Lecture 4:
General BGG sequences

Andreas Čap
(joint minicourse with J.M. Landsberg and M.G. Eastwood)

University of Vienna
Faculty of Mathematics

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- I will start by completing the discussion of the deformation sequence discussed in the last lecture by passing to the underlying structure. The crucial ingredient here is the consequences of Kostant's Hodge decomposition which are available on this level.
- The basic technical ingredients generalize immediately to the case of an arbitrary tractor bundle endowed with its canonical (linear) tractor connection. This leads to a vast number of examples of geometric overdetermined systems.
- In contrast to the case of the deformation sequence, the machinery only provides a partial prolongation of the overdetermined system. At the same time, it singles out a special subclass of solutions called *normal solutions*.
- I will finish by briefly discussing full prolongations, as well as some recent applications of the machinery.

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Structure

1. The deformation sequence
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Recall that we had ...

- A principal P–bundle $\mathcal{G} \to M$ endowed with a normal Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ which encodes an underlying geometric structure.
- The *adjoint tractor bundle* $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$ of the geometry with a nice interpretation of the spaces $\Omega^i(M, \mathcal{A}M)$ for $i = 0, 1, 2$ as infinitesimal principal bundle automorphisms of $\mathcal{G}$, infinitesimal deformations of Cartan connections and infinitesimal changes of Cartan curvature, respectively.
- A linear connection $\tilde{\nabla}$ on $\mathcal{A}M$ such that the twisted de–Rham sequence $(\Omega^*(M, \mathcal{A}M), d^{\tilde{\nabla}})$ admits an interpretation as a deformation sequence.
- Natural bundle maps $\partial^* : \Lambda^k T^*M \otimes \mathcal{A}M \to \Lambda^{k-1} T^*M \otimes \mathcal{A}M$ induced by the Kostant codifferential with $\partial^* \circ \partial^* = 0$, so we get natural subbundles $\operatorname{im}(\partial^*) \subset \ker(\partial^*) \subset \Lambda^k T^*M \otimes \mathcal{A}M$.

and we observed that ...

- The bundles $\mathcal{H}_k := \ker(\partial^*)/\operatorname{im}(\partial^*) \cong \mathcal{G} \times_P H_k(\mathfrak{g}_+, \mathfrak{g})$ consist of much simpler geometric object than the adjoint tractor bundle, their form can be deduced from Kostant's theorem.
- Both ∂^* and $d^{\tilde{\nabla}}$ are compatible with a natural filtration on the bundles $\Lambda^k T^*M \otimes \mathcal{A}M$ induced from the grading of $\mathfrak{g}$ we started with. Morover the lowest homogeneous component of $d^{\tilde{\nabla}}$ is tensorial and induced by the Lie algebra cohomology differential ∂.
- This implies that certain aspects of Kostant's Hodge decomposition carry over to the curved setting. Defining $\tilde{\square}^R := d^{\tilde{\nabla}}\partial^* + \partial^* d^{\tilde{\nabla}}$, which is an operator on $\Omega^k(M, \mathcal{A}M)$, we get:

Theorem

1. $\ker(\tilde{\Box}^R) = \ker(\partial^*) \cap \ker(\partial^* d^{\tilde{\nabla}})$.
2. Given $\varphi \in \Omega^k(M, \mathcal{A}M)$ there is $\psi \in \Omega^{k-1}(M, \mathcal{A}M)$ such that $\partial^*(\varphi + d^{\tilde{\nabla}}\psi) = 0$. One may use $\psi = \tilde{Q}(\partial^*\varphi)$ for a differential operator $\tilde{Q}$ on $\Gamma(\mathrm{im}(\partial^*))$ which is a polynomial in $\tilde{\Box}^R$.
3. Given $\alpha \in \Gamma(\mathcal{H}_k)$ there is a unique section $\psi \in \ker(\tilde{\Box}^R)$ such that $\pi_H(\psi) = \alpha$. This defines a differential operator $\tilde{S} : \Gamma(\mathcal{H}_k) \to \ker(\tilde{\Box}^R) \subset \Omega^k(M, \mathcal{A}M)$ which can be written as a (universal) polynomial in $\tilde{\Box}^R$. ("splitting operators")

Most of this follows directly from Kostant's Hodge theory. For example, if $0 = \tilde{\Box}^R(\varphi)$, one applies ∂^* to conclude that $\partial^* d^{\tilde{\nabla}} \partial^* \varphi = 0$. If $\partial^*\varphi$ were non–zero, then its lowest non–vanishing homogeneous component would be annihilated by $\partial^*\partial$ which contradicts Kostant's Hodge decompositon. Thus $\partial^*\varphi = 0$ and hence $\partial^* d^{\tilde{\nabla}} \varphi = 0$.

Having the splitting operators at hand, we can define the *BGG operators* $\tilde{D} = \tilde{D}_k : \Gamma(\mathcal{H}_k) \to \Gamma(\mathcal{H}_{k+1})$ by $\tilde{D}(\alpha) := \pi_H(d^{\tilde{\nabla}}\tilde{S}(\alpha))$.
Having these at hand, we can now interpret the deformation sequence in terms of the underlying structure:
First we need a fact specific to this case, expressing that trivial deformations of normal geometries are normal:

Having the splitting operators at hand, we can define the *BGG operators* $\tilde{D} = \tilde{D}_k : \Gamma(\mathcal{H}_k) \to \Gamma(\mathcal{H}_{k+1})$ by $\tilde{D}(\alpha) := \pi_H(d^{\tilde{\nabla}}\tilde{S}(\alpha))$. Having these at hand, we can now interpret the deformation sequence in terms of the underlying structure:

For $s \in \Gamma(\mathcal{A}M)$, we have $\partial^* d^{\tilde{\nabla}}\tilde{\nabla}s = 0. \qquad (*)$

Now we can prove that $\tilde{D}_0$ is the infinitesimal automorphism operator and $(\tilde{\nabla}, \tilde{S})$ provides a prolongation:

$\{s \in \Gamma(\mathcal{A}M) : \tilde{\nabla}s = 0\} \cong \ker(\tilde{D}_0)$ via π_H and $\tilde{S}$

Proof: For $s \in \Gamma(\mathcal{A}M)$ we always have $\partial^* s = 0$, so we can form $\sigma = \pi_H(s) \in \Gamma(\mathcal{H}_0)$. If $\tilde{\nabla}s = 0$, then $\tilde{\Box}^R(s) = 0$, so $s = \tilde{S}(\sigma)$ and $\tilde{D}_0(\sigma) = \pi_H(\tilde{\nabla}s) = 0$. Conversely, for $\sigma \in \Gamma(\mathcal{H}_0)$ consider $s = \tilde{S}(\sigma)$, which by definition satisfies $0 = \partial^*\tilde{\nabla}s$. But then $(*)$ implies that $\tilde{\Box}^R(\tilde{\nabla}s) = 0$, so $\tilde{\nabla}s = \tilde{S}(\pi_H(\tilde{\nabla}s)) = \tilde{S}(D_0(\sigma))$.

At the level of one–forms, we can first show:

For $\alpha \in \Gamma(\mathcal{H}_1)$, $\tilde{S}(\alpha) \in \Omega^1(M, \mathcal{A}M)$ is a normal infinitesimal deformation, and $\tilde{D}_1(\alpha)$ is the infinitesimal change of harmonic curvature caused by this. Morover, any normal deformation is equivalent modulo trivial deformations to one of this form.

Proof: $0 = \partial^* d^{\tilde{\nabla}} \tilde{S}(\alpha)$ holds by definition, and since $\tilde{D}_1(\alpha) = \pi_H(d^{\tilde{\nabla}} \tilde{S}(\alpha))$ the first part follows. For the second part, given $\varphi \in \Omega^1(M, \mathcal{A}M)$, put $\psi := \varphi + \tilde{\nabla} \tilde{Q}(\partial^* \varphi)$, which by construction satisfies $\partial^* \psi = 0$. Moreover, by $(*)$ the deformation ψ is again normal. But this shows $\tilde{\Box}^R(\psi) = 0$ and thus $\psi = \tilde{S}(\pi_H(\psi))$.
Having this at hand, similar arguments as before show that

The quotient of normal infinitesimal deformations by trivial ones can be identified with the quotient $\Gamma(\mathcal{H}_1)/\operatorname{im}(\tilde{D}_0)$.

If we start from a flat Cartan connection (which is automatically normal), then the linear connection $\tilde{\nabla}$ is flat and the twisted de–Rham sequence is a complex defining a fine resultion of the sheaf of infinitesimal automorphims. Here one easily proves:

In this case, also the BGG–sequence $(\Gamma(\mathcal{H}_*), \tilde{D})$ is a complex and a fine resolution of the same sheaf.

Here one obtains a deformation complex on the level of the underlying structures generalizing the one for locally conformally flat manifolds constucted by Gasqui and Goldschmidt.

There are cases in which one may construct deformation complexes in categories of semi–flat geometries using the BGG machinery. In particular, this works for self–dual conformal structures in dimension 4, for quaternionic structures, (integrable) CR–structures and quaternionic contact structures.

Structure

Tractor bundles and tractor connections

A *tractor bundle* is a bundle of the form $\mathcal{V}M := \mathcal{G} \times_P V$, where V is a representation of G (viewed as a representation of P). As geometric objects, these bundles are rather unusual, but they inherit linear connections from the Cartan connection ω.

For a section $s \in \Gamma(\mathcal{V}M)$ consider the corresponding P–equivariant function $f : \mathcal{G} \to V$ and for a vector field $\xi \in \mathfrak{X}(M)$ take a P–invariant lift $\tilde{\xi} \in \mathfrak{X}(\mathcal{G})$. Then the function

$$u \mapsto (\tilde{\xi} \cdot f)(u) + \omega(\tilde{\xi}(u)) \cdot f(u)$$

(with the dot in the last term denoting the action of $\mathfrak{g}$ on V) is P–equivariant and depends only on ξ, so we can define $\nabla_\xi s \in \Gamma(\mathcal{V}M)$ to be the corresponding section.

Remark

In the case $V = \mathfrak{g}$, we return to the adjoint tractor bundle $\mathcal{A}M$ used in the case of the deformation sequence. However, the connection ∇ is different from $\tilde{\nabla}$:
If $s \in \Gamma(\mathcal{A}M)$ corresponds to $\tilde{\eta} \in \mathfrak{X}(\mathcal{G})^P$, then by defintion $\tilde{\nabla} s$ corresponds to $\mathcal{L}_{\tilde{\eta}}\omega$, while the equivariant function $\mathcal{G} \to \mathfrak{g}$ corresponding to s is $\omega(\tilde{\eta})$. Then for $\xi \in \mathfrak{X}(M)$ and a lift $\tilde{\xi} \in \mathfrak{X}(\mathcal{G})^P$, we see that $\tilde{\nabla}_\xi s$ corresponds to the function

$$(\mathcal{L}_{\tilde{\eta}}\omega)(\tilde{\xi}) = \tilde{\eta} \cdot \omega(\tilde{\xi}) - \omega([\tilde{\eta}, \tilde{\xi}]) = d\omega(\tilde{\eta}, \tilde{\xi}) + \tilde{\xi} \cdot \omega(\tilde{\eta}) = \kappa(\eta, \xi) + [\omega(\tilde{\xi}), \omega(\tilde{\eta})] + \tilde{\xi} \cdot \omega(\tilde{\eta})$$

The last two summands correspond to $\nabla_\xi s$, so ∇ and $\tilde{\nabla}$ differ by a curvature term. The BGG machinery applies to both since κ is homogeneous of positive degree.

As before, we obtain a twisted de–Rham sequence $(\Omega^*(M,\mathcal{V}M),d^\nabla)$. Also, the Kostant codifferential induces bundle maps $\partial^*:\Lambda^kT^*M\otimes\mathcal{V}M\to\Lambda^{k-1}T^*M\otimes\mathcal{V}M$ and thus natural subbundles $\operatorname{im}(\partial^*)\subset\ker(\partial^*)\subset\Lambda^kT^*M\otimes\mathcal{V}M$ such that $\mathcal{H}_k^V:=\ker(\partial^*)/\operatorname{im}(\partial^*)\cong\mathcal{G}\times_P H_k(\mathfrak{g}_+,V)$.
To imitate the case of the deformation sequence, we first observe that V carries a natural P–invariant filtration

$$V=V^0\supset V^1\supset\dots\supset V^N\supset V^{N+1}=0$$

defined by $V^N=\{v\in V:\mathfrak{g}_+\cdot v=0\}$ and inductively $V^{i-1}=\{v\in V:\mathfrak{g}_+\cdot v\in V^i\}$.

Both ∂^* and d^∇ are compatible with the induced filtration on $\Omega^*(M,\mathcal{V}M)$ and the lowest homogeneous component of d^∇ is tensorial and induced by the Lie algebra differential ∂.

The next steps are completely parallel to the deformation case: Defining $\Box^R := d^\nabla \partial^* + \partial^* d^\nabla$, one proves

Theorem

1. $\ker(\Box^R) = \ker(\partial^*) \cap \ker(\partial^* d^\nabla)$.
2. Given $\varphi \in \Omega^k(M, \mathcal{V}M)$ there is $\psi \in \Omega^{k-1}(M, \mathcal{V}M)$ such that $\partial^*(\varphi + d^\nabla \psi) = 0$. One may use $\psi = Q(\partial^* \varphi)$ for a differential operator Q on $\Gamma(\operatorname{im}(\partial^*))$ which is a polynomial in $\Box^R$.
3. Given $\alpha \in \Gamma(\mathcal{H}_k^V)$ there is a unique section $\psi \in \ker(\Box^R)$ such that $\pi_H(\psi) = \alpha$. This defines a differential operator $S : \Gamma(\mathcal{H}_k^V) \to \ker(\Box^R) \subset \Omega^k(M, \mathcal{V}M)$ which can be written as a (universal) polynomial in $\Box^R$. ("splitting operators")

The BGG–operators $D = D_k : \Gamma(\mathcal{H}_k) \to \Gamma(\mathcal{H}_{k+1})$ are again defined as $D(\alpha) = \pi_H(d^\nabla S(\alpha))$.

Structure

It turns out that the first operator D_0 in each BGG sequence defines a natural overdetermined system of PDEs. It is defined on sections of the bundle $\mathcal{H}_0^V = \mathcal{V}M/\mathcal{V}^1 M$, the quotient of the tractor bundle by the largest filtration component. The target bundle can be easily described using Kostant's theorem. The general machinery easily shows that we obtain a partial prolongation of D_0:

The projection π_H and the splitting operator S restrict to inverse bijections between $\{s \in \Gamma(\mathcal{A}M) : \nabla s \in \Gamma(\mathrm{im}(\partial^*))\}$ and $\ker(D_0)$.

This means that for $\alpha \in \Gamma(\mathcal{H}_0)$ the equation $D_0(\alpha) = 0$ is equivalent to $\nabla S(\alpha) = \partial^* \varphi$ for some $\varphi \in \Omega^2(M, \mathcal{V}M)$, and the right hand side can be computed from $S(\alpha)$. This was used to obtain a full prolongation, even for quasilinear operators with the same principal symbol in many cases.

Normal solutions

The BGG construction automatically leads to a subclass of solutions of the first BGG operator:

A *normal solution* of D_0 is a section $\alpha \in \Gamma(\mathcal{H}_0^V)$ such that $\nabla S(\alpha) = 0$. (Note that this implies $D_0(\alpha) = 0$.)

Remarks: (1) In many cases of interest, the BGG sequence constructed for the adjoint tractor bundle $\mathcal{A}M$ from the connection ∇ produces the same splitting operator and the same BGG operator in degree zero, so one again gets the infinitesimal automorphism operator. Normal solutions in this case are infinitesimal automorphisms which in addition insert trivially into the Cartan curvature.
(2) It turns out that one can obtain a natural full prolongation of the first BGG operator in all cases.

Normal solutions II

- On the homogeneous model G/P any solution is normal and solutions can be described in terms of algebraic geometry or even linear algebra.
- Normal solutions on general geometries are locally conjugate to solutions on G/P via a diffeomorphism. This in particular gives information on their zero set.
- This fits into the much more general context of holonomy reductions of Cartan geometries.
- If the zero set is always an embedded hypersurface, then one can looks at the situation of a manifold with boundary such that the zero set coincides with the boundary. This leads to notions of geometric compactifications. A model case is Poincaré–Einstein manifolds.
- . . .

Thanks for your attention

Happy Birthday, Robert!