Poisson transforms, the BGG complex and discrete series representations

Andreas Čap

University of Vienna
Faculty of Mathematics

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- This talk reports on joint work with C. Harrach (Vienna) and P. Julg (Orleans).
- The main motivation for this work comes from Julg's program for proving the Baum–Connes conjecture for discrete subgroups of semi-simple Lie groups of real rank one. Details of this motivation are not important, it all boils down to constructing Poisson transforms with specific properties.
- Our current results concern transform defined on the Rumin complex of the CR sphere and hence concern the group $SU(n+1,1)$, but there are good chances that things carry over to $Sp(n+1,1)$ for which the conjecture is open.
- Some of the results are valid in a much more general setting and even have the potential for curved analogs. They fit into the general framework of understanding geometric compactifications, boundary asymptotics, etc.

Contents

The idea to define Poisson transforms for differential forms goes back to work of P.Y. Gaillard that aimed at constructing co-closed harmonic differential forms on real hyperbolic space. Putting $G = SO_0(n+1,1)$ hyperbolic space is G/K and the conformal sphere is G/P, where K is maximally compact and P is parabolic.

Gaillard used the Poincaré ball model to geometrically define G-invariant differential forms φ_k on $G/K \times G/P$ (a "Poisson kernel") of bi-degree $(k, n-k)$ for $k = 0, \dots, n$. Such a form then induces a G-equivariant transform $\Phi_k : \Omega^k(G/P) \to \Omega^k(G/K)$ the sends α to the integral over the fiber G/P of $\varphi_k \wedge \alpha$. It is then possible to analyze $\Phi_k \circ d_P$, $d_K \circ \Phi_k$ and $\delta_K \circ \Phi_k$ and thus prove that one indeed obtains co-closed, harmonic forms.

Using representation theory, Gaillard also analyzed the boundary asymptotics of the values of the transform, which crucially depends on the degree of forms. For low degrees, they are smooth up to the boundary, while for high degrees they go to infinity.

The behavior is particularly interesting in the case that $n = 2m-1$ is odd (so the interior has even dimension) in the middle degree. It turns out that is this case the values of $d \circ \Phi_{m-1} : \Omega^{m-1}(G/P) \to \Omega^m(G/K)$ are not bounded but L^2. This is remarkable since under weak assumptions L^2-harmonic forms can only occur in middle degree on non-compact, even dimensional manifolds.

The subspace of L^2-harmonic forms in $\Omega^m(G/K)$ is G-invariant, thus providing a unitary representation of G. It turns out that this splits into a direct sum of two irreducible components (SD and ASD forms) which are exactly the two irreducible discrete series representations of $SO_0(2m,1)$ with trivial infinitesimal character.

Building on Gaillard's work, J. Lott used the Poisson transform to describe these discrete series representations as a space of differential forms on the sphere of Sobolev regularity $H^{-1/2}$.

There were attempts to generalize Gaillard's work to the complex case, but it quickly turns out that the real case is deceivingly simple. The algebraic tools that we will introduce next imply that the forms used by Gaillard are essentially unique, which explains why they are so nicely compatible with the available operations.

Already in the complex case, there are several possible choices for Poisson kernels of fixed bi-degree and the resulting transforms do not produce harmonic forms in general. Using the algebraic approach described below, C. Harrach studied the available transforms in the complex case in detail in his thesis.

Based on these results, a recent article of ours constructs a family of transforms for the complex case that produces co-closed harmonic values and descends to the Rumin complex in the sense discussed below. However, this is heavy on computations, so the possibility of generalizing to the quaternionic case is questionable and more efficient methods are needed.

Harmonic values and BGG (C. Harrach)

This works for general G and P, even after twisting with tractor bundles. The filtration on $T := T(G/P)$ gives rise to natural bundle maps $\partial^* : \Lambda^k T^* \to \Lambda^{k-1} T^*$, the subquotient bundles $\mathcal{H}_k := \ker(\partial^*)/\operatorname{im}(\partial^*)$ ("homology bundles") and tensorial operations on $\Omega^*(G/P)$.

If $\Phi : \Omega^k(G/P) \to \Omega^\ell(G/K)$ satisfies $\Phi \circ \partial^* = 0$, then it descends to $\underline{\Phi} : \Gamma(\mathcal{H}_k) \to \Omega^\ell(G/K)$. BGG calculus shows that if in addition $\Phi \circ d \circ \partial^* = 0$, then Φ is completely determined by $\underline{\Phi}$.

The Laplacian on G/K is closely related to the Casimir element of G. On G/P, the relation of the Casimir to BGG is described in joint work of V. Souček and myself. Using this, C. Harrach proved that all the values of a Poisson transform Φ are harmonic if and only if $\Phi \circ \partial^* = 0$ and $\Phi \circ d \circ \partial^* = 0$.

An algebraic approach

For any (G,P), it is well known that K acts transitively on G/P. Hence $G/K \times G/P = G/M$, where $M = K \cap P$, so Poisson kernels admit an algebraic description. The product structure leads to $(\mathfrak{g}/\mathfrak{m})^* = (\mathfrak{g}/\mathfrak{k})^* \oplus (\mathfrak{g}/\mathfrak{p})^*$ and a splitting of $\Lambda^r(\mathfrak{g}/\mathfrak{m})^*$ by bi-degree.

Poisson kernels of bi-degree $(k, n-\ell)$ thus are in bijective correspondence with the space of M-invariants
$(\Lambda^{\ell,n-k}(\mathfrak{g}/\mathfrak{m})^*)^M \cong \mathrm{Hom}_M(\Lambda^k(\mathfrak{g}/\mathfrak{p})^*, \Lambda^\ell(\mathfrak{g}/\mathfrak{k})^*)$.
The isomorphisms follows since M is contained in the semisimple part of the Levi factor $G_0 \subset P$ and hence acts trivially on $\Lambda^n(\mathfrak{g}/\mathfrak{p})^*$.

Thus Poisson kernels can be completely described via representation theory. In addition, the action of any G-invariant differential operator on a G-invariant form admits an algebraic description, which helps understanding the properties of Poisson transforms.

The relation to the BGG machinery can be nicely phrased in this algebraic language. Since $M \subset G_0$, Kostant's harmonic theory gives rise to an M-invariant decomposition $\Lambda^k(\mathfrak{g}/\mathfrak{p})^* = \operatorname{im}(\partial) \oplus \ker(\Box) \oplus \operatorname{im}(\partial^*)$. We show that a transform has harmonic values iff the corresponding M-equivariant map is non-zero on the middle summand only.

If $\mathfrak{g}$ has real rank one, then one directly gets a family of distinguished homomorphisms with this property. Via the Killing form of $\mathfrak{g}$, one gets $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$ and $(\mathfrak{g}/\mathfrak{k})^* = \mathfrak{q}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ is the Cartan decomposition. For real rank one, $\mathfrak{a} := \mathfrak{q} \cap \mathfrak{g}_0$ is the line spanned by the grading element, and via $Z \mapsto \theta(Z) - Z$, $\mathfrak{q} \cong \mathfrak{a} \oplus \mathfrak{p}_+$ as a representation of M.

This gives an M-homomorphism $\Lambda^k(\mathfrak{g}/\mathfrak{p})^* \to \Lambda^k(\mathfrak{g}/\mathfrak{k})^*$. Moreover, it is well known that $\ker(\Box)$ is completely reducible as a representation of G_0, so it splits into a sum $\oplus_i \mathbb{E}^i_k$ of irreducibles.

The distinguished transforms

By restriction, we obtain a homomorphism $\mathbb{E}^i_k \to \Lambda^k(\mathfrak{g}/\mathfrak{k})^*$, which induces a Poisson transform $\Phi^i_k : \Omega^k(G/P) \to \Omega^k(G/K)$ that has harmonic values. The induced transform on $\Gamma(\mathcal{H}_k)$ is non-zero on one irreducible subbundle only. Further properties of these transforms can be verified algebraically:

Theorem

For the distinguished transforms Φ^i_k from the above, the values are co-closed (and harmonic). Moreover, there is a direct (simple) description for the homomorphism inducing the composition $d \circ \Phi^i_k$.

All these results work uniformly for all real rank one case, in particular they also apply in the quaternionic case, in which the Baum-Connes conjecture is still open.

The setup

Here $P \subset G$ is the stabilizer of an isotropic line in $\mathbb{C}^{n+2}$ and $P \cong CSU(n) \ltimes \mathfrak{h}_n$, where $\mathfrak{h}_n$ is the complex Heisenberg algebra of dimension $2n+1. G/P \cong S^{2n+1}$ is naturally as CR manifold (from embedding into $\mathbb{C}^{n+1}$). Thus we have a complex rank n subbundle $H \subset T := T(G/P)$ and we put $Q := T/H$. In addition H is contact, so the map $\mathcal{L} : H \times H \to Q$ induced by the Lie bracket of vector fields is non-degenerate.

Putting $\Lambda^2_0 H^* := \ker(\mathcal{L})$, there is a canonical complementary line subbundle in $\Lambda^2 H^*$, which gets identified with Q by $\mathcal{L}$. Dualizing the inverse map, one obtains a bundle map $\Lambda^2 H^* \to Q^*$. This extends to a map $\Lambda^k H^* \to \Lambda^{k-2} H^* \otimes Q^*$ which is surjective for $k \leq n$ and injective for $k \geq n$ and defines $\mathfrak{d}^* : \Lambda^k T^* \to \Lambda^{k-1} T^*$. This shows that $\mathcal{H}_k$ is isomorphic to $\Lambda^k_0 H^*$ for $k \leq n$ and to a quotient of $\Lambda^{k-1} H^* \otimes Q^*$ for $k > n$.

Passing to complex valued forms, the bundles $\Lambda^k H^*$ decompose and accordingly, we get $\mathcal{H}_k = \oplus_{p+q=k}\mathcal{H}_{p,q}$ via (p,q)-types. Moreover, it turns out that each of the bundles $\mathcal{H}_{p,q}$ is induced by an irreducible representation of P. The BGG operators turn out to decompose accordingly as $D = \mathcal{D} + \overline{\mathcal{D}}$.

Our general construction thus provides us with Poisson transforms $\Phi_{p,q}$ which map $(p+q)$-forms on G/P to co-closed harmonic forms on G/K, which turn out to lie in $\Omega^{(p,q)}(G/K)$. Via the algebraic methods I have discussed, one can analyze the compositions $\partial\circ\Phi_{p,q}$ and $\overline{\partial}\circ\Phi_{p,q}$ as well as $\Phi_{p,q}\circ\mathcal{D}$ and $\Phi_{p,q}\circ\overline{\mathcal{D}}$.

The first result that we obtain is that we can form appropriate linear combinations Φ_k of the transforms $\Phi_{p,q}$ with $p+q=k$ in such a way that Φ_k sends real forms on G/P to real forms on G/K and descends such that the induced maps on the real BGG complex satisfy $d\circ\Phi_{k-1} = \Phi_k\circ D$ for all $k\leq n$.

The Poincaré ball model

This realizes G/K as the unit ball in $\mathbb{C}^{n+1}$ (with the Bergmann metric) and G/P as the unit sphere. (It is the model for adding a boundary at infinity to a complete Kähler-Einstein metric.) In this picture, one can then study the asymptotic behavior of the values of the Poisson transforms we construct.

Using the polar decomposition of G/K, harmonic analysis for K-irreducible components in the sections of the BGG complex, and ODEs implied by forms being harmonic, we prove

- For $k \leq n$, any value of Φ_k allow a continuous extension to the boundary. On the BGG bundles the boundary values recover the initial form (up to a constant multiple).
- The transform $d \circ \Phi_n$ has values in L^2-harmonic forms on G/K (which exist only in this degree).

Now it is again known that L^2-harmonic forms on G/K realize the direct sum of the discrete series representations of $SU(n+1,1)$ with trivial infinitesimal character. After complexification, this decomposes into $n+2$ irreducibles according to (p,q)-type. Using this we arrive at our final result.

Theorem

The transform $d \circ \Phi_n$ descends to an isomorphism from $\Gamma(\mathcal{H}_n)/\operatorname{im}(D_n)$ onto a dense subspace of the space of L^2-harmonic forms on G/K. Hence the discrete series representations of $SU(n+1,1)$ with trivial infinitesimal character can be realized as a completion of a space of differtial forms on G/P with respect to an appropriate (explicitly computable) norm.

Using this to “cut off” the BGG complex, one obtains the complex needed for the applications towards the Baum-Connes conjecture.