ALMOST CR-MANIFOLDS OF CR-DIMENSION AND CODIMENSION TWO

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This talk discusses applications of the general theory of parabolic geometries to certain almost CR–manifolds of CR–dimension and codimension two. We will show that not only this theory leads to a construction of a canonical Cartan connection for such manifolds, but it also provides efficient and powerful tools for a direct geometric interpretation of the curvature of this Cartan connection, which is well known to be a complete obstruction to local isomorphism with the flat model. Moreover, general ideas about parabolic geometries lead to a surprising relation between certain projective structures on almost complex manifolds and almost CR–structures. The talk is based on the paper [7] and on recent joint work of G. Schmalz and myself.

1.1. **Partially integrable almost CR–structures.** Let $(M, T^{CR}M, J)$ be an almost CR–manifold of CR–dimension k and codimension ℓ, i.e. a smooth manifold of real dimension $2k+\ell$ equipped with a subbundle $T^{CR}M$ of the tangent bundle of real rank $2k$ provided with an almost complex structure $\tilde{J} : T^{CR}M \to T^{CR}M$. (The reason for using the tilde in the notation is that we shall meet another, more important almost complex structure on that subbundle later on.) By QM we denote the quotient bundle $TM/T^{CR}M$, and by $q : TM \to QM$ the canonical projection. Then the Lie bracket of vector fields on M induces a skew symmetric bundle map $\mathcal{L} : T^{CR}M \times T^{CR}M \to QM$, the *Levi–bracket*. The almost CR–structure is called *partially integrable* if and only if the Levi–bracket is totally real, i.e. $\mathcal{L}(\tilde{J}\xi, \tilde{J}\eta) = \mathcal{L}(\xi, \eta)$ for all tangent vectors ξ and η on M.

If this condition is satisfied, then for any two sections ξ, η of $T^{CR}M$, also $[\tilde{J}\xi, \eta] + [\xi, \tilde{J}\eta]$ is a section of $T^{CR}M$, so we may define

$$\tilde{N}(\xi, \eta) = [\xi, \eta] - [\tilde{J}\xi \tilde{J}\eta] + \tilde{J}\big([\tilde{J}\xi, \eta] + [\xi, \tilde{J}\eta]\big).$$

The usual computation shows that this expression is bilinear over smooth functions, so it defines a bundle map $\tilde{N} : T^{CR}M \times T^{CR}M \to T^{CR}M$, the *Nijenhuis tensor* of M. One immediately verifies that $\tilde{N}$ is skew symmetric and conjugate linear in both arguments. The almost CR–structure is called a *CR–structure* if and only if this Nijenhuis tensor vanishes identically.

Partial integrability as well as being CR have simple interpretations in terms of the complexified tangent bundle. Indeed, in the complexification we have $T^{CR}_{\mathbb{C}}M = T^{CR}_{1,0}M \oplus T^{CR}_{0,1}M$ and partial integrability is equivalent to the bracket of two sections of $T^{CR}_{0,1}$ being a section of $T^{CR}_{\mathbb{C}}M$, while being CR is equivalent to integrability of $T^{CR}_{0,1}M$.

Under the hypothesis of partial integrability, the Levi–bracket $\mathcal{L}$ is exactly the imaginary part of the classical Levi–form, so non–degeneracy can be defined in terms of $\mathcal{L}$: A partially integrable almost CR–structure is called *non–degenerate* at a point $x \in M$, if and only if
(i) $\mathcal{L}(\xi,\eta)=0$ for all $\eta \in T^{CR}_xM$ implies $\xi=0$, and
(ii) for any nonzero element $\psi \in Q^*_xM$, the map $\mathcal{L}^\psi=\psi\circ\mathcal{L}: T^{CR}_xM\times T^{CR}_xM\to\mathbb{R}$ is nonzero.

Note that the second condition is just a coordinate-free version of the requirement that the ℓ components of $\mathcal{L}$ should be linearly independent. Visibly, non–degeneracy is an open condition, so for local problems one may restrict to manifolds which are non–degenerate at all their points, which we will do in the sequel.

For general CR–dimension and codimension, even under this strong non–degeneracy hypothesis, the classification of equivalence classes of possible Levi–brackets at a point is highly nontrivial and one may have continuously varying isomorphism classes. Indeed, there are only a few cases in which there is only a discrete family of possible isomorphisms classes of $\mathcal{L}$ and thus the chance to have a uniform local model (on an infinitesimal level). The classical case is the case $\ell=1$, i.e. the case of hypersurface type. In this case the isomorphism class of $\mathcal{L}$ is characterised by the signature, and the classical constructions of Cartan (in dimension three) and Tanaka (see [8]) and Chern–Moser (see [3]) for general dimensions lead to a canonical normal Cartan connection for CR manifolds in this case. In [2] it has been shown that there also is a canonical normal Cartan connection for partially integrable almost CR manifolds of hypersurface type.

1.2. **The case of CR–dimension and codimension two.** This talk is concerned with one of the few other manageable cases, namely the case $k=\ell=2$ (so the standard examples for this case are certain codimension two submanifolds in $\mathbb{C}^4$). Thus let us assume that $(M,T^{CR}M,\tilde{J})$ is a non–degenerate partially integrable almost CR manifold of dimension six such that $T^{CR}M$ has rank four. The first step towards understanding such manifolds is the determination of possible equivalence classes of the Levi–bracket at a point. So let $x\in M$ be a point and consider a nonzero element $\psi\in Q^*_xM = L(Q_xM,\mathbb{R})$. Since $\mathcal{L}$ is totally real, the nullspace H^ψ of $\mathcal{L}^\psi=\psi\circ\mathcal{L}$ is a complex subspace of T^{CR}_xM so by non–degeneracy it is either zero or of complex dimension one. Note that H^ψ depends only on the class $[\psi]$ of ψ in the projectivisation $\mathcal{P}(Q^*_xM)\cong\mathbb{R}P^1$, so we will also denote it by $H^{[\psi]}$.

Proposition. *Suppose that $(M,T^{CR}M,\tilde{J})$ is a partially integrable almost CR manifold of CR–dimension and codimension two. Then for a point $x\in M$ there are three possibilities:*
(1) There are two different points $[\psi_1]\neq[\psi_2]\in\mathcal{P}(Q^*_xM)$ such that for $0\neq\psi\in Q^*_xM$ the bilinear form $\mathcal{L}^\psi$ is degenerate if and only if $\psi\in[\psi_1]$ or $\psi\in[\psi_2]$. In this case, $T^{CR}_xM=H^{[\psi_1]}\oplus H^{[\psi_2]}$ and the point x is called hyperbolic.
(2) There is one point $[\psi_0]\in\mathcal{P}(Q^*_xM)$ such that $\mathcal{L}^\psi$ is degenerate for $\psi\neq0$ if and only if $\psi\in[\psi_0]$. In this case, the point x is called exceptional.
(3) $\mathcal{L}^\psi$ is non degenerate for all $\psi\neq0$. In this case the point x is called elliptic.

Sketch of proof. In fact, we only have to show that there are at most two different points $[\psi]\in\mathcal{P}(Q^*_xM)$ such that $H^{[\psi]}\neq\{0\}$. It is elementary to show that if $[\psi_1]\neq[\psi_2]$ are two such points, then $T^{CR}_xM = H^{[\psi_1]} \oplus H^{[\psi_2]}$. Using this, one then shows that the existence of a third point $[\psi_3]$ with the same property, which is different from $[\psi_1], [\psi_2]$ would contradict non–degeneracy of $\mathcal{L}$. □

Remarks. (1) From the definition it is clear that hyperbolic and elliptic are open properties, i.e. any hyperbolic (elliptic) point has a neighbourhood consisting entirely of hyperbolic (elliptic) points. Thus, to understand the local behaviour at hyperbolic (elliptic) points, one may restrict to manifolds all of whose points are hyperbolic (elliptic). In the sequel, we will restrict to the elliptic case, the hyperbolic case is parallel to that and simpler.
(2) With exceptional points, the situation is much more complicated. While there are manifolds consisting entirely of parabolic points (as the example of an appropriate quadric shows), they also may form lower dimensional submanifolds. To my knowledge, no way is known up to now to study these points.
(3) To put the results discussed here into perspective, note that in [4] the authors gave a very involved construction for canonical principal bundles equipped with an absolute parallelism over elliptic and hyperbolic CR–manifolds of CR–dimension and codimension two, and partly gave geometric interpretations of the associated curvature. This parallelism is not a Cartan connection in general, since it lacks the appropriate equivariancy properties.

1.3. While in the hyperbolic case it follows almost immediately from the characterisation in proposition 1.2 that oriented hyperbolic partially integrable almost CR manifolds are exactly the parabolic geometries of type $(PSU(2,1) \times PSU(2,1), B \times B)$, where $B \subset PSU(2,1)$ is the Borel subgroup, an additional step is necessary in the elliptic case:

Proposition. *Let $(M, T^{CR}M, \tilde{J})$ be an oriented elliptic partially integrable almost CR manifold.*
(1) There is a unique almost complex structure J^Q on the bundle QM which is compatible with the orientation of M and has the property that for each point $x \in M$ there is a nonzero element $\eta \in T^{CR}_xM$ such that $\mathcal{L}_x(\tilde{J}\xi, \eta) = J^Q\mathcal{L}_x(\xi, \eta)$ for all $\xi \in T^{CR}_xM$.
(2) If we define $T^{CR\pm}_xM$ to be the subspaces consisting of all $\eta \in T^{CR}_xM$ such that $\mathcal{L}_x(\tilde{J}\xi, \eta) = \pm J^Q\mathcal{L}_x(\xi, \eta)$ for all $\xi \in T^{CR}_xM$, then these subspaces fit together to form smooth subbundles $T^{CR\pm}M \subset T^{CR}M$, which both are complex line bundles and have the property that $T^{CR}M = T^{CR+}M \oplus T^{CR-}M$.
(3) If we define a new almost complex structure J on $T^{CR}M$ by $J|_{T^{CR+}M} = -\tilde{J}$ and $J|_{T^{CR-}M} = \tilde{J}$, then with respect to the almost complex structures J and J^Q the Levi bracket $\mathcal{L} : T^{CR}M \times T^{CR}M \to QM$ is complex bilinear.

Sketch of proof. This is similar to the proof of proposition 1.2 but in a complexified setting: First, one extends the Levi bracket $\mathcal{L} : T^{CR}M \times T^{CR}M \to QM$ to a Hermitian form $\mathcal{H}$ with values in $QM \otimes \mathbb{C}$. For an element $\psi \in L_{\mathbb{C}}(Q^*_xM \otimes \mathbb{C}, \mathbb{C})$, we can now form $\mathcal{H}^\psi$ similarly as before, and consider the nullspace H^ψ. Similarly as before, one sees that H^ψ is either zero or a complex subspace of dimension one, and it depends only on the class of ψ in the complex projectivisation of $Q^*_xM \otimes \mathbb{C}$, which is a complex projective line. Now it turns out that in that projective line there are two different points (which are conjugate to each other) such that $H^{[\psi]} \neq \{0\}$, and each of those restricts to an $\mathbb{R}$–linear isomorphism $Q_xM \to \mathbb{C}$ (defined up to complex multiples). Exactly one of these classes of isomorphisms is compatible with the orientation, which leads to the almost complex structure J^Q. The rest then follows by rather direct arguments. □

1.4. Proposition 1.3 is actually all one needs to get the machinery of parabolic geometries going. This proposition shows that an elliptic partially integrable almost CR–structure $(M, T^{CR}M, \tilde{J})$ on an oriented manifold M gives us two transversal complex line bundles $(T^{CR\pm}M, J) \subset TM$ plus an almost complex structure J^Q on $TM/T^{CR}M$ such that $\mathcal{L} : T^{CR}M \times T^{CR}M \to QM$ is complex bilinear (with respect to J) and non–degenerate. If we conversely assume that M is a six–dimensional manifold equipped with such a configuration of subbundles of the tangent bundle and almost complex structures, then we can make it into an elliptic partially integrable almost CR manifold by defining $T^{CR}M$ to be the sum of the two complex line bundles with the almost complex structure obtained by flipping the almost complex structure on one of the line bundles. Moreover, one easily sees that these constructions actually describe an equivalence of categories.

From this, it is fairly easy to proceed: Considering the Lie algebra $\mathfrak{g} := \mathfrak{sl}(3, \mathbb{C})$ as a real Lie algebra and its Borel subalgebra $\mathfrak{b}$, we get a $|2|$–grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \dots \oplus \mathfrak{g}_2$ such that $\mathfrak{b} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Further, put $G = PSL(3, \mathbb{C})$, and define $B_0 \subset B \subset G$ as the subgroups of those elements whose adjoint action respects the grading respectively the corresponding filtration of $\mathfrak{g}$. One easily verifies that B_0 is a product of two copies of $\mathbb{C} \setminus \{0\}$. Moreover, as B_0–modules $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^- \oplus \mathfrak{g}_{-1}^+$ with $\mathfrak{g}_{-1}^\pm \cong \mathbb{C}$, $\mathfrak{g}_{-2} = \mathbb{C}$, and the Lie bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is complex bilinear and non–degenerate. Now one can apply one of the procedures for constructing canonical Cartan connections, e.g. the ones of [2] or [6], or (with a slight reinterpretation of the structure) the one in [9] to the case of a six manifold endowed with two transversal complex line bundles sitting in the tangent bundle and an almost complex structure on the quotient bundle such that the Levi-bracket is complex bilinear and non–degenerate, which together with the above leads to

Theorem. *The category of elliptic partially integrable almost CR–manifolds is equivalent to the category of smooth six–dimensional manifolds M endowed with a principal B–bundle $\mathcal{G} \to M$ and a normal Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, i.e. the category of normal parabolic geometries of type $(PSL(3, \mathbb{C}), B)$.*

The curvature of the Cartan connection ω can be interpreted as a function $\mathcal{G} \to C^2(\mathfrak{g}_-, \mathfrak{g})$, the space of Lie algebra two–cochains, where $\mathfrak{g}_-$ is the negative part in the grading of $\mathfrak{g}$. On this space of two–cochains besides the Lie algebra differential ∂, there is also a canonical adjoint ∂^*, the codifferential, and thus a resulting Hodge–theory and in particular a harmonic subspace isomorphic to the cohomology space $H^2(\mathfrak{g}_-, \mathfrak{g})$. The normalisation condition on ω is that its curvature has values in the kernel of ∂^*. However, it has been shown by Tanaka (see [9]) that then even the harmonic part of the curvature is a complete obstruction against local isomorphism with the flat model (which in our case is the full flag manifold G/B). Thus, to understand the obstructions against local flatness, one has to give a geometric interpretation of the harmonic part of the curvature.

1.5. Next, we describe three essential tools for the geometric interpretation of curvatures, which are available for any parabolic geometry: The first of these tools is Kostant's version of the Bott–Borel–Weil theorem, see [5]. This theorem gives a complete description of the $\mathfrak{g}_0$–module structure of the cohomology $H^*(\mathfrak{g}_-, \mathfrak{g})$ in the case where $\mathfrak{g}$ is complex and simple, so applying some basic tricks of the trade, one also can describe this cohomology if $\mathfrak{g}$ is real and semisimple.

The second main tool is the following

Lemma. *Let $(p : \mathcal{G} \to M, \omega)$ be a normal parabolic geometry with curvature function $\kappa : \mathcal{G} \to C^2(\mathfrak{g}_-, \mathfrak{g})$, let $x \in M$ be a point and $u \in \mathcal{G}$ such that $p(u) = x$. Then there is an open neighbourhood U of x in M and an extension operator $\xi \mapsto \tilde{\xi}$ from the tangent space T_xM to the space of local vector fields on M defined on U, which is compatible with all structures induced by the parabolic geometry and has the following property: If $\xi, \eta \in T_xM$, $X \in \mathfrak{g}_-$ is the unique element such that $T_up \cdot \omega_u^{-1}(X) = \xi$ and $Y \in \mathfrak{g}_-$ is the corresponding element for η, then we have*

$$[\tilde{\xi}, \tilde{\eta}](x) = T_up \cdot \omega_u^{-1}([X, Y] - \kappa(u)(X, Y)).$$

The upshot of this lemma is that any tensorial expression which can be written in terms of Lie brackets allows a direct interpretation in terms of κ.

The third essential tool is the Bianchi–identity, see e.g. [2, 4.9], which reads as

$$(\partial \circ \kappa)(X, Y, Z) = \sum_{\text{cycl}} \left(\kappa(\kappa_-(X, Y), Z) + (\omega^{-1}(X) \cdot \kappa)(Y, Z) \right),$$

where the sum in the right hand side is over all cyclic permutations of the arguments, and $\kappa_-(X, Y)$ is just the $\mathfrak{g}_-$–component of $\kappa(X, Y)$. The main point here is that if one splits κ according to homogeneous degree, then both ∂ and ∂^* preserve homogeneous degrees and the Bianchi identity expresses the composition of ∂ with some homogeneous component in terms of homogeneous components of lower degree. (This also is an essential input for showing that the curvature vanishes if its harmonic part vanishes.) The main application of the Bianchi identity is that it allows to draw consequences from the vanishing of certain components of the curvature.

1.6. To apply the general tools described in 1.5 above to our case, we have to start by computing the *real* cohomology $H^2(\mathfrak{g}_-, \mathfrak{g})$ using Kostant's version of the Bott–Borel–Weil theorem. Since we are dealing with the real cohomology of a complex Lie algebra, this splits as $H^2(\mathfrak{g}_-, \mathfrak{g}) = \oplus_{p+q=2;\ell} H^{p,q}_{(\ell)}(\mathfrak{g}_-, \mathfrak{g})$, where the upper indices refer to (p, q)–types and the lower index refers to the homogeneity. Note that in our case B_0 is abelian, so any irreducible component in a B_0–representation is one–dimensional.

The computation of the cohomology is carried out in [7], although the splitting into (p, q)–types is not noted explicitly there. The result is, that there are eight nontrivial irreducible components in $H^2(\mathfrak{g}_-, \mathfrak{g})$, six of which consist of maps having values in $\mathfrak{g}_-$, so they correspond to parts of κ which have the character of torsions. The remaining two components are contained in $H^{2,0}_{(4)}(\mathfrak{g}_-, \mathfrak{g})$ and represented by maps $\mathfrak{g}_{-2} \times \mathfrak{g}^{\pm}_{-1} \to \mathfrak{g}^{\pm}_1$. They are similar to the two curvatures of a 3–dimensional CR–manifold of hypersurface type. They could be interpreted using analogues of Webster–Tanaka connections, but we will not go into this aspect here. Rather we will concentrate on the analysis of the torsion type components, which can be described in more detail as follows: There are four nonzero irreducible components in $H^{1,1}_{(1)}(\mathfrak{g}_-,\mathfrak{g})$, which are represented on one hand by maps $\Lambda^2\mathfrak{g}^{\pm}_{-1}\to\mathfrak{g}^{\mp}_{-1}$ (which automatically must be totally real) and on the other hand by linear maps $\mathfrak{g}^+_{-1}\otimes\mathfrak{g}^-_{-1}\to\mathfrak{g}^-_{-1}$ and $\mathfrak{g}^-_{-1}\otimes\mathfrak{g}^+_{-1}\to\mathfrak{g}^+_{-1}$, which are complex linear in the first, and conjugate linear in the second variable. Finally, there are two nonzero irreducible components contained in $H^{0,2}_{(1)}(\mathfrak{g}_-,\mathfrak{g})$, which are represented by maps $\mathfrak{g}_{-2}\otimes\mathfrak{g}^{\pm}_{-1}\to\mathfrak{g}_{-2}$.

1.7. To start interpretation of the torsion–type components let us first consider the Nijenhuis tensor $\tilde N: T^{CR}M\times T^{CR}M\to T^{CR}M$ introduced in 1.1. As we we have noted there, it is skew symmetric and conjugate linear with respect to $\tilde J$ in both arguments. In particular, this implies that it restricts to zero on $T^{CR+}M\times T^{CR+}M$ and $T^{CR-}M\times T^{CR-}M$, so we are left with a linear map $\tilde N: T^{CR+}M\otimes T^{CR-}M\to T^{CR}M$. Splitting according to values, we get $\tilde N=\tilde N^+ +\tilde N^-$, and taking into account the definition of the almost complex structure J it follows that conjugate linearity in both arguments with respect to $\tilde J$ implies that the linearity properties of $\tilde N^{\pm}$ with respect to J are exactly the same as the linearity properties of the two sesquilinear components of $H^{1,1}_{(1)}(\mathfrak{g}_-,\mathfrak{g})$. Since the Nijenhuis tensor is an algebraic expression obtained as a combination of Lie brackets we can directly use lemma 1.5 to see that (up to a nonzero factor) the tensors $\tilde N^{\pm}$ exactly represent the two harmonic components of κ corresponding to the two sesquilinear components of $H^{1,1}_{(1)}(\mathfrak{g}_-,\mathfrak{g})$. In particular, vanishing of both these harmonic curvature components is equivalent to M being CR.

Next, by construction, $T^{CR+}M\subset T^{CR}M$ is isotropic for the Levi–bracket $\mathcal{L}$, which implies that for smooth sections ξ and η of $T^{CR+}M$, the Lie bracket $[\xi,\eta]$ is a section of $T^{CR}M$. Consequently, we may project this bracket to $T^{CR-}M$, thus obtaining a skew symmetric tensorial map $T^+: T^{CR+}M\times T^{CR+}M\to T^{CR-}M$. Again, this is an algebraic operation constructed from Lie brackets, so lemma 1.5 can be directly applied to show that T^+ represents the harmonic component of κ corresponding to totally real maps $\Lambda^2\mathfrak{g}^+_{-1}\to\mathfrak{g}^-_{-1}$. Clearly, vanishing of this component is equivalent to integrability of the subbundle $T^{CR+}M\subset T^{CR}M\subset TM$. Exchanging $+$ and $-$ one gets a tensor T^- which represents the last harmonic component of κ corresponding having values in $H^{1,1}(\mathfrak{g}_-,\mathfrak{g})$.

1.8. The interpretation of the harmonic curvature components of type $(0,2)$ is more difficult. The first thing to observe here is, that from the fact that we have a Cartan connection with values in a complex Lie algebra on a principal bundle over M with complex structure group, it follows that we get an induced almost complex structure J on M, which is compatible with the structures J on $T^{CR}M$ and J^Q on QM. If $\hat J$ is any almost complex structure compatible with J and J^Q, then for a vector field ξ on M and a smooth section η of $T^{CR}M$ consider the expression $q([\hat J\xi,\eta])-J^Qq([\xi,\eta])$. One immediately verifies that this is bilinear over smooth functions and depends only on $q(\xi)$ so via the splitting of $T^{CR}M$ it induces two bundle maps $S^{\pm}: QM\otimes T^{CR\pm}M\to QM$ (associated to the almost complex structure $\hat J$). By construction, for any $\hat J$ the maps $S^{\pm}$ are conjugate linear in the first variable.

Proposition. *The almost complex structure J on M induced by the canonical Cartan connection is the unique almost complex structure compatible with J on $T^{CR}M$ and J^Q on QM such that the corresponding tensors $S^\pm$ are conjugate linear in the second argument. Moreover, up to a nonzero factor, these two tensors exactly represent the harmonic components of κ corresponding to $H^{0,2}(\mathfrak{g}_-, \mathfrak{g})$. Finally, the almost complex structure J on M is integrable, i.e. M is a complex manifold, if and only if both these harmonic curvature components vanish.*

Sketch of proof. Since the tensors $S^\pm$ are constructed from Lie brackets, one may apply lemma 1.5 directly to see on one hand that the almost complex structure J induced by the Cartan connection has the property that $S^\pm$ are conjugate linear in both arguments and on the other hand that they represent the appropriate harmonic curvature components. That this characterises the almost complex structure J can be easily verified directly.

To verify the last statement, one first shows that the Nijenhuis tensor of J is exactly induced by the part of κ which is conjugate linear in both arguments and has values in $\mathfrak{g}_-$.The above observations and the Bianchi identity imply that vanishing of that part is equivalent to vanishing of S^+ and S^-. □

It should be remarked at this place that there are special results in the embedded case. If M is a real analytic submanifold of $\mathbb{C}^4$ of codimension two, such that the induced CR–structure on M is non–degenerate and elliptic. Then clearly $\tilde{N}^\pm = 0$, but it turns out that one also must have $T^\pm = 0$. This is proved using an osculation by the flat model (a quadric) provided by a simple normal form argument, see [7], where this osculation is used as the basis for the construction of the parabolic geometry. Thus, in the embedded case, the tensors $S^\pm$ (and hence integrability of the almost complex structure J, which is not directly related to the ambient complex structure) are the only torsion–type obstructions against local flatness.

1.9. To conclude the discussion of torsions, we give a geometric interpretation of torsion freeness, i.e. the fact that κ has values in $\mathfrak{b}$, which by the discussion above is equivalent to simultaneous vanishing of $\tilde{N}^\pm$, $T^\pm$ and $S^\pm$. To interpret this, note first that the Cartan connection provides a linear isomorphism between each tangent space of the total space $\mathcal{G}$ of the canonical principal bundle and a complex vector space, and thus an almost complex structure $J^{\mathcal{G}}$ on $\mathcal{G}$.

Theorem. *Let $p : \mathcal{G} \to M$ be the canonical B–principal bundle. If $\tilde{N}^\pm = T^\pm = S^\pm = 0$, then the almost complex structure $J^{\mathcal{G}}$ is integrable, so $\mathcal{G}$ is a complex manifold. Moreover, in this case $p : \mathcal{G} \to M$ is a holomorphic principal B–bundle. Finally in this case the Cartan connection $\omega \in \Omega^{1,0}(\mathcal{G}, \mathfrak{g})$ is a holomorphic form, so $(p : \mathcal{G} \to M, \omega)$ is a complex parabolic geometry of type $(PSL(3, \mathbb{C}), B)$. Conversely, any complex parabolic geometry of that type is torsion free when viewed as a real parabolic geometry.*

Sketch of proof. It is easy to see that the Nijenhuis tensor of $J^{\mathcal{G}}$ is induced by the component of κ which is conjugate linear in both arguments. By the Bianchi identity, torsion freeness implies that nonzero homogeneous components of κ have degree at least four and the the degree four part is complex bilinear, see 1.6. The only possible homogeneity for a component which is conjugate linear in both arguments would be five, but a simple application of the Bianchi identity shows that this component has to vanish. Holomorphicity of the principal bundle is then a simple consequence of the construction. Holomorphicity of ω is easily seen to be equivalent to the fact that κ has values in complex bilinear mappings only. Proving this requires a pretty involved application of the Bianchi identity. The last statement is a simple consequence of the fact that the complex cohomology of $\mathfrak{g}_-$ with values in $\mathfrak{g}$ coincides with $H^{2,0}(\mathfrak{g}_-,\mathfrak{g})$. $\square$

1.10. **Relations to projective structures.** To finish, we outline a surprising relation between certain projective structures and elliptic partially integrable almost CR–structures. First we need a few definitions on projective structures.

Let (N, J^N) be an almost complex manifold and let ∇ and $\hat{\nabla}$ be linear connections on the tangent bundle TN of N. Then ∇ and $\hat{\nabla}$ are said to be *projectively equivalent* if and only if there is a smooth $(1,0)$–form Υ on N such that $\hat{\nabla}_\xi\eta = \nabla_\xi\eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$. Note that this differs from the usual (real) version of projective equivalence, since $\Upsilon(\xi), \Upsilon(\eta) \in \mathbb{C}$. By $[\nabla]$ we denote the projective equivalence class of ∇. From the definition it easily follows that projectively equivalent connections have the same torsion, and if $\nabla J^N = 0$ then the same is true for all projectively equivalent connections, so we can talk about projective equivalence classes which are compatible with J^N.

A *normal projective structure* on an almost complex manifold (N, J^N) is then defined to be the choice of a projective equivalence class $[\nabla]$ which is compatible with J^N and whose torsion $TN \times TN \to TN$ is conjugate linear in both arguments.

Elementary arguments show that on any almost complex manifold there exist many normal projective structures. More precisely, these structures form an affine space modelled on the space of smooth sections of the bundle $(S^2_{\mathbb{C}}T^*N \otimes TN)_0$, so these structures are very easy to understand (compared for example to elliptic partially integrable almost CR–structures).

Finally, if (N, J^N) is an almost complex manifold, then we define the *correspondence space* $\mathcal{C}N$ of N to be the complex projectivisation of the tangent bundle of N. Thus, $\mathcal{C}N \to N$ is a locally trivial fiber bundle with fiber a complex projective space and a point $u \in \mathcal{C}N$ lying over $x \in N$ is just a complex line $u \subset T_xN$.

Theorem. *Let (N, J^N) be a smooth almost complex manifold of real dimension four with correspondence space $\mathcal{C}N$. Then any choice of a normal projective structure $[\nabla]$ on N makes $\mathcal{C}N$ canonically into an elliptic partially integrable almost CR manifold of CR–dimension and codimension two, which has the property that $\tilde{N}^\pm$, T^-, S^- and the harmonic component of κ corresponding to complex bilinear maps $\mathfrak{g}_{-2}\otimes\mathfrak{g}_{-1}^- \to \mathfrak{g}_1^-$ vanish. The three remaining harmonic components of the curvature of $\mathcal{C}N$ are directly related to projective curvatures and the torsion on N. Finally, the group of CR–automorphisms of $\mathcal{C}N$ coincides with the group of projective automorphisms of N.*

Sketch of proof. Besides the $|2|$–grading induced by the Borel subalgebra, the Lie algebra $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$ also has a $|1|$–grading corresponding to the parabolic subalgebra $\mathfrak{p}$ given by block upper triangular matrices with blocks of sizes 1 and 2. If $P \subset G$ is the corresponding parabolic subgroup (so $G/P = \mathbb{C}P^2$), one proves that a real parabolic geometry of type (G, P) on a smooth manifold N is the same thing as an almost complex structure J^N plus a normal projective structure $[\nabla]$ on (N, J^N).

In particular, this implies that given a normal projective structure $[\nabla]$ on (N, J^N) we get a P–principal bundle $\mathcal{G} \to N$ endowed with a normal Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Now by construction, $B \subset P$ is a closed subgroup, so we may form the orbit space $\mathcal{G}/B$, which turns out to be canonically isomorphic to $\mathcal{C}N$. The canonical projection $\mathcal{G} \to \mathcal{G}/B$ is a B–principal bundle and ω is a Cartan connection on that bundle, which is easily seen to be normal. By theorem 1.4, $\mathcal{C}N$ is an elliptic partially integrable almost CR manifold of CR–dimension and codimension two. Moreover, the curvature of the projective structure on N and of the almost CR structure on $\mathcal{C}N$ are represented by the same function κ on $\mathcal{G}$. The automatic vanishing of CR curvature components is then due to the fact that some tangent vectors are not vertical on $\mathcal{G} \to \mathcal{C}N$ but are vertical on $\mathcal{G} \to N$, so κ has to vanish on them.

Finally, both the projective automorphisms of N and the CR–automorphisms of $\mathcal{G}/B$ coincide with the group of diffeomorphisms $\Phi : \mathcal{G} \to \mathcal{G}$ such that $\Phi^*\omega = \omega$. □

Remarks. (1) The almost CR structure on $\mathcal{C}N$ induced by a normal projective structure on N can be easily described explicitly.
(2) There are strong indications that a converse of the above theorem holds as well, i.e. that if $(M, T^{CR}M, \tilde{J})$ is an elliptic partially integrable almost CR manifold of CR–dimension and codimension two which satisfies the curvature restrictions stated in the theorem, then it is locally CR–diffeomorphic to the correspondence spaces of an almost complex manifold endowed with a normal projective structure.

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