Lecture 3: The infinitesimal automorphism equation for parabolic geometries

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- I will start by describing the general version of Kostant's Hodge theory for analyzing certain Lie algebra cohomology groups.
- This Hodge theory plays a central role in the construction of canonical Cartan connections, which I will describe in detail for conformal structures and outline for general parabolic geometries.
- This setup directly leads to a twisted de–Rham sequence which controls infinitesimal automorphisms and deformations on the level of Cartan geometries. Interpreting this sequence in terms of the underlying structures provides an archtypical example of the machinery of BGG sequences.

Contents

Structure

Let $\mathfrak{g}$ be a semisimple Lie algebra endowed with a grading

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+ = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k.$$

This is (loosely speaking) equivalent to specifying the *parabolic subalgebra* $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_+$.
We will later look at geometric structures determined by $(\mathfrak{g}, \mathfrak{p})$, important examples with the corresponding geometries are:

$\mathfrak{so}(n+1,1) = \mathbb{R}^n \oplus \mathfrak{cso}(n) \oplus \mathbb{R}^{n*}$ (conformal)
$\mathfrak{sl}(n+1,\mathbb{R}) = \mathbb{R}^n \oplus \mathfrak{gl}(n,\mathbb{R}) \oplus \mathbb{R}^{n*}$ (projective)
$\mathfrak{su}(n+1,1) = (\mathbb{R} \oplus \mathbb{C}^n) \oplus \mathfrak{csu}(n) \oplus (\mathbb{C}^n \oplus \mathbb{R})$ (CR).

Given a representation V of $\mathfrak{g}$, the playground for Kostant's Hodge structure is the space $\Lambda^*(\mathfrak{g}/\mathfrak{p})^* \otimes V$, and the key observation is that this admits two different Lie theoretic interpretations.

First, via the Killing form of $\mathfrak{g}$, we get $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{g}_+$ which leads to the Lie algebra homology differentials (traditionally called "Kostant codifferential") $\partial^* : \Lambda^k \mathfrak{g}_+ \otimes V \to \Lambda^{k-1}\mathfrak{g}_+ \otimes V$.

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$$\begin{aligned}\partial^*(Z_1 \wedge \cdots \wedge Z_k \otimes v) := {} & \textstyle\sum_i (-1)^i Z_1 \wedge \dots \widehat{Z_i} \dots \wedge Z_k \otimes Z_i \cdot v \\ & + \textstyle\sum_{i<j} (-1)^{i+j} [Z_i, Z_j] \wedge Z_1 \wedge \dots \widehat{Z_i} \dots \widehat{Z_j} \dots \wedge Z_k \otimes v\end{aligned}$$

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On the other hand, linearly and as a $\mathfrak{g}_0$–module, we have $(\mathfrak{g}/\mathfrak{p}) \cong \mathfrak{g}_-$. Hence we can identify $\Lambda^k(\mathfrak{g}/\mathfrak{p})^* \otimes V$ with multilinear alternating V–valued maps on $\mathfrak{g}_-$ and obtain the *Lie algebra cohomology differential* $\partial : \Lambda^k \mathfrak{g}_-^* \otimes V \to \Lambda^{k+1} \mathfrak{g}_-^* \otimes V$.

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$$\begin{aligned}\partial\varphi(X_0, \dots, X_k) &:= \textstyle\sum_i (-1)^i X_i \cdot \varphi(X_0 \dots, \widehat{X_i}, \dots, X_k)\\ &\quad + \textstyle\sum_{i<j} (-1)^{i+j} \varphi([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k).\end{aligned}$$

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Together we have two sequences of operators

$$\cdots \Lambda^{k-1}(\mathfrak{g}/\mathfrak{p})^* \otimes V \overset{\longleftarrow}{\longrightarrow} \Lambda^k(\mathfrak{g}/\mathfrak{p})^* \otimes V \overset{\longleftarrow}{\longrightarrow} \Lambda^{k+1}(\mathfrak{g}/\mathfrak{p})^* \otimes V \cdots$$

with leftward pointing arrows being ∂^*'s and rightward pointing arrows being ∂'s. Moreover, $\partial \circ \partial = 0$ and $\partial^* \circ \partial^* = 0$.

Kostant's results

- The operators ∂ and ∂^* are adjoint with respect to a natural inner product.
- Defining the *Kostant Laplcian* $\Box = \Box_k := \partial^* \circ \partial + \partial \circ \partial^*$ on $\Lambda^k(\mathfrak{g}/\mathfrak{p})^* \otimes V$ one obtains the Hodge decomposition

$$\Lambda^k(\mathfrak{g}/\mathfrak{p})^* \otimes V = \mathrm{Im}(\partial^*) \oplus \ker(\Box_k) \oplus \mathrm{Im}(\partial).$$

- The first two summands in the Hodge decomposition add up to $\ker(\partial^*)$, the last two to $\ker(\partial)$. In particular,

$$H_k(\mathfrak{g}_+, V) \cong \ker(\Box_k) \cong H^k(\mathfrak{g}_-, V),$$

  and this can be described algorithmically as a representation of $\mathfrak{g}_0$.

Structure

1. Kostant's Hodge theory
2. Canonical Cartan connections
3. A BGG sequence related to deformations

The conformal sphere

Consider $G := SO(n+1,1)$, the special orthogonal group of a Lorentzian vector space of dimension $n+2$, and let $P \subset G$ be the stabilizer of an isotropic line in $\mathbb{R}^{n+1,1}$. On the Lie algebra level, one gets a grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

The homogeneous space G/P can be identified with the space of null lines in $\mathbb{R}^{n+1,1}$ which is isomorphic to the sphere S^n, and G is the group of conformal isometries of S^n.

The *Poincaré conformal group* P is the group of conformal isometries of S^n which fix a distinguished point o. Taking the derivative in o, defines a surjective homomorphism $P \to CO(n) =: G_0$. The kernel of this is a subgroup $G_1 \subset P$ with Lie algebra $\mathfrak{g}_1$. One can also identify G_0 with the subgroup of those elements of P whose adjoint action preserves the grading of $\mathfrak{g}$ and $P = G_0 \ltimes G_1$.

Now suppose that M is an n–manifold endowed with a conformal structure $[g]$. Then there is a natural frame bundle $\mathcal{G}_0 \to M$ with structure group $CO(n) = G_0$. Now we tautologically extend the structure group of this bundle to P by forming $\mathcal{G} := \mathcal{G}_0 \times_{G_0} P$.

Next, look for *Cartan connections* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, i.e. P–equivariant trivializations of $T\mathcal{G}$, which reproduce the generators of fundamental vector fields. Via $\mathcal{G}_0 = \mathcal{G}/G_1$ any Cartan connection on $\mathcal{G}$ descends to a soldering form $\theta \in \Omega^1(\mathcal{G}_0, \mathfrak{g}/\mathfrak{p})$, and one requires that ω descends to the canonical soldering form on the frame bundle.

Remark: In standard EDS–language, one would view θ (and its pull–back to $\mathcal{G}$) as an n–tuple (θ^i) of 1–forms and ω as a matrix (ω^i_j) of 1–forms of size $n+2$. The fact that ω is $\mathfrak{g}$–valued is then expressed by relations among the ω^i_j. The fact that ω descends to θ is expressed by formulae for some ω^i_j in terms of θ^k's.

Local existence of such Cartan connections is easily seen. To ensure uniqueness (which then also implies that local connections fit together) on has to impose *normalization conditions* on the curvature.

Definition

The *curvature* $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is defined by $K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$.
This is horizontal and P–equivariant, and thus completely captured by the *curvature function* $\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, which is characterized by $K(\xi, \eta) = \kappa(\theta(\xi), \theta(\eta))$.

If ω is a Cartan connection, then all Cartan connections lifting the same soldering form are parametrised by P–equivariant functions $\mathcal{G} \to L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g}_0 \oplus \mathfrak{g}_1)$.

Denoting the function by Φ, the relation is determind by

$$\hat{\omega}(\xi) = \omega(\xi) + \Phi(\theta(\xi)).$$

By construction $\hat{\omega} \equiv \omega \mod \mathfrak{p}$ and thus also $d\hat{\omega}(\xi, \eta) \equiv d\omega(\xi, \eta)$. On the other hand, modulo $\mathfrak{p}$ we get

$$[\hat{\omega}(\xi), \hat{\omega}(\eta)] - [\omega(\xi), \omega(\eta)] \equiv [\omega(\xi), \Phi(\theta(\eta))] + [\Phi(\theta(\xi)), \omega(\eta)].$$

We may further replace the ω's in the right hand side by θ's and the Φ's by their components Φ_0 in $L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g}_0)$, so modulo $\mathfrak{p}$, we get:

$$\hat{\kappa}(X, Y) \equiv \kappa(X, Y) + (\partial \circ \Phi_0)(X, Y) \mod \mathfrak{p}.$$

Using Kostant's hodge theory, one sees that, given ω, the map Φ can be chosen in such a way that $\hat{\kappa}$ is congruent to an element of $\ker(\partial^*)$ modulo $\mathfrak{p}$–valued maps.

Assuming that ω already has this property, one can do the same thing with $\Phi : \mathcal{G} \to L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g}_1)$, computing curvatures modulo $\mathfrak{g}_1$. This shows that $\hat{\kappa}$ can be made congruent to an element of $\ker(\partial^*)$ modulo $\mathfrak{g}_1$–valued maps. Since these are automatically in $\ker(\partial^*)$ we get:

Any first order G_0–structure is induced by a Caran connection ω which is *normal* in the sense that $\partial^* \circ \kappa = 0$.

To study uniqueness, assume that we have two principal P–bundles endowed with normal Cartan connections. Using a cohomological condition which can be verified using Kostant's theorem one deduces:

Any morphism of the underlying G_0–principal bundles preserving the induced soldering forms uniquely lifts to an morphism of the Cartan geometries. Thus passing to the normal Cartan geometry is an equivalence of categories.

Harmonic curvature

If ω is a normal Cartan connection on $\mathcal{G}$, then by definition its curvature function $\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ has values in the subspace $\ker(\partial^*)$. Thus we can project its values into $\ker(\partial^*)/\operatorname{im}(\partial^*) \cong H^2(\mathfrak{g}_{-1}, \mathfrak{g})$.

The resulting geometric object is called the *harmonic curvature* κ_H of the geometry. Here, this produces the Weyl curvature respectively the Cotton–York tensor.

Using the Bianchi–identity one shows that the lowest non–zero component of κ always has values in $\ker(\partial)$ thus obtaining:

Proposition (Tanaka)

If κ_H vanishes, then κ vanishies, so the geormetry is locally isomorphic to G/P.

Tanaka theory

In pioneering work in the 1960's and 70's N. Tanaka showed that the Kostant codifferential provides an appropriate normalization condition for general parabolic subalgebras in semisimple Lie algebras. This leads to a class of geometric structures called *parabolic geometries*. Some modifications are necessary:

- The underlying structure now involves a filtration of the tangent bundle with prescribed symbol algebras and a reduction of the associated graded of the tangent bundle.
- The filtrations involved lead to a natural notion of *homogeneity* for $\mathfrak{g}$–valued differential forms. In the procedure one has to look at components of some fixed homogeneity rather than sorting components by their values.
- The iterative process has to be run more often.

Structure

We have already observed that the space of Cartan connections on a principal P–bundle has an affine structure. More formally:

If ω and $\hat{\omega}$ are Cartan connections on $\mathcal{G}$ then $\hat{\omega} - \omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is horizontal and P–equivariant.
Conversely, if $\psi \in \Omega^1_{hor}(\mathcal{G}, \mathfrak{g})^P$ then $\omega + \psi$ is a Cartan connection provided that its restriction to each tangent space is injective.

Likewise, we have seen that the curvature K of ω lies in $\Omega^2_{hor}(\mathcal{G}, \mathfrak{g})$.

Fixing a Cartan connection ω, we get an isomorphism

$$\Omega^k_{hor}(\mathcal{G}, \mathfrak{g})^P \cong \Omega^k(M, \mathcal{A}M) \quad \text{where } \mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}.$$

$\mathcal{A}M$ is called the *adjoint tractor bundle*. Via ω, we can identify $\Gamma(\mathcal{A}M) \cong \mathfrak{X}(\mathcal{G})^P$, the space of P–invariant vector fields on $\mathcal{G}$.

Given ω, we can thus identify $\Gamma(\mathcal{A}M)$ with the space of ifnititesimal principal bundle automorphisms and $\Omega^1(M, \mathcal{A}M)$ with the tangent space at ω to the space of all Cartan connections. The infinitesimal change of ω caused by $\xi \in \mathfrak{X}(\mathcal{G})^P$ is of course given by the Lie derivative $\mathcal{L}_\xi\omega$. Now we get

Proposition

For $\xi \in \mathfrak{X}(\mathcal{G})^P$ we get $\mathcal{L}_\xi\omega \in \Omega^1_{hor}(\mathcal{G}, \mathfrak{g})^P$. Viewing $\xi \mapsto \mathcal{L}_\xi\omega$ as an operator $\Gamma(\mathcal{A}M) \to \Omega^1(M, \mathcal{A}M)$, it defines a linear connection $\tilde{\nabla}$ on $\mathcal{A}M$.

Likewise, one can interpret $\varphi \in \Omega^1(M, \mathcal{A}M)$ as an infinitesimal deformation of the Cartan connection ω and compute the resulting infinitesimal change of curvature, which is an element of $\Omega^2(M, \mathcal{A}M)$. It turns out that this is $d^{\tilde{\nabla}}\varphi$, where $d^{\tilde{\nabla}}$ is the covariant exterior derivative associated to $\tilde{\nabla}$.

Thus the *twisted de–Rham sequence* $(\Omega^*(M, \mathcal{A}M), d^{\tilde{\nabla}})$ associated to the linear connection $\tilde{\nabla}$ admits an interpretation as a deformation sequence on the level of the Cartan geometry $(\mathcal{G}, \omega)$:

- The kernel of the first operator is the space of infinitesimal automorphism while its image is the space of trivial infinitesimal deformations.
- If $\kappa = 0$, then $\tilde{\nabla}$ is flat, so we obtain a complex. The kernel of the second operator is the space of infinitesimal *flat* deformations, and the cohomology in degree one is the formal tangent space to the moduli space of flat structures.

Suppose that we start with a normal Cartan connection determined by some underlying structure. Then the categorical equivalence implies that deformations in the sub–category of normal Cartan connections should be equivalent to deformations of the underlying structure. This is implemented by the BGG machinery.

To formalize this, we first observe that ∂^* defines $\mathfrak{p}$–equivariant maps on the spaces $\Lambda^k \mathfrak{g}_+ \otimes \mathfrak{g}$, so there are induced bundle maps $\partial^* : \Lambda^k T^*M \otimes \mathcal{A}M \to \Lambda^{k-1} T^*M \otimes \mathcal{A}M$ and hence

- $\operatorname{im}(\partial^*) \subset \ker(\partial^*) \subset \Lambda^k T^*M \otimes \mathcal{A}M$ are natural subbundles
- the quotient bundle $\mathcal{H}_k := \ker(\partial^*)/\operatorname{Im}(\partial^*)$ is isomorphic to $\mathcal{G} \times_P H^k(\mathfrak{g}_-, \mathfrak{g})$.

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In the conformal case, $\mathcal{H}_0 = TM$, $\mathcal{H}_1 \cong S^2_0 T^*M$ and $\mathcal{H}_2$ is the bundle of Weyl tensors respectively Cotton–York tensors.

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From what we have discussed above, it is clear that $\varphi \operatorname{im} \Omega^1(M, \mathcal{A}M)$ is a normal infinitesimal deformation iff $\partial^*(d^{\tilde{\nabla}}\varphi) = 0$. Denoting by $\pi_H : \ker(\partial^*) \to \mathcal{H}_k$ the natural projections, $\pi_H(d^{\tilde{\nabla}}\varphi)$ is the infinitesimal change of harmonic curvature caused by φ.
The second crucial ingredient is that $d^{\tilde{\nabla}}$ preserves homogeneities and its lowest homogeneous component is tensorial and induced by ∂. (Technically speaking, the "lowest homogeneous component" is the induced operator on sections of the associated graded bundles.)

This implies that certain aspects of Kostant's Hodge decomposition carry over to the curved setting. Defining $\tilde{\square}^R := d^{\tilde{\nabla}}\partial^* + \partial^* d^{\tilde{\nabla}}$, which is an operator on $\Omega^r(M, \mathcal{A}M)$, one shows

Theorem

1. $\ker(\tilde{\square}^R) = \ker(\partial^*) \cap \ker(\partial^* d^{\tilde{\nabla}})$.
2. Given $\varphi \in \Omega^k(M, \mathcal{A}M)$ there is $\psi \in \Omega^{k-1}(M, \mathcal{A}M)$ such that $\partial^*(\varphi + d^{\tilde{\nabla}}\psi) = 0$. One may use $\psi = \tilde{Q}(\partial^*\varphi)$ for a differential operator $\tilde{Q}$ on $\Gamma(\operatorname{im}(\partial^*))$ which is a polynomial in $\tilde{\square}^R$.
3. Given $\alpha \in \Gamma(\mathcal{H}_k)$ there is a unique section $\psi \in \ker(\tilde{\square}^R)$ such that $\pi_H(\psi) = \alpha$. This defines a differential operator $\tilde{S} : \Gamma(\mathcal{H}_k) \to \ker(\tilde{\square}^R) \subset \Omega^k(M, \mathcal{A}M)$ which can be written as a (universal) polynomial in $\tilde{\square}^R$. ("splitting operators")

Having the splitting operators at hand, we can define the *BGG operators* $\tilde{D} = \tilde{D}_k : \Gamma(\mathcal{H}_k) \to \Gamma(\mathcal{H}_{k+1})$ by $\tilde{D}(\alpha) := \pi_H(d^{\tilde{\nabla}}\tilde{S}(\alpha))$.
Having these at hand, we can now interpret the deformation sequence in terms of the underlying structure:
First we need a fact specific to this case, expressing that trivial deformations of normal geometries are normal:

Having the splitting operators at hand, we can define the *BGG operators* $\tilde{D} = \tilde{D}_k : \Gamma(\mathcal{H}_k) \to \Gamma(\mathcal{H}_{k+1})$ by $\tilde{D}(\alpha) := \pi_H(d^{\tilde{\nabla}}\tilde{S}(\alpha))$. Having these at hand, we can now interpret the deformation sequence in terms of the underlying structure:

For $s \in \Gamma(\mathcal{A}M)$, we have $\partial^* d^{\tilde{\nabla}}\tilde{\nabla}s = 0. \qquad (*)$

Now we can prove that $\tilde{D}_0$ is the infinitesimal automorphism operator and $(\tilde{\nabla}, \tilde{S})$ provides a prolongation:

$\{s \in \Gamma(\mathcal{A}M) : \tilde{\nabla}s = 0\} \cong \ker(\tilde{D}_0)$ via π_H and $\tilde{S}$

Proof: For $s \in \Gamma(\mathcal{A}M)$ we always have $\partial^* s = 0$, so we can form $\sigma = \pi_H(s) \in \Gamma(\mathcal{H}_0)$. If $\tilde{\nabla}s = 0$, then $\tilde{\square}^R(s) = 0$, so $s = \tilde{S}(\sigma)$ and $\tilde{D}_0(\sigma) = \pi_H(\tilde{\nabla}s) = 0$. Conversely, for $\sigma \in \Gamma(\mathcal{H}_0)$ consider $s = \tilde{S}(\sigma)$, which by definition satisfies $0 = \partial^*\tilde{\nabla}s$. But then $(*)$ implies that $\tilde{\square}^R(\tilde{\nabla}s) = 0$, so $\tilde{\nabla}s = \tilde{S}(\pi_H(\tilde{\nabla}s)) = \tilde{S}(D_0(\sigma))$.

At the level of one–forms, we can first show:

For $\alpha \in \Gamma(\mathcal{H}_1)$, $\tilde{S}(\alpha) \in \Omega^1(M, \mathcal{A}M)$ is a normal infinitesimal deformation, and $\tilde{D}_1(\alpha)$ is the infinitesimal change of harmonic curvature caused by this. Morover, any normal deformation is equivalent modulo trivial deformations to one of this form.

Proof: $0 = \partial^* d^{\tilde{\nabla}} \tilde{S}(\alpha)$ holds by definition, and since $\tilde{D}_1(\alpha) = \pi_H(d^{\tilde{\nabla}} \tilde{S}(\alpha))$ the first part follows. For the second part, given $\varphi \in \Omega^1(M, \mathcal{A}M)$, put $\psi := \varphi + \tilde{\nabla} \tilde{Q}(\partial^* \varphi)$, which by construction satisfies $\partial^* \psi = 0$. Moreover, by $(*)$ the deformation ψ is again normal. But this shows $\tilde{\Box}^R(\psi) = 0$ and thus $\psi = \tilde{S}(\pi_H(\psi))$.

Having this at hand, similar arguments as before show that

The quotient of normal infinitesimal deformations by trivial ones can be identified with the quotient $\Gamma(\mathcal{H}_1)/\operatorname{im}(\tilde{D}_0)$.

If we start from a flat Cartan connection (which is automatically normal), then the linear connection $\tilde{\nabla}$ is flat and the twisted de–Rham sequence is a complex defining a fine resultion of the sheaf of infinitesimal automorphims. Here one easily proves:

In this case, also the BGG–sequence $(\Gamma(\mathcal{H}_*), \tilde{D})$ is a complex and a fine resolution of the same sheaf.

Here one obtains a deformation complex on the level of the underlying structures generalizing the one for locally conformally flat manifolds constucted by Gasqui and Goldschmidt.

There are cases in which one may construct deformation complexes in categories of semi–flat geometries using the BGG machinery. In particular, this works for self–dual conformal structures in dimension 4, for quaternionic structures, (integrable) CR–structures and quaternionic contact structures.