-1}(U) \to U \times \mathbb{R}^n$ defined on an open neighborhood U of $x \in M$ and a number $k \leq n$ such that such that φ restricts to a vector bundle chart $p^{-1}(U) \cap \tilde{E} \to U \times \mathbb{R}^k$. To see this, start from a vector bundle chart for $\tilde{E}$ defined on a neighborhood of x and consider the corresponding local frame for E. Then the values of the elements of this frame in x form a basis for $\tilde{E}_x \subset E_x$. Extend them to a local basis of E_x and then extend these vectors to local sections of E. Then on some neighborhood of x, these sections together with the local frame for $\tilde{E}$ form a local frame for E, thus defining a vector bundle chart with the required properties.

At this point we can already invoke the general machinery. Let $H \subset GL(n, \mathbb{R})$ be the stabilizer of $\mathbb{R}^k \subset \mathbb{R}^n$. This consists of block matrices $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ where A and C are invertible matrices of sizes $k \times k$ and $(n-k) \times (n-k)$, respectively. Obviously, on can restrict the representation of H on $\mathbb{R}^n$ to a representation on $\mathbb{R}^k$, corresponding to the A–block. On the other hand, H also has a natural representation on $\mathbb{R}^n/\mathbb{R}^k \cong \mathbb{R}^{n-k}$ corresponding to the C–block.

Given the bundle E and the subbundle $\tilde{E} \subset E$, we obtain a natural principal bundle $p : P \to M$ with structure group H, by taking the fiber P_x to be the set of all those linear isomorphisms $\mathbb{R}^n \to E_x$, which map the subspace $\mathbb{R}^k \subset \mathbb{R}^n$ to $\tilde{E}_x \subset E_x$ (and thus restrict to an isomorphism between these subspaces).

As in Example 2.8 (2), this readily implies that $E \cong P \times_H \mathbb{R}^n$ and $\tilde{E} \cong P \times_H \mathbb{R}^k$, and we define the *quotient bundle* $E/\tilde{E}$ as $P \times_H (\mathbb{R}^n/\mathbb{R}^k)$. This is a vector bundle with fibers of dimension $n - k$. Moreover, the natural surjection $\mathbb{R}^n \to \mathbb{R}^{n-k}$ is H–equivariant by definition, so it induces a vector bundle homomorphism $q : E \to E/\tilde{E}$. Evidently, this restricts to a surjection $E_x \to (E/\tilde{E})_x$ on each fiber with kernel $\tilde{E}_x$, so $(E/\tilde{E})_x \cong E_x/\tilde{E}_x$ for each $x \in M$.

The quotient bundle has a similar universal property as the quotient of a vector space by a subspace. Let $p : E \to M$ and $\tilde{p} : \tilde{E} \to \tilde{M}$ be a vector bundles, $F : E \to \tilde{E}$ a homomorphism with base map f and let $V \subset E$ be a subbundle. Suppose that $F(v) = 0$ for any $v \in V \subset E$. Then there is a unique vector bundle homomorphism $\underline{F} : E/V \to \tilde{E}$ with base map f, such that $\underline{F} \circ q = f$. Here the existence of $\underline{F}$ follows readily from the universal property of the quotient of vector spaces while smoothness follows from the obvious fact that q is a surjective submersion.

Consider vector bundles $p : E \to M$ and $\tilde{p} : \tilde{E} \to \tilde{M}$ and a homomorphism $F : E \to \tilde{E}$ of vector bundles with base map f. Then for each $x \in M$, the restriction $F_x := F|_{E_x} : E_x \to \tilde{E}_{f(x)}$ is a linear map, and hence has a well defined rank. We say that F *is a vector bundle homomorphism of constant rank* if and only if this rank is the same for all points of M. In this case, we define the *kernel* $\ker(F)$ of F to be the union of the spaces $\ker(F_x)$ and the *image* $\operatorname{im}(F)$ of F to be the union of the spaces $\operatorname{im}(F_{f(x)})$.

PROPOSITION 2.11. *Let $p : E \to M$ and $\tilde{p} : \tilde{E} \to \tilde{M}$ be vector bundles and let $F : E \to \tilde{E}$ be a homomorphism of constant rank with base map $f : M \to \tilde{M}$. Then $\ker(F)$ is a smooth subbundle of E. If $M = \tilde{M}$ and $f = \mathrm{id}$, $\operatorname{im}(F) \subset \tilde{E}$ is a smooth subbundle and F induces an isomorphism $E/\ker(F) \to \operatorname{im}(F)$ of vector bundles.*

Proof. Let us first consider the case that $\tilde{M} = M$ and $f = \mathrm{id}_M$. Take a point $x \in M$ and a local frame $\xi_1, \dots, \xi_n$ for E defined on some open neighborhood U of x in M. Then $F \circ \xi_1, \dots, F \circ \xi_n$ are smooth sections of $\tilde{E}$. Denoting by r the constant rank of F, we know that the vectors $F(\xi_i(x))$ span a subspace of $\tilde{E}_x$ of dimension r, so renumbering if necessary, we can assume that $F(\xi_1(x)), \dots, F(\xi_r(x))$ are linearly independent. In a vector bundle chart for $\tilde{E}$, the just means that we have r smooth functions $U \to \mathbb{R}^m$ whose values in x are linearly independent. This implies that they are linearly independent locally around x, so shrinking U, we may assume that $F(\xi_1(y)), \dots, F(\xi_r(y))$ are linearly independent for all $y \in U$. But since the rank is constant, the sections $F \circ \xi_1, \dots, F \circ \xi_r$ form a local frame for $\mathrm{im}(F)$ on U, so we see that $\mathrm{im}(F) \subset \tilde{E}$ is a smooth subbundle.

Replacing $\tilde{E}$ by $\mathrm{im}(F)$ we may without loss of generality assume that all the maps F_y for $y \in M$ are surjective. Let us again start from $x \in M$ and choose a basis for E_x such that the last $n - r$ elements for a basis for $\ker(F_x)$. Extend this basis to a local frame $\xi_1, \dots, \xi_n$ for E defined on an open neighborhood U of x. Possibly shrinking U, we may assume that the sections $\eta_1 := F \circ \xi_1, \dots, \eta_r := F \circ \xi_r$ form a local frame for $\tilde{E}$ on all of U. But then by definition $F(\xi_j) = \eta_j$ for all $j = 1, \dots, r$ while $F(\xi_j) = \sum_{\ell=1}^{r} c_j^\ell \eta_\ell$ for some smooth functions $c_j^\ell : U \to \mathbb{R}$. But this readily implies that the the smooth sections $\tilde{\xi}_j := \xi_j - \sum_{\ell=1}^{r} c_j^\ell \xi_\ell$ for $j = r+1, \dots, n$ have values in $\ker(F)$. The construction also shows that these elements remain linearly independent, so they form a local smooth frame for $\ker(F)$. Having observed that $\ker(F)$ is a smooth subbundle, it now follows that F induces a vector bundle homomorphism $\underline{F} : E/\ker(F) \to \mathrm{im}(F)$. Linear algebra implies that this induces a linear isomorphism in each fiber. By Proposition 2.6, $\underline{F}$ is an isomorphism of vector bundles, so the proof in the case $f = \mathrm{id}$ is complete.

In the general case, we can apply the first part of the proof to the vector bundle homomorphism $\hat{F} : E \to f^*\tilde{E}$ induced by F, see Proposition 2.6. By construction $(f^*\tilde{E})_x = \tilde{E}_{f(x)}$ and $\hat{F}_x = F_x$, so $\ker(F) = \ker(\hat{F}) \subset E$ is a smooth subbundle. $\square$

2.12. Constructions with vector bundles. Next we show that functorial constructions with vector spaces, which can be used to define constructions with representations of Lie groups, can also be used to define constructions with vector bundles. The simplest example is provided by functors involving only one argument.

The dual bundle. Let $p : E \to M$ be a $\mathbb{K}$–vector bundle with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with typical fiber $\mathbb{K}^n$. Then we can form the frame bundle $p : P \to M$ for E, which is a principal bundle with structure group $G := GL(n, \mathbb{K})$. Now the standard representation of G on $\mathbb{K}^n$ induces the so–called *dual* or *contragradient* representation of G on the dual space $\mathbb{K}^{n*} = L_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K})$. Explicitly, this is given by $(A \cdot \lambda)(x) = \lambda(A^{-1}x)$ for $A \in GL(n, \mathbb{K})$, $x \in \mathbb{K}^n$ and $\lambda \in \mathbb{K}^{n*}$. Since inversion in $GL(n, \mathbb{K})$ is smooth, it follows readily that this defines a smooth representation. Using this, we define the *dual bundle* E^* to E as $P \times_G \mathbb{K}^{n*}$.

We claim that for each $x \in M$, the fiber E_x^* can be naturally identified with $(E_x)^*$. To see this, recall that the fiber P_x consists of all linear isomorphisms $u : \mathbb{R}^n \to E_x$ and from Example 2.8 we know that that $[u, v] \mapsto u(v)$ induces an isomorphism $P \times_G \mathbb{R}^n \to E$. For $x \in M$ and $\lambda \in \mathbb{K}^{n*}$ we can now map $(u, \lambda) \in P \times \mathbb{K}^{n*}$ to the map $E_x \to \mathbb{K}$ defined by $\xi \mapsto \lambda(u^{-1}(\xi))$. Evidently, this is a linear functional and for $(u \cdot A, A^{-1} \cdot \lambda)$, we obtain $\xi \mapsto \lambda(AA^{-1}u^{-1}(\xi))$ and hence the same functional. Hence this descends to a map which associates to each element of E_x^* a linear functional on E_x, and it readily follows that this is an isomorphism.

Similarly, for sections $\sigma \in \Gamma(E)$ and $\tau \in \Gamma(E^*)$, one may consider the point–wise pairing $x \mapsto \tau(x)(\sigma(x))$. This gives rise to a map $\Gamma(E) \times \Gamma(E^*) \to C^\infty(M, \mathbb{K})$, which is bilinear over smooth functions. Moreover, it is easy to see that a function τ, that assigns to each point $x \in M$ a linear functional on E_x, defines a smooth section of E^* if and only if for each $\sigma \in \Gamma(E)$ the function $x \mapsto \tau(x)(\sigma(x))$ is smooth. In particular, for the tangent bundle of a smooth manifold, the dual bundle is exactly the usual *cotangent bundle* T^*M. Alternatively, the dual pairing can be encoded into a morphism $E \times_M E^* \to M \times \mathbb{K}$ of fiber bundles, whose restriction to each fiber is bilinear.

Consider vector bundles $p : E \to M$ and $\tilde{p} : \tilde{E} \to \tilde{M}$ and a vector bundle homomorphism $F : E \to \tilde{E}$ whose base map $f : M \to \tilde{M}$ is a diffeomorphism. Then for each $x \in M$, $F_x : E_x \to \tilde{E}_{f(x)}$ is a linear map, so there is a dual map $(F_x)^* : (\tilde{E}_{f(x)})^* \to (E_x)^*$. These maps fit together to define map $F^* : \tilde{E}^* \to E^*$ with base map f^{-1}. It is easy to verify in local vector bundle charts that this is a smooth homomorphism of vector bundles.

Tensor powers, symmetric powers and exterior powers. Given a bundle $p : E \to M$, one constructs bundles $\otimes^k E$, $S^k E$ and $\Lambda^k E$ using the natural representations of $G = GL(n, \mathbb{K})$ on the kth tensor power $\otimes^k \mathbb{K}^n$, the kth symmetric power $S^k\mathbb{K}^n$ and that kth exterior power $\Lambda^k\mathbb{K}^n$, respectively. The actions on this spaces are all induced by $A\cdot(x_1 \otimes \cdots \otimes x_n) := Ax_1 \otimes \cdots \otimes Ax_n$. As above, one verifies that $(\otimes^k E)_k = \otimes^k(E_x)$ and so on. Given sections $\sigma_1, \dots, \sigma_k \in \Gamma(E)$, one obtains a section $si_1 \otimes \cdots \otimes \sigma_k$ of $\Gamma(\otimes^k E)$, and any smooth section of $\otimes^k E$ can be written as a finite sum of sections of this form. In the picture of the equivariant functions $f_1, \dots, f_k : P \to \mathbb{K}^n$ corresponding to the σ_i, the section $si_1 \otimes \cdots \otimes \sigma_k$ simply corresponds to the function $u \mapsto f_1(u) \otimes \cdots \otimes f_k(u)$.

The universal properties of these objects from linear algebra carry over to this setting. For example, there is a natural morphism $E \times_M \ldots \times_M E \to \otimes^k E$ of fiber bundles (with k–factors in the fibered product) whose value in each fiber is k–linear. This has the universal property that any morphism $F : E \times_M \ldots \times_M E \to \tilde{E}$ whose restriction to each fiber is k–linear comes from a unique homomorphism $\hat{F} : \otimes^k E \to \tilde{E}$ of vector bundles. In particular, this can be used to associate to a vector bundle homomorphism $F : E \to \tilde{E}$, vector bundle homomorphisms $\otimes^k F : \otimes^k E \to \otimes^k \tilde{E}$ and likewise $S^k F$ and $\Lambda^k F$. It also follows readily that the canonical isomorphism from linear algebra, like $(\otimes^k \mathbb{R}^n)^* \cong \otimes^k(\mathbb{R}^{n*})$ are isomorphisms of representations and hence continue to hold for the constructions with vector bundles.

Observe at this point that all the constructions discussed so far can be done more directly if we start with an associated bundle. Suppose that $E = P \times_G V$ for a principal G–bundle P and a linear representation of G on a vector space V. Then we can define the dual representation of G on V^* and likewise we get canonical representations on $\otimes^k V$, $S^k V$, and $\Lambda^k V$. Of course, the associated bundles with respect to these representations are naturally isomorphic to E^*, $\otimes^k E$, $S^k E$, and $\Lambda^k E$, respectively. To prove this formally, one just has to observe that forming the frame bundle $\tilde{P}$ of E with structure group $GL(V)$, one gets an obvious morphism $P \to \tilde{P}$ of principal bundles over the homomorphism $\rho : G \to GL(V)$ defining the representation and with base map the identity. Now for any representation of $GL(V)$, one obtains a representation of G, and this is exactly how the representations on V^*, $\otimes^k V$, $S^k V$, and $\Lambda^k V$ are obtained. From 2.10 we know that the principal bundle homomorphism induces a homomorphism on the associated vector bundles with respect to any representation of $GL(V)$. By construction, each of these homomorphisms has the identity as its base map and induces a linear isomorphism on each fiber, and hence is an isomorphism of vector bundles by Proposition 2.6.

Bundles associated to bifunctors. To extend the above ideas to functors involving several vector spaces, we only need one more observation. To get this, let us reconsider the definition of the Whitney sum from Section 2.7. Given two vector bundles E and $\tilde{E}$, we defined $E \oplus \tilde{E}$ as the fibered product $E \times_M \tilde{E}$. Now let P and $\tilde{P}$ be principal bundles with structure groups G and $\tilde{G}$ such that $E = P \times_G V$ and $\tilde{E} = \tilde{P} \times_{\tilde{G}} \tilde{V}$. (As before we can either start with given associated bundles or take the full linear frame bundles.) Then we can form the fibered product $P \times_M \tilde{P}$ and in Section 2.7 we have observed that this is a principal bundle with structure group $G \times \tilde{G}$.

Since the projections from $G \times \tilde{G}$ onto G and $\tilde{G}$ are group homomorphisms, we obtain representations of $G \times \tilde{G}$ on V and $\tilde{V}$. Moreover, these define a representation of $G \times \tilde{G}$ on $V \oplus \tilde{V}$, which is explicitly given by $(g, \tilde{g}) \cdot (v, \tilde{v}) = (g \cdot v, \tilde{g} \cdot \tilde{v})$. Hence we can form the bundle $(P \times_M \tilde{P}) \times_{G \times \tilde{G}} (V \oplus \tilde{V})$. Denoting by $q : P \times V \to E$ and $\tilde{q} : \tilde{P} \times \tilde{V} \to \tilde{E}$ the canonical maps, we obtain a map $(P \times_M \tilde{P}) \times (V \oplus \tilde{V}) \to E \times_M \tilde{E}$ defined by $((u, \tilde{u}), (v, \tilde{v})) \mapsto (q(u, v), \tilde{q}(\tilde{u}, \tilde{v}))$. One immediately verifies that this descends to an isomorphism $(P \times_M \tilde{P}) \times_{G \times \tilde{G}} (V \oplus \tilde{V}) \to E \oplus \tilde{E}$.

Having this point of view at hand, things easily generalize. The representations of G on V and of $\tilde{G}$ on $\tilde{V}$ give rise to several other representations. Taking the tensor product $V \otimes \tilde{V}$, one defines a representation of $G \times \tilde{G}$ by $(g, \tilde{g}) \cdot (v \otimes \tilde{v}) := (g \cdot v) \otimes (\tilde{g} \cdot \tilde{v})$. Similarly, one obtains a representation of $G \times \tilde{G}$ on $L(V, \tilde{V})$ defined by $((g, \tilde{g}) \cdot f)(v) := \tilde{g} \cdot (f(g^{-1} \cdot v))$. Forming associated bundles with respect to this representations, we obtain the *tensor product* $E \otimes \tilde{E}$ of vector bundles and the *bundle of linear maps* $L(E, \tilde{E})$. By construction, the fibers of these bundles are given by $(E \otimes \tilde{E})_x = E_x \otimes \tilde{E}_x$ and $L(E, \tilde{E})_x = L(E_x, \tilde{E}_x)$, respectively.

The tensor product of vector bundles inherits an analog of the universal property of the tensor product of vector spaces. The bilinear map $V \times \tilde{V} \to V \otimes \tilde{V}$ induces a morphism $E \times_M \tilde{E} \to E \otimes E$ by Section 2.10. This is not a homomorphism of vector bundles, since the restrictions to the fibers of $E \times_M \tilde{E}$ are bilinear rather than linear. Now suppose that L is any vector bundle and $F : E \times_M \tilde{E} \to L$ is a fiber bundle morphism with base map f such that for each x, the induced map $F_x : E_x \times \tilde{E}_x \to L_{f(x)}$ is bilinear. Then one can take the induced linear map $E_x \otimes \tilde{E}_x \to L_{f(x)}$ and piece these maps together to a homomorphism $\tilde{F} : E \otimes \tilde{E} \to L$ of vector bundles.

Similarly, for a vector bundle homomorphism $F : E \to \tilde{E}$ with base map f, the linear map $F_x : E_x \to \tilde{E}_x$ defines an element of $L(E, \tilde{E})_X$ and these fit together to a smooth section of the bundle $L(E, \tilde{E})$. This establishes an isomorphism between the space $\Gamma(L(E, \tilde{E}))$ of smooth sections of the bundle $L(E, \tilde{E})$ and the space of vector bundle homomorphisms $E \to \tilde{E}$ with base map the identity.

As before natural isomorphisms between vector spaces like $(V \otimes \tilde{V})^* \cong (V^*) \otimes (\tilde{V}^*)$ and $V^* \otimes \tilde{V} \cong L(V, V^*)$ are isomorphisms of representations, so the corresponding isomorphisms work for the constructions with vector bundles.

Constructions with vector bundles provide an interesting connection to algebraic topology. This is usually formulated in the setting of continuous vector bundles over (sufficiently nice) topological spaces. The basic idea here is to consider the set of isomorphisms classes of $\mathbb{K}$–vector bundles over M (the more common choice is $\mathbb{K} = \mathbb{C}$). The Whitney sum makes this set into a commutative semi–group with unit element 0 (given by $id_M : M \to M$, viewed as a vector bundle with 0–dimensional fibers).

Similarly as the integers are constructed from $\mathbb{N}$, one can construct an abelian group $K(M)$ out of this semi–group (the "Grothendieck group"): One takes the set of pairs $(E, \tilde{E})$ and defines an equivalence relation by $(E, \tilde{E}) \sim (F, \tilde{F})$ if and only if $E \oplus \tilde{F} \cong F \oplus \tilde{E}$. The fiber–wise Whitney sum gives rise to a well defined associative operation on the set $K(M)$ of equivalence classes, for which $(0, 0)$ is a neutral element. Moreover, by construction $(\tilde{E}, E)$ is an inverse for $(E, \tilde{E})$, so $K(M)$ is a commutative group. In particular, for a vector bundle $E \to M$, we denote by $[E] \in K(M)$ the class of $(E, 0)$. Then $-[E]$ is the class of $(0, E)$ and any element in $K(M)$ can be written as $[E] - [\tilde{E}]$ for bundles E and $\tilde{E}$.

There is a substantial difference to the case of integers, however. Passing from a semi–group to a group enforces the cancellation rule, which is satisfied in $\mathbb{N}$ anyway, but does not hold for vector bundles in general. Suppose that E, $\tilde{E}$ and F are vector bundles, such that $E \oplus F \cong \tilde{E} \oplus F$ ("stably isomorphic bundles"). Then in $K(M)$ we have $[E] + [F] = [\tilde{E}] + [F]$ and hence $[E] = [\tilde{E}]$. However, it may happen that stably isomorphic vector bundles are not isomorphic, so $K(M)$ does not contain the full information on isomorphism classes of vector bundles over M.

For example, consider S^2 as the unit sphere in $\mathbb{R}^3$. Then the fiber of the tangent bundle at $x \in S^2$ is given by $T_xS^2 = x^\perp$. On the other hand, consider $S^2 \times \mathbb{R}^3$ and in there the subset $\{(x, y) : y \in \mathbb{R} \cdot x\}$. This clearly is isomorphic to $S^2 \times \mathbb{R}$ an we see that the Whitney sum of TS^2 and this bundle is the trivial bundle $S^2 \times \mathbb{R}^3$ and hence isomorphic to the Whitney sum of $S^2 \times \mathbb{R}^2$ with $S^2 \times \mathbb{R}$. But it is well known that the bundle TS^2 is not trivial, since there is not even one nowhere–vanishing vector field on S^2.

The tensor product of vector bundles induces a multiplication on $K(M)$, making it into a commutative ring with unit (the trivial bundle $M \times \mathbb{K} \to M$). Moreover, for a map $f : M \to N$, the pullback of vector bundles induces a homomorphism $f^* : K(N) \to K(M)$ of rings. This is *topological K–theory*, which turns out to be a so–called generalized cohomology theory in the sense of algebraic topology.

3. Homogeneous bundles and invariant sections

3.1. Homogeneous bundles. Following Section 1.4.2 of [**book**].

Definition 3.1. Homogeneous fiber bundles, vector bundles, principal bundles, and their morphisms.

Example 3.1. (1) Natural bundles like TM, T^*M and tensor bundles.
(2) $p : G \to G/H$ and associated bundles.

3.2. Classification of homogeneous bundles. Following Section 1.4.3 of [**book**]

Proposition 3.2. *Functorial isomorphism between H–spaces and homogeneous fiber bundles over G/H. Interpretation for subclasses of homogeneous bundles.*

Example 3.2. Description of tensor bundles over G/H. Underlying reduction of the full frame bundle of G/H.

3.3. Sections of homogeneous bundles. Following Section 1.4.4 of [**book**]: The natural action of G on sections of a homogeneous bundle. Induced representations both in the picture of sections and of equivariant functions.

Theorem 3.3. *Consider the homogeneous bundle $\pi : E \to G/H$ corresponding to a given left action $H \times S \to S$. Then $\sigma \mapsto \sigma(o)$ induces a bijection between G–invariant elements in $\Gamma(E)$ and H–invariant elements in S.*

More generally, consider a left action of G on a set X. Then the evaluation map $ev_o : \Gamma(E) \to E_o$ at $o = eH \in G/H$ induces a bijection between the set of G–equivariant maps $X \to \Gamma(E)$ and the set of H–equivariant maps $X \to E_o$.

Corollary 3.3 (Frobenius reciprocity). *Let V be a finite dimensional representation of H and $Ind_H^G(V)$ the induced representation of G on $\Gamma(G \times_H V)$. Further, let W be any representation of G and let $Res_H^G(W)$ be the restriction of W to H. Then*

$$\mathrm{Hom}_G(W, Ind_H^G(V)) \cong \mathrm{Hom}_H(Res_H^G(W), V)$$

3.4. Applications. (1) Invariant Riemannian metrics as in Example 1.4.4 of [**book**].

As a specific example consider Euclidean space as a homogeneous space of $G = \mathrm{Euc}(n)$ as in 1.3. There we have seen that this realization is given by $E^{=} \mathrm{Euc}(n)/O(n)$. Moreover, as a representation of $H = O(n)$, the Lie algebra $\mathfrak{euc}(n)$ decomposes as $\mathfrak{o}(n) \oplus \mathbb{R}^n$. In particular, the representation $\underline{\mathrm{Ad}}$ of H on $\mathfrak{g}/\mathfrak{h}$ is just the standard representation of $O(n)$ on $\mathbb{R}^n$. Hence the standard inner product gives rise to a G–invariant Riemannian metric on E^n, which is just the usual flat metric. Linear algebra shows that the standard inner product is the unique $O(n)$–invariant inner product on $\mathbb{R}^n$ up to a positive multiple. Consequently, the G–invariant metric on E^n is unique up to a positive constant multiple, too.

We can also use this example to illustrate a phenomenon called *mutation.* The simplest realization of the sphere as a homogeneous space comes from the fact that the standard action of $G := O(n+1)$ on $\mathbb{R}^{n+1}$ preserves the norm of vectors and thus restricts to an action on the unit sphere S^n. Linear algebra shows that this action is transitive and picking a base point, say e_{n+1}, the isotropy group $H := G_{e_{n+1}}$ is isomorphic to $O(n)$, compare with Section 1.17 of [**LG**]. Explicitly, this isomorphism comes from the action of the isotropy group on ${e_{n+1}}^{\perp} = T_{e_{n+1}} S^n$. Hence Proposition 3.2 and Example 3.2 show that also in this case $\mathfrak{g}/\mathfrak{h} \cong \mathbb{R}^n$ as a representation of $H = O(n)$. The corresponding $O(n+1)$–invariant Riemann metric on S^n of course is the usual (round) metric.

This discussion shows that from the point of view of invariant tensor fields, there is absolutely no difference between the compact homogeneous space $S^n = O(n+1)/O(n)$ and the non–compact homogeneous space $E^n = \mathrm{Euc}(n)/O(n)$. Therefore, these two homogeneous spaces are sometimes called mutations of each other. There actually is a third mutation in this case. Put $G := O(n,1)$, the orthogonal group of a Lorentzian inner product b on $\mathbb{R}^{n+1}$. One defines the n–dimensional *hyperbolic space* $\mathcal{H}^n$ to be the set $\{x \in \mathbb{R}^{n+1} : b(x,x) = -1\}$. Linear algebra shows that the standard action of G on $\mathbb{R}^{n+1}$ restricts to a transitive action on $\mathcal{H}^n$. Moreover, similarly to S^n, one gets $T_x\mathcal{H}^n = \{y \in \mathbb{R}^{n+1} : b(x,y) = 0\}$ and $G_x \cong O(n)$ via the induced action on $T_x\mathcal{H}^n$. Hence we again get $\mathfrak{g}/\mathfrak{h} \cong \mathbb{R}^n$ as a representation of $H = O(n)$, so $\mathcal{H}^n$ carries a unique (up to a positive constant factor) $O(n,1)$–invariant Riemannian metric and the same invariant tensor fields as S^n and E^n. Geometrically, $\mathcal{H}^n$ is a two–sheeted hyperboloid, so it consists of two connected components, each of which is diffeomorphic to an open n–ball.

From the point of view of Riemannian geometry, E^n, S^n and $\mathcal{H}^n$ are exactly the complete Riemannian *space forms*, i.e. Riemannian manifolds with constant sectional curvature, compare with Section 2.11 of [**Riem**]. This constant curvature is positive for S^n, negative for H^n, and zero for E^n.

(2) **Decomposing spaces of functions or sections**: This is a short outlook on how one proceeds in understanding induced representations. For simplicity, let us focus on the example of $S^n = SO(n+1)/SO(n)$. The simplest example of an induced

representation then is provided by starting from the trivial representation $\mathbb{R}$ of $SO(n)$. By definition, $\mathrm{Ind}_{SO(n)}^{SO(n+1)}(\mathbb{R}) = C^\infty(S^n, \mathbb{R})$ with the action of $SO(n+1)$ given by $g \cdot f := f \circ \ell_{g^{-1}}$. Frobenius reciprocity tells us that for any representation V of $SO(n+1)$, the space $\mathrm{Hom}_G(V, C^\infty(S^n, \mathbb{R}))$ is isomorphic to $\mathrm{Hom}_H(V, \mathbb{R})$. By compactness of $SO(n+1)$, we may assume that V is irreducible. Restricting the representation V to $SO(n)$, it is not irreducible any more, but it splits into a direct sum of irreducible representations. The space $\mathrm{Hom}_H(V, \mathbb{R})$ is simply $\mathbb{R}^{n(V)}$, where $n(V)$ is the number of trivial factors in this decomposition.

For example, taking $V = \mathbb{R}$, we of course have $n(\mathbb{R}) = 1$ so

$$\mathrm{Hom}_G(\mathbb{R}, C^\infty(S^n, \mathbb{R})) = \mathbb{R}.$$

This corresponds to the fact that the constant functions on S^n are the only G–invariant functions. Next, consider the defining representation $V = \mathbb{R}^{n+1}$. Restricted to $SO(n)$, this decomposes as $\mathbb{R}^n \oplus \mathbb{R}$, so $n(\mathbb{R}^{n+1}) = 1$, and $\mathrm{Hom}_G(\mathbb{R}^{n+1}, C^\infty(S^n, \mathbb{R})) = \mathbb{R}$. The image of any nonzero homomorphism in this family (which then is independent of the choice) is an $n+1$–dimensional subrepresentation of $C^\infty(S^n, \mathbb{R})$ isomorphic to $\mathbb{R}^{n+1}$. Of course, this subrepresentation is spanned by the the components of the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$.

More generally, for each $k \in \mathbb{N}$, the representation $S^k\mathbb{R}^{n+1}$ restricted to $SO(n)$ decomposes as $\oplus_{i=0}^k S^i\mathbb{R}^n$. In particular, each of these representations contains exactly one copy of the trivial representation. Hence for each k, the representation $C^\infty(S^n, \mathbb{R})$ contains a unique subrepresentation isomorphic to $S^k\mathbb{R}^{n+1}$. This is spanned by the restrictions to S^n of homogeneous polynomials of degree k on $\mathbb{R}^{n+1}$.

More complete information can be obtained using the Peter–Weyl theorem. Since $SO(n+1)$ is compact, the functions in which lie in a finite dimensional $SO(n+1)$–invariant subspace are dense in $C^\infty(S^n, \mathbb{R})$. By complete reducibility, any such function can be written as a finite sum of elements in the image of a G–equivariant map from an appropriate finite dimensional irreducible representation V to $C^\infty(S^n, \mathbb{R})$. Hence these ideas lead to a complete description of a dense subspace of the representation $C^\infty(S^n, \mathbb{R})$. Similar arguments apply to more general induced representations.

3.5. Invariant differential forms. Invariant differential forms can be described using the general result in Theorem 3.3. There are special aspects here, however, due to the existence of the exterior derivative. Moreover, the relation to de Rham cohomology leads to interesting results even in cases where the determination of invariant forms itself is rather trivial.

Let us first make things a bit more explicit to obtain a relation to the calculus on differential forms. Consider a homogeneous space G/H and for $g \in G$ let us write $\ell_g : G/H \to G/H$ for the left action by g. In particular, given a differential form $\varphi \in \Omega^k(G/H)$, we can form the pullback $(\ell_g)^*\varphi$. By definition, for vector fields $\xi_1, \dots, \xi_k \in \mathfrak{X}(G/H)$ we have

$$(\ell_g)^*\varphi(\xi_1, \dots, \xi_k)(g') = \varphi(gg')(T_{g'}\ell_g \cdot \xi_1, \dots, T_{g'}\ell_g \cdot \xi_k).$$

This shows that the natural action of G on $\Omega^k(G/H)$ as introduced in Section 3.3 is given by $g \cdot \varphi = (\ell_{g^{-1}})^*\varphi$. In particular, φ is G–invariant if and only if $(\ell_g)^*\varphi = \varphi$ for all $g \in G$. In particular, putting $H = \{e\}$ one recovers the usual notion of a *left–invariant differential form* on G. Theorem 3.3 in this case just recovers the fact that these left invariant forms are in bijective correspondence with $\Lambda^k\mathfrak{g}^*$.

Returning to the general case of G/H, Theorem 3.3 shows that the space $\Omega^k(G/H)^G$ of G–invariant k–forms on G/H is isomorphic to the space $(\Lambda^k(\mathfrak{g}/\mathfrak{h})^*)^H$ of H–invariant elements. For a k–linear, alternating map $\alpha : (\mathfrak{g}/\mathfrak{h})^k \to \mathbb{R}$, the H–action is given by

$$(h \cdot \alpha)(X_1 + \mathfrak{h}, \dots, X_k + \mathfrak{h}) = \alpha(\mathrm{Ad}(h^{-1})(X_1) + \mathfrak{h}, \dots, \mathrm{Ad}(h^{-1})(X_1) + \mathfrak{h}).$$

Now it is a standard result in differential geometry that pullbacks of differential forms are compatible with both the wedge product of differential forms and with the exterior derivative. This shows that the space $\Omega^*(G/H)^G$ of all invariant forms is a subalgebra of the graded commutative algebra $\Omega^*(G/H)$ and closed under the exterior derivative. But via Theorem 3.3 we see that the exterior derivative must give rise to linear maps $(\Lambda^k(\mathfrak{g}/\mathfrak{h})^*)^H \to (\Lambda^{k+1}(\mathfrak{g}/\mathfrak{h})^*)^H$.

To describe these maps, define a linear map $\partial : \Lambda^k\mathfrak{g}^* \to \Lambda^{k+1}\mathfrak{g}^*$ by

$$\partial\alpha(A_0, \dots, A_k) := \sum_{i<j}(-1)^{i+j}\alpha([A_i, A_j], A_0, \dots, \widehat{A_i}, \dots, \widehat{A_j}, \dots, A_k),$$

where the hats denote omission. It is easy to see directly the $\partial\alpha$ is again alternating (although this will also follow from the next result). Likewise, one can prove directly that $\partial \circ \partial = 0$. The map ∂ is called the *Lie algebra cohomology differential* (for trivial coefficients).

Proposition 3.5. *The map ∂ induces a well defined linear map $\partial_H : (\Lambda^k(\mathfrak{g}/\mathfrak{h})^*)^H \to (\Lambda^{k+1}(\mathfrak{g}/\mathfrak{h})^*)^H$. This map represents the exterior derivative on invariant k–forms in the sense that if $\varphi \in \Omega^k(G/H)^G$ corresponds to $\alpha \in (\Lambda^k(\mathfrak{g}/\mathfrak{h})^*)^H$, then $d\varphi$ corresponds to $\partial_H\alpha$.*

Proof. Let us first consider the case that $H = \{e\}$, so we are dealing with left invariant forms on G. Theorem 3.3 in this case says that if $\omega \in \Omega^k(G)^G$ is such a form, then it is uniquely determined by $\omega(e)$, which is a k–linear alternating map $\mathfrak{g}^k \to \mathbb{R}$. Explicitly, we get

$$\omega(g)(\xi_1, \dots, \xi_k) = \omega(e)(T_g\lambda_{g^{-1}} \cdot \xi_1, \dots, T_g\lambda_{g^{-1}} \cdot \xi_1).$$

Otherwise put, the left invariant form ω generated by $\varphi : \mathfrak{g}^k \to \mathbb{R}$ is characterized by the fact that for $X_1, \dots, X_k \in \mathfrak{g}$ and the corresponding left invariant vector fields $L_{X_1}, \dots, L_{X_k}$, the function $\omega(L_{X_1}, \dots, L_{X_k})$ is constant and equal to $\varphi(X_1, \dots, X_k)$. Using this, the standard formula for the exterior derivative reduces to

$$d\omega(L_{x_0}, \dots, L_{X_k}) = \textstyle\sum_{i<j}(-1)^{i+j}\omega([L_{X_i}, L_{X_j}], L_{X_1}, \dots, \widehat{L_{X_i}}, \dots, \widehat{L_{X_i}}, \dots, L_{X_k}),$$

which shows that $d\omega$ corresponds to $\partial\varphi$.

The general case then follows readily from general facts on the calculus of differential forms. Consider the projection $p : G \to G/H$. Given a differential form $\omega \in \Omega^k(G/H)$, we can form the pullback $p^*\omega \in \Omega^k(G)$, and $p^*d\omega = dp^*\omega$ by naturality of the exterior derivative. For $g \in G$, we have $p \circ \lambda_g = \ell_g \circ p$. If $\omega \in \Omega^k(G/H)^G$, then $(\ell_g)^*\omega = \omega$ for all $g \in G$, which implies $p^*\omega = p^*(\ell_g)^*\omega = (\lambda_g)^*p^*\omega$. Hence we see that $p^*\omega$ is a left invariant form on G, so $dp^*\omega = p^*d\omega$ is induced by $\partial\tilde{\varphi}$, where $\tilde{\varphi} = p^*\omega(e)$. Denoting by $\varphi : (\mathfrak{g}/\mathfrak{h})^k \to \mathbb{R}$ the map corresponding to ω, it is clear that $\tilde{\varphi}(X_1, \dots, X_k) = \varphi(X_1 + \mathfrak{h}, \dots, X_k + \mathfrak{h})$. Hence we see that for an H–equivariant map φ, the map $\partial\tilde{\varphi}$ has to descend to an H–equivariant map $(\mathfrak{g}/\mathfrak{h})^{k+1} \to \mathbb{R}$.

Alternatively, it is a good exercise to verify directly the fact that $\partial\tilde{\varphi}$ descends to an equivariant map directly. The main step toward this is observing that the infinitesimal version of equivariancy of φ is that for $X \in \mathfrak{h}$ and $X_1, \dots, X_k \in \mathfrak{g}$, one gets

$$0 = \textstyle\sum_{i=1}^k \varphi(X_1 + \mathfrak{h}, \dots, [X, X_i] + \mathfrak{h}, \dots, X_k + \mathfrak{h}),$$

which then implies that $\partial\tilde{\varphi}$ vanishes upon insertion of one element of $\mathfrak{h}$, and hence descends to a map $(\mathfrak{g}/\mathfrak{h})^{k+1} \to \mathbb{R}$. $\square$

EXAMPLE 3.5. Let G be any Lie group with Lie algebra $\mathfrak{g}$. Then we can consider the adjoint representation $\mathrm{Ad} : G \to \mathfrak{gl}(\mathfrak{g})$, and the corresponding dual representation $\mathrm{Ad}^* : G \to \mathfrak{gl}(\mathfrak{g}^*)$, which is usually called the *coadjoint representation.* If G is compact or semisimple, then there is a non–degenerate, G–invariant bilinear form on $\mathfrak{g}$ and thus $\mathfrak{g} \cong \mathfrak{g}^*$ as a representation of G, but this is not important here.

By a *coadjoint orbit* of G, one means one of the G–orbits in $\mathfrak{g}^*$. Given $\lambda \in \mathfrak{g}^*$, we can consider the isotropy group $G_\lambda = \{g \in G : \lambda \circ \mathrm{Ad}(g) = \lambda\}$ and the orbit $G \cdot \lambda$ can be identified with the homogeneous space G/G_λ. The importance of the coadjoint orbits lies in the fact that they admit a canonical G–invariant *symplectic structure*, i.e. a G–invariant 2–form ω, such that for each x, the value $\omega(x) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is non–degenerate as a bilinear form and such that $d\omega = 0$.

Symplectic structure in this sense form the mathematical framework for the Hamiltonian formulation of classical mechanics. Given a symplectic manifold (M, ω) and a smooth function $H : M \to \mathbb{R}$ (the "Hamiltonian"), we can consider the one–form $dH \in \Omega^1(M)$. Non–degeneracy of ω then implies that there is a unique field $X_H \in \mathfrak{X}(M)$ such that $dH(\xi) = \omega(X_H, \xi)$ for all $\xi \in TM$. This is the Hamiltonian vector field governing the evolution of the system in time. Symmetries of such a system can be encoded into an action of a Lie group G on M which is compatible with both the Hamiltonian H and the symplectic from ω. Via so–called moment maps and symplectic reduction, coadjoint orbits of G (with their canonical symplectic structure) can be used to describe G–orbits on M. Hence coadjoint orbits provide the fundamental examples of mechanical systems with symmetries.

So let us assume that we have given G and $0 \neq \lambda \in \mathfrak{g}^*$. Let us first describe the Lie algebra $\mathfrak{g}_\lambda$ of the stabilizer G_λ. Differentiating $(\exp(tX) \cdot \lambda)(Y) = \lambda(\mathrm{Ad}(\exp(-tX)(Z)))$ we get $(X \cdot \lambda)(Y) = -\lambda([X, Y])$, so $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : \lambda \circ \mathrm{ad}(X) = 0\}$. Otherwise put, $\mathfrak{g}_\lambda$ is exactly the null–space of the skew symmetric bilinear form $\partial\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Thus we conclude that $\partial\lambda$ descends to a non–degenerate, skew symmetric bilinear map $\mathfrak{g}/\mathfrak{g}_\lambda \times \mathfrak{g}/\mathfrak{g}_\lambda \to \mathbb{R}$. (This in particular implies that the dimension of $\mathfrak{g}/\mathfrak{g}_\lambda$ must be even, which is not clear from the beginning.) For $h \in G_\lambda$ we further get

$$\begin{aligned}(h \cdot \partial\lambda)(X, Y) &= \partial\lambda(\mathrm{Ad}(h^{-1})(X), \mathrm{Ad}(h^{-1})(Y)) \\ &= \lambda([\mathrm{Ad}(h^{-1})(X), \mathrm{Ad}(h^{-1})(Y)]) = \lambda(\mathrm{Ad}(h^{-1})([X, Y])) = \partial\lambda(X, Y).\end{aligned}$$

This implies that the descended map is G_λ–equivariant and thus induces a G–invariant 2–form $\omega \in \Omega^2(G/G_\lambda)^G$, which by construction is non–degenerate in each point. This is called the *Kirillov–Kostant–Souriau form* or *KKS–form* on the coadjoint orbit.

To see that $d\omega = 0$, we can either directly compute via Proposition 3.5. Alternatively, on can observe that the construction can be rephrased as follows. The functional λ induces a left–invariant one–form $\alpha \in \Omega^1(G)^G$. While α itself does not descend to G/G_λ, its exterior derivative $d\alpha$ descends and $d\alpha = p^*\omega$. But this readily implies that $0 = dd\alpha = dp^*\omega = p^*d\omega$ and by construction p^* is injective, so $d\omega = 0$.

3.6. Invariant cohomology. In certain cases, invariant differential forms make it possible to compute the de Rham cohomology of homogeneous spaces and, more generally, of manifolds endowed with group actions. Recall that the exterior derivative d on differential forms satisfies $d^2 = d \circ d = 0$. Hence for any smooth manifold M and each $k \in \mathbb{N}$, the space $B^k(M) := \{d\beta : \beta \in \Omega^{k-1}(M)$ of *exact* k–forms on M is contained in the space $Z^k(M) := \{\alpha \in \Omega^k(M) : d\alpha = 0\}$ of closed k–forms on M. The quotient space $H^k(M) := Z^k(M)/B^k(M)$ is called the kth *de Rham cohomology* of M.

The de Rham cohomology groups (which are vector spaces) are fundamental topological invariants of M.

If M is endowed with a smooth action $G \times M \to M$ of a Lie group G, there is an obvious notion of G–invariant differential forms. One calls $\alpha \in \Omega^k(M)$ invariant, if and only if $(\ell_g)^*\alpha = \alpha$ for all $g \in G$, and one writes $\Omega^k(M)^G$ for the space of of G–invariant k–forms. If M is the homogeneous space G/H, then of course this coincides with the definition from 3.2. As in Section 3.5 it follows that also in this more general setting the exterior derivative of an invariant form is invariant, too. Hence one defines the *invariant cohomology* $H^k_G(M)$ of M as the quotient of the space $Z^k(M)^G$ of closed invariant k–forms on M by the subspace $d(\Omega^{k-1}(M)^G)$.

Now suppose that G is a compact Lie group, $H \subset G$ is a closed subgroup and consider the homogeneous space G/H. Then $\Lambda^k T^*M = G \times_H \Lambda^k(\mathfrak{g}/\mathfrak{h})^*$, and hence by Corollary 2.8, we can identify $\Omega^k(M)$ with the space $C^\infty(G, \Lambda^k(\mathfrak{g}/\mathfrak{h})^*)^H$ of H–equivariant smooth functions. We have also seen in 3.2 that in this picture the G–action is given by $(g \cdot f)(\tilde{g}) = f(g^{-1}\tilde{g})$. Similarly as in the construction of invariant inner products on representations of G in Section 3.4 of [**LG**], compactness of G implies that one can average such a function to obtain a smooth function $I(f) : G \to \Lambda^k(\mathfrak{g}/\mathfrak{h})^*$ which is still H–equivariant but also satisfies $g \cdot I(f) = I(f)$ for all $g \in G$. Otherwise put, one obtains an operator $I : \Omega^k(G/H) \to \Omega^k(G/H)^G$. Using a so–called fiber integral, one similarly defines an operator mapping forms to invariant forms on general manifolds endowed with an action of a compact Lie group G.

It turns out that for the action of a compact group, the invariant cohomology coincides with the de Rham cohomology. We will not give a full proof, but only sketch how the proof works in the special case of a homogeneous space of a compact Lie group.

THEOREM 3.6. *Let M be a smooth manifold endowed with an action of a compact Lie group G. Then the inclusions of invariant forms into all differential forms induces an isomorphism $H^k_G(M) \cong H^k(M)$ for all k.*

SKETCH OF PROOF. For $\alpha \in Z^k(M)^G$, we have $d\alpha = 0$, so we can form the class $[\alpha] \in H^k(M)$. By definition, this factorizes to a linear map $i_* : H^k_G(M) \to H^k(M)$. On the other hand, since G is compact, there is the the map $I : \Omega^k(M) \to \Omega^k(M)^G$.

From now on, we specialize to the case of a homogeneous space $M = G/H$, the analogous results in the general case are proved using properties of integration along the fiber. For $M = G/H$, the operator I is induced by averaging the functions $g \cdot f : G \to \Lambda^k(\mathfrak{g}/\mathfrak{h})^*$ over $g \in G$. As in Section 3.5, the function f corresponding to a form α is explicitly given by

$$f(\tilde{g})(X_1 + \mathfrak{h}, \dots, X_k + \mathfrak{h}) = p^*\alpha(L_{X_1}, \dots, L_{X_k})(\tilde{g}).$$

Using this it is easy to directly compute the function corresponding to $d\alpha$ in terms of f and its derivatives with respect to left invariant vector fields. Using this, one in turn easily verifies that $dI(\alpha) = I(d\alpha)$ holds for all forms α. Hence for a closed form α, also $I(\alpha) \in \Omega^k(M)^G$, so we can map α to the class $[I(\alpha)] \in H^k_G(M)$. As above, it is easy to see that this descends to a well defined linear map $I_* : H^k(M) \to H^k_G(M)$.

By construction I is the identity on invariant forms, so $I_* \circ i_*$ is the identity on $H^k_G(M)$. Hence the proof can be completed by showing that $i_* \circ I_*$ is the identity on $H^k(M)$, which implies that the two maps are inverse isomorphisms. To do this, one constructs a so–called chain homotopy $h : \Omega^k(M) \to \Omega^{k-1}(M)$, which has the property that for all $\alpha \in \Omega^k(M)$ one has $\alpha - I(\alpha) = dh(\alpha) + h(d\alpha)$. This readily implies that for $\alpha \in Z^k(M)$, one has $\alpha = I(\alpha) + dh(\alpha)$, so α and $I(\alpha)$ represent the same class in

$H^k(M)$. The construction of h is beyond the scope of this course, it needs several facts on integration on manifolds and on de Rham cohomology. □

In some cases, this result can be used to determine the de Rham cohomology of a manifold almost without computation. For example, consider the sphere S^n as a homogeneous space of the compact Lie group $SO(n+1)$. Then $\mathfrak{g}/\mathfrak{h} \cong \mathbb{R}^n$ as a representation of $H = SO(n)$. Now $\Lambda^0\mathbb{R}^{n*}$ and $\Lambda^n\mathbb{R}^{n*}$ are trivial representations, but it is well known (and not difficult to prove directly) that for $1 \le k < n$, there are no H–invariant elements in the $\Lambda^k\mathbb{R}^{n*}$. Hence invariant forms are only available in degree 0 and n, so we readily see that $H^k(S^n) = H^k_G(S^n)$ is isomorphic to $\mathbb{R}$ for $k = 0, n$ and zero for all other degrees.

The results on invariant cohomology are also interesting in the case of a compact Lie group G viewed as a homogeneous space $G = G/\{e\}$ where one deals with left invariant forms. There is a second very interesting way to view a compact Lie group G as a homogeneous space, however. Namely, consider $G \times G$ acting on G by multiplications from both sides, i.e. $(g, h) \cdot \tilde{g} := g\tilde{g}h^{-1}$. Evidently, this action is transitive and the stabilizer of e is $G_\Delta := \{(g, g) : g \in G\} \cong G$. Hence we can realize G as $(G \times G)/G_\Delta$, and this leads to the concept of *bi–invariant forms*, i.e. $\Omega^k(G)^{G\times G}$ consists of those forms $\varphi \in \Omega^k(G)$, for which both $(\lambda_g)^*\varphi$ and $(\rho^g)^*\varphi$ coincide with φ for all $g \in G$.

Of course, the Lie algebra of $G \times G$ is $\mathfrak{g} \oplus \mathfrak{g}$ with component–wise operations and the Lie algebra of the isotropy group is identified with $\{(X, X) : X \in \mathfrak{g}\} =: \mathfrak{g}_\Delta$. The quotient $(\mathfrak{g} \oplus \mathfrak{g})/\mathfrak{g}_\Delta$ is linearly isomorphic to $\mathfrak{g}$ with an isomorphism induced by $(X, Y) \mapsto X - Y$. Moreover, the adjoint action of $G \times G$ on $\mathfrak{g} \oplus \mathfrak{g}$ is of course given by $\mathrm{Ad}(g, h)(X, Y) = (\mathrm{Ad}(g)(X), \mathrm{Ad}(h)(Y))$. This easily shows that under the isomorphism with $\mathfrak{g}$, the action $\underline{\mathrm{Ad}}$ of G_Δ on $(\mathfrak{g} \oplus \mathfrak{g})/\mathfrak{g}_\Delta$ simply corresponds to the adjoint action of G on $\mathfrak{g}$. Hence Theorem 3.3 tells us that bi–invariant k–forms on G are in bijective correspondence with elements in $(\Lambda^k\mathfrak{g}^*)^G$, with the G–action being induced by the adjoint action of G on $\mathfrak{g}$.

To formulate the next result, we just need one more concept. Recall that the map $\partial : \Lambda^k\mathfrak{g}^* \to \Lambda^{k+1}\mathfrak{g}^*$ has the property that $\partial \circ \partial = 0$. Hence for each k, the space $B^k(\mathfrak{g}) := \partial(\Lambda^{k-1}\mathfrak{g}^*)$ is a subspace of $Z^k(\mathfrak{g}) := \{\alpha \in \Lambda^k\mathfrak{g}^* : \partial\alpha = 0\}$. The quotient space $H^k(\mathfrak{g}) := Z^k(\mathfrak{g})/B^k(\mathfrak{g})$ is called the kth *Lie algebra cohomology* group (or Chevalley–Eilenberg cohomology group) of $\mathfrak{g}$ (with coefficients in the trivial representation). This is computable (in principle) by linear algebra. Using this, we now formulate:

Corollary 3.6. *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Then*

(1) The de Rham cohomology $H^*(G)$ is isomorphic to the Lie algebra cohomology $H^*(\mathfrak{g})$.

(2) The de Rham cohomology $H^k(G)$ is isomorphic to the space $\Omega^k(G)^{G\times G}$ of bi–invariant k–forms on G.

Proof. (1) View G as the homogeneous space $G/\{e\}$. Then by Theorem 3.3, $\Omega^k(G)^G \cong \Lambda^k\mathfrak{g}^*$, and the exterior derivative on left invariant forms is induced by ∂. Thus the result is an immediate consequence of theorem 3.6.

(2) We view G as the homogeneous space $(G\times G)/G_\Delta$ and we claim that the exterior derivative is trivial on bi–invariant differential forms. In the quotient $(\mathfrak{g} \oplus \mathfrak{g})/\mathfrak{g}_\Delta$, an element (X, Y) is congruent to $\frac{1}{2}(X - Y, Y - X)$ and hence to an element of the form $(Z, -Z)$. But for two such elements the bracket is given by $[(Z, -Z), (W, -W)] = ([Z, W], [Z, W]) \in \mathfrak{g}_\Delta$. This already implies that ∂ has to descend to the zero map on $(\Lambda^k((\mathfrak{g} \oplus \mathfrak{g})/\mathfrak{g}_\Delta)^*)^G$. □

3.7. Example: Special Riemannian structures on spheres. Let us illustrate the applications of Theorem 3.3 by discussing various ways to make spheres into homogeneous spaces and discuss some related examples. This gives rise to several special geometries on spheres of appropriate dimensions, which all include the round Riemannian metric as part of the structure. It is interesting to observe that from the point of view of these special structures, the sphere is different from Euclidean and hyperbolic space as discussed in Example 3.4 (1).

To start the discussion, we remark that the presentation of S^n as the homogeneous space $O(n+1)/O(n)$ is equivalent to the round Riemannian metric. This can be seen from the fact that $O(n)$ coincides with the group of isometries of the metric. More generally, one may even consider two connected open subsets $U, V \subset S^n$ with the induced Riemannian metrics. Then it turns out that any isometry $f : U \to V$ of these Riemannian metrics is the restriction of the actions of a unique element of $O(n+1)$. We will discuss this in more detail when having invariant connections at hand.

Now suppose that $G \subset O(n+1)$ is a subgroup such that the restriction of the $O(n+1)$–action on S^n to G is still transitive. Then we can define $H := G \cap O(n)$ and get an identification $S^n \cong G/H$. We will interpret this as specifying an additional geometric structure to the round metric such that G consists exactly of those isometries of S^n which preserve this additional structure, although we will not always prove this fact.

At the first glance, it may look like there could be lots of appropriate subgroups $G \subset O(n+1)$ which act transitively on S^n. However, it turns out that the groups with these properties can be completely classified, and that the list is not too long. The first step towards this is that transitivity of the action on S^n implies that $\mathbb{R}^{n+1}$ is an irreducible representation of G, which already strongly restricts the structure of G. The most obvious choice for G is $SO(n+1)$, and for this it is also easy to see the structure that is preserved: As a submanifold of $\mathbb{R}^{n+1}$ with a global unit normal field, S^n is orientable, and the actions of elements of $SO(n+1)$ are exactly the orientation–preserving isometries of S^n.

The Sasakian sphere. The next obvious choice for G only exists for odd values of n. Putting $n = 2m-1$, we can view S^{2m-1} as the unit sphere in $\mathbb{C}^m$ and the standard inner product on that space as the real part of the standard Hermitian inner product. This leads to an inclusion $U(m) \hookrightarrow SO(2m)$ and linear algebra shows that $U(m)$ acts transitively on the unit sphere S^{2m-1}. Since $U(m)$ consists of complex linear maps, an element $A \in U(m)$ which stabilizes a unit vector, say e_m, acts as the identity on the whole complex line spanned by that vector. Linear algebra then tells us that the stabilizer of e_m is identified with $U(m-1)$ via the action on the complex orthocomplement of e_m. Hence we get a presentation of S^{2m-1} as the homogeneous space $U(m)/U(m-1)$.

Now the tangent space $T_{e_m}S^{2m-1}$ is the *real* orthocomplement of e_m, so as a representation of $U(m-1)$, this is isomorphic to a trivial representation on $\mathbb{R}$ (spanned by ie_m) and the standard representation $\mathbb{C}^{m-1}$. In particular, the $U(m-1)$–invariant element ie_m gives rise to a $U(m)$–invariant vector field $\xi \in \mathfrak{X}(S^{2m-1})^{U(m)}$ of length 1. Likewise, the subspace $\mathbb{C}^{m-1}$ corresponds to a smooth subbundle $H \subset TS^{2m-1}$ with fibers of real codimension one, which is invariant under the action of $U(m)$. The fact that this is indeed a complex vector subbundle can be most easily encoded as follows: Multiplication by i defines a $U(m-1)$–invariant element in $L_{\mathbb{R}}(\mathbb{C}^{m-1}, \mathbb{C}^{m-1})$. This gives rise to a $U(m)$–invariant section of $L(H, H)$, which can be equivalently interpreted as a $U(m)$–invariant bundle map $J : H \to H$ such that $J \circ J = -\operatorname{id}_H$. Finally, the property that the standard inner product is Hermitian with respect to J implies that the same holds for the round metric, i.e. $g(J\eta_1, J\eta_2) = g(\eta_1, \eta_2)$ for all $\eta_1, \eta_2 \in H$. We can uniquely extend J to a $\binom{1}{1}$–tensor field $\tilde{J}$ on S^{2m-1} by requiring $\tilde{J}(\xi) = 0$.

To understand the subbundle H better, let us pass from the invariant vector field ξ to the dual one–form $\alpha \in \Omega^1(S^{2m-1})^{U(m)}$. This corresponds to the linear functional $v \mapsto \operatorname{Re}(\langle v, ie_m\rangle)$ on the real orthocomplement of e_m. In particular, H is the (point–wise) kernel of α. To compute the exterior derivative $d\alpha$, we have to make the description of $\mathfrak{h} \subset \mathfrak{g}$ more explicit. In terms of complex matrices, $\mathfrak{g}$ consists of all matrices of the form $\begin{pmatrix} X & v \\ -v^* & ir \end{pmatrix}$, where $X \in M_{m-1}(\mathbb{C})$ satisfies $X^* = -X$, $v \in \mathbb{C}^{m-1}$ is any vector, and r is any real number. The subalgebra $\mathfrak{h}$ corresponds to the elements with $v = 0$ and $r = 0$, which readily shows that $\mathfrak{g}/\mathfrak{h} \cong \mathbb{R} \oplus \mathbb{C}^m$. Using this, we can directly compute that the function inducing $d\alpha$ vanishes if one of its entries is from the $\mathbb{R}$–factor and coincides with the imaginary part of the standard inner product on $\mathbb{C}^{m-1}$, so it is non–degenerate there. This shows that $d\alpha$ restricts to a non–degenerate bilinear form on $H = \ker(\alpha)$, so α is a so–called *contact form* and $H \subset TS^{2m-1}$ is a *contact distribution.*

Now one can turn things around and view the one–form α and the $\binom{1}{1}$–tensor field $\tilde{J}$ (which satisfy appropriate compatibility conditions) as the main ingredients of the geometry. From these ingredients, the remaining data can be recovered as follows: The distribution H is the kernel of α and the compatibility conditions mentioned above imply that $\tilde{J}$ can be restricted to $J : H \to H$ such that $J \circ J = -\operatorname{id}$. Next, it turns out that ξ is uniquely determined by that facts that $\alpha(\xi) = 1$ and $i_\xi d\alpha = 0$. Finally, the (round) Riemannian metric on S^{2m-1} can be recovered as a combination of $\alpha \vee \alpha$ and $(\eta_1, \eta_2) \mapsto d\alpha(\eta_1, \tilde{J}(\eta_2))$. This structure is called the canonical *Sasaki structure* on an odd dimensional sphere.

As a small variation, we can also use $G = SU(m) \subset SO(2m)$ and then obtain $H = SU(m-1)$ and hence $S^{2m-1} \cong SU(m)/SU(m-1)$. Geometrically, this can be understood as the canonical Sasaki structure plus an additional choice of a "complex volume form". Since H is a complex vector bundle (with respect to J), we can form the top complex exterior power $\Lambda^{m-1}_{\mathbb{C}} H$. This is a complex line bundle induced by the one–dimensional representation $\Lambda^{m-1}\mathbb{C}^{m-1}$ of $U(m)$, which is via the complex determinant. Hence we see that, up to a constant complex multiple, there is a unique $SU(m)$–invariant section of $\Lambda^{m-1}_{\mathbb{C}} H$, and $SU(m) \subset U(m)$ can be characterized as those automorphisms of the Sasaki structure which in addition preserve this section.

The Fubini Study metric. Viewing S^{2m-1} as a homogeneous space of $U(m)$ respectively $SU(m)$ is closely related to a structure on complex projective space $\mathbb{C}P^{m-1}$, which is of fundamental importance for several areas in mathematics. We start directly from $SU(m)$, since in contrast to $U(m)$, this leads to an effective action on $\mathbb{C}P^{m-1}$. As above, consider the standard representation of $G = SU(m)$ on $\mathbb{C}^m$, but now let $K \subset G$ be the stabilizer of the complex line spanned by the unit vector e_m. Then any element of K also stabilizes the complex orthocomplement of e_m and linear algebra shows that the action on the orthocomplement induces an isomorphism $K \cong U(m-1)$. (The action on the distinguished line is determined by the action on the orthocomplement, since the overall transformation is required to have complex determinant one.) Denoting by H the stabilizer of e_m, we get $H \cong SU(m-1)$ and $H \subset K$ and $K/H \cong U(1)$.

The quotient G/K can be identified with the space $\mathbb{C}P^{m-1}$ of one–dimensional complex subspaces in $\mathbb{C}^m$. The obvious projection $p : S^{2m-1} \to \mathbb{C}P^{m-1}$ mapping any unit vector to the complex line it spans is called the *Hopf fibration.* It is a $SU(m)$–equivariant map $G/H \to G/K$.

It is easy to see that the representation $\mathfrak{g}/\mathfrak{k}$ of $K = U(m-1)$ now is exactly the standard representation $\mathbb{C}^{m-1}$ of $U(m-1)$. In particular, the tangent bundle $T\mathbb{C}P^{m-1}$ now is a complex vector bundle, so one has an invariant $\binom{1}{1}$–tensor field J such that $J \circ J = -\operatorname{id}$. Moreover, the (real part of the) standard inner product on $\mathbb{C}^{m-1}$ induces an $SU(m)$–invariant Riemannian metric g on $\mathbb{C}P^{m-1}$ which is Hermitian with respect to J. This is called the *Fubini Study metric* on $\mathbb{C}P^{m-1}$. This can also be constructed from the round metric on S^{2m-1} via the Hopf fibration.

Now the imaginary part of the standard inner product defines a $U(m-1)$–invariant, skew symmetric bilinear map $\mathbb{C}^{m-1} \times \mathbb{C}^{m-1} \to \mathbb{R}$, so this gives rise to a $SU(m)$–invariant two–form ω on $\mathbb{C}P^{m-1}$. Up to a factor, this can be written as $\omega(\xi, \eta) = g(\xi, J\eta)$, so it is called the *fundamental two–form* of the Hermitian metric g. More generally, for $k = 2, \dots, m-1$ the k–fold wedge product ω^k defines a non–zero element of $\Omega^{2k}(\mathbb{C}P^{m-1})$ and up to a multiple, these are the only invariant forms on $\mathbb{C}P^{m-1}$, so in particular $d\omega^k = 0$ for all k. For $k = 1$, this shows that g is a *Kähler metric* on $\mathbb{C}P^{m-1}$. This also implies that J is really induced from the structure of a complex manifold on $\mathbb{C}P^{m-1}$, i.e. there are local charts with values in open subsets of $\mathbb{C}^{m-1}$ for which the chart changes are holomorphic maps. On the other hand, since there are no invariant forms of odd degree, we can readily read of the invariant cohomology and hence by Theorem 3.6 the de Rham cohomology of $\mathbb{C}P^{m-1}$. Namely, $H^k(\mathbb{C}P^{m-1})$ is isomorphic to $\mathbb{R}$ for $k = 0, 2, 4, \dots, 2m-2$ and zero in all other degrees, so $\mathbb{C}P^{m-1}$ is topologically significantly more complicated than a sphere.

The 3–Sasakian sphere. Let $\mathbb{H}$ be the skew–field of quaternions. The simplest construction of $\mathbb{H}$ is as the space of all complex 2×2–matrices of the form $\begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}$ for $z, w \in \mathbb{C}$. One immediately verifies that this subspace is closed under matrix multiplication, so it is a unital associative algebra. Moreover, a non–zero matrix of this form has determinant $|z|^2 + |w|^2 \neq 0$ and the inverse is of the same form. Hence each element of $\mathbb{H}$ admits a multiplicative inverse, so $\mathbb{H}$ is a skew–field (it has all properties of a field except of commutativity of multiplication). In fact, for such a matrix A, the inverse is given by $A^{-1} = \frac{1}{\det(A)} A^*$, and this equation is equivalent to the defining form for non–zero matrices.

It is easy to see that one can choose a basis of the real vector space $\mathbb{H}$ of the form $\mathbb{I} =: 1,\ i,\ j,\ k$ such that $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$ (from which all other multiplicative relations between these elements follow). This is the classical definition of the quaternions, for which associativity of the multiplication needs a non–trivial verification, however. Similarly as for $\mathbb{C}$, one can define a real part and a (three–dimensional) imaginary part of a quaternion, and a quaternionic version of conjugation. This satisfies $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$ and $q^{-1} = \frac{1}{|q|^2} \bar{q}$ for the obvious Euclidean norm.

The basics of linear algebra extend to skew–fields. However, one has to decide from which side scalars are multiplied onto vectors and to obtain the usual conventions for matrix multiplication, one has to view $\mathbb{H}^n$ as a *right* vector space over $\mathbb{H}$. Doing this, $\mathbb{H}$–linear maps $\mathbb{H}^n \to \mathbb{H}^n$ can be described via multiplication by quaternionic $n \times n$–matrices, with composition corresponding to the usual matrix multiplication. Moreover, there is a quaternionic version of (positive definite) Hermitian forms, and there is a unique (up to isomorphism) form of this type on $\mathbb{H}^n$. This is defined by

$$\langle (p_1, \dots, p_n), (q_1, \dots, q_n) \rangle = \textstyle\sum_\ell \bar{p}_\ell q_\ell,$$

and its real part is just the standard Euclidean inner product on $\mathbb{R}^{4n}$. Hence the group $Sp(n)$ of quaternionically linear isomorphisms of $\mathbb{H}^n$, which in addition preserve the quaternionic Hermitian form, is naturally a subgroup of $SO(4n)$ and it is easy to see that $SP(n)$ acts transitively on the unit sphere $S^{4n-1} \subset \mathbb{H}^n$. Any map which stabilizes a unit vector acts as the identity on the quaternionic line spanned by the unit vector. Via the action on the quaternionic orthocomplement, the stabilizer of such a vector is then identified with $Sp(n-1)$, so one gets an isomorphism $S^{4n-1} \cong Sp(n)/Sp(n-1)$. For $n = 1$, $Sp(1)$ is simply the set of quaternions of unit length, which clearly is isomorphic to S^3.

This now leads to a similar but slightly more complicated picture as in the complex case above. The tangent bundle TS^{4n-1} decomposes into a direct sum of a subbundle H of corank three and a three–dimensional trivial bundle spanned by three invariant vector fields corresponding to $e_n i$, $e_n j$ and $e_n k$. There is a corresponding three–dimensional space of invariant one–forms, whose common kernel equals H. The exterior derivatives of these forms can be computed similarly as above to give the (three–dimensional) imaginary part of the quaternionic Hermitian form. This shows that H is maximally non–integrable in a certain sense. Moreover, one gets three bundle maps $I, J, K : H \to H$ which satisfy the quaternion relations, so H defines a so–called *quaternionic contact structure*. Similarly as above, these can be extended to $\binom{1}{1}$–tensor fields $\tilde{I}$, $\tilde{J}$, and $\tilde{K}$, which together with the three invariant one–forms can be viewed as an equivalent encoding of the whole structure. This is called the standard 3*–Sasakian structure* on S^{4n-1}.

Quaternionic projective space. Quaternionic projective space $\mathbb{H}P^{n-1}$ is defined as the space of 1–dimensional quaternionic subspaces of $\mathbb{H}^n$. There is an evident transitive action of $G := Sp(n)$ on $\mathbb{H}P^{n-1}$, which identifies $\mathbb{H}P^{n-1}$ with G/K, where $K = Sp(n-1) \times Sp(1)$, thus making it into a smooth manifold. (This is slightly simpler than in the complex case, since there is no quaternionic determinant.) Moreover, there is an obvious $Sp(n)$–equivariant projection $\pi : S^{4n-1} \to \mathbb{H}P^{n-1}$, which maps any unit vector to the quaternionic line it spans. This map is called the *quaternionic Hopf fibration* and it turns out to be a principal fiber bundle with structure group $Sp(1) \cong SU(2)$, so as a manifold, this is diffeomorphic to S^3. The simplest case is $n = 2$, where it is easy to show that $\mathbb{H}P^1 \cong S^4$ so one obtains a quaternionic Hopf fibration $\pi : S^7 \to S^4$ with fiber S^3.

Working with matrices, one obtains $\mathfrak{g}/\mathfrak{k} \cong \mathbb{H}^{n-1}$. The action of K on this space is built up from $Sp(1)$ acting by quaternionic scalar multiplications and $Sp(n-1)$ acting by quaternionically linear maps. (Observe that non–comutativity of $\mathbb{H}$ implies that quaternionic scalar multiplications are not quaternionically linear.) This shows that the tangent spaces of $\mathbb{H}P^{n-1}$ are not quite quaternionic vector spaces, because of the presence of the $Sp(1)$–factor. There is a more complicated description of the resulting structure which is called the standard (almost) *quaternionic structure* on quaternionic projective space. Still the K–action preserves the standard (real) inner product on $\mathbb{H}^{n-1}$, so one obtains a Riemannian metric on $\mathbb{H}P^{n-1}$ which is nicely compatible with the quaternionic structure. This is called the standard *quaternion–Kähler metric*.

The subgroups of $O(n+1)$ acting transitively on S^n that we have met so far almost exhaust the list of all such subgroups. In addition to $Sp(n) \subset SO(4n)$, this list also contains the group $Sp(n)Sp(1)$ that we have met in the discussion of quaternionic projective space. However, this does not give rise to a "new" homogeneous geometry on S^n, since the whole $Sp(1)$–factor is contained in the isotropy subgroup of a point in S^{4n-1}, so the resulting homogeneous space can be naturally identified with $Sp(n)/Sp(n-1)$.

The rest of the list consists only of three isolated examples, each of which exists in only one dimension. The simplest of these examples is a group $G \subset SO(7)$ whose Lie algebra is the compact real form of the exceptional Lie algebra of type G_2. This is related to the fact that beyond the quaternions, there is also a division algebra $\mathbb{O}$ in dimension 8 called the *octonions*, which however is not associative. Still the automorphism of $\mathbb{O}$ is a Lie group, and it turns out that any automorphism is determined by its restriction to the 7–dimensional subspace of purely imaginary octonions. This leads to the group G used above. Correspondingly, one can make the sphere S^6 into a homogeneous space G/H. The resulting structure on S^6 is called the canonical *nearly Kähler structure*. Apart from the round metric, this includes an almost complex structure on S^6, which however does not come from the structure of a complex manifold. (The problem, whether S^6 can be made into a complex manifolds, was open for a long time. Recently, it was claimed that there is a proof that this is impossible, but it seems that this proof is not generally accepted so far.)

The remaining two groups in the list are the spin groups $Spin(7)$ and $Spin(9)$, i.e. the universal covering groups of $SO(7)$ and $SO(9)$, respectively. All spin groups have a special representation, called the spin–representation, which does not descend to SO. In these two examples, the spin representations have dimension 8 and 16, respectively, and carry an invariant inner product. It turns out that these two spin groups act transitively on the unit spheres of the spin representations, which gives an additional homogeneous geometry on S^7 and S^{15}, respectively.

3.8. Example: The conformal sphere and the CR–sphere. To contrast the last examples, we discuss to representations of spheres as homogeneous spaces which do not carry invariant Riemannian metrics but only a weaker structure, which has important geometric applications.

For $n \geq 3$ consider the space $\mathbb{R}^{n+2}$ endowed with the Lorentzian inner product

$$\langle (x_0, \dots, x_{n+1}), (y_0, \dots, y_{n+1}) \rangle := \textstyle\sum_{i=0}^{n} x_i y_i - x_{n+1} y_{n+1}.$$

For a vector x and $\lambda \in \mathbb{R}$, we get $\langle \lambda x, \lambda x \rangle = \lambda^2 \langle x, x \rangle$, so we see that the restriction of $\langle\ ,\ \rangle$ to a line can be either positive definite, or negative definite or zero. In the last case, the line is called *isotropic*. Now writing $x = (x', t)$ with $x' \in \mathbb{R}^{n+1}$ we see that $\langle x, x \rangle = |x'|^2 - t^2$ with the Euclidean norm in the first summand. This shows that for any isotropic vector x, the t–coordinate has to be non–zero, and any isotropic line in $\mathbb{R}^{n+2}$ contains a unique point of the form $(x', 1)$ with $x' \in S^n \subset \mathbb{R}^{n+1}$. This shows that we can identify S^n with the space of isotropic lines in $\mathbb{R}^{n+2}$.

Now let $G := SO_0(n+1, 1) \subset GL(n+2, \mathbb{R})$ be the connected component of the identity of the (pseudo–) orthogonal group of $\langle\ ,\ \rangle$. Clearly, for any isotropic line ℓ and $A \in G$, also $A(\ell)$ is an isotropic line, so one obtains an action of G on S^n. Moreover, $SO(n+1)$ sits as a subgroup in G (as those maps fixing the last basis vector) and the restriction of the G–action to this subgroup leads to the standard action of $O(n+1)$ on S^n. Hence G acts transitively on S^n, so $S^n = G/H$, where H is the isotropy group of a point in S^n.

This isotropy group can be determined explicitly using some basic linear algebra. For an isotropic line $\ell \subset \mathbb{R}^{n+2}$ the orthogonal space $\ell^\perp = \{y : \forall x \in \ell \langle x, y \rangle = 0\}$ is a codimension–one subspace of $\mathbb{R}^{n+2}$ which contains ℓ. The restriction of $\langle\ ,\ \rangle$ to $\ell^\perp$ is degenerate with null–space $\ell \subset \ell^\perp$. Hence there is an induced inner product on $\ell^\perp/\ell$, which is easily seen to be positive definite and thus non–degenerate. Now for $A \in H$, we must have $A(\ell) \subset \ell$ and $A(\ell^\perp) \subset \ell^\perp$. It turns out that A can act by multiplication by any non–zero factor a on ℓ, but then for any $x \in \mathbb{R}^{n+2}$, one must have $Ax - a^{-1}x \in \ell^\perp$. Finally, the automorphism of $\ell^\perp/\ell$ induced by A must be orthogonal.

On the level of Lie algebras $\mathfrak{g}$ consists of all linear maps X on $\mathbb{R}^{n+2}$, which are skew symmetric with respect to $\langle\ ,\ \rangle$, and $\mathfrak{h}$ consists of those skew symmetric maps which preserve ℓ. Now for $X \in \mathfrak{g}$, skew symmetry implies that $X(\ell) \subset \ell^\perp$. Composing with the quotient projection, we can associate to X the linear map $\ell \to \ell^\perp/\ell$ induced by X. This defines a linear map $\mathfrak{g} \to L(\ell, \ell^\perp/\ell)$ which vanishes on $\mathfrak{h}$ by construction. One verifies that this actually induces a linear isomorphism $\mathfrak{g}/\mathfrak{h} \to L(\ell, \ell^\perp/\ell)$. Fixing an element of ℓ, we can identify that space with $\ell^\perp/\ell$ and one can describe the action of $A \in H$ on that space explicitly. In the notation from above, it is given by a^{-1} times the orthogonal endomorphism of $\ell^\perp/\ell$ induced by A. This easily implies that one obtains a surjection from H onto the group $CSO(\ell^\perp/\ell)$ of all endomorphisms B of $\ell^\perp/\ell$ such that B^*B is a positive multiple of the the identity. This is the *conformal group* of $\ell^\perp/\ell \cong \mathbb{R}^n$. The kernel of this homomorphism is a normal subgroup of H isomorphic to the additive group $\mathbb{R}^n$ (essentially given by linear maps $\mathbb{R}^{n+2}/\ell^\perp \to \ell^\perp/\ell$ respectively $\ell^\perp/\ell \to \ell$).

The group $CO(n)$ evidently contains all multiples of the identity, so it is not compact. Thus Example (1) of 3.4 shows that there is no Riemannian metric on S^n which is invariant under the action of $SO(n+1,1)$. It turns out that the left actions of elements of $SO(n+1,1)$ are exactly the *conformal diffeomorphisms* of the round metric of S^n. Here a diffeomorphism $\varphi : S^n \to S^n$ is called conformal iff there is a positive smooth function $f : S^n \to \mathbb{R}_+$ such that $\varphi^*g = fg$. There is another "new" phenomenon showing up in this case, namely that the representation $\underline{\mathrm{Ad}} : H \to GL(\mathfrak{g}/\mathfrak{h})$ is not injective. This corresponds to the fact that there exist conformal diffeomorphisms of S^n which fix a point $x_0 \in S^n$ to first order, i.e. they satisfy $\varphi(x_0) = x_0$ and $T_{x_0}\varphi = \mathrm{id}$. For an isometry of a Riemannian metric on a connected manifold, these two properties already imply that it coincides with the identity, but this is no more true for conformal diffeomorphisms. The presentation of S^n as a homogeneous space of $SO(n+1,1)$ is one of the basic ingredients of conformal differential geometry.

One of the reasons why the conformal geometry of S^n is interesting is the relation to hyperbolic space $\mathcal{H}^{n+1}$ as discussed in Example 3.4. There we viewed $\mathcal{H}^{n+1}$ as $\{x \in \mathbb{R}^{n+2} : \langle x, x\rangle = -1\}$, thus exhibiting it as a homogeneous space of $G = SO_0(n+1,1)$, which is the isometry group of the hyperbolic metric on $\mathcal{H}^{n+1}$. Now each $x \in \mathcal{H}^{n+1}$ spans a line which is negative for $\langle\ ,\ \rangle$, and it is easy to see that an element of G that preserves that negative line already preserves the vector x. Hence we can also view $\mathcal{H}^{n+1}$ as the space of negative lines in $\mathbb{R}^{n+2}$. As above, any vector in a negative line has to have non–zero last coordinate, so any negative linear contains a unique point of the form $(x', 1)$. In contrast to above, we must have $\|x'\| < 1$ here, so we see that $\mathcal{H}^{n+1}$ is diffeomorphic to the open unit disk in $\mathbb{R}^{n+1}$ and the sphere S^n from above can be naturally identified with the boundary of that disk. However, from the point of view of Riemannian geometry, $\mathcal{H}^{n+1}$ is infinitely large, since the hyperbolic metric is complete. Hence one can view S^n as a "boundary at infinity" attached to $\mathcal{H}^{n+1}$ and any isometry of $\mathcal{H}^{n+1}$ "extends" uniquely to a conformal diffeomorphism of the boundary at infinity. This is a fundamental example of a conformally compact metric and of a compactification of a symmetric space.

Again there is a complex analog of this construction, which is very interesting. Here one starts from $\mathbb{C}^{n+2}$ endowed with a Lorentzian Hermitian form

$$\langle (z_0, \dots, z_{n+1}), (w_0, \dots, w_{n+1})\rangle := \textstyle\sum_{j=0}^n z_j \bar{w}_j - z_{n+1}\bar{w}_{n+1}.$$

Again the restriction of this form to a complex subspace of dimension one can be positive definite, negative definite, or zero, and we consider the space of isotropic complex lines in $\mathbb{C}^{n+2}$. As above, this can be identified with the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$, since any isotropic complex line contains a unique point of the form $(z', 1)$ with $|z'| = 1$. This leads to a transitive action of the special unitary group $SU(n+1,1)$ on S^{2n+1}.

The description of the isotropy group is parallel to the real case, but slightly more complicated. For a complex isotropic line $\ell \subset \mathbb{C}^{n+2}$, we have to consider the real orthocomplement $\ell^{\perp_{\mathbb{R}}}$ (i.e. the orthocomplement with respect to the real part of $\langle\ ,\ \rangle$), which of course contains the complex orthocomplement $\ell^{\perp}$, that in turn contains ℓ. Now for a point $z \in S^{2n+1}$ corresponding to an isotropic line ℓ, one obtains a natural identification of T_zS^{2n+1} with the quotient space $\ell^{perp_{\mathbb{R}}}/\ell$. This naturally contains $\ell^{\perp}/\ell$, which is a complex vector space of dimension n. The resulting spaces together with their complex structure coincide with the contact distribution $H \subset TS^{2n+1}$ which we have met in the description of the canonical Sasaki structure on the sphere in 3.7.

An element $A \in SU(n+1,1)$, which stabilizes ℓ, also stabilizes $\ell^{\perp_{\mathbb{R}}}$ and $\ell^{\perp}$, so it induces a (complex) linear automorphism of the space $\ell^{\perp}/\ell$, on which $\langle\ ,\ \rangle$ induces a positive definite Hermitian form. This defines a surjective homomorphism from the stabilizer of ℓ to the conformal unitary group $CU(n)$ of this form (i.e. the group generated by $U(n)$ and multiples of the identity). The kernel of this homomorphism turns out to be a nilpotent normal subgroup isomorphic to $\mathbb{C}^n \oplus \mathbb{R}$ with Lie algebra coming from the imaginary part of the standard positive definite Hermitian form on $\mathbb{C}^n$. This is called a *complex Heisenberg group*.

Using the description of the stabilizer, one easily verifies that in this case, there is no $SU(n+1,1)$–invariant complement to $H \subset TS^{2n+1}$. Indeed, it turns out that all the geometry preserved by $SU(n+1,1)$ on S^{2n+1} can be recovers from the contact distribution $H \subset TS^{2n+1}$ and its complex structure $J : H \to H$. The pair (H, J) is called the *standard CR–structure* on S^{2n+1}, with "CR" being an abbreviation for "Cauchy–Riemann". This structure is *strictly pseudoconvex*, which means that for any choice of $\alpha \in \Omega^1(S^{2n+1})$ the restriction of the exterior derivative $d\alpha$ to $H \times H$ is the imaginary part of a positive definite Hermitian form.

The name "Cauchy–Riemann" suggests a relation to complex analysis, and this is indeed the reason why CR structures are important. This is connected on the complex analog of the relation to hyperbolic space. As in the real case, we can consider the space of negative complex lines in $\mathbb{C}^{n+2}$ and each such line contains a unique point of the form $(z', 1)$ with $|z'| < 1$, so this space is identified with the open unit ball $D^{n+1} \subset C^{n+1}$ and the sphere S^{2n+1} is the boundary this domain. The CR structure on S^{2n+1} can be directly obtained from this description, since the tangent space T_zS^{2n+1} is a real hyperplane in $\mathbb{C}^{n+1}$ and H_z is exactly the maximal complex subspace in this real hyperplane. In the language of complex analysis, the condition on positive definiteness of the exterior derivative of a contact form is equivalent to the fact that the domain D^{n+1} is strictly pseudoconvex. As for the sphere we get a natural action of $SU(n+1,1)$ on D^{n+1} from the interpretation as negative lines. It turns out that the resulting transformations are exactly the *biholomorphisms* of D^{n+1}, i.e. the holomorphic diffeomorphisms for which also the inverse is holomorphic. So biholomorphisms of the domain D^{n+1} are equivalent to CR–diffeomorphisms of its boundary S^{2n+1}.

Indeed, for any strictly pseudoconvex domain $U \subset \mathbb{C}^{n+1}$ with smooth boundary $M = \partial U$, the boundary inherits a CR–structure $(H \subset TM, J)$ in a similar way. Studying the relation between properties of the domain and the CR geometry of its boundary is an important topic in complex analysis.

Let us finally mention that there is also a complex version of the hyperbolic metric and the interpretation of the sphere as a boundary at infinity. The space of negative lines in $\mathbb{C}^{n+2}$ naturally inherits a Riemannian metric g which is Hermitian with respect to the standard complex structure. This can be viewed as a metric on D^{n+1} (called the Bergmann metric in complex analysis), but this metric is complete, so the space is infinitely large from the metric point of view. In this picture, it is usually called *complex hyperbolic space* $\mathcal{H}^{n+1}_{\mathbb{C}}$ and the metric is called the complex hyperbolic metric. This is an example of a Kähler–Einstein metric, and the CR structure on S^{2n+1} can also be viewed as a complex version of the conformal infinity. Again, the study of boundaries at infinity for complete Kähler–Einstein metrics is an active area of current research.

Let us remark that there is also a quaternionic version of this construction, i.e. a transitive action of $Sp(n+1,1)$ on the unit sphere $S^{4n+3} \subset \mathbb{H}^{n+1}$. It turns out that the geometric structure preserved by this action is exactly the quaternionic contact structure $H \subset TS^{4n+3}$ from the discussion of the 3–Sasakian sphere in Section 3.7. This is a model for the quaternionic contact structure which the boundary at infinity of a quaternion–Kähler metrics inherits.

4. Connections

In this chapter, we briefly discuss the concept of connections in the language of linear connections on vector bundles and in the language of principal connections. Any connection has a basic invariant called its curvature, for linear connections on the tangent bundle of a manifold, there is an additional invariant called the torsion. In the case of a homogeneous space G/H, we then discuss existence of a G–invariant principal connection on $G \to G/H$, which leads to invariant connections on all associated bundles. In the case such a connection does not exist, we briefly discuss the classification of G–invariant linear connections on homogeneous vector bundles.

4.1. Linear connections and their curvature. Let M be a smooth manifold and let $p : E \to M$ be a vector bundle over M. Then a *linear connection* on E is a bilinear operator $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ written as $(\xi, s) \mapsto \nabla_\xi s$ such that for any smooth function $f : M \to \mathbb{R}$, one has $\nabla_{f\xi} s = f\nabla_\xi s$ and $\nabla_\xi(fs) = (\xi \cdot f)s + f\nabla_\xi s$.

The intuitive idea for such a connection is to define an analog of a directional derivative for sections of E. The direction is determined by the vector field ξ, and the property that $\nabla_{f\xi} s = f\nabla_\xi s$ implies that $\nabla_\xi s(x)$ depends only on the value of ξ in x. On the other hand, the Leibniz rule in s shows that s is differentiated once.

Theorem 4.1. *On any smooth vector bundle $p : E \to M$, there exists a linear connection ∇ and the space of all linear connections is an affine space modeled on the (infinite dimensional) vector space $\Gamma(T^*M \otimes L(E,E))$ of one–forms on M with values in $L(E,E)$.*

Proof. A smooth section A of the bundle $T^*M \otimes L(E,E)$ can be equivalently interpreted as defining a bilinear operator $A : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$, which is linear over smooth functions in both variables. This shows that for a linear connection ∇ on E, also $\hat{\nabla}_\xi s = \nabla_\xi s + A(\xi, s)$ defines a linear connection on E. Conversely, if ∇ and $\hat{\nabla}$ are linear connections on E, the define $A(\xi, s) := \hat{\nabla}_\xi s - \nabla_\xi s$. Evidently, this is linear over smooth functions in the first variable, but since the Leibniz rule is the same for both connections, it is also linear over smooth functions in the second variable. Hence A defines a smooth section of $T^*M \otimes L(E,E)$, and it only remains to prove existence of one linear connection.

To do this, let us first consider the case that $E = M \times \mathbb{R}^n$. Then a section of E is a smooth function $s = (s_1, \dots, s_n) : M \to \mathbb{R}^n$ and we can simply define $\nabla_\xi s := (\xi \cdot s_1, \dots, \xi \cdot s_n)$. One immediately verifies that this defines a linear connection on E. In the general case, take a vector bundle atlas $(U_\alpha, \varphi_\alpha)$ for E and for sections $s \in \Gamma(E)$ with support in U_α and $x \in U_\alpha$ define $\nabla^\alpha_\xi s(x) := (\varphi_\alpha)^{-1}(x, (\xi \cdot s_1(x), \dots, \xi \cdot s_n(x)))$, where $\varphi_\alpha \circ s(x) = (x, (s_1(x), \dots, s_n(x)))$. Further define $\nabla^\alpha_\xi s(y) = 0$ for $y \notin U_\alpha$. As before, this is linear over smooth maps and satisfies $\nabla^\alpha_\xi (fs) = (\xi \cdot f)s + f\nabla^\alpha_\xi s$ on all of M (if s has support in U_α). Now take a partition (f_α) of unity subordinate to the covering U_α of M and define $\nabla_\xi s := \sum_\alpha \nabla^\alpha_\xi (f_\alpha s)$. Then this is smooth since the sum is locally finite and one immediately verifies that it defines a linear connection on E. □

The defining properties of a linear connection show that there is a way to construct a tensorial quantity out of a connection, which is called its *curvature*. This turns out to be a fundamental invariant of a linear connection.

PROPOSITION 4.1. *Let $p : E \to M$ be a vector bundle and ∇ be a linear connection on E. Then there is a unique sections $R \in \Gamma(\Lambda^2 T^*M \otimes L(E, E))$ which is characterized by the fact that for $\xi, \eta \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$, one has*

$$R(\xi, \eta)(s) = \nabla_\xi \nabla_\eta s - \nabla_\eta \nabla_\xi s - \nabla_{[\xi,\eta]} s.$$

PROOF. The defining expression for R evidently is trilinear over $\mathbb{R}$ and skew symmetric in ξ and η, so the complete the proof we only have to shows that, viewed as an operator $\mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$, it is linear over smooth functions in all variables. By skew symmetry, it suffices to check this for η and s. Now for $f \in C^\infty(M, \mathbb{R})$, we get $[\xi, f\eta] = (\xi \cdot f)\eta + f[\xi, \eta]$. Using this and the defining properties of ∇, we conclude that $R(\xi, f\eta)(s) - fR(\xi, \eta)(s)$ is given by $(\xi \cdot f)\nabla_\eta s - \nabla_{(\xi \cdot f)\eta} s = 0$.

On the other hand, $\nabla_\xi \nabla_\eta (fs) = \nabla_\xi (f\nabla_\eta s + (\eta \cdot f)s)$, which implies that $\nabla_\xi \nabla_\eta (fs) - f\nabla_\xi \nabla_\eta s$ can be written as

$$(\xi \cdot f)\nabla_\eta s + (\eta \cdot f)\nabla_\xi s + (\xi \cdot (\eta \cdot f))s$$

Subtracting the same expression with ξ and η exchanged, the first two summands cancel, and then subtracting $\nabla_{[\xi,\eta]} fs - f\nabla_{[\xi,\eta]} s = ([\xi, \eta] \cdot f)s$ we get zero by definition of the Lie bracket. □

Giving a detailed geometric interpretation of the curvature of a connection is beyond the scope of this course, since it would require to develop quite a bit of theory on concepts like parallel transport.

4.2. Affine connections and their torsion. An *affine connection* on a smooth manifold M is just a linear connection ∇ on the tangent bundle $p : TM \to TM$, so this is like an abstract version of directional derivative for vector fields. Apart from the curvature R of ∇, which in this case can be interpreted as a $\binom{1}{3}$–tensor field on M, there is a second fundamental invariant of a such a connection, which is called the *torsion* of ∇ and which is even simpler than the curvature.

PROPOSITION 4.2. *Let M be a smooth manifold and let ∇ be an affine connection on M. Then there is a smooth section T of the bundle $\Lambda^2 T^*M \otimes TM$ characterized by the fact that for $\xi, \eta \in \mathfrak{X}(M)$, we have*

$$T(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta].$$

PROOF. The defining equation for T is evidently bilinear over $\mathbb{R}$ and skew–symmetric in ξ and η, so by Proposition 2.3 it suffices to verify that it is linear over $C^\infty(M, \mathbb{R})$ in

η to complete the proof. But this follows readily from the defining properties of ∇ and the fact that $[\xi, f\eta] = (\xi \cdot f)\eta + f[\xi, \eta]$. □

An affine connection ∇ on M is called *torsion–free* if its torsion vanishes identically. It is easy to see that on any smooth manifold there exist torsion–free affine connections. Indeed, starting from an arbitrary affine connection $\hat{\nabla}$ with torsion T, one may simply define $\nabla_\xi \eta := \hat{\nabla}_\xi \eta - \frac{1}{2}T(\xi, \eta)$. This is a linear connection on TM by Theorem 4.1 and one immediately verifies that its torsion vanishes identically.

A fundamental result of Riemannian geometry is that given a Riemannian metric γ on M, there is a unique torsion–free affine connection ∇ on TM which is compatible with the metric γ in the sense that $\xi \cdot \gamma(\eta, \zeta) = \gamma(\nabla_\xi \eta, \zeta) + \gamma(\eta, \nabla_\xi \zeta)$ for all vector fields $\xi, \eta, \zeta \in \mathfrak{X}(M)$. This is called the Levi–Civita connection of the metric γ.

4.3. Connections and constructions with vector bundles. Given a linear connection on a vector bundle $p : E \to ME$, there are corresponding linear connections on bundles obtained by natural constructions from E. Likewise, this works with constructions involving several bundles. We only sketch this briefly, because it all can be deduced via principal connections and induced connections.

As a simple example, consider the dual bundle E^*. Fix a section $\lambda \in \Gamma(E^*)$ and a vector field $\xi \in \mathfrak{X}(M)$. Then for a section $s \in \Gamma(E)$ we can form $\lambda(s) \in C^\infty(M, \mathbb{R})$ and consider the operator $\Gamma(E) \to C^\infty(M, \mathbb{R})$ defined by $s \mapsto \xi \cdot (\lambda(s)) - \lambda(\nabla_\xi s)$, where ∇ is the given linear connection on E. Then this is immediately seen to be linear over smooth functions, thus defining a section $\nabla^*_\xi \lambda \in \Gamma(E^*)$. From the definitions, one then verifies that ∇^* indeed defines a linear connection on E^*.

Similarly, suppose we have given two bundles E and F over the same manifold M and linear connections ∇^E and ∇^F on the bundles. Then we can define connections $\nabla^{E\oplus F}$ on $E \oplus F$ and $\nabla^{E\otimes F}$ on $E \otimes F$ characterized by $\nabla^{E\oplus F}_\xi (s_1, s_2) = (\nabla^E_\xi s_1, \nabla^F_\xi s_2)$ and by $\nabla^{E\otimes F}_\xi (s_1 \otimes s_2) = (\nabla^E_\xi s_1) \otimes s_2 + s_1 \otimes (\nabla^F_\xi s_2)$, respectively. Together with the above, we can use the isomorphism $L(E, F) \cong E^* \otimes F$ to obtain a linear connection $\nabla^{L(E,F)}$ which is characterized by $(\nabla^{L(E,F)}_\xi f)(s_1) = \nabla^F_\xi (f(s_1)) - f(\nabla^E_\xi s_1)$, and so on.

In all these cases, it is also possibly to directly compute the action of the curvature. For example, for the curvature R^* of the dual connection ∇^* corresponding to ∇ on E, one obtains $(R^*(\xi, \eta)(\lambda))(s) = -\lambda(R(\xi, \eta)(s))$, where R is the curvature of ∇.

4.4. Principal connections. Let $p : E \to M$ be any fiber bundle. Then in the tangent bundle TE, there is a natural subbundle VE, called the *vertical subbundle*. This is defined by $V_yE := \ker(T_yp) \subset T_yE$, so intuitively, it consists of those tangent vectors at y which are tangent to the fiber E_x, where $x = p(y)$. Correspondingly, one calls a vector field $\xi \in \mathfrak{X}(E)$ *vertical* if its values always lie in the vertical subbundle. Dually to the concept of vertical vector fields, there is the concept of *horizontal differential forms* on a fiber bundle. Here we call a form horizontal if and only if it vanishes upon insertion of any vertical vector field.

Since the bundle projection $p : E \to M$ is a submersion, T_yp induces a linear isomorphism $T_yE/V_yE \to T_{p(y)}M$. Now a (general) connection on the fiber bundle $p : E \to M$ is given by the choice of a distribution $H \subset TE$, the *horizontal distribution* of the connection, which is complementary to VE. In the case of a principal fiber bundle, one in addition requires that this distribution is invariant under the principal right action:

DEFINITION 4.4. Let $p : P \to M$ be a principal fiber bundle with structure group G. Then a *principal connection* is given by a smooth distribution $H \subset TP$ such that

$T_uP = H_u \oplus V_uP$ and $T_ur^g \cdot H_u = H_{u\cdot g}$ hold for all $u \in P$ and all $g \in G$, with r^g denoting the principal right action by g.

A principal connection can be equivalently encoded as the so–called *vertical projection*, which is a smooth section $\Pi \in \Gamma(L(TP, VP))$ such that Π restricts to the identity on $VP \subset TP$. Given H, one defines $\Pi(u)$ as the projection onto the second summand in $T_uP = H_u \oplus V_uP$. Conversely, given Π, one defines $H_u := \ker(\Pi(u))$. The condition of equivariancy can also be phrased in the language of the vertical projection. Since $p \circ r^g = p$, we see that $T_u(r^g)(V_uP) = V_{u\cdot g}P$ holds for all $u \in P$ and $g \in G$, so one may simply require that $(r^g)^*\Pi = \Pi$ for all $g \in G$, in the sense that $\Pi(u \cdot g) \circ T_ur^g = T_ur^g \circ \Pi(u)$.

From this description, we can already deduce a fundamental technical result about principal connections:

Lemma 4.4. *Let $p : P \to M$ be a principal fiber bundle with structure group G and let $H \subset TP$ be a principal connection on P.*

Then for each vector field $\xi \in \mathfrak{X}(M)$, there is a unique vector field $\xi^h \in \mathfrak{X}(P)$ such that for all $u \in P$ we have $\xi^h(u) \in H_u \subset T_uP$ and $T_up \cdot \xi^h(u) = \xi(p(u))$, so ξ^h is p–related to ξ. Moreover, ξ is G–invariant in the sense that $(r^g)^*\xi^h = \xi^h$ for all $g \in G$.

Proof. For $u \in P$, we know that by definition H_u is complementary to $V_uP = \ker(T_up)$ in T_uP, so T_up restricts to a linear isomorphism $H_u \to T_{p(u)}M$. Thus given a vector field $\xi \in \mathfrak{X}(M)$, we get for each $x \in M$ and $u \in P$ with $p(u) = x$ a unique tangent vector $\xi^h(u) \in H_u \subset T_uP$ which projects onto $\xi(x)$. Moreover, for $g \in G$, we get $p \circ r^g = p$, so $T_{u\cdot g}p \cdot T_ur^g \cdot \xi^h(u) = \xi(x)$. Moreover, since T_ur^g maps H_u to $H_{u\cdot g}$, so we conclude that $T_ur^g \cdot \xi^h(u) = \xi^h(u \cdot g)$. To complete the proof, it thus remains to show that these tangent vectors fit together smoothly.

This may be proved locally on M, so choose a principal bundle chart (U, φ) with $x \in U$. Then φ is a diffeomorphism $p^{-1}(U) \to U \times G$ with $pr_1 \circ \varphi = p$. Hence also $T\varphi : TP|_{p^{-1}(U)} \to T(U \times G) \cong TM|_U \times TG$ is a diffeomorphism and the first component of $T\varphi$ coincides with Tp. This shows that $\tilde{\xi}(u) := (T\varphi)^{-1}(\xi(p(u)), 0)$ is a smooth vector field on $p^{-1}(U)$ such that $T_up \cdot \tilde{\xi}(u) = \xi(p(u))$ for all $u \in p^{-1}(U)$.

Now let Π be the vertical projection associated to the principal connections. Since this a smooth bundle map, also $u \mapsto \tilde{\xi}(u) - \Pi(u)(\tilde{\xi}(u))$ is a smooth vector field on $p^{-1}(U)$. But since $\Pi(u)$ has values in V_uP, this still projects onto $\xi(p(u))$ and by construction it lies in H_uP, so it coincides with $\xi^h(u)$. □

The operation described in the Lemma is called the *horizontal lift*. From the proof it is clear that the horizontal lift is defined both for tangent vectors and for smooth vector fields.

To get to the simplest description of a principal connection, one more step is needed. Let $\mathfrak{g}$ be the Lie algebra of the structure group G. For $u \in P$, the map $g \mapsto u \cdot g$ defines a diffeomorphism from G to the fiber $P_{p(u)}$ of P through u. Differentiating this defines a map $\mathfrak{g} \to V_uP$, which is easily seen to be a linear isomorphism. Explicitly, this map $X \in \mathfrak{g}$ to $\zeta_X(u) := \frac{d}{dt}|_{t=0} u \cdot \exp(tX)$. It turns out that for each $X \in \mathfrak{g}$, this gives rise to a smooth vector field $\zeta_X \in \mathfrak{X}(P)$, called the *fundamental vector field* generated by X. This easily follows from the fact that in a principal bundle chart ζ_X corresponds to the left invariant vector field L_X generated by X.

Using this on can now encode a principal connection on P as a *connection form* $\gamma \in \Omega^1(P, \mathfrak{g})$. This is characterized by the fact that the vertical projection is given by $\Pi(u)(\xi) = \zeta_{\gamma(\xi)}(u)$. So by definition, we must have $\gamma(u)(\zeta_X(u)) = X$ for any $X \in \mathfrak{g}$, so γ "reproduces the generators of fundamental vector fields". On the other hand, we have to see what equivariancy of Π means for γ. This easily follows from the fact that $g^{-1}\exp(tX)g = \exp(t\,\mathrm{Ad}(g^{-1})(X))$ and hence $\zeta_{\mathrm{Ad}(g^{-1})(X)}(u\cdot g) = T_u r^g \cdot \zeta_u(X)$. This implies that equivariancy of Π is equivalent to the fact that $(r^g)^*\gamma = \mathrm{Ad}(g^{-1})\circ\gamma$ for all $g\in G$, which is referred to as γ being *G–equivariant.*

Summing up, we see that a principal connection on P can be equivalently encoded by a G–equivariant one–form $\gamma\in\Omega^1(P,\mathfrak{g})$ which reproduces the generators of fundamental vector fields. This is the standard definition of a principal connection. The horizontal distribution defined by γ is then simply given by $H_u = \ker(\gamma(u))$, where we view $\gamma(u)$ as a linear map $T_uP\to\mathfrak{g}$.

In the language of connection forms, it is easy to prove existence of principal connections and also to describe all such connections on a given principal bundle:

Proposition 4.4. *Let $p: P\to M$ be a principal fiber bundle with structure group G. Then there is a principal connection form $\gamma\in\Omega^1(P,\mathfrak{g})$. Moreover, the space of all such forms is an affine space modeled on the space $\Omega^1_{hor}(P,\mathfrak{g})^G$ of horizontal, G–equivariant one–forms. The latter space is isomorphic to the space of sections of the vector bundle $T^*M\otimes(P\times_G\mathfrak{g})\to M$.*

Proof. Consider the trivial bundle $M\times G\to M$ and let $\omega\in\Omega^1(G,\mathfrak{g})$ the left Maurer–Cartan form on G. Then by definition $\omega(\xi) = T_g\lambda_{g^{-1}}\cdot\xi$ for $\xi\in T_gG$. This easily implies that for the right translation $\rho^g: G\to G$ by g one obtains $(\rho^g)^*\omega = \mathrm{Ad}(g^{-1})\circ\omega$, see Proposition 2.7 of [**LG**]. This readily implies that $(pr_2)^*\omega\in\Omega^1(M\times G,\mathfrak{g})$ is a principal connection form on $M\times G$.

For a general principal fiber bundle $p: P\to M$ consider a principal bundle atlas $(U_\alpha,\varphi_\alpha)$. For each α, consider $\gamma_\alpha\in\Omega^1(p^{-1}(U_\alpha),\mathfrak{g})$ defined as $\varphi_\alpha^*((pr_2)^*\omega)$. Since $\varphi_\alpha: p^{-1}(U_\alpha)\to U_\alpha\times G$ is an isomorphism of principal bundles, it follows readily that γ_α is a connection form on $p^{-1}(U_\alpha)$. Now choose a partition (f_α) on M which is subordinate to the covering U_α and define $\gamma := \sum_\alpha (f_\alpha\circ p)\gamma_\alpha\in\Omega^1(P,\mathfrak{g})$. Then for $X\in\mathfrak{g}$, we get $\gamma(\zeta_X(u)) = \sum_\alpha f_\alpha(p(u))X = X$. Similarly, since $(f_\alpha\circ p)$ is G–invariant, one easily verifies that γ is G–equivariant and thus a connection form.

Suppose that γ is a connection form and $\varphi\in\Omega^1(P,\mathfrak{g})$ is horizontal and G–equivariant. Then of course, $\gamma+\varphi$ is G–equivariant and since φ vanishes on fundamental vector fields, it is again a connection form. Conversely, for connection forms γ_1 and γ_2, the difference $\gamma_2-\gamma_1$ is G–equivariant and vanishes on fundamental vector fields and thus is horizontal.

To obtain the last isomorphism, consider $\varphi\in\Omega^1(P,\mathfrak{g})^G$ and for a vector field $\xi\in\mathfrak{X}(M)$. Let ξ^h be the horizontal lift of ξ from Lemma 4.4 and consider the function $f := \varphi(\xi^h): G\to\mathfrak{g}$. From Lemma 4.4, we know that $\xi^h(u\cdot g) = T_ur^g\cdot\xi^h(u)$, which shows that $f(u\cdot g) = ((r^g)^*\varphi)(\xi^h)(u) = \mathrm{Ad}(g^{-1})(f(u))$. Hence f is G–equivariant, thus corresponding to a smooth section of $P\times_G\mathfrak{g}$ by Corollary 2.8. Hence we have obtained an operator $\mathfrak{X}(M)\to\Gamma(P\times_G\mathfrak{g})$, but the construction readily implies that this is linear over $C^\infty(M,\mathbb{R})$, thus defining a section of the bundle $T^*M\otimes P\times_G\mathfrak{g}$. It is easy to verify that this construction actually gives rise to an isomorphism. □

4.5. The curvature of a principal connection. The *curvature* of a principal connection defined by $\gamma\in\Omega^1(P,\mathfrak{g})$ is then the form $\Omega\in\Omega^2(P,\mathfrak{g})$ defined by $\Omega(\xi,\eta) := d\gamma(\xi,\eta)+[\gamma(\xi),\gamma(\eta)]$ where the last bracket is in $\mathfrak{g}$. We can easily deduce basic properties of the curvature:

Proposition 4.5. *Let $p: P\to M$ be a principal fiber bundle with structure group G and let $\gamma\in\Omega^1(P,\mathfrak{g})$ be a principal connection form with curvature $\Omega\in\Omega^2(P,\mathfrak{g})$. Then we have:*

(1) The curvature form Ω is horizontal and G–equivariant, i.e. for $\xi \in \Gamma(VP)$ we have $i_\xi\Omega = 0$ and for $g \in G$, we have $(r^g)^*\Omega = \mathrm{Ad}(g^{-1}) \circ \Omega$. Hence Ω can be interpreted as a section of the vector bundle $\Lambda^2 T^*M \otimes (P \times_G \mathfrak{g})$ over M.

(2) The curvature form Ω vanishes identically if and only if the horizontal distribution $H \subset TP$ determined by γ is involutive and hence integrable.

Proof. (1) Using $(r^g)^*\gamma = \mathrm{Ad}(g^{-1}) \circ \gamma$ one easily verifies that $(r^g)^*d\gamma = \mathrm{Ad}(g^{-1}) \circ d\gamma$. Using that $\mathrm{Ad}(g^{-1})$ is a homomorphism of Lie algebras, this together with a simple direct computation proves equivariancy of Ω. On the other hand, for $X \in \mathfrak{g}$, equivariancy of γ shows that $(r^{\exp(tX)})^*\gamma = \mathrm{Ad}(\exp(-tX)) \circ \gamma$. Now in the left hand side, $r^{\exp(tX)}$ is the flow of the fundamental vector field ζ_X up to time t. Hence differentiating the left hand side with respect to t at $t = 0$, we get the Lie derivative $\mathcal{L}_{\zeta_X}\gamma$. Using Cartan's formula we may write this as $i_{\zeta_X}d\gamma + di_{\zeta_X}\gamma$, and since $i_{\zeta_X}\gamma$ is constant, we are left with $i_{\zeta_X}d\gamma$. For the right hand side, we simply get $-\mathrm{ad}(X) \circ \gamma$. Hence we obtain $d\gamma(\zeta_X, \eta) = -[X, \gamma(\eta)]$, and since $\gamma(\zeta_X) = X$, this exactly says that $\Omega(\zeta_X, \eta) = 0$. Since the fundamental vector fields span V_uP, this shows that Ω is horizontal.

The interpretation as a section of a bundle over M works similarly as in the proof of Proposition 4.4: Given vector fields ξ and η on M, $\Omega(\xi^h, \eta^h)$ defines a G–equivariant functions $P \to \mathfrak{g}$ and thus a smooth section of $P \times_G \mathfrak{g}$. Viewed as an operator, this is bilinear over smooth functions on M, thus defining a section as claimed.

(2) Since Ω is horizontal by part (1), vanishing of Ω is equivalent to the fact that $\Omega(\xi^h, \eta^h) = 0$ for all vector fields $\xi, \eta \in \mathfrak{X}(M)$. But since γ vanishes on horizontal lifts, the formula for Ω simplifies in this case to $\Omega(\xi^h, \eta^h) = d\gamma(\xi^h, \eta^h) = -\gamma([\xi^h, \eta^h])$. So vanishing of Ω is equivalent to the fact that the Lie bracket $[\xi^h, \eta^h]$ is horizontal for all $\xi, \eta \in \mathfrak{X}(M)$. Since H admits local frames consisting of horizontal lifts, this implies the claim. □

4.6. Induced connections. The key feature of principal connections is that a single principal connection on the bundle $p : P \to M$ gives rise to linear connections on all vector bundles associated to P.

Theorem 4.6. *Let $p : P \to M$ be a principal fiber bundle with structure group G and consider a representation of G on V. Then a principal connection on P defines a linear connection ∇^V on the associated bundle $P \times_G V$. These connections are compatible (in the sensed discussed in Section 4.3) with all natural constructions arising from constructions with representations. Moreover, the bundle map between two associated bundles induced by a G–equivariant linear map between the corresponding representations is preserved by the induced connections.*

Proof. By Corollary 2.8, sections of $P \times_G V$ are in bijective correspondence with smooth functions $f : P \to V$ such that $f(u \cdot g) = g^{-1} \cdot f(u)$, where in the right hand side the action is via the given representation of G on V. Now given a vector field $\xi \in \mathfrak{X}(M)$, we can consider the horizontal lift ξ^h with respect to the given principal connection on P. For a smooth section $s \in \Gamma(P \times_G V)$ consider the corresponding function f and the derivative $\xi^h \cdot f : P \to V$. Now from Proposition 4.4 we know that $\xi^h(u \cdot g) = T_u(r^g) \cdot \xi(u)$. Using this to differentiate f, we obtain the derivative of $f \circ r^g$ in direction $\xi(u)$. But by equivariancy $f \circ r^g = \rho(g^{-1}) \circ f$, where we denote the representation $G \to GL(V)$ by ρ. Since $\rho(g^{-1})$ is a linear map, we can simply differentiate through it and obtain $(\xi^h \cdot f)(u \cdot g) = \rho(g^{-1})(\xi^h \cdot f(u))$. Hence the function $\xi^h \cdot f$ is again equivariant, thus corresponding to a section $\nabla^V_\xi s \in \Gamma(P \times_G V)$.

So we have defined an operator $\nabla^V : \mathfrak{X}(M) \times \Gamma(P \times_G V) \to \Gamma(P \times_G V)$ and we can verify its compatibility with multiplication by a smooth function $\varphi : M \to \mathbb{R}$. From the definition of the horizontal lift, we immediately see that $(\varphi\xi)^h = (\varphi \circ p)\xi^h$. Likewise, if a section s corresponds to $f : P \to V$, then φs corresponds to $(\varphi \circ p)f$. Hence the function corresponding to $\nabla^V_{\varphi\xi} s$ is $((\varphi \circ p)\xi^h) \cdot f = (\varphi \circ p)(\xi^h \cdot f)$, which in turn corresponds to $\varphi \nabla^V_\xi s$. Likewise, $\nabla^V_\xi(\varphi s)$ corresponds to $\xi^h \cdot (\varphi \circ p)f = ((Tp \cdot \xi^h) \cdot \varphi) \circ p)f + (\varphi \circ p)\xi^h \cdot f$. This exactly shows that $\nabla^V_\xi(\varphi s) = (\xi \cdot \varphi)s + \varphi \nabla^V_\xi s$.

The compatibility with natural constructions is easy to verify directly. For example, consider the dual representation V^* corresponding to V. By definition $(g \cdot \lambda)(v) = \lambda(g^{-1} \cdot v)$ for $\lambda \in V^*$ and $v \in V$, which is equivalent to $(g \cdot \lambda)(g \cdot v) = \lambda(v)$. Now consider sections $s \in \Gamma(P \times_G V)$ corresponding to $f : P \to V$ and $\sigma \in \Gamma(P \times_G V^*)$ corresponding to $\varphi : P \to V^*$. From above we see that the function $u \mapsto \varphi(u)(f(u))$ is G–invariant, so it descends to M and of course, this equals the pairing $\sigma(s)$. This shows that $\xi \cdot (\sigma(s))$ can be computed as $\xi^h \cdot (\varphi(f)) = (\xi^h \cdot \varphi)(f) + \varphi(\xi^h \cdot f)$, where we have used that the evaluation map $V \times V^* \to \mathbb{R}$ is bilinear. But the right hand side clearly represents $(\nabla^{V^*}_\xi \sigma)(s) + \sigma(\nabla^V_\xi s)$, which is exactly the property used in Section 4.3.

Likewise, a tensor product of sections of two associated bundles corresponds to the point–wise tensor product of the corresponding functions, which leads to the compatibility of the connections ∇^V, ∇^W and $\nabla^{V\otimes W}$ from Section 4.3.

For the last statement, let $\varphi : V \to W$ be a morphism of representations of G, so as an element of $L(V, W)$, the map φ satisfies $g \cdot \varphi = \varphi$ for all $g \in G$. Let E and F be the associated bundles defined by V and W, and put $\Phi := P[\varphi] : P \times_G V \to P \times_G W$ as in Section 2.12. Then Φ defines a smooth section of the bundle $L(E, F)$ and the corresponding smooth map $P \to L(V, W)$ of course is the constant map φ. Hence by definition $\nabla^{L(V,W)}_\xi \Phi = 0$ for any $\xi \in \mathfrak{X}(M)$. By compatibility with natural constructions, this means that for any $s \in \Gamma(E)$, we get $\nabla^W_\xi(\Phi(s)) = \Phi(\nabla^V_\xi s)$. □

Usually, one simply denotes all the induced connections simply by the symbol ∇. This is justified by the compatibility proved in the Proposition. It is also easy to compute the curvature of induced connections. Recall the curvature of a principal connection γ on P is a two–form $\Omega \in \Omega^2(P, \mathfrak{g})$. Given a representation of G on V, the infinitesimal representation defines a bilinear map $\cdot : \mathfrak{g} \times V \to V$.

Proposition 4.6. *Let $p : P \to M$ be a principal G–bundle and let γ be a principal connection on P with curvature $\Omega \in \Omega^2(P, \mathfrak{g})$. Let ∇ be the induced linear connection on the bundle $E := P \times_G V$ and let R be its curvature. Then for a section $s \in \Gamma(E)$ corresponding to the equivariant function $f : P \to V$ and $\xi, \eta \in \mathfrak{X}(M)$, the section $R(\xi, \eta)(s) \in \Gamma(E)$ corresponds to the function $\mapsto \Omega(\xi^h(u), \eta^h(u)) \cdot f(u)$, where the dot denotes the infinitesimal action of $\mathfrak{g}$ on V.*

Proof. By definition of the curvature of a linear connection, the function corresponding to $R(\xi, \eta)(s)$ is given by

$$\xi^h \cdot (\eta^h \cdot f) - \eta^h \cdot (\xi^h \cdot f) - [\xi, \eta]^h \cdot f = ([\xi^h, \eta^h] - [\xi, \eta]^h) \cdot f.$$

Since ξ^h is p–related to ξ and η^h is p–related to η, we have $T_u p \cdot [\xi^h, \eta^h](u) = [\xi, \eta](u)$. This shows that $[\xi^h, \eta^h] - [\xi, \eta]^h$ is a vertical vector field on P and since $[\xi, \eta]^h$ is horizontal, it can be computed as the vertical projection of $[\xi^h, \eta^h]$. By definition of the connection form, this equals $\zeta_{\gamma([\xi^h,\eta^h])} \cdot f$ and the infinitesimal version of equivariancy of f says that in a point $u \in P$, this equals $-[\gamma([\xi^h, \eta^h])(u), f(u)]$.

But since $\gamma(\xi^h) = \gamma(\eta^h) = 0$, we can compute $\gamma([\xi^h, \eta^h])$ as $-d\gamma(\xi^h, \eta^h) = -\Omega(\xi^h, \eta^h)$, where we have used horizontality in the second equality once more. □

Using the affine structures on the spaces of connections, one easily shows that any linear connection ∇ on a vector bundle $p : E \to M$ is induced from a principal connection on the frame bundle $p : P \to M$ of E (with structure group $G = GL(n, \mathbb{R})$). Indeed, we can choose any principal connection on P, let $\hat{\gamma} \in \Omega^1(P, \mathfrak{g})$ be its connection form and $\hat{\nabla}$ the induced linear connection on $E = P \times_G \mathbb{R}^n$. Then by Theorem 4.1, $A(\xi, s) := \hat{\nabla}_\xi s - \nabla_\xi s$ defines a smooth section A of the bundle $T^*M \otimes L(E, E)$. Since $G = Gl(n, \mathbb{R})$, we get $\mathfrak{g} = L(\mathbb{R}^n, \mathbb{R}^n)$ and hence $L(E, E) = P \times_G \mathfrak{g}$. In Proposition 4.4, we have seen that $\Gamma(T^*M \otimes (P \times_G \mathfrak{g})) \cong \Omega^1_{hor}(P, \mathfrak{g})$, and we denote by $\tilde{A}$ the one–form corresponding to A. Again by Proposition 4.4 we know that $\gamma := \gamma + \tilde{A}$ is again a principal connection form on P, and it is easy to verify that the linear connection induced by γ coincides with ∇.

As discussed in Section 2, reductions of structure group of the frame bundle of a vector bundle E can be interpreted as defining additional structures on E. Linear connections on E induced from the resulting principal bundles can then be interpreted as the linear connections on E which are compatible with the additional structures. As the simplest example, consider the case of the frame bundle of a smooth manifold M. Then a reduction of structure group to the orthogonal group is equivalent to a Riemannian metric on M for which one obtains the orthonormal frame bundle. Now it is easy to see that a linear connection on TM is induced by a principal connection on the orthonormal frame bundle if and only if it is metric in the sense discussed in Section 4.2.

4.7. The case of homogeneous spaces. We first observe that there are simple notions of invariance for connections. Let $p : P \to M$ be a principal fiber bundle and $F : P \to P$ a principal bundle isomorphism (with any base map). Then equivariancy of F readily implies that $T_uF \cdot \zeta_X(u) = \zeta_Z(F(u))$ for each X and u, so $F^*\zeta_X = \zeta_X$. For a principal connection form $\gamma \in \Omega^1(P, \mathfrak{g})$, this implies that $(F^*\gamma)(\zeta_X) = \zeta_X$ and equivariancy of F and γ shows that $F^*\gamma$ is equivariant, too, so $F^*\gamma$ is a principal connection form. In the language of horizontal distributions, this of course means that $T_uF(H_u) = H_{F(u)}$. In particular, having given a homogeneous space G/H and a homogeneous principal bundle $p : P \to G/H$, there is the notion of a G–invariant principal connection on P.

In the case of linear connections, one can phrase things similarly, but for our purposes it is easier to use the action of G on sections. Recall that for a homogeneous vector bundle $E = G \times_H V \to G/H$ corresponding to a representation V of H, there is a natural representation of G on the space $\Gamma(E)$ of sections, see Section 3.3. This is defined by $(g \cdot s)(\tilde{g}H) = g \cdot (s(g^{-1}\tilde{g}H))$ and in the language of equivariant function corresponds to $(g \cdot f)(\tilde{g}) = f(g^{-1}g)$. Since the tangent bundle $T(G/H)$ is a homogeneous vector bundle, too, we also have an action of g on vector fields. Now one calls a linear connection ∇ on E *G–invariant* if and only if for any $g \in G$, $\xi \in \mathfrak{X}(G/H)$ and $s \in \Gamma(E)$, we have $\nabla_{g \cdot \xi}(g \cdot s) = g \cdot (\nabla_\xi s)$.

The first result we discuss is the existence of an invariant principal connection on the principal fiber bundle $p : G \to G/H$. In view of Theorems 3.3 and 4.6 such a connection gives rise to linear connections on all homogeneous vector bundles over G/H, which are compatible with all homomorphisms of homogeneous bundles.

THEOREM 4.7. *Let G be a Lie group and $H \subset G$ a closed subgroup. Then G–invariant principal connections on $p : G \to G/H$ are in bijective correspondence with H–invariant linear subspaces $\mathfrak{m} \subset \mathfrak{g}$, which are complementary to $\mathfrak{h} \subset \mathfrak{g}$. In particular, this always exists if H is compact.*

For a G–invariant principal connection on $p : G \to H$, the linear connections induced on homogeneous vector bundles, are G–invariant, too, and they are compatible with all homomorphisms of homogeneous vector bundles.

Proof. Of course, $T_eG = \mathfrak{g}$ and $V_eG = \mathfrak{h} \subset \mathfrak{g}$, so for a principal connection on $p : G \to G/H$, the horizontal subspace H_e is a linear subspace of $\mathfrak{g}$ which is complementary to $\mathfrak{h}$. Moreover, a linear subspace $H_e \subset \mathfrak{g}$ uniquely extends to a G–invariant distribution $H \subset TG$ via $H_g := T_e\lambda_g(H_e)$. Since $V_gG = T_e\lambda_g(\mathfrak{h})$, we see that H_g is always complementary to V_gG, so we only have to analyze the condition that H defines a principal connection.

This is clearly equivalent to $T_g\rho^h(H_g) = H_{gh}$ for all $h \in H$. Applied at $g = e$, this says that $T_e\rho^h(H_e) = H_h = T_e\lambda_h(H_e)$ and bringing one of the translations to the other side, we get $\mathrm{Ad}(h)(H_e) \subset H_e$. Hence for a G–invariant connection, the subspace $\mathfrak{m} := H_e$ is H–invariant.

Conversely, suppose that H_e is H–invariant. Differentiating $\rho^h \circ \lambda_g = \lambda_{gh}\,\mathrm{conj}_{h^{-1}}$ at e, we get $T_g\rho^h \circ T_e\lambda_g = T_e\lambda_{gh} \circ \mathrm{Ad}(h^{-1})$. The image of H_e under the left hand side by definition is $T_g\rho^h(H_g)$, while for the left hand side, invariance implies that we simply get H_{gh}. Hence we obtain an invariant connection.

Invariance of the principal connection readily implies equivariancy of the horizontal lift. For $\xi \in \mathfrak{X}(G/H)$ let $\xi^h \in \mathfrak{X}(G)$ be the horizontal lift. For $g \in G$ we get $T_{g^{-1}\tilde{g}}\lambda_g \cdot \xi^h(g^{-1}\tilde{g}) \in H_{g\tilde{g}}$ and projecting this tangent vector to G/H, we get $T_{\tilde{g}H}\ell_g \cdot \xi(g^{-1}\tilde{g}H) = (g \cdot \xi)(\tilde{g}H)$, so this describes the horizontal lift of $(g \cdot \xi)$. If $f : G \to V$ is a smooth function, then we compute

$$(g \cdot \xi)^h \cdot f(g^{-1}\tilde{g}) = T_{g^{-1}\tilde{g}}\lambda_g \cdot \xi^h(g^{-1}\tilde{g}) \cdot f = (\xi^h \cdot f)(g^{-1}\tilde{g}),$$

which, for equivariant f, exactly is invariance of the induced linear connection on $E = G \times_H V$. The remaining claims then follows from the general results on induced connections in Theorem 4.6. □

Having given an H–invariant complement $\mathfrak{m}$ to $\mathfrak{h}$ in $\mathfrak{g}$, one of course gets an isomorphism $\mathfrak{m} \cong \mathfrak{g}/\mathfrak{h}$ of H–modules. Hence the tangent bundle $T(G/H)$ can be identified with the associated bundle $G \times_H \mathfrak{m}$. Moreover, we can restrict the Lie bracket on $\mathfrak{g}$ to $\mathfrak{m}$, and then decompose according to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ to obtain H–equivariant, skew symmetric bilinear maps $[\ ,\ ]_{\mathfrak{h}} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{h}$ and $[\ ,\ ]_{\mathfrak{m}} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$. It follows readily from the definitions the the curvature of an invariant principal connection is an invariant section of $\Lambda^2T^*(G/H) \otimes (G \times_H \mathfrak{h})$, so by Theorem 3.3, it corresponds to an H–invariant element in $\Lambda^2\mathfrak{m}^* \otimes \mathfrak{h}$. It is easy to verify directly, that this is exactly $-[\ ,\ ]_{\mathfrak{h}}$. Likewise, the torsion of the induced linear connection on $T(G/H)$ is an invariant tensor field, thus corresponding to an H–invariant element in $\Lambda^2\mathfrak{m}^* \otimes \mathfrak{m}$. It turns out that this element is exactly $-[\ ,\ ]_{\mathfrak{m}}$.

Example 4.7. (1) Consider $G = O(n+1)$ and $H = O(n)$, so $G/H \cong S^n$, compare with example (1) of 3.4. There we have notices that $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}^n$ as a representation of H, so $\mathfrak{m} = \mathbb{R}^n$ is an H–invariant complement to $\mathfrak{h}$ in $\mathfrak{g}$. In fact, it is easy to see that this is the unique H–invariant complement as follows. Suppose that $\mathfrak{m}' \subset \mathfrak{h} \oplus \mathfrak{m}$ is an H–invariant complement, then the restriction to $\mathfrak{m}'$ of the second projection defines a linear isomorphism $\mathfrak{m}' \to \mathfrak{m}$. Inverting this and composing with the restriction of the first projection, we obtain a linear map $\mathfrak{m} \to \mathfrak{h}$, which by construction is H–equivariant. But it is easy to see that there is no non–zero $O(n)$–equivariant map $\mathbb{R}^n \to \mathfrak{o}(n)$.

It is also easy to see that in this case $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, so the induced linear connection on TS^n is torsion free. Moreover, the G–invariant Riemannian metric on S^n is a natural section of $S^2T^*S^n$ and hence parallel for the induced connection. Thus we obtain the Levi–Civita connection of the round metric on S^n in this case.

(2) On the Sasaki sphere $S^{2n+1} \cong U(n+1)/U(n)$, we have also observed in Section 3.7 that we obtain an $U(n)$–invariant decomposition $\mathfrak{u}(n+1) = \mathfrak{u}(n) \oplus \mathfrak{m}$ with $\mathfrak{m} \cong \mathbb{C}^n \oplus \mathbb{R}$. This again gives rise to an invariant principal connection on $U(n+1) \to S^{2n+1}$, which induces a linear connection on TS^{2n+1} which preserves the round metric on the sphere. However, in this case, the restriction of the Lie bracket to $\mathfrak{m} \times \mathfrak{m}$ has a substantial non–trivial part in $\mathfrak{m}$. Indeed, this vanishes on $\mathbb{R} \times \mathfrak{m}$ and has values in $\mathbb{R} \subset \mathfrak{m}$, but the restriction to $\mathbb{C}^n \times \mathbb{C}^n$ is the imaginary part of the standard Hermitian inner product, and thus non–degenerate. Hence we obtain a connection with torsion on TS^{2n+1}, which thus has to be different from the Levi–Civita connection.

This is not surprising, since the induced connection ∇ on TS^{2n+1} has to preserve sections of the subbundle $H \subset TS^{2n+1}$. For $\xi, \eta \in \Gamma(H)$, we know that $[\xi, \eta] \notin \Gamma(H)$ in general, since H is a contact distribution. But in such a case, $\nabla_\xi \eta - \nabla_\eta \xi \in \Gamma(H)$, so the torsion cannot be trivial.

(3) It is easy to verify that in the case of the conformal sphere and the CR sphere discussed in Example 3.8, there is no H–invariant complement to $\mathfrak{h}$ in $\mathfrak{g}$, so there are no G–invariant principal connections on $G \to G/H$ in these cases.

4.8. Invariant linear connections. In the case that there is no invariant principal connection on $G \to G/H$, one may still have invariant connections on some associated vector bundles.

Theorem 4.8. *Consider a Lie group G, a closed subgroup $H \subset G$ and a representation $\rho : H \to GL(V)$ with derivative $\rho' : \mathfrak{h} \to \mathfrak{gl}(V)$. Then G–invariant linear connections on the homogeneous vector bundle $E := G \times_H V$ are in bijective correspondence with H–equivariant maps $\alpha : \mathfrak{g} \to \mathfrak{gl}(V)$ such that $\alpha|_{\mathfrak{h}} = \rho'$. Here H acts on $\mathfrak{g}$ via Ad_G and on $\mathfrak{gl}(V)$ via $\mathrm{Ad}_{GL(V)} \circ \rho$.*

In particular, if non–empty, then the space of all G–invariant connections on E is affine with modeling vector space $\mathrm{Hom}_H(\mathfrak{g}/\mathfrak{h}, \mathfrak{gl}(V))$.

Proof. One can either prove this by passing through the frame bundle of E. This is the associated bundle $G \times_H GL(V)$, where $h \in H$ acts on $GL(V)$ via left multiplication by $\rho(h)$. The tangent space of this bundle in $[e, \mathrm{id}_V]$ is the quotient space of $\mathfrak{g} \times \mathfrak{gl}(V)$ by the linear subspace $\{(X, -\rho'(X)) : X \in \mathfrak{h}\}$. The quotient projection restrict to an injection on $\{0\} \times \mathfrak{gl}(V)$ and the image of this subspace is the vertical subspace in $[e, \mathrm{id}_V]$. Given a map α as in the theorem, one obtains an horizontal subspace as the image of $\{(Y, -\alpha(Y)) : Y \in \mathfrak{g}\}$ in the quotient. This subspace is invariant under the action of H given by Ad in the first component and $\mathrm{Ad}_{GL(V)} \circ \rho$ in the second component.

Conversely, having given a horizontal subspace with this invariance property, we can associate to $Y \in \mathfrak{g}$ the horizontal projection of $(Y, 0)$. This can be uniquely written as the class of $(Y, -\alpha(Y))$ for some linear map $\alpha : \mathfrak{g} \to \mathfrak{gl}(V)$ and one easily verifies that this has the required property. The equivalence between horizontal subspace in $[e, \mathrm{id}_V]$ with the given invariance property and G–invariant principal connections is then easily verified directly.

Alternatively, there is a more direct description. Given an H–equivariant function $f : G \to V$, consider the map $\psi : G \to L(\mathfrak{g}, V)$ defined by $\tilde{\psi}(g)(X) = L_X(g) \cdot f + \alpha(X)(f(g))$. Here L_X denotes the left invariant vector field generated by $X \in \mathfrak{g}$. It is easy to verify that this function is H–equivariant and equivariancy of f together with the properties of α shows that $\tilde{\psi}(g)(X) = 0$ for $X \in \mathfrak{h}$. Therefore, $\tilde{\psi}$ descends to an H–equivariant map $G \to L(\mathfrak{g}/\mathfrak{h}, V)$, which corresponds to a smooth section of the bundle $T^*M \otimes E$.

For the section s corresponding to f, on then defines $\nabla_\xi s$ as the evaluation of that section on ξ. Again, this procedure can be reversed.

The affine structure is clear from the fact that the difference of two invariant linear connections is an invariant tensorial object. $\square$

Of course, the curvature of an invariant linear connection on E is a G–invariant section of the bundle $\Lambda^2 T^*M \otimes L(E,E)$, thus corresponding to an H–equivariant, skew symmetric bilinear map $\mathfrak{g}/\mathfrak{h} \times \mathfrak{g}/\mathfrak{h} \to \mathfrak{gl}(V)$. Now assume that $\alpha : \mathfrak{g} \to \mathfrak{gl}(V)$ is a map as in Theorem 4.8, and consider the map $(X,Y) \mapsto [\alpha(X), \alpha(Y)] - \alpha([X,Y])$, where the first bracket is in $\mathfrak{g}$ and the second one in $\mathfrak{gl}(V)$. Then the infinitesimal version of equivariancy of α is $\alpha \circ \mathrm{ad}(X) = \mathrm{ad}(\alpha(X)) \circ \alpha$ which shows that this descends to $\mathfrak{g}/\mathfrak{h}$. It turns out that this descended map induces the curvature.

Likewise, taking $\rho = \underline{\mathrm{Ad}} : H \to GL(\mathfrak{g}/\mathfrak{h})$ one gets $E = TM$, and then the torsion of a G–invariant vector field is a G–invariant tensor field, hence corresponding to a skew symmetric, bilinear map $\mathfrak{g}/\mathfrak{h} \times \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$. Given $\alpha : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ as above, consider the map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ defined by $(X,Y) \mapsto \alpha(X)(Y+\mathfrak{h}) - \alpha(Y)(X+\mathfrak{h}) - [X,Y] + \mathfrak{h}$. Again this is easily seen to descend to $\mathfrak{g}/\mathfrak{h}$ and it induces the torsion.

Finally, one can also describe the G–invariant linear connections induced by a G–invariant principal connection on $p : G \to G/H$ in this picture. Given an H–invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, we denote by $\pi_{\mathfrak{h}}$ the projection onto the first factor. Given any representation $\rho : H \to GL(V)$ with derivative $\rho' : \mathfrak{h} \to \mathfrak{gl}(V)$, we can define $\alpha : \mathfrak{g} \to \mathfrak{gl}(V)$ by $\alpha := \rho' \circ \pi_{\mathfrak{h}}$. Of course, this coincides with ρ' on $\mathfrak{h} \subset \mathfrak{g}$ and is H–equivariant by construction. It is easy to verify that this map α exactly corresponds to the induced linear connection on $G \times_H V$. This gives an alternative proof for the description of the curvature and torsion of these induced linear connections from Section 4.7.

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