On Automorphism Groups of Parabolic Geometries

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Contents

Preface

The aim of this work is to determine the second largest possible dimension of automorphism groups of regular parabolic geometries of a certain type.

In the first chapter we will draw our attention to the concept of a Cartan geometry. This concept was introduced by Elie Cartan in the 1920's under the name "éspace généralisé" in order to give a common generalization of two theories, which seemed to be largely incompatible, namely geometry in the sense of Felix Klein's Erlangen program and differential geometry.

Klein noticed that the non-Euclidean geometries, which were discovered in the 19-th century, can be viewed as smooth manifolds endowed with a transitive smooth action of a Lie group, which leaves invariant the properties of objects studied in this geometry. Having fixed a base point such a geometry is given by a certain homogeneous space. So his definition of a geometry as a smooth manifold together with a smooth transitive action of a Lie group offered a common approach to Euclidean geometry as well as to the various non-Euclidean ones.

It was not clear at that time how this definition of geometry can be reconciled with the generalization of Euclidean geometry due to Bernhard Riemann (1854), which nowadays is called Riemannian geometry. It was Cartan, who recognized that smooth manifolds endowed with a certain geometric structure can be seen as "curved analogs" of homogeneous spaces, such as Riemannian manifolds can be viewed as deformations of Euclidean spaces. In modern language these geometric structures are called Cartan geometries. For a Lie group G and a closed subgroup H a Cartan geometry of type (G, H) is given by a smooth manifold M of the same dimension as G/H together with a principal H-bundle $\mathcal{G} \to M$ and a 1-form on $\mathcal{G}$ with values in the Lie algebra $\mathfrak{g}$ of G, which is H-equivariant, reproduces fundamental vector fields and trivializes the tangent bundle of $\mathcal{G}$. This concept, which associates to a homogeneous space G/H the notion of a Cartan geometry of type (G, H), really provides a common generalization of Klein geometry and Riemannian geometry. Indeed, the principal bundle of a Cartan geometry of type (G, H) generalizes the principal bundle $G \to G/H$ given by the natural projection associated to the Klein geometry G/H, whereas the 1-form of a Cartan geometry generalizes the Maurer-Cartan form

on G. For a Riemannian manifold M the principal bundle is the orthogonal frame bundle of M endowed with the soldering form on this bundle and the Levi-Civita connection. The main references concerning the general theory of Cartan geometries presented in the first chapter are **[2]** and **[8]**.
Having introduced the concept of a Cartan geometry we will consider the automorphism group $Aut(\mathcal{G}, \omega)$ of a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, H). Due to a theorem of Richard Palais **[6]**, which characterizes Lie transformation groups, the automorphism group of a Cartan geometry of type (G, H) over a connected manifold can be made into a Lie group, whose dimension is as most the dimension of G. We will see that the Lie algebra of $Aut(\mathcal{G}, \omega)$ can be described in terms of the Lie algebra of G and the curvature of the geometry only, cf. **[1]**.
In the second chapter we will turn to a certain class of Cartan geometries, so called parabolic geometries. By a parabolic geometry one understands a Cartan geometry of type (G, P), where G is a semisimple Lie group and P a parabolic subgroup. We will show that a parabolic geometry can be equivalently described as a Cartan geometry of type (G, P), where G is a semisimple Lie group, whose Lie algebra $\mathfrak{g}$ is endowed with a $|k|$-grading and P the subgroup of G preserving the filtration associated to the grading on $\mathfrak{g}$, by establishing a correspondence between the standard parabolic subalgebras of $\mathfrak{g}$ and all possible $|k|$-gradings on $\mathfrak{g}$. References for this chapter are **[10]** and **[3]** and for the background on the structure theory of semisimple Lie algebras, which we assume to be known, **[4]** and **[5]**.
In the third and last chapter we will look more closely at the automorphism group of a regular parabolic geometry of type (G, P) and will show how the description of its Lie algebra in terms of the Lie algebra of G and the curvature, can be used to study possible dimensions of automorphism groups of regular parabolic geometries of some fixed type. Finally, we will use this to determine the second largest possible dimension of automorphism groups of regular parabolic geometries of some fixed type. We will consider the cases, where G is $SO(n+1, n)$, G_2, $Sp(n+1, 1)$ or $Sp(6, \mathbb{R})$.

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CHAPTER 1

Cartan geometries

As we mentioned in the preface in this chapter we will introduce the notion of a Cartan geometry of type (G, H), which is a differential geometric structure on a manifold M of the same dimension as the homogeneous space G/H. Such a geometric structure on M is given by a principal H-bundle $\mathcal{G} \to M$ and a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Further we will show that the group of automorphisms of any Cartan geometry over a connected manifold can be made into a Lie group, whose Lie algebra can be described in terms of the Lie algebra $\mathfrak{g}$ of G and the curvature of the Cartan geometry only.

1.1. Definitions and basic facts

First we recall the definition of a principal fiber bundle. Throughout this work all smooth manifolds are supposed to be Hausdorff and second countable.
Let E, B and F be smooth manifolds and $p : E \to B$ a smooth map. The quadruple (E, B, p, F) is called a *(smooth) fiber bundle with standard fiber* F, if for each point $b \in B$, there is an open neighborhood U in B and a diffeomorphism $\varphi : p^{-1}(U) \xrightarrow{\sim} U \times F$ such that $pr_1 \circ \varphi = p|_{p^{-1}(U)}$, i.e. such that the following diagram commutes

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

where $pr_1 : U \times F \to U$ denotes the projection on the first factor. B is called the basis, E the total space and $p^{-1}(b)$ the fiber over b. Note that in this case p is in particular a surjective submersion. The pair (U, φ) is called a *(fiber bundle) chart*. A *fiber bundle atlas* is a collection of charts $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$, such that $\{U_\alpha\}$ cover B. For two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, one can consider the coordinate change $\varphi_\alpha \circ \varphi_\beta^{-1} : U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F$, which is given by $\varphi_\alpha(\varphi_\beta^{-1}(x, y)) = (x, \phi_{\alpha\beta}(x, y))$ for a smooth function $\phi_{\alpha\beta} : U_{\alpha\beta} \times F \to F$, called the *transition function*.
A *fiber bundle homomorphism* between (E, B, p, F) and (E', B', p', F') is a smooth map $\psi : E \to E'$ such that there exists a smooth map $\overline{\psi}$

which makes the following diagram commute:

$$\begin{tikzcd} E \arrow[r, "\psi"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "\overline{\psi}"] & B' \end{tikzcd}$$

So ψ has to be fiber respecting, i.e. ψ maps $p^{-1}(b)$ to $p'^{-1}(\overline{\psi}(b))$ for all $b \in B$.

Let (E, B, p, V) be a fiber bundle with standard fiber a finite dimensional vector space V. A *vector bundle atlas* on (E, B, p, V) is a fiber bundle atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ such that the transition functions are linear in the second variable. Two vector bundle atlases are equivalent, if there union is again a vector bundle atlas. A *vector bundle* is then defined to be a fiber bundle with standard fiber a vector space endowed with an equivalence class of vector bundle atlases. Let (E, B, p, V) be a vector bundle, $b \in B$ and $(U_\alpha, \varphi_\alpha)$ a chart with $b \in U_\alpha$. Then the fiber $p^{-1}(b)$ over b can be canonically given the structure of a vector space such that $pr_2 \circ \varphi|_{p^{-1}(b)} : p^{-1}(b) \to V$ is a linear isomorphism, where $pr_2 : U_\alpha \times V \to V$ denotes the projection of the second factor. A *vector bundle homomorphism* between two vector bundles is a fiber bundle homomorphism, where the restriction to each fiber is a linear map.

Let H be a Lie group and (E, B, p, H) a fiber bundle with standard fiber H. A *principal (fiber) bundle atlas* on (E, B, p, H) is a fiber bundle atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$, where the transition functions are given by $\phi_{\alpha\beta} : (x, h) \mapsto \varphi_{\alpha\beta}(x)h$ for smooth functions $\varphi_{\alpha\beta} : U_{\alpha\beta} \to H$. Two principal bundle atlases are equivalent if their union is again a principal bundle atlas. A *principal bundle with structure group H* or simply a *principal H-bundle* is then a fiber bundle with standard fiber H together with an equivalence class of principal bundle atlases.

Each principal H-bundle admits a free right action of H on E, whose orbits are the fibers of p. This so called *principal right action* $r : E \times H \to E$ is given by $r(u, h) = \varphi_\alpha^{-1}((x, ah))$ for $u \in p^{-1}(U_\alpha)$ with $\varphi_\alpha(u) = (x, a)$. The map is well defined, i.e. independent of the local trivialization, since left and right action of H commute. It is obviously free and for $u \in p^{-1}(x)$ one has the diffeomorphism $r_u : H \xrightarrow{\sim} p^{-1}(x)$, where $r_u(h) = r(u, h)$.

Given a principal H-bundle, the family of mappings $\{\varphi_{\alpha\beta}\}$, whose elements are now called the *transition functions (of the principal bundle)*, satisfies the cocycle condition $\varphi_{\alpha\beta}(x)\varphi_{\beta\gamma}(x) = \varphi_{\alpha\gamma}(x)$ for all $x \in U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$ and $\varphi_{\alpha\beta}^{-1}(x) = \varphi_{\beta\alpha}(x)$ for all $x \in U_{\alpha\beta}$. One can show that a H-bundle is determined up to isomorphism by its cocycle of transition functions. Conversely, given an open covering $\{U_\alpha\}$ of a manifold B and a family of smooth mappings $\{\varphi_{\alpha\beta} : U_{\alpha\beta} \to H\}$

satisfying the two conditions above, one can construct a principal H-bundle having exactly these as transition functions, which is unique up to isomorphism.
Observe that for a Lie group G and a Lie subgroup H, the projection $p : G \to G/H$ is a principal H-bundle, where the principal right action is just the multiplication from the right of H on G.
A *principal bundle homomorphism* between (E, B, p, H) and (E', B', p', H) is a fiber bundle homomorphism $\psi : E \to E'$ such that ψ is H-equivariant.
Given a principal H-bundle (E, B, p, H) and a smooth manifold N together with a smooth left action $l : H \times N \to N$ of H, then we can define a right action of H on $E \times N$ by $(x, n) \cdot h = (r(x, h), l(h^{-1}, n))$. The orbit space $(E \times N)/H$, which we will denote by $E \times_H N$, is a fiber bundle $\overline{p} : E \times_H N \to B$ over B with standard fiber N, where $\overline{p}$ is defined by

$$\begin{tikzcd} E \times N \arrow[r, "q"] \arrow[d, "pr_1"'] & E \times_H N \arrow[d, "\overline{p}"] \\ E \arrow[r, "p"] & B \end{tikzcd}$$

with $q : E \times N \to (E \times N)/H$ the orbit projection.
We call $(E \times_H N, B, \overline{p}, N)$ the *associated bundle* to (E, B, p, H) and (N, l, H). If N is a vector space and l is a representation of H on N, the associated bundle $E \times_H N$ carries the structure of a vector bundle.
The last ingredient we need, to introduce the concept of a Cartan geometry, is the notion of fundamental vector fields. Given a smooth right action r of a Lie group G on a manifold M and let $\mathfrak{g}$ be the Lie algebra of G, then one can define the *fundamental vector field generated by* $X \in \mathfrak{g}$ by $\zeta_X(x) = T_{(x,e)}r(0, X) = \frac{d}{dt}|_{t=0} r(x, exp(tX))$ for all $x \in M$. The map $X \mapsto \zeta_X$ is a Lie algebra homomorphism between $\mathfrak{g}$ and $\mathfrak{X}(M)$.
For a smooth map $f : M \to N$ between two manifolds and a k-form $\varphi \in \Omega^k(N, V)$ on N with values in a finite dimensional vector space V let us denote by $f^*\varphi \in \Omega^k(M, V)$ the pullback of φ along f: $f^*\varphi(x)(\xi_1, ..., \xi_k) = \varphi(f(x))(T_x f \xi_1, ..., T_x f \xi_k)$

Definition. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and H a Lie subgroup of G with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$.*
A Cartan geometry of type (G, H) on a smooth manifold M is a principal H-bundle $p : \mathcal{G} \to M$ together with a one form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, called the Cartan connection, such that:
(1) ω is H-equivariant: $(r^h)^*\omega = Ad(h)^{-1} \circ \omega$ for all $h \in H$, where r^h denotes the principal right action of h and $Ad : H \to GL(\mathfrak{h})$ the adjoint representation of H on $\mathfrak{h}$.
(2) ω reproduces the fundamental vector fields: $\omega(\zeta_A) = A$ for all

$A \in \mathfrak{h}$, where ζ_A denotes the fundamental vector field with generator A.
(3) $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

Note that $\dim(M) = \dim(G/H)$ follows immediately from the definition, since $\dim(G) = \dim(\mathcal{G}) = \dim(M) + \dim(H)$.
The basic example of a Cartan geometry of type (G, H) is given by the principal H-bundle $p : G \to G/H$ together with the Maurer-Cartan form $\omega_{MC} \in \Omega^1(G, \mathfrak{g})$, $\omega_{MC}(g)(\xi) = T_g\lambda_{g^{-1}}\xi$ for $\lambda_{g^{-1}} : G \to G$ being the left multiplication by g^{-1}. This principal bundle is called the *homogenous model* for Cartan geometries of type (G, H) and plays a distinguished role in the study of Cartan geometries. One can see Cartan geometries of type (G, H) as a generalization of the homogenous model, since they are derived from the homogenous model by replacing the projection $G \to G/H$ by an arbitrary principal H-bundle and the Maurer-Cartan form by a 1-form satisfying the three properties from above.
Given a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, H), one has the notion of the constant vector fields: for $X \in \mathfrak{g}$ the corresponding constant vector field $\widetilde{X}$ is characterized by $\omega(\widetilde{X}(u)) = X$ for all $u \in \mathcal{G}$. For $X \in \mathfrak{h}$ we obtain $\widetilde{X} = \zeta_X$.
A *morphism* between two Cartan geometries $(\mathcal{G} \to M, \omega)$ and $(\mathcal{G}' \to M', \omega')$ of type (G, H), is a principal bundle homomorphism $\psi : \mathcal{G} \to \mathcal{G}'$ such that $\psi^*\omega' = \omega$. Note that ψ is a local diffeomorphism, since ω and ω' define isomorphisms between each tangent space $T_u\mathcal{G}$ respectively $T_u\mathcal{G}'$ and $\mathfrak{g}$.
For a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, H) we have the notion of its curvature. The *curvature form* $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ is given by $K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$ for $\xi, \eta \in \mathfrak{X}(\mathcal{G})$, where d denotes the exterior derivative. Since ω by definition trivializes the the tangent bundle $T\mathcal{G}$, K is already determined by its values on the constant vector fields. So we have an equivalent description of the curvature given by the *curvature function* $\kappa : \mathcal{G} \to L(\Lambda^2\mathfrak{g}, \mathfrak{g})$ defined by $\kappa(u)(X, Y) = K(u)(\widetilde{X}(u), \widetilde{Y}(u))$ for all $u \in \mathcal{G}$ and all $X, Y \in \mathfrak{g}$.
Observe that the fundamental vector fields trivialize the vertical bundle $V\mathcal{G} := ker(Tp) \subset T\mathcal{G}$. Indeed, consider the map $\mathcal{G} \times \mathfrak{h} \to T\mathcal{G}$ $(u, X) \mapsto (u, \zeta_X(u))$. Since the right action of H on $\mathcal{G}$ is fiber preserving and the dimension of each fiber equals the dimension of $\mathfrak{h}$, $X \mapsto \zeta_X(u)$ is a linear isomorphism between $\mathfrak{h}$ and $V_u(\mathcal{G})$ for all $u \in \mathcal{G}$. So the map $(u, X) \mapsto (u, \zeta_X(u))$ defines a fiber bundle isomorphism between $\mathcal{G} \times \mathfrak{h}$ and $V\mathcal{G}$. Using the H-equivariancy of ω and that $\mathcal{G} \times \mathfrak{h} \simeq V\mathcal{G}$, we obtain that K is horizontal, i. e. $K(\xi, \eta) = 0$ for $\eta \in \mathfrak{X}(\mathcal{G})$ with values in the vertical bundle and any $\xi \in \mathfrak{X}(\mathcal{G})$, and consequently the curvature function can be seen as map having values in $L(\Lambda^2\mathfrak{g}/\mathfrak{h}, \mathfrak{g})$.
The equivariancy of ω immediately implies the equivariancy of K:

$(r^h)^*K = Ad(h^{-1}) \circ K$. A short computation shows that κ is also H-equivariant, where the action of H on $L(\Lambda^2\mathfrak{g}, \mathfrak{g})$ is the action induced from the adjoint action of G. So K is a horizontal H-equivariant 2-form on $\mathcal{G}$.
The curvature of the homogenous model $(G \to G/H, \omega_{MC})$ is zero, since the Maurer-Cartan form satisfies the equation

$$d\omega_{MC}(\xi, \eta) + [\omega_{MC}(\xi), \omega_{MC}(\eta)] = 0$$

for all $\xi, \eta \in \mathfrak{X}(G)$. A manifold M endowed with a Cartan geometric structure of type (G, H) can therefore be viewed as curved analog of the homogenous space G/H, where the curvature measures the extent of the deformation.
Now we want to show that for a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, H) the tangent bundle TM is isomorphic to the associated vector bundle $\mathcal{G} \times_H \mathfrak{g}/\mathfrak{h}$.
We consider the map $\Phi : \mathcal{G} \times \mathfrak{g} \to TM$ defined by
$(u, X) \mapsto (p(u), T_u p\widetilde{X}(u))$. Since p is a surjective submersion and ω trivializes the tangent bundle $T\mathcal{G}$, Φ is surjective. For $X \in \mathfrak{h}$ we have $\widetilde{X} = \zeta_X$ and we remarked above that the fundamental vector fields trivialize the vertical bundle $V\mathcal{G}$, therefore Φ factors to a map $\overline{\Phi} : \mathcal{G} \times \mathfrak{g}/\mathfrak{h} \to TM$ such that $X + \mathfrak{h} \mapsto T_u p\widetilde{X}(u)$ defines a linear isomorphism between $\mathfrak{g}/\mathfrak{h}$ and $T_{p(u)}M$ for fixed $u \in \mathcal{G}$.
Observe that H acts from the left on $\mathfrak{g}/\mathfrak{h}$, by the action induced from the restriction to H of the adjoint action of G on $\mathfrak{g}$. Since ω is H-equivariant, $\overline{\Phi}$ induces a vector bundle homomorphism between $\mathcal{G} \times_H \mathfrak{g}/\mathfrak{h}$ and TM, which covers the identity on M and defines a linear isomorphism between the fibers. Hence this map is an isomorphism of vector bundles.
An important observation concerning Cartan geometries is that they can be restricted to open subsets. In fact, given a Cartan geometry $(\mathcal{G} \to M, \omega)$ and an open subset $U \subset M$ the restriction $(p|_{p^{-1}(U)} : p^{-1}(U) \to U, \omega|_{p^{-1}(U)})$ defines a Cartan geometry such that the curvature is the restriction of the original curvature. One can show that the curvature of a Cartan geometry vanishes identically if and only if it is locally isomorphic to the homogenous model, see [**8**].

1.2. The automorphism group of a Cartan geometry

Let $(p : \mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, H) and suppose that M is connected.
Denote by

$$Aut(\mathcal{G}, \omega) := \{\psi : \mathcal{G} \to \mathcal{G} : \psi \quad H\text{-equiv. diffeomorphism s. t. } \psi^*\omega = \omega\}$$

the group of automorphisms of $(p : \mathcal{G} \to M, \omega)$.
An *infinitesimal automorphism* of $(p : \mathcal{G} \to M, \omega)$ is a vector field $\xi \in \mathfrak{X}(\mathcal{G})$ such that the Lie derivative $\mathcal{L}_\xi\omega := \frac{d}{dt}|_{t=0}(Fl^\xi_t)^*\omega = 0$ $(Fl^\xi$

the flow of the vector field ξ) and $(r^h)^*\xi = \xi$ for all $h \in H$. For a complete vector field ξ these two conditions are obviously equivalent to $Fl_t^\xi \in Aut(\mathcal{G}, \omega)$ for all $t \in \mathbb{R}$. We denote by $\mathfrak{inf}(\mathcal{G}, \omega)$ the space of all infinitesimal automorphism. The space $\mathfrak{inf}(\mathcal{G}, \omega)$ is a Lie subalgebra of $\mathfrak{X}(\mathcal{G})$, since $\mathcal{L}_{[\xi,\eta]} = [\mathcal{L}_\xi, \mathcal{L}_\eta]$ and $(r^h)^*[\xi, \eta] = [(r^h)^*\xi, (r^h)^*\eta]$.
Now we want to show that $Aut(\mathcal{G}, \omega)$ is a Lie group whose Lie algebra is given by all complete vector fields in $\mathfrak{inf}(\mathcal{G}, \omega)$. Therefore we need the following theorem due to Palais, [**6**]:

THEOREM 1.1. *Let K be a group of diffeomorphisms of a smooth manifold N and let $\mathfrak{k} \subset \mathfrak{X}(N)$ be the space of all complete vector fields ξ whose flows Fl_t^ξ lie in K for all $t \in \mathbb{R}$. If the Lie subalgebra of $\mathfrak{X}(N)$ generated by $\mathfrak{k}$ is finite dimensional, then it coincides with $\mathfrak{k}$ and K can be made into a Lie group with Lie algebra $\mathfrak{k}$.*

For $\xi \in \mathfrak{inf}(\mathcal{G}, \omega)$ and $\widetilde{X}$ any constant vector field, we have $0 = (\mathcal{L}_\xi\omega)(\widetilde{X}) = -\omega([\xi, \widetilde{X}])$. So, by the third property of the Cartan connection, we obtain $[\xi, \widetilde{X}] = 0$ for all constant vector fields $\widetilde{X}$. This means that the flows of all constant vector fields commute with ξ: $T_u Fl_t^{\widetilde{X}}(\xi(u)) = \xi(Fl_t^{\widetilde{X}}(u))$ for all $u \in \mathcal{G}$ and for all $t \in \mathbb{R}$ for which the flow is defined. Therefore the value of ξ in u uniquely determines ξ locally around u, since the constant vector fields span each tangent space. Since ξ is H-invariant and M is connected we conclude that $\xi(u)$ already uniquely defines ξ on the whole of M. Hence for any $u \in \mathcal{G}$ the map $\xi \mapsto \omega(\xi(u))$ defines a linear isomorphism form $\mathfrak{inf}(\mathcal{G}, \omega)$ onto a linear subspace $\mathfrak{a}$ of $\mathfrak{g}$. In particular, the dimension of $\mathfrak{inf}(\mathcal{G}, \omega)$ is at most the dimension of $\mathfrak{g}$.
Let now $\mathfrak{aut}(\mathcal{G}, \omega)$ be the subset of complete vector fields in $\mathfrak{inf}(\mathcal{G}, \omega)$, then the Lie subalgebra of $\mathfrak{X}(\mathcal{G})$ generated by $\mathfrak{aut}(\mathcal{G}, \omega)$ must be finite dimensional, since it must be contained in $\mathfrak{inf}(\mathcal{G}, \omega)$. So we can directly apply the theorem of Palais to the group of diffeomorphism $Aut(\mathcal{G}, \omega)$ and obtain:

THEOREM 1.2. *The automorphism group $Aut(\mathcal{G}, \omega)$ of a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, H) over a connected manifold M is a Lie group with Lie algebra $\mathfrak{aut}(\mathcal{G}, \omega)$. Moreover the dimension of $Aut(\mathcal{G}, \omega)$ is at most the dimension of G.*

We are able to describe the Lie bracket on $\mathfrak{aut}(\mathcal{G}, \omega)$ in terms of $\mathfrak{g}$ and the curvature of the Cartan geometry only.
The Lie bracket on $\mathfrak{aut}(\mathcal{G}, \omega)$ is induced by the negative of the Lie bracket on $\mathfrak{X}(\mathcal{G})$. This bracket makes also sense on $\mathfrak{inf}(\mathcal{G}, \omega)$
For $\xi \in \mathfrak{inf}(\mathcal{G}, \omega)$ and $\eta \in \mathfrak{X}(\mathcal{G})$ we have

$$0 = (\mathcal{L}_\xi\omega)(\eta) = \xi \cdot \omega(\eta) - \omega([\xi, \eta]) = d\omega(\xi, \eta) + \eta \cdot \omega(\xi) =$$

$$= \kappa(\omega(\xi), \omega(\eta)) - [\omega(\xi), \omega(\eta)] + \eta \cdot \omega(\xi)$$

If η is also an infinitesimal automorphism we get $\eta \cdot \omega(\xi) = -\omega([\xi, \eta])$. Hence $\omega([\xi, \eta]) = \kappa(\omega(\xi), \omega(\eta)) - [\omega(\xi), \omega(\eta)]$ for $\xi, \eta \in \mathfrak{inf}(\mathcal{G}, \omega)$.
Therefore we obtain that for any $u \in \mathcal{G}$ the bracket on $\mathfrak{inf}(\mathcal{G}, \omega)$ (induced from the negative of the bracket on $\mathfrak{X}(\mathcal{G})$) corresponds to the operation $(X, Y) \mapsto [X, Y] - \kappa(u)(X, Y)$ on $\mathfrak{a} = \{\omega(\xi(u)) : \xi \in \mathfrak{inf}(\mathcal{G}, \omega)\} \subset \mathfrak{g}$.
For the homogenous model $(G \to G/H, \omega_{MC})$ with G/H connected, the group of automorphisms $Aut(G, \omega_{MC})$ coincides with the set of all left translation $\lambda_g : G \to G$ for $g \in G$. In fact, since left translation and right translation commute we see that any left translation is an element in $Aut(G, \omega_{MC})$. To see the converse, we need the following lemma:

Lemma 1.1. *Let K be a Lie group, N a smooth connected manifold, $f_1, f_2 : N \to K$ smooth functions such that $(f_1)^*\omega_{MC} = (f_2)^*\omega_{MC}$. Then there is an element $k \in K$ such that $f_2 = \lambda_k \circ f_1$.*

Proof. Consider the smooth function $h := f_1(\nu \circ f_2) : N \to K$, where $\nu : k \mapsto k^{-1}$ is the inversion on K and set $\delta f := f^*\omega_{MC}$ for every smooth function $f : N \to K$.
Then

$$\delta h(x) = \delta(\nu \circ f_2)(x) + Ad(f_2) \circ \delta f_1(x) = Ad(f_2(x))(\delta f_1(x) - \delta f_2(x)) = 0$$

where we used the following rules, which can be shown by straightforward computations:

$$\delta(fg)(x) = \delta g(x) + Ad(g(x)^{-1}) \circ \delta f(x)$$

and

$$\delta(\nu \circ f)(x) = -Ad(f(x)) \circ \delta f(x)$$

for smooth functions $f, g : N \to K$.
By the injectivity of the Maurer-Cartan form in each point, $\delta h = 0$ implies $T_x h = 0$ for all $x \in N$. Since N is connected, h is constant. This means there is an element $k \in K$ such that $f_1(x) = k f_2(x)$ for all $x \in N$. $\square$

Now let $\psi \in Aut(G, \omega_{MC})$. Then we have $\psi^*\omega_{MC} = \omega_{MC} = id^*\omega_{MC}$ and therefore by the lemma that ψ is given by a left translation on each connected component on G. Since ψ is a principal bundle map and G/H is supposed to be connected, $\psi = \lambda_g : G \to G$ for some $g \in G$. So the Cartan geometry on G/H, given by the the homogenous model, is a geometric structure on G/H, whose group of automorphisms is just G.
The last fact we will need from the theory of general Cartan geometries is the Liouville theorem for the homogenous model $(G \to G/H, \omega_{MC})$:

Theorem 1.3. *Let U and V be connected open subsets of the homogenous space G/H. Then any isomorphism between the restrictions*

$(p^{-1}(U) \to U, \omega_{MC}|_{p^{-1}(U)})$ and $(p^{-1}(V) \to V, \omega_{MC}|_{p^{-1}(V)})$ uniquely extends to an automorphism of the homogenous model $(p : G \to G/H, \omega_{MC})$.

Proof. Let $\Psi : p^{-1}(U) \to p^{-1}(V) \subset G$ be an isomorphism between the restrictions of the Cartan geometry to U and V respectively. In particular Ψ is a smooth map from $p^{-1}(U) \to G$ with $\Psi^*\omega_{MC} = \omega_{MC}|_{p^{-1}(U)} = i^*\omega_{MC}$, where $i : p^{-1}(U) \to G$ is the inclusion. So Ψ differs from the inclusion by a left translation, using again lemma 1.1. Hence the claim follows. $\square$

CHAPTER 2

Parabolic geometries

A parabolic geometry is a Cartan geometry of type (G, P), where G is a semisimple Lie group, whose Lie algebra $\mathfrak{g}$ is endowed with a $|k|$-grading and P is the subgroup of G preserving the filtration associated to the grading on $\mathfrak{g}$. We will show that P is a so called parabolic subgroup and that all parabolic subgroups of G are obtained as stabilizers of the filtration associated to some grading on $\mathfrak{g}$, by establishing a correspondence between the standard parabolic subalgebras of $\mathfrak{g}$ and all possible $|k|$-gradings on $\mathfrak{g}$, cf. **[10]**. In the presentation of this correspondence we follow **[3]**.

2.1. Gradings on semisimple Lie algebras

DEFINITION. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $k \in \mathbb{N}$.*
A $|k|$-grading on $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ of $\mathfrak{g}$ into a direct sum of subspaces, such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ $\forall i, j \in \mathbb{Z}$, where $\mathfrak{g}_i := \{0\}$ for $|i| > k$, and such that the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_{-1}$ is generated as Lie algebra by $\mathfrak{g}_{-1}$.
A filtration of $\mathfrak{g}$ is an increasing sequence of subspaces $\mathfrak{g}^k \subseteq \mathfrak{g}^{k-1} \subseteq ...$ with union $\mathfrak{g}$ and such that $[\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j}$ for all $i, j \in \mathbb{Z}$, where $\mathfrak{g}^m := \{0\}$ for all $m \geq k$.

Now we collect some facts and properties about gradings on semisimple Lie algebras, which we will need in the sequel.
First of all we note that if $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ is a $|k|$-grading on a semisimple Lie algebra $\mathfrak{g}$, then $\mathfrak{g}_0$, $\mathfrak{p} := \mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_k$, $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_k$, and $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_{-1}$ are subalgebras of $\mathfrak{g}$ by the grading property. In particular, $\mathfrak{p}_+$ is a nilpotent ideal in $\mathfrak{p}$ and $\mathfrak{g}_-$ a nilpotent subalgebra of $\mathfrak{g}$.
Every $\mathfrak{g}_i$ can be seen as $\mathfrak{g}_0$-module, where the action of $\mathfrak{g}_0$ on $\mathfrak{g}_i$ is given by the adjoint action. Moreover the linear mappings $\varphi_{i,j} : \mathfrak{g}_i \otimes \mathfrak{g}_j \to \mathfrak{g}_{i+j}$ induced from the bracket $[\ ,\ ] : \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$ are $\mathfrak{g}_0$-homomorphisms. Note that since $\mathfrak{g}_0 = \mathfrak{p}/\mathfrak{p}_+$, we can regard every $\mathfrak{g}_0$-module also as a $\mathfrak{p}$-module with trivial action of $\mathfrak{p}_+$.
When $\mathfrak{g}$ is a $|k|$-graded Lie algebra, there is a natural filtration of $\mathfrak{g} = \mathfrak{g}^{-k} \supset \mathfrak{g}^{-k+1} \supset ... \supset \mathfrak{g}^k$, where $\mathfrak{g}^i$ is given by $\mathfrak{g}^i := \bigoplus_{j \geq i} \mathfrak{g}_j$, called the *associated filtration*. By definition $\mathfrak{p} = \mathfrak{g}^0$ and $\mathfrak{p}_+ = \mathfrak{g}^1$. By

the grading property we see that every filtration component $\mathfrak{g}^i$ is a $\mathfrak{p}$-submodule of $\mathfrak{g}$, so the filtration is $\mathfrak{p}$-invariant.
When $(\mathfrak{g}, \{\mathfrak{g}^i\})$ is a filtered Lie algebra, the associated graded vector space $\mathrm{gr}(\mathfrak{g}) = \bigoplus_j \mathfrak{g}^j/\mathfrak{g}^{j+1}$ can be made naturally to a Lie algebra, where the Lie bracket $[\ ,\ ] : \mathfrak{g}^i/\mathfrak{g}^{i+1} \times \mathfrak{g}^j/\mathfrak{g}^{j+1} \to \mathfrak{g}^{i+j}/\mathfrak{g}^{i+j+1}$ is defined as $[X_i + \mathfrak{g}^{i+1}, X_j + \mathfrak{g}^{j+1}] = [X_i, X_j] + \mathfrak{g}^{i+j+1}$. We call $\mathrm{gr}(\mathfrak{g})$ the *associated graded Lie algebra*.
When the filtration comes from a grading on $\mathfrak{g}$, $\mathrm{gr}(\mathfrak{g})$ and $\mathfrak{g}$ are canonically isomorphic as graded Lie algebras. By the filtration property $\mathrm{gr}_i(\mathfrak{g}) = \mathfrak{g}^i/\mathfrak{g}^{i+1}$ is a $\mathfrak{p}$-module with trivial action of $\mathfrak{p}_+$. So $\mathrm{gr}_i(\mathfrak{g})$ is $\mathfrak{g}_i$ with the action of $\mathfrak{g}_0$ trivially extended to $\mathfrak{p}$. Hence $\mathrm{gr}(\mathfrak{g})$ can be seen as $\mathfrak{p}$-module with trivial action of $\mathfrak{p}_+$.

Proposition 2.1. *Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ be a $|k|$-grading on a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ be a non-degenerate invariant (i.e. $B([X,Y],Z) = B(X,[Y,Z])$) bilinear form. Then we get:*
(1) There is a unique element $E \in \mathfrak{g}$ such that $[E,X] = jX$ $\forall X \in \mathfrak{g}_j$, $j = -k,...,k$. The element E lies in the center $\mathfrak{z}(\mathfrak{g}_0)$ of the Lie subalgebra $\mathfrak{g}_0$ and is called the grading element.
(2) Let $\mathfrak{g} = \mathfrak{a}_1 \oplus ... \oplus \mathfrak{a}_l$ be the decomposition of $\mathfrak{g}$ into simple ideals. Then the $|k|$-grading on $\mathfrak{g}$ induces a $|k_i|$- grading on each $\mathfrak{a}_i$, where $k_i \leq k$ Hence $\mathfrak{g}$ is the direct sum of $|k_i|$-graded Lie algebras, $k_i \leq k$.
(3) The isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ given by B is compatible with the grading and the associated filtration. In particular, B induces an isomorphism of $\mathfrak{g}_0$-modules between $\mathfrak{g}_{-i}$ and $\mathfrak{g}_i^*$ for all $i = 1,...,k$ and the filtration component $\mathfrak{g}^i$ is the annihilator of $\mathfrak{g}^{-i+1}$ with respect to B. Therefore we have a duality of $\mathfrak{g}_0$-modules between $\mathfrak{g}_i$ and $\mathfrak{g}_{-i}$ and a duality of $\mathfrak{p}$-modules between $\mathfrak{g}/\mathfrak{g}^{-i+1}$ and $\mathfrak{g}^i$ for $i = -k,...,k$.
(4) For $i < 0$ we have $[\mathfrak{g}_{i+1}, \mathfrak{g}_{-1}] = \mathfrak{g}_i$. If no simple factor of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$, this holds also for $i = 0$.
(5) Let $A \in \mathfrak{g}_i$ with $i > 0$ an element such that $[A,X] = 0$ $\forall X \in \mathfrak{g}_{-1}$. Then $A = 0$. If no simple factor of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$, this holds also for $i = 0$.

Proof. (1) We consider the linear map $D : \mathfrak{g} \to \mathfrak{g}$ defined by $D(X) = jX$ for $X \in \mathfrak{g}_j$, $j = -k,...,k$. From $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ it follows that D is a derivation, i. e. $D([X,Y]) = [D(X),Y] + [X,D(Y)]$ $\forall X,Y \in \mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, there must be a unique element $E \in \mathfrak{g}$ such that $ad(E) = D$. When we decompose E into $E = E_{-k} + ... + E_k$ with $E_i \in \mathfrak{g}_i$, we get $0 = [E,E] = \sum_{j=-k}^{k} [E, E_j] = \sum_{j=-k}^{k} jE_j$ and therefore $E = E_0 \in \mathfrak{g}_0$. $E \in \mathfrak{z}(\mathfrak{g}_0)$, since $0 = D(A) = [E,A]$ $\forall A \in \mathfrak{g}_0$.
(2) We have $\mathfrak{a}_r = \bigoplus_{j=-k}^{k} (\mathfrak{a}_r \cap \mathfrak{g}_j)$ for $r = 1,...,l$. This is a $|k_r|$-grading on $\mathfrak{a}_r$ with $k_r \leq k$, since $[\mathfrak{a}_r \cap \mathfrak{g}_i, \mathfrak{a}_r \cap \mathfrak{g}_j] \subset \mathfrak{a}_r \cap \mathfrak{g}_{i+j}$ for $r = 1,...,l$ and for $i,j = -k,...,k$ and since obviously $\mathfrak{a}_r \cap \mathfrak{g}_{-1}$ generates $\mathfrak{g}_- \cap \mathfrak{a}_r$ for $r = 1,...,l$.

(3) The invariance of B implies in particular that for the grading element E

$$B([E,X],Y) = -B(X,[E,Y])$$

for all $X, Y \in \mathfrak{g}$. For $X \in \mathfrak{g}_i$ $Y \in \mathfrak{g}_j$ we get $0 = (i+j)B(X,Y)$ and therefore $B(X,Y) = 0$ unless $i + j = 0$. Non-degeneracy of B implies immediately that the restrictions of B to $\mathfrak{g}_0 \times \mathfrak{g}_0$ and to $\mathfrak{g}_j \times \mathfrak{g}_{-j}$ for $j = 1, ..., k$ are non-degenerate. Hence the isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ given by $X \mapsto B(X, \ )$ is compatible with the grading. The invariance of B implies that this isomorphism is a $\mathfrak{g}_0$-module isomorphism, where the action of $\mathfrak{g}_0$ on $\mathfrak{g}^*$ is given by the action that is induced from the adjoint action on $\mathfrak{g}$. In particular, using invariance and non-degeneracy of B, one gets an $\mathfrak{g}_0$-module isomorphism between the $\mathfrak{g}_0$-modules $\mathfrak{g}_{-i}$ and $\mathfrak{g}_i^*$.
The compatibility of B with the grading implies that the restriction of B to $\mathfrak{g}^{-i+1} \times \mathfrak{g}^i$ vanishes, so it induces a bilinear form $\mathfrak{g}/\mathfrak{g}^{-i+1} \times \mathfrak{g}^i \to \mathbb{K}$. This bilinear form is non-degenerate, since the non-degeneracy of B implies that for any $X \in \mathfrak{g}^i$ one can find an element $Y \in \mathfrak{g}$ such that $B(Y,X) \neq 0$. Since $\mathfrak{g}^i = \mathfrak{g}_i \oplus ... \oplus \mathfrak{g}_k$ and $\mathfrak{g}/\mathfrak{g}^{-i+1}$ is as vector space isomorphic to $\mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_{-i}$, it follows from $\mathfrak{g}_{-i} \cong (\mathfrak{g}_i)^*$ that these vector spaces have the same dimensions. Therefore B induces a linear isomorphism between $\mathfrak{g}/\mathfrak{g}^{-i+1}$ and $(\mathfrak{g}^i)^*$, and the invariance of B implies that this isomorphism is an isomorphism of $\mathfrak{p}$-modules.
(4) Since $\mathfrak{g}_{-1}$ generates $\mathfrak{g}_-$, the statement is true for $i \leq -2$. From $[E,X] = -X$ for $X \in \mathfrak{g}_{-1}$ and $E \in \mathfrak{g}_0$ the grading element, we get the statement for $i = -1$. For the case $i = 0$ we look at $\mathfrak{a} := [\mathfrak{g}_1, \mathfrak{g}_{-1}] \oplus \bigoplus_{i \neq 0} \mathfrak{g}_i$. $\mathfrak{a}$ is a non-trivial ideal in $\mathfrak{g}$, which can be seen by using the fact that $\mathfrak{g}_-$ is generated by $\mathfrak{g}_{-1}$. If no simple factor lies in $\mathfrak{g}_0$ this is only possible if $\mathfrak{a} = \mathfrak{g}$, since every ideal in a semisimple Lie algebra admits a complementary ideal, which by (2)is the sum of some of its homogeneous components.
(5) Let $K(X,Y) = tr(ad(X) \circ ad(Y))$ be the Killing form. K is non-degenerate and invariant. For any $Z \in \mathfrak{g}_{-i+1}$ and $X \in \mathfrak{g}_{-1}$ we have $0 = K([A,X],Z) = K(A,[X,Z])$, so A is orthogonal to $[\mathfrak{g}_{-1}, \mathfrak{g}_{-i+1}]$ with respect to the Killing form. Since $[\mathfrak{g}_{-1}, \mathfrak{g}_{-i+1}] = \mathfrak{g}_{-i}$ by (4) for $i > 0$ (for $i \geq 0$ if no simple factor is contained in $\mathfrak{g}_0$) and since K restricted to $\mathfrak{g}_i \times \mathfrak{g}_{-i}$ is non-degenerate by (3), the result follows. □

2.2. Parabolic subalgebras and gradings

2.2.1. The complex case. As one knows from the structure theory of Lie algebras, complex semisimple Lie algebras can be completely classified up to isomorphism by a simple subsystem Δ^0 of the root system Δ which is associated to a complex semisimple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.
Having fixed a Cartan subalgebra $\mathfrak{h}$ of a complex semisimple Lie algebra $\mathfrak{g}$, we can choose an total ordering on $\mathfrak{h}_0^*$, where $\mathfrak{h}_0 \subset \mathfrak{h}$ is the

subspace on which all roots are real, in the following way: Choose an ordered basis $\{H_1, ..., H_n\}$ of $\mathfrak{h}_0$ and define a linear functional $\phi \in \mathfrak{h}_0^*$ to be positive, if for some $i \in \{1, ..., n\}$ one has $\phi(H_j) = 0$ for $j < i$ and $\phi(H_i) > 0$. Then we obtain an total ordering on $\mathfrak{h}_0^*$ by defining $\phi \leq \psi$ if and only if $\psi - \phi$ is positive. In particular we obtain the set of positive roots Δ^+ and the set of simple roots $\Delta^0 \subset \Delta^+$, which by definition consists of those positive roots, which can not be written as the sum of two positive ones.
Starting with a complex semisimple Lie algebra $\mathfrak{g}$, the two choices, of a Cartan subalgebra first and then of a positive subsystem of the associated root system, can be equivalently described as the choice of a *Borel subalgebra*, which is a maximal solvable subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. Once one has chosen a Cartan subalgebra $\mathfrak{h}$ and a set of positive roots $\Delta^+ \subseteq \Delta(\mathfrak{g}, \mathfrak{h})$, the associated Borel subalgebra is given by $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ where $\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ is the direct sum of all root spaces corresponding to positive roots $\alpha \in \Delta^+$. It can be easily seen that $\mathfrak{b}$ is a maximal solvable subalgebra, called the *standard Borel subalgebra associated to* $\mathfrak{h}$ *and* Δ^+.
The fact that Cartan subalgebras and positive subsystems of roots are unique up to conjugation can then be translated to the fact that any two Borel subalgebras of a complex semisimple Lie algebra $\mathfrak{g}$ are conjugate by an inner automorphism of $\mathfrak{g}$. By an inner automorphism of a Lie algebra $\mathfrak{l}$ we understand an element of the connected virtual Lie subgroup of the automorphism group $Aut(\mathfrak{l})$ corresponding to $ad(\mathfrak{l})$. We will the denote this Lie group of inner automorphisms in the sequel by $Int(\mathfrak{l})$. If $\mathfrak{l}$ is semisimple, the derivations of l coincide with $ad(\mathfrak{l})$ and $Int(\mathfrak{l})$ equals the connected component of the identity of $Aut(\mathfrak{l})$.

DEFINITION. *Let* $\mathfrak{g}$ *be a complex semisimple Lie algebra,* $\mathfrak{h}$ *a Cartan subalgebra of* $\mathfrak{g}$, $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ *the associated root system and* $\Delta^+ \subseteq \Delta$ *a system of positive roots.*
A parabolic subalgebra $\mathfrak{p}$ *of* $\mathfrak{g}$ *is a subalgebra of* $\mathfrak{g}$ *which contains a Borel subalgebra.*
A standard parabolic subalgebra is a subalgebra $\mathfrak{p}$ *of* $\mathfrak{g}$ *that contains the standard Borel subalgebra associated to* $\mathfrak{h}$ *and* Δ^+.

The fact that any two Borel subalgebras of a complex semisimple Lie algebra are conjugate says in particular that, once you have fixed a Cartan subalgebra and a positive system of roots or equivalently a standard Borel subalgebra, every Borel subalgebra is conjugate to the standard one and therefore also every parabolic subalgebra is conjugate to a standard parabolic subalgebra. To give a complete description of all parabolic subalgebras (up to conjugation) of a given semisimple Lie algebra, it therefore suffices to describe all standard parabolic subalgebras. The standard parabolic subalgebras in turn can be described by subsets of simple roots, which is shown in the following proposition.

PROPOSITION 2.2. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the associated root system and Δ^0 the set of simple roots for the choice of some system of positive roots Δ^+.*
Then there is a bijection Φ between the standard parabolic subalgebras $\mathfrak{p}$ of $\mathfrak{g}$ and the subsets $\Sigma \subseteq \Delta^0$ of simple roots which is given by:

$$\Phi : \mathfrak{p} \mapsto \Sigma_{\mathfrak{p}} \qquad \Phi^{-1} : \Sigma \mapsto \mathfrak{p}_{\Sigma}$$

where $\Sigma_{\mathfrak{p}} = \{\alpha \in \Delta^0 : \mathfrak{g}_{-\alpha} \nsubseteq \mathfrak{p}\}$ and $\mathfrak{p}_{\Sigma}$ is the sum of the standard Borel subalgebra $\mathfrak{b}$ and all those negative root spaces corresponding to roots which can be written as a linear combination of elements of $\Delta^0 \backslash \Sigma$.

PROOF. First we have to show that for an arbitrary subset Σ of Δ^0 $\mathfrak{p}_{\Sigma}$ is a standard parabolic subalgebra. Using the fact that for $\alpha, \beta \in \Delta$ with $\alpha + \beta \in \Delta$ we have $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] = \mathfrak{g}_{\alpha+\beta}$, one can immediately see that $\mathfrak{p}_{\Sigma}$ is a subalgebra. Since by definition $\mathfrak{b} \subseteq \mathfrak{p}_{\Sigma}$, it follows that $\mathfrak{p}_{\Sigma}$ is a standard parabolic subalgebra.
Let now $\mathfrak{p} \subset \mathfrak{g}$ be a standard parabolic subalgebra. Since by definition $\mathfrak{b} \subset \mathfrak{p}$ it follows that $\mathfrak{p}$ is the direct sum of $\mathfrak{b}$ and some negative root spaces (every subalgebra of $\mathfrak{g}$ that contains $\mathfrak{h}$ must be the direct sum of $\mathfrak{h}$ and some of the root spaces, since the projections on the eigenspaces of an operator are polynomials in the operator). Let $\Psi \subset \Delta^+$ be the set of all roots α such that $\mathfrak{g}_{-\alpha} \subset \mathfrak{p}$. Note first that for $\alpha \in \Psi$ and $\beta \in \Delta^+$ such that $\alpha - \beta \in \Delta$, we have $\mathfrak{g}_{-(\alpha-\beta)} = [\mathfrak{g}_{-\alpha}, \mathfrak{g}_{\beta}]$ and therefore $\alpha - \beta \in \Psi$. When we now write $\Delta^0 = \{\alpha_1, ..., \alpha_r\}$ and suppose that $\alpha \in \Psi$ decomposes as $\alpha = a_1\alpha_1 + ... + a_r\alpha_r$, then we can conclude inductively from the above that if $a_i \neq 0$, $\alpha_i \in \Psi$. Conversely, if $\alpha, \beta \in \Psi$ such that $\alpha + \beta \in \Delta$, $\alpha + \beta$ must be contained in Ψ, since $\mathfrak{g}_{-\alpha-\beta} = [\mathfrak{g}_{-\alpha}, \mathfrak{g}_{-\beta}] \subset \mathfrak{p}$. Hence we get that Ψ is closed under integral linear combinations (i. e. any integral linear combination of roots contained in Ψ which itself is a root has to be also an element of Ψ) and completely determined by $\Psi \cap \Delta_0$. Now we see that $\Sigma_{\mathfrak{p}} = \Delta^0 \setminus (\Psi \cap \Delta^0)$ and $\mathfrak{p} = \mathfrak{p}_{\Sigma_{\mathfrak{p}}}$. □

The description of standard parabolic subalgebras by subsets of the simple roots can now be used to establish the correspondence between the standard parabolic subalgebras of a semisimple Lie algebra $\mathfrak{g}$ and all possible $|k|$-gradings of $\mathfrak{g}$.

DEFINITION. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, Δ the corresponding root system, Δ^0 the set of simple roots for some choice of positive roots Δ^+ and $\Sigma \subset \Delta^0$ a subset of simple roots.*
For $\alpha \in \Delta$ the Σ-height $ht_{\Sigma}(\alpha)$ of α is the sum of all coefficients of elements of Σ in the representation of α as linear combination of simple roots.

The decomposition of $\mathfrak{g}$ *according to* Σ*-height is given by*

$$\mathfrak{g}_i := \bigoplus_{\alpha : ht_\Sigma(\alpha)=i} \mathfrak{g}_\alpha$$

for $i \in \mathbb{Z} \setminus \{0\}$ *and*

$$\mathfrak{g}_0 := \mathfrak{h} \oplus \bigoplus_{\alpha : ht_\Sigma(\alpha)=0} \mathfrak{g}_\alpha$$

If r *is the* Σ*-height of the maximal root in* Δ*, we get* $\mathfrak{g}_i = \{0\}$ *for* $|i| > r$ *and hence* $\mathfrak{g} = \mathfrak{g}_{-r} \oplus ... \oplus \mathfrak{g}_r$*.*

THEOREM 2.1. *Let* $\mathfrak{g}$ *be a complex semisimple Lie algebra,* $\mathfrak{h}$ *a Cartan subalgebra,* Δ *the corresponding root system and* Δ^0 *the set of simple roots for some choice of positive roots* Δ^+*.*
(1) For any standard parabolic subalgebra $\mathfrak{p}$ *of* $\mathfrak{g}$ *corresponding to the subset* $\Sigma \subset \Delta^0$ *the decomposition* $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ *according to* Σ*-height defines a* $|k|$*-grading on* $\mathfrak{g}$ *such that* $\mathfrak{p} = \mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_k$ *and where* k *is the* Σ*-height of the maximal root in* Δ*. In addition, the subalgebra* $\mathfrak{g}_0 \subset \mathfrak{g}$ *is reductive and the dimension of its center* $\mathfrak{z}(\mathfrak{g}_0)$ *coincides with the number of elements in* Σ*.*
(2) For any $|k|$*- grading* $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ *with* $k \in \mathbb{N}$*, the subalgebra* $\mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_k$ *is parabolic. Choosing the Cartan subalgebra and the positive roots such that* $\mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_k$ *is a standard parabolic subalgebra, the grading is given by* Σ*-height.*

PROOF. (1) The inclusion $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ follows immediately from the properties of the root decomposition $[\mathfrak{h}, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha$, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ for $\alpha, \beta \in \Delta$ with $\alpha + \beta \in \Delta$ (in the latter case $ht_\Sigma(\alpha+\beta) = ht_\Sigma(\alpha) + ht_\Sigma(\beta)$). Next we have to show that $\mathfrak{g}_{-1}$ generates $\mathfrak{g}_-$: Let $\mathfrak{a} \subset \mathfrak{g}_-$ be the subalgebra generated by $\mathfrak{g}_{-1}$. Suppose $\mathfrak{a} \neq \mathfrak{g}_-$, then there must be a root α such that $\mathfrak{g}_\alpha \not\subseteq \mathfrak{a}$. Choose α to be the root of maximal hight with this property. Since $-\alpha$ is positive but not simple, there exists a simple root $\alpha_i \in \Delta^0 = \{\alpha_1, ..., \alpha_n\}$ such that $\alpha + \alpha_i \in \Delta$. The root $\alpha + \alpha_i$ has bigger height than α, so $\mathfrak{g}_{\alpha+\alpha_i}$ is contained in $\mathfrak{a}$. Since $\mathfrak{g}_\alpha = [\mathfrak{g}_{-\alpha_i}, \mathfrak{g}_{\alpha+\alpha_i}]$ and $[\mathfrak{g}_l, \mathfrak{a}] \subset \mathfrak{a}$ for $l = 0, -1$, we get a contradiction from the fact that for $\alpha_i \in \Sigma$ we have $\mathfrak{g}_{-\alpha_i} \subset \mathfrak{g}_{-1}$ and for $\alpha_i \notin \Sigma$ we have $\mathfrak{g}_{-\alpha_i} \subset \mathfrak{g}_0$. So $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ makes $\mathfrak{g}$ into a $|k|$-graded Lie algebra.
The subalgebra $\mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_k$ consists of $\mathfrak{h}$, all positive root spaces and those negative root spaces corresponding to roots, which can be written as linear combination of elements in $\Delta^0 \setminus \Sigma$, so $\mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_k = \mathfrak{p}$.
To prove the last statement, consider the subspace

$$\mathfrak{h}' := \{H \in \mathfrak{h} : \alpha_i(H) = 0 \quad \forall \alpha_i \in \Delta^0 \setminus \Sigma\}$$

of $\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha : ht_\Sigma(\alpha)=0} \mathfrak{g}_\alpha$. Then for $H \in \mathfrak{h}'$ and all roots α with $ht_\Sigma(\alpha) = 0$ one has $\alpha(H) = 0$ and thus $\mathfrak{h}' \subset \mathfrak{z}(\mathfrak{g}_0)$. Since the simple roots form a basis of $\mathfrak{h}^*$, the dimension of $\mathfrak{h}'$ is $|\Sigma|$. It remains to

show that $\mathfrak{h}' = \mathfrak{z}(\mathfrak{g}_0)$ and that $\mathfrak{g}_0$ is reductive. Let K be the Killing form. For $\alpha_i \in \Delta^0$ define $H_{\alpha_i} := \frac{2}{K(h_{\alpha_i}, h_{\alpha_i})} h_{\alpha_i}$ where h_{α_i} is the unique element which is characterized by $K(H, h_{\alpha_i}) = \alpha_i(H)$ for all $H \in \mathfrak{h}$. These elements form a basis of $\mathfrak{h}$. Let $\mathfrak{h}''$ be the span of the H_{α_i} with $\alpha_i \in \Delta^0 \setminus \Sigma$. Since $\alpha_i(H_{\alpha_i}) = 2$, we have $\mathfrak{h}' \cap \mathfrak{h}'' = \{0\}$ and $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$ by dimensional reasons. To prove that $\mathfrak{g}_0$ is reductive we show that any solvable ideal of $\mathfrak{g}_0$ is contained in $\mathfrak{z}(\mathfrak{g}_0)$. Let I be an ideal with $I \not\subseteq \mathfrak{h}'$. Since the decomposition $\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha : ht_\Sigma(\alpha)=0} \mathfrak{g}_\alpha$ is the decomposition into eigenspaces for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}_0$ and the projection on an eigenspace of an operator is a polynomial in the operator, I must be the direct sum of a subspace of $\mathfrak{h}$ and some of the root spaces. If I is not contained in $\mathfrak{h}'$, then it has to contain at least one of the root spaces $\mathfrak{g}_\alpha \subset \mathfrak{g}_0$: Suppose I contains an element $\neq 0$ of $\mathfrak{h} \setminus \mathfrak{h}'$, then there is a root space on which this element acts non-trivially and this root space must lie in I, since I is an ideal.
But if $\mathfrak{g}_\alpha$ is contained in I, also $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is in I and therefore also $\mathfrak{g}_{-\alpha} = [[\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha], \mathfrak{g}_{-\alpha}]$. Thus I must contains the subalgebra $\mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_\alpha \cong \mathfrak{sl}(2, \mathbb{C})$ and I can not be solvable. Hence any solvable ideal must be contained in $\mathfrak{h}' \subset \mathfrak{z}(\mathfrak{g}_0)$, so $\mathfrak{g}_0$ is reductive.
Since $\mathfrak{z}(\mathfrak{g}_0)$ is a solvable ideal in $\mathfrak{g}_0$, $\mathfrak{z}(\mathfrak{g}_0) \subset \mathfrak{h}'$ and so $\mathfrak{z}(\mathfrak{g}_0) = \mathfrak{h}'$.
(2) Suppose that $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ is a $|k|$-grading. By proposition 2.1., there is a grading element $E \in \mathfrak{z}(\mathfrak{g}_0)$ which satisfies $ad(E)(X) = jX$ for $X \in \mathfrak{g}_j$. In particular, $ad(E)$ is diagonalizable and we can extend $\mathbb{C}E$ to a maximal abelian subalgebra consisting of semisimple elements, thus we obtain a Cartan subalgebra $\mathfrak{h}$ with $E \in \mathfrak{h}$. Since the eigenvalues of $ad(E)$ are $\{-k, ..., k\}$, E lies in the subspace $\mathfrak{h}_0$ of $\mathfrak{h}$, on which all roots real. Take a basis of $\mathfrak{h}_0$ that starts with E, then the resulting positive roots have non-negative values on E. Since $[E, \mathfrak{h}] = 0$, $\mathfrak{h} \subset \mathfrak{g}_0$ and all positive root spaces lie in $\mathfrak{g}^0 := \mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_k$, therefore $\mathfrak{g}^0$ is a standard parabolic subalgebra. E acts by a scalar on each root space, since $E \in \mathfrak{h}$, thus each root space must be contained in a grading component. For $\alpha \in \Delta^0$, $\mathfrak{g}_{-\alpha}$ is contained in $\mathfrak{g}_i$ for some $i \leq 0$. From the fact that $\mathfrak{g}_{-1}$ generates $\mathfrak{g}_-$ and α is simple, we get that $i = 0$ or $= -1$. That means all simple root spaces are in $\mathfrak{g}_0$ or $\mathfrak{g}_1$. Define $\Sigma := \{\alpha \in \Delta^0 : \mathfrak{g}_\alpha \subset \mathfrak{g}_1\}$. Then $\mathfrak{p}_\Sigma = \mathfrak{g}^0$ and the grading is given by Σ-height. □

This theorem gives us immediately supplementary information about gradings on semisimple Lie algebras:

Corollary 2.1. *Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ be a $|k|$-graded semisimple Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then we get:*
(1) For $i > 0$: $[\mathfrak{g}_{i-1}, \mathfrak{g}_1] = \mathfrak{g}_i$ and the filtration component $\mathfrak{g}^i$ is the $i-th$ power of $\mathfrak{p}_+ = \mathfrak{g}^1$, hence $\mathfrak{p}_+ \supset \mathfrak{g}^2 \supset ... \supset \mathfrak{g}^k$ is the lower central series of $\mathfrak{p}_+$.
(2) If there is an element $X \in \mathfrak{g}_i$ for some $i < 0$ such that $[X, Z] = 0$

for all $Z \in \mathfrak{g}_1$, then $X = 0$. If no simple factor is contained in $\mathfrak{g}_0$, this holds for $i = 0$.

Proof. We prove (1) and (2) together. In the complex case changing the sign of the grading just means to use the opposite order for the roots. Therefore, changing the sign of the grading leads to an isomorphic $|k|$-graded Lie algebra. For a real $|k|$-graded Lie algebra $\mathfrak{g}$ the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is a $|k|$-graded complex Lie algebra, where the grading is given by $(\mathfrak{g}_{\mathbb{C}})_i = \mathfrak{g}_i \otimes_{\mathbb{R}} \mathbb{C}$. So the same is true in the real case. (1) and (2) follow therefore from (4) and (5) in proposition 2.1. $\square$

Now we want to introduce a notation for the standard parabolic subalgebras of a complex semisimple Lie algebra. Therefore let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra and Δ^0 the simple roots for some choice of positivity. Then we know that a standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is uniquely determined by a subset $\Sigma \subset \Delta^0$. Hence we can denote $\mathfrak{p}$ or equivalently the corresponding $|k|$-grading by Σ-height, by the Dynkin diagram of $\mathfrak{g}$, where we represent the elements of Σ by a cross instead of a dot.

There is another important description of parabolic subalgebras of complex semisimple Lie algebras as stabilizers of lines and flags, which gives a more geometric interpretation of parabolic subalgebras.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra and Δ be the associated root system. Choose an ordering on $\mathfrak{h}_0^*$, where $\mathfrak{h}_0$ is the subspace on which all roots are real and denote by Δ^+ the corresponding set of positive roots. Suppose V is a complex finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight λ and $v_0 \in V$ a highest weight vector. Then the stabilizer of the line through v_0 is the standard parabolic subalgebra of $\mathfrak{g}$ corresponding to the set $\Sigma = \{\alpha \in \Delta^0 : \langle \lambda, \alpha \rangle \neq 0\}$, where $\langle\ ,\ \rangle$ is the non-degenerate complex bilinear form on $\mathfrak{h}^* \times \mathfrak{h}^*$ induced from the Killing form K on $\mathfrak{g} \times \mathfrak{g}$.

Indeed, the stabilizer of the line through v_0 contains $\mathfrak{h}$ and all positive root spaces, since by definition of a highest weight vector $H \cdot v_0 = \lambda(H) v_0$ for all $H \in \mathfrak{h}$ and for $X \in \mathfrak{g}_\alpha$ with $\alpha \in \Delta^+$ we have $X \cdot v_0 = 0$. So the stabilizer is a standard parabolic subalgebra, which we denote by $\mathfrak{p}$. By proposition 2.2. $\mathfrak{p}$ corresponds to the subset of simple roots $\Sigma := \{\alpha \in \Delta^0 : \mathfrak{g}_{-\alpha} \nsubseteq \mathfrak{p}\}$. So it remains to show that for a simple root α and a nonzero element $X \in \mathfrak{g}_{-\alpha}$ one has $X \cdot v_0 = 0$ if and only if $\langle \lambda, \alpha \rangle = 0$.

Suppose X is a nonzero element in $\mathfrak{g}_{-\alpha}$ with $\alpha \in \Delta^0$ such that $X \cdot v_0 = 0$ and choose $Y \in \mathfrak{g}_\alpha$ such that $[Y, X] = h_\alpha$, where h_α denotes the dual element to α with respect to the Killing form. By using $Y \cdot v_0 = 0$, we obtain that $0 = Y \cdot X \cdot v_0 = [Y, X] \cdot v_0 = \lambda(h_\alpha) v_0 = \langle \lambda, \alpha \rangle v_0$ and therefore $\langle \lambda, \alpha \rangle = 0$.

Conversely, assume $\langle \lambda, \alpha \rangle = 0$ for some $\alpha \in \Delta^0$ and let X be a

nonzero element in $\mathfrak{g}_{-\alpha}$. If now $X \cdot v_0 \neq 0$, then it has to be a weight vector of weight $\lambda - \alpha$. Since the Weyl group acts on the set of weights, we obtain that also $s_\alpha(\lambda - \alpha) = s_\alpha(\lambda) + \alpha$ is a weight of V, where $s_\alpha : \mathfrak{h}_0^* \to \mathfrak{h}_0^*$ is the reflection on the hyperplane $\alpha^\perp$ given by $s_\alpha(\mu) = \mu - 2\frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$. By assumption $\langle \lambda, \alpha \rangle = 0$, so $s_\alpha(\lambda) = \lambda$ and $\lambda + \alpha$ is a weight, which contradicts the maximality of λ. Hence $X \cdot v_0 = 0$.
If $\Delta^0 = \{\alpha_1, ..., \alpha_n\}$, we denote by $\omega_{\alpha_1}, ..., \omega_{\alpha_n} \in \mathfrak{h}_0^*$ the fundamental weights given by $\frac{2\langle \omega_{\alpha_i}, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}$.
We remember that the highest weight λ as a dominant algebraically integral weight can be written as a linear combination of the fundamental weights with non-negative integral coefficients. The coefficient of ω_{α_i} is $2\frac{\langle \lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}$ and therefore Σ is exactly the set of those simple roots for which the corresponding fundamental weights have a nonzero coefficient in the expansion of λ. Given a standard parabolic subalgebra $\mathfrak{p}$ corresponding to the set $\Sigma \subseteq \Delta^0$ we can therefore realize $\mathfrak{p}$ as stabilizer of a highest weight line by taking V to be the unique (up to isomorphism) finite dimensional irreducible representation corresponding to $\mu := \sum_{\alpha_i \in \Sigma} \omega_{\alpha_i}$.
Since the standard parabolic subalgebra $\mathfrak{p}_\Sigma$ corresponding to $\Sigma = \Sigma_1 \cup \Sigma_2$ for two subsets Σ_1, Σ_2 of Δ^0 is given by the intersection of $\mathfrak{p}_{\Sigma_1}$ and $\mathfrak{p}_{\Sigma_2}$, one can also describe all standard parabolic subalgebras as stabilizers of certain flags. We will see some examples in the next chapter.

2.2.2. The real case. First we collect some facts about the structure theory of real semisimple Lie algebras. Proofs and more details on the structure thoery can be found in [**5**].
Real semisimple Lie algebras can be studied as real forms of complex semisimple Lie algebras, since a real Lie algebra is semisimple if and only if its complexification is semisimple. By a *real form* of a complex semisimple Lie algebra $\mathfrak{g}$ we understand a real Lie subalgebra $\mathfrak{g}_0$ such that $\mathfrak{g}_\mathbb{R}$ decomposes as vector space into $\mathfrak{g}_\mathbb{R} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$, where $\mathfrak{g}_\mathbb{R}$ is the real Lie algebra obtained from $\mathfrak{g}$ by restricting the scalar multiplication to $\mathbb{R}$. The *conjugation of* $\mathfrak{g}$ *with respect to the real form* $\mathfrak{g}_0$ is the $\mathbb{R}$ linear map $\mathfrak{g} \to \mathfrak{g}$ that is 1 on $\mathfrak{g}_0$ and -1 on $i\mathfrak{g}_0$. Hence the conjugation is an involutive automorphism of $\mathfrak{g}_\mathbb{R}$.
A *Cartan subalgebra* of a real semisimple Lie algebra $\mathfrak{s}$ is a subalgebra, whose complexification is a Cartan subalgebra of $\mathfrak{s}_\mathbb{C}$.
For any complex semisimple Lie algebra $\mathfrak{g}$ there are two distinguished types of real forms: split real forms and compact real forms.
A *split real form* of $\mathfrak{g}$ is a real form $\mathfrak{g}_0$ that contains a Cartan subalgebra $\mathfrak{h}$ on which all roots of $\mathfrak{g}$ with respect to $\mathfrak{h}_\mathbb{C}$ are real.
A *compact real form* of $\mathfrak{g}$ is a real form $\mathfrak{u}$ such that $\mathfrak{u}$ is compact, i. e. $\mathfrak{u}$ is the Lie algebra of a compact Lie group. For any complex semisimple Lie algebra there is up to isomorphism exactly one compact real form.

It can be shown that for a semisimple Lie algebra to be compact is equivalent to the fact that the Killing form is negative definite.
Let $\mathfrak{g}$ be a real semisimple Lie algebra, $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ the Killing form and $\theta : \mathfrak{g} \to \mathfrak{g}$ a *Cartan involution*, i. e. an involutive automorphism such that $K_\theta(X, Y) := -K(X, \theta(Y))$ is positive definite. For any real semisimple Lie algebra such an involution exists and is unique up to conjugation with an inner automorphism of $\mathfrak{g}$. Define $\mathfrak{k}$ and $\mathfrak{q}$ as the eigenspaces of θ to the eigenvalues 1 and -1 respectively. Then $\mathfrak{g}$ decomposes as vector space into $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$. In addition we have that $\mathfrak{k}$ is a subalgebra and that $[\mathfrak{k}, \mathfrak{q}] \subset \mathfrak{q}$ and $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{k}$. Note that the restriction of the Killing form K to $\mathfrak{q}$ is positive definite and to $\mathfrak{k}$ it is negative definite. So $\mathfrak{k}$ is compact. One can easily see that $ad(\theta(X)) = -ad(X)^t$ with respect to K_θ. Thus $ad(X)$ is skew symmetric for $X \in \mathfrak{k}$ and symmetric for $X \in \mathfrak{q}$. That means in particular, that $ad(X)$ is never diagonalizable over $\mathbb{R}$ for $X \in \mathfrak{k}$ and always for $X \in \mathfrak{q}$. Therefore to get an analog of the root decomposition in the complex case, let $\mathfrak{a} \subset \mathfrak{q}$ be a maximal abelian subalgebra. Then the maps $ad(A)$ for $A \in \mathfrak{a}$ form a family of commuting symmetric linear maps which is thus simultaneously diagonalizable and the eigenvalue on the corresponding eigenspace depends linearly on $A \in \mathfrak{a}$. For a linear functional $\lambda : \mathfrak{a} \to \mathbb{R}$ we set $\mathfrak{g}_\lambda := \{X \in \mathfrak{g} : ad(A)(X) = \lambda(A)X \ \forall A \in \mathfrak{a}\}$. The nonzero functionals λ with $\mathfrak{g}_\lambda \neq \{0\}$ are called the *restricted roots* of $\mathfrak{g}$ with respect to $\mathfrak{a}$ and will be denoted by $\Delta_r = \Delta_r(\mathfrak{g}, \mathfrak{a})$. Finally one obtains a decomposition of $\mathfrak{g}$ into an orthogonal (with respect to K_θ) direct sum $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Delta_r} \mathfrak{g}_\lambda$, where $\mathfrak{g}_0$ is the direct sum of $\mathfrak{a}$ and of the centralizer $Z_\mathfrak{k}(\mathfrak{a})$ of $\mathfrak{a}$ in the $\mathfrak{k}$. One can show that $\Delta_r \subset \mathfrak{a}^*$ is an abstract root system, which need not to be reduced in general.
If $\mathfrak{t}$ is a maximal abelian subalgebra of $Z_\mathfrak{k}(\mathfrak{a})$, then $\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{t}$ is a maximally non-compact θ-stable Cartan subalgebra, i.e. a Cartan subalgebra such that $\theta(\mathfrak{h}) \subset \mathfrak{h}$ and the dimension of $\mathfrak{h} \cap \mathfrak{q}$ is the maximal possible among θ-stable Cartan subalgebras. Any two maximally non-compact θ-stable Cartan subalgebras of $\mathfrak{g}$ are conjugate by an element of K, where $K \subset G$ is the Lie subgroup corresponding to $\mathfrak{k}$.
Having chosen a maximally non-compact θ-stable Cartan subalgebra $\mathfrak{h}$, we get that $\mathfrak{h} \cap \mathfrak{q} =: \mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{q}$. Then consider the root system $\Delta = \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ associated to $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ and the system of restricted roots Δ_r associated to $(\mathfrak{g}, \mathfrak{a})$. By construction, a restricted root space $\mathfrak{g}_\lambda$ is just the intersection of $\mathfrak{g}$ with the direct sum of those root spaces $(\mathfrak{g}_\mathbb{C})_\alpha$ with $\alpha \mid_\mathfrak{a} = \lambda$. Therefore the restricted roots are really the nonzero restrictions to $\mathfrak{a}$ of elements in Δ.
Let σ be the conjugation of $\mathfrak{g}_\mathbb{C}$ with respect to the real form $\mathfrak{g}$. Then σ induces an involutive automorphism $\sigma^* : \Delta \to \Delta$ defined by $\sigma^*\alpha(H) := \overline{\alpha(\sigma(H))} \ \forall H \in \mathfrak{h}_\mathbb{C}$. We define the set of *compact roots* $\Delta_c \subset \Delta$ as the set of those roots such that $\sigma^*\alpha = -\alpha$. We note that all roots have real values on $i\mathfrak{t} \oplus \mathfrak{a}$ and that $\alpha \in \Delta_c$ if and only if the restriction of α

to $\mathfrak{h}$ has purely imaginary values. Hence the compact roots can also be described as $\Delta_c = \{\alpha \in \Delta : \alpha|_{\mathfrak{a}} = 0\}$. One can show that for $\alpha \in \Delta_c$ the root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{k}_{\mathbb{C}}$, which explains the name compact roots.
Finally, we want to introduce the Satake diagram of $\mathfrak{g}$ and describe its relation to the restricted roots Δ_r. A positive subsystem $\Delta^+ \subset \Delta$ is called *admissible* if for $\alpha \in \Delta^+$ either $\sigma^*\alpha = -\alpha$ or $\sigma^*\alpha \in \Delta^+$. This means that for an admissible positive subsystem $\Delta^+ \subset \Delta$ we have $\sigma^*\alpha \in \Delta^+$ for all $\alpha \in \Delta^+ \setminus \Delta_c$. One can show that $\Delta_c^0 := \Delta_c \cap \Delta^0$ is a simple subsystem of Δ_c, where Δ^0 is the set of simple roots induced by Δ^+, and that for any $\alpha \in \Delta^0 \setminus \Delta_c^0$ there exists a unique $\alpha' \in \Delta^0 \setminus \Delta_c^0$ with $\sigma^*\alpha - \alpha' \in \Delta_c$. This induces an involutive automorphism of $\Delta^0 \setminus \Delta_c^0$ given by $\alpha \mapsto \alpha'$. The Satake diagram is then defined as follows: Take the Dynkin diagram of Δ^0 and represent elements of Δ_c^0 by a black dot $\bullet$ and elements of $\Delta^0 \setminus \Delta_c^0$ by a white dot $\circ$. In addition, connect α and α' by an arrow for any element $\alpha \in \Delta^0 \setminus \Delta_c^0$ with $\alpha \neq \alpha'$.
Since for any α the restrictions to $\mathfrak{h}$ of α and $\sigma^*\alpha$ are conjugate, we get $\sigma^*\alpha|_{\mathfrak{a}} = \alpha|_{\mathfrak{a}}$. Hence for an admissible positive subsystem $\Delta^+ \subset \Delta$, the image of Δ^+ under the surjective restriction map $r : \Delta \setminus \Delta_c \to \Delta_r$ is a positive subsystem of Δ_r. From this it follows that the induced simple system Δ_r^0 is the quotient of $\Delta^0 \setminus \Delta_c^0$ obtained by identifying each α with α'.
Now we have all what we need to describe the standard parabolic subalgebras of a given real semisimple Lie algebra in a similar way as in the complex case.

DEFINITION. *Let $\mathfrak{g}$ be a real semisimple Lie algebra with complexification $\mathfrak{g}_{\mathbb{C}}$, θ a Cartan involution, $\mathfrak{h}$ a θ-stable maximally non-compact Cartan subalgebra, Δ the root system corresponding to $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and $\Delta^+ \subset \Delta$ an admissible positive subsystem.*
A Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called a standard parabolic subalgebra with respect to $\mathfrak{h}$ and Δ^+, if the complexification $\mathfrak{p}_{\mathbb{C}}$ of $\mathfrak{p}$ is a standard parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$ and Δ^+.

PROPOSITION 2.3. *Let $\mathfrak{g}$ be a real semisimple Lie algebra, θ a Cartan involution with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$, $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ a maximally non-compact θ-stable Cartan subalgebra. Let Δ be the root system associated to $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, σ^* the involutive automorphism of Δ induced by the conjugation σ with respect to $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ and $\Delta^+ \subset \Delta$ an admissible positive subsystem. Then we get:*
(1) Let $\mathfrak{n}$ be the direct sum of all positive restricted root spaces and $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$. Then $\mathfrak{p}_0 := \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is a subalgebra and the standard parabolic subalgebras of $\mathfrak{g}$ are exactly the subalgebras of $\mathfrak{g}$ containing $\mathfrak{p}_0$.
(2) Let Δ^0 be the set of simple roots and Δ_r^0 the induced set of simple restricted roots. Then there is a bijection between the subsets of Δ_r^0 and

the subsets of $\Delta^0 \setminus \Delta_c^0$ that are stable under the involution $\eta : \alpha \mapsto \alpha'$ induced by σ^*. The subsets of $\Delta^0 \setminus \Delta_c^0$ that are stable under η are in turn in bijective correspondence with the set of all standard parabolic subalgebras of $\mathfrak{g}$. Namely, the parabolic subalgebra corresponding to the η-stable subset $\Sigma \subset \Delta^0 \setminus \Delta_c^0$ is the sum of $\mathfrak{p}_0$ and the restricted root spaces which correspond to those negative restricted roots in whose representation as a sum of simple restricted roots no element of the image of Σ in Δ_r^0 occurs with a nonzero coefficient.

PROOF. (1) By definition $\mathfrak{m}$ and $\mathfrak{a} \oplus \mathfrak{n}$ are subalgebras of $\mathfrak{g}$. Using the Jacobi identity and that $[\mathfrak{a}, \mathfrak{m}] = 0$, we get that $ad(m)$ maps any restricted root space to itself for all $m \in \mathfrak{m}$. So in particular $[\mathfrak{m}, \mathfrak{n}] \subset \mathfrak{n}$ and thus $\mathfrak{p}_0$ a subalgebra of $\mathfrak{g}$.
To prove the second statement we consider the complexification $(\mathfrak{p}_0)_{\mathbb{C}}$ and show that $(\mathfrak{p}_0)_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta^+}((\mathfrak{g}_{\mathbb{C}})_\alpha \oplus (\mathfrak{g}_{\mathbb{C}})_{\sigma^*\alpha})$.
By construction $\mathfrak{h} \subset \mathfrak{m} \oplus \mathfrak{a} \subset \mathfrak{p}_0$ and hence $\mathfrak{h}_{\mathbb{C}} \subset (\mathfrak{p}_0)_{\mathbb{C}}$. For $\alpha \in \Delta^+$ we have $\sigma^*\alpha \mid_{\mathfrak{a}} = \alpha \mid_{\mathfrak{a}}$. We already know that the restriction of α to $\mathfrak{a}$ is either 0 or a positive restricted root. If $\alpha \in \Delta_c$, then $\mathfrak{g}_\alpha \subset \mathfrak{k}_{\mathbb{C}}$, and so we get $\mathfrak{l} := \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta_c}(\mathfrak{g}_{\mathbb{C}})_\alpha \subset \mathfrak{k}_{\mathbb{C}}$. Since σ is an automorphism of the real Lie algebra $\mathfrak{g}_{\mathbb{C}}$, σ maps $(\mathfrak{g}_{\mathbb{C}})_\alpha$ to $(\mathfrak{g}_{\mathbb{C}})_{\sigma^*\alpha}$. So we have $\sigma(\mathfrak{l}) \subset \mathfrak{l}$ and hence $\mathfrak{l}$ is the complexification of $\mathfrak{l} \cap \mathfrak{g}$. Any element of $\mathfrak{l}$ commutes with any element of $\mathfrak{a}$, hence $\mathfrak{l} \cap \mathfrak{g} \subset \mathfrak{m}$ and so $\mathfrak{l} \subset \mathfrak{m}_{\mathbb{C}}$. One can easily see that actually $\mathfrak{l} = \mathfrak{m}_{\mathbb{C}}$.
If $\alpha \in \Delta^+$ such that $\sigma^*\alpha \in \Delta^+$, then the subspace $(\mathfrak{g}_{\mathbb{C}})_\alpha \oplus (\mathfrak{g}_{\mathbb{C}})_{\sigma^*\alpha}$ is σ-stable and $\mathfrak{g} \cap ((\mathfrak{g}_{\mathbb{C}})_\alpha \oplus (\mathfrak{g}_{\mathbb{C}})_{\sigma^*\alpha})$ is a subspace of a positive restricted root space. Conversely, for a positive restricted root λ, the complexification of $\mathfrak{g}_\lambda$ has to be contained in $\bigoplus_{\alpha : \alpha|_{\mathfrak{a}} = \lambda}(\mathfrak{g}_{\mathbb{C}})_\alpha$. Now we can conclude that $(\mathfrak{p}_0)_{\mathbb{C}}$ is the sum of the standard Borel subalgebra and its conjugate, $(\mathfrak{p}_0)_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta^+}((\mathfrak{g}_{\mathbb{C}})_\alpha \oplus (\mathfrak{g}_{\mathbb{C}})_{\sigma^*\alpha})$.
(2) From (1) we know that the standard parabolic subalgebras of $\mathfrak{g}$ are exactly the intersections of $\mathfrak{g}$ with σ-stable standard parabolic subalgebras of $\mathfrak{g}_{\mathbb{C}}$. As we saw in proposition 2.2. parabolic subalgebras of $\mathfrak{g}_{\mathbb{C}}$ are in bijective correspondence with subsets of Δ^0, hence we now have to describe subsets of Δ^0 that correspond to σ-stable parabolic subalgebras.
Suppose $\mathfrak{p}$ is a σ-stable standard parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Then for $\alpha \in \Delta$ with $(\mathfrak{g}_{\mathbb{C}})_\alpha \subset \mathfrak{p}$, we have $(\mathfrak{g}_{\mathbb{C}})_{\sigma^*\alpha} \subset \mathfrak{p}$. So the subset of simple roots corresponding to $\mathfrak{p}$, denoted by Σ, must be disjoint from Δ_c^0, since for $\alpha \in \Delta_c^0$ we have $\sigma^*\alpha = -\alpha$. For $\alpha \in \Sigma \subset \Delta^0 \setminus \Delta_c^0$ there exists a unique $\alpha' \in \Delta^0 \setminus \Delta_c^0$ such that $\sigma^*\alpha - \alpha' \in \Delta_c$ and therefore $ht_\Sigma(\sigma^*\alpha) = ht_\Sigma(\alpha')$. Thus Σ is stable under the involutive automorphism η induced by σ^* $\alpha \mapsto \alpha'$.
Conversely, if $\Sigma \subset \Delta^0$ is disjoint from Δ_c^0 and stable under η, then we have $ht_\Sigma(\alpha) = ht_\Sigma(\sigma^*\alpha)$ for all $\alpha \in \Delta^0$. It follows immediately that this holds for all $\alpha \in \Delta$ and so the corresponding parabolic subalgebra

needs to be stable under σ.
The correspondence between η-stable subsets of $\Delta^0 \setminus \Delta^0_c$ and subsets of Δ^0_r follows from the fact that Δ^0_r can be identified with the quotient of $\Delta^0 \setminus \Delta^0_c$ obtained by identifying α with α'. $\square$

Now there is, as in the complex case, a relation between $|k|$-gradings and standard parabolic subalgebras.

THEOREM 2.2. *Let $\mathfrak{g}$ be a real semisimple Lie algebra, θ a Cartan involution, $\mathfrak{h}$ a θ-stable maximally non-compact Cartan subalgebra $\mathfrak{h}$ and $\Delta^+ \subset \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ an admissible positive subsystem. Then we have*
(1) For any standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ corresponding to a subset $\Sigma \subset \Delta^0_r$, the decomposition of $\mathfrak{g}$ according to Σ-height makes $\mathfrak{g}$ into a $|k|$-graded Lie algebra with $\mathfrak{g}^0 = \mathfrak{p}$, where k is the Σ-height of the maximal restricted root.
(2) For any $|k|$-grading of $\mathfrak{g}$, there is an inner automorphism $\phi \in Int(\mathfrak{g})$ such that $\phi(\mathfrak{g}^0)$ is a standard parabolic subalgebra. If we put $\Sigma \subset \Delta^0_r$ the subset corresponding to $\phi(\mathfrak{g}^0)$, then the given grading is the image of the grading by Σ-height under ϕ.

PROOF. (1) follows in the same way as in theorem 2.1.
(2) A $|k|$- grading on $\mathfrak{g}$ induces a $|k|$-grading on the complexification of $\mathfrak{g}$, where the i-th component is given by $(\mathfrak{g}_{\mathbb{C}})_i = \mathfrak{g}_i \otimes_{\mathbb{R}} \mathbb{C}$. The grading element E of $\mathfrak{g}$ is then also the grading element of $\mathfrak{g}_{\mathbb{C}}$ by identifying E and $E \otimes 1$. Let $\mathfrak{h}'$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing E (this is possible, since $ad(E)$ is diagonalizable on $\mathfrak{g}_{\mathbb{C}}$). Now one can construct a compact real form from $\mathfrak{h}'$

$$\mathfrak{u} := i\mathfrak{h}'_0 \oplus \bigoplus_{\alpha \in \Delta^+} (\mathbb{R}(X_\alpha - X_{-\alpha}) + i\mathbb{R}(X_\alpha + X_{-\alpha})),$$

see theorem 6.11 in [**5**]. Denote by τ' and σ the conjugations of $\mathfrak{g}_{\mathbb{C}}$ corresponding to the real forms $\mathfrak{u}$ and $\mathfrak{g}$ respectively. Since τ' is a conjugation with respect to a compact real form, τ' is a Cartan involution of the real semisimple Lie algebra $(\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}}$ and therefore $K_{\tau'}$ a positive inner product on $(\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}}$. The automorphism $\sigma\tau' : (\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}} \to (\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}}$ is a symmetric linear map, since

$$\begin{aligned} K_{\tau'}(\sigma\tau'(X), Y) = -K(\sigma\tau'(X), \tau'(Y)) \\ = -K(\tau'(X), \sigma\tau'(Y)) = K_{\tau'}(X, \sigma\tau'(Y)), \end{aligned}$$

where we used the invariance of the Killing form. Hence the automorphism $\sigma\tau'\sigma\tau'$ is diagonalizable with positive eigenvalues. For such an automorphism we can form $(\sigma\tau'\sigma\tau')^r$ for all $r \in \mathbb{R}$, by taking powers of the eigenvalues. We define $\psi := (\sigma\tau'\sigma\tau')^{\frac{1}{4}}$. Since all eigenvalues of $ad(E)$ are real, we must have $\tau'(E) = -E$ and we obtain $\sigma\tau'\sigma\tau'(E) = E$. It can be easily seen that the map $\theta' := \psi\tau'\psi^{-1}$ commutes with σ and that it is the conjugation with respect to the compact real form $\psi(\mathfrak{u})$. Therefore θ' restricts to a Cartan involution on $\mathfrak{g}$ and

by construction $\theta'(E) = -E$. Since any two Cartan involutions are conjugate by an inner automorphism, we can find $\phi_1 \in Int(\mathfrak{g})$ such that $\phi_1(E)$ lies in the (-1)-eigenspace $\mathfrak{q}$ of θ. Now we chose a maximal abelian subspace $\mathfrak{a}' \subset \mathfrak{q}$ that contains $\phi_1(E)$. As one knows from the structure theory of real semisimple Lie algebras, any two such maximal abelian subspaces are conjugate, so we can find $\phi_2 \in Int(\mathfrak{g})$, which commutes with θ such that $\phi_2(E) \in \mathfrak{a} = \mathfrak{h} \cap \mathfrak{q}$. Since also any two choices of positive restricted roots are conjugate, we finally get $\phi \in Int(\mathfrak{g})$ such that $\phi(E) \in \mathfrak{a}$ and $\alpha(\phi(E)) \geq 0$ for all positive restricted roots α.
As $ad(E)$ is diagonalizable on $\mathfrak{g}$ with real eigenvalues, the same is true for $ad(\phi(E)) = \phi \circ ad(E) \circ \phi_{-1}$. So we see that ϕ maps $\mathfrak{g}^0$ to the sum of all eigenspaces of $ad(\phi(E))$ corresponding to non-negative eigenvalues. This means that $\phi(\mathfrak{g}^0)$ contains the minimal standard parabolic subalgebra $\mathfrak{p}_0$ and therefore is a standard parabolic subalgebra. □

There is also an analog of the presentation of complex parabolic subalgebras by crossed Dynkin diagrams for real parabolic subalgebras: Let $\mathfrak{p}$ be a standard parabolic subalgebra of a real semisimple Lie algebra. Then we consider the Satake diagram of $\mathfrak{g}$ and denote all simple roots in Σ by crosses, where Σ is the subset of simple roots corresponding to $\mathfrak{p}$.
As in the complex case we can describe real standard parabolic subalgebras as stabilizers of lines and flags. In fact: Let $\mathfrak{g}$ be a real semisimple Lie algebra. One can choose an inclusion $\mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}$ such that a θ-stable maximally non-compact Cartan subalgebra for some Cartan involution θ complexifies to the standard Cartan subalgebra in $\mathfrak{g}_{\mathbb{C}}$ and such that the usual positive root system is admissible. Then the complexification of a standard parabolic subalgebra of $\mathfrak{g}$ is the complex standard parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ corresponding to the same set of roots. So we can use the description in the complex case to obtain one in the real case.

2.3. On parabolic geometries

In the introduction of this chapter we mentioned that a *parabolic geometry* is a Cartan geometry of type (G, P), where G is a real or complex semisimple Lie group, whose Lie algebra $\mathfrak{g}$ is endowed with a $|k|$-grading and P is the subgroup of all elements of G, whose adjoint action preserves the filtration associated to the $|k|$-grading on $\mathfrak{g}$.
Now we want to show that

$$P := \{g \in G : Ad(g)(\mathfrak{g}^i) \subset \mathfrak{g}^i \text{ for } i = -k, ..., k\}$$

is a closed subgroup with Lie algebra $\mathfrak{p} = \mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_k$ and that

$$G_0 := \{g \in G : Ad(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \text{ for } i = -k, ..., k\}$$

is a closed subgroup with Lie algebra $\mathfrak{g}_0$.

Proposition 2.4. *Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ be a $|k|$-graded semisimple Lie algebra over $\mathbb{R}$ or $\mathbb{C}$ and G a Lie group with Lie algebra $\mathfrak{g}$.*
Then the subgroups P and G_0 of G are closed with Lie algebras $\mathfrak{p}$ and $\mathfrak{g}_0$ respectively.

Proof. Since $G_0 = \bigcap_{i=-k}^{k} N_G(\mathfrak{g}_i)$ and $P = \bigcap_{i=-k}^{k} N_G(\mathfrak{g}^i)$ are intersections of normalizers, which by definition are closed subgroups, G_0 and P are closed subgroup and therefore a Lie subgroups of G. In particular, the notion of a parabolic geometry is well defined.
Observe that the the subalgebras $\mathfrak{g}_0$ and $\mathfrak{p}$ satisfy $[\mathfrak{g}_0, \mathfrak{g}_i] \subset \mathfrak{g}_i$ and $[\mathfrak{p}, \mathfrak{g}^i] \subset \mathfrak{g}^i$. In fact, they are already characterized by these properties: For $X \in \mathfrak{g}$ denote by $X = X_{-k} + ... + X_k$ the decomposition according to the grading. We obtain $[E, X] = -kX_{-k} - ... - X_{-1} + X_1 + ... + kX_k$, where $E \in \mathfrak{g}_0$ is the grading element. So we see that $[E, X]$ is contained in $\mathfrak{g}_0$ (resp. in $\mathfrak{p}$) if and only if $X = X_0$ ($X_j = 0$ for $j < 0$, which means $X \in \mathfrak{p}$). In particular, we have $\mathfrak{p} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}) := \{X \in \mathfrak{g} : [X, Y] \in \mathfrak{p} \quad \text{for all} \quad Y \in \mathfrak{p}\}$.
The Lie algebra of P is given by the set

$$\{X \in \mathfrak{g} : exp(tX) \in P \quad \forall t\} =$$
$$= \{X \in \mathfrak{g} : Ad(exp(X))(\mathfrak{g}^j) = e^{ad(X)}(\mathfrak{g}^j) \subset \mathfrak{g}^j \text{ for } j = -k, ..., k\} =$$
$$= \{X \in \mathfrak{g} : ad(X)(\mathfrak{g}^j) \subset \mathfrak{g}^j \text{ for } j = -k, ..., k\}.$$

So the Lie algebra of P is just $\mathfrak{p}$. In the same way we obtain that the Lie algebra of G_0 is $\mathfrak{g}_0$. □

Since we know from the last two sections that $\mathfrak{p}$ is a parabolic subalgebra, we see that P is a *parabolic subgroup* of G, i. e. a closed subgroup of G, whose Lie algebra is a parabolic subalgebra.
Using the bijection between standard parabolic subalgebras and all possible $|k|$-gradings of a semisimple Lie algebra, we can equivalently say that a parabolic geometry is a Cartan geometry of type (G, P), where G is a semisimple Lie group and P is a parabolic subgroup.
A parabolic geometry $(\mathcal{G} \to M, \omega)$ is called *regular*, if the curvature function satisfies $\kappa(u)(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1}$ for all $u \in \mathcal{G}$ and all $i, j = -k, ..., k$.
One can show that a regular parabolic geometry, whose curvature satisfies a certain normalization condition, is always equivalent to some simpler underlying geometric structure. For regular normal parabolic geometries these underlying structures include for example conformal and projective structures, quaternionic structures, such as CR structures and generic rank two distributions in manifolds of dimension 5. So parabolic geometries offer a unified approach to many geometric structures. The automorphism group of a regular normal parabolic geometry is then naturally isomorphic to the automorphism group of the underlying equivalent structure. A description of the equivalence between regular normal parabolic geometries and underlying geometric

structures can be found in [**2**].
In the next chapter we want to look more closely at the group of automorphism of a parabolic geometry.

CHAPTER 3

Automorphism groups of parabolic geometries

Now we will show how the description of the Lie algebra of the automorphism group of a Cartan geometry, which we gave in section 1.2., can be improved to determine/estimate the second largest possible dimension of automorphism groups of regular parabolic geometries of a fixed type, see [**1**]. Further we will use this to estimate the second largest possible dimension of automorphism groups of regular parabolic geometries in some concrete cases.

Let $(\mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P) over a connected manifold M and denote by $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ the $|k|$-graded Lie algebra of G. From section 1.2. we know that the automorphism group of $(\mathcal{G} \to M, \omega)$, denoted by $Aut(\mathcal{G}, \omega)$, is a Lie group with Lie algebra $\mathfrak{aut}(\mathcal{G}, \omega) = \{\xi \in \mathfrak{inf}(\mathcal{G}, \omega) : \xi \quad \text{is complete}\}$. For any $u \in \mathcal{G}$ we saw that the map $\xi \mapsto \omega(\xi(u))$ defines a linear isomorphism between $\mathfrak{inf}(\mathcal{G}, \omega)$ and the subspace $\mathfrak{a} := \{\omega(\xi(u)) : \xi \in \mathfrak{inf}(\mathcal{G}, \omega)\} \subset \mathfrak{g}$. The subspace $\mathfrak{a}$ endowed with the bracket $[[X, Y]] := [X, Y] - \kappa(u)(X, Y)$ is a Lie algebra.
We can define a vector space filtration on $\mathfrak{a}$ by restricting the filtration associated to $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ to $\mathfrak{a}$: $\mathfrak{a}^i := \mathfrak{g}^i \cap \mathfrak{a}$.
If $(\mathcal{G} \to M, \omega)$ is a regular parabolic geometry, the bracket $[[\ ,\ ]] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ makes $(\mathfrak{a}, \{\mathfrak{a}^i\})$ even into a filtered Lie algebra.
From now on we assume that $(\mathcal{G} \to M, \omega)$ is a regular parabolic geometry over a connected manifold M.
We can consider the inclusion $\mathfrak{a} \hookrightarrow \mathfrak{g}$. It is a filtration preserving linear map and so it induces a linear map $\mathrm{gr}(\mathfrak{a}) \to \mathrm{gr}(\mathfrak{g})$ between the associated graded Lie algebras. By regularity this map is also an Lie algebra homomorphism. The filtration on $\mathfrak{g}$ comes from a grading, so we can identify the graded Lie algebras $\mathfrak{g}$ and $\mathrm{gr}(\mathfrak{g})$ (see also section 2.1.). Therefore $\mathrm{gr}(\mathfrak{a})$, which has the same dimension as $\mathfrak{a}$, is isomorphic to a graded subalgebra in $\mathfrak{g}$.
The second largest possible dimension of automorphism groups of regular parabolic geometries of a fixed type (G, P) can therefore be estimated by determining the largest possible dimension of proper graded subalgebras of $\mathfrak{g}$.
Now we turn to some concrete cases.

3.1. The case $SO(n+1,n)$

First we have to fix some notation. We denote by $\{e_1,...,e_n\}$ the standard basis of $\mathbb{R}^n$, by $M(n,\mathbb{K})$ the $n\times n$ matrices over a field $\mathbb{K}$, by I_n the identity matrix of dimension n, by $E_{i,j}\in M(n,\mathbb{K})$ the matrix with all entries zero except the (i,j) entry, which is $1_{\mathbb{K}}$ and by $\varepsilon_i : M(n,\mathbb{K}) \to \mathbb{K}$ the linear functional that extracts from an $(n\times n)$ matrix the i-th diagonal entry.
For $n\geq 3$ let G be $SO(n+1,n)$, the closed subgroup of $SL(2n+1,\mathbb{R})$ that preserves the symmetric non-degenerate bilinear form

$$Q:\mathbb{R}^{2n+1}\times\mathbb{R}^{2n+1}\to\mathbb{R}$$

given by $Q(v,w)=v^tJw$, where

$$J:=\begin{pmatrix} 0 & & I_n \\ & 1 & \\ I_n & & 0\end{pmatrix}\in M(2n+1,\mathbb{R})$$

Since for $X\in SL(2n+1,\mathbb{R})$ we have $Q(Xv,Xw)=Q(v,w)$ for all $v,w\in\mathbb{R}^{2n+1}$ if and only if $X^tJX=J$, we obtain

$$G=SO(n+1,n)=\{X\in SL(2n+1,\mathbb{R}):X^tJX=J\}.$$

The Lie algebra $\mathfrak{g}$ of G is obviously

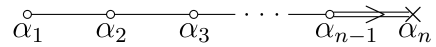
$$\mathfrak{so}(n+1,n):=\{M\in\mathfrak{gl}(2n+1,\mathbb{R}):M^tJ=-JM\}=$$
$$=\left\{\begin{pmatrix} A & v & B\\ -w^t & 0 & -v^t\\ C & w & -A^t\end{pmatrix}: A\in\mathfrak{gl}(n,\mathbb{R}), C,B\in\mathfrak{o}(n) \text{ and } v,w\in\mathbb{R}^n\right\}.$$

The real Lie algebra $\mathfrak{so}(n+1,n)$ is a real form of $\mathfrak{so}(2n+1,\mathbb{C}) = \{M\in\mathfrak{gl}(2n+1,\mathbb{C}) : M^tJ=-JM\}$ of dimension $2n^2+n$. Let σ be the conjugation of $\mathfrak{so}(2n+1,\mathbb{C})$ with respect to $\mathfrak{so}(n+1,n)$ and $\theta:\mathfrak{so}(n+1,n)\to\mathfrak{so}(n+1,n)$ be the Cartan involution given by $\theta(X)=-X^t$. The set of all diagonal matrices in $\mathfrak{so}(n+1,n)$ is a θ-stable Cartan subalgebra of $\mathfrak{so}(n+1,n)$, which we denote by $\mathfrak{h}$, whose complexification $\mathfrak{h}_{\mathbb{C}}$ is the standard Cartan subalgebra of diagonal matrices in $\mathfrak{so}(2n+1,\mathbb{C})$. Moreover the restrictions to $\mathfrak{h}$ of all roots of $\mathfrak{so}(2n+1,\mathbb{C})$ with respect to $\mathfrak{h}_{\mathbb{C}}$ are real. So $\mathfrak{so}(n+1,n)$ is a split real form of $\mathfrak{so}(2n+1,\mathbb{C})$. The restricted roots with respect to $\mathfrak{h}$ are just the restrictions to $\mathfrak{h}$ of the roots

$$\Delta=\Delta(\mathfrak{so}(2n+1,\mathbb{C}),\mathfrak{h}_{\mathbb{C}})=\{\pm\varepsilon_i\pm\varepsilon_j:1\leq i<j\leq n\}\cup\{\pm\varepsilon_j:1\leq j\leq n\}$$

and σ^* acts as the identity on the real roots. So the Satake diagram of $\mathfrak{so}(n+1,n)$ coincides with the Dynkin diagram of $\mathfrak{so}(2n+1,\mathbb{C})$.
We choose the total ordering on $\mathfrak{h}^*$, which is induced from the basis $\{H_1,...,H_n\}$ of $\mathfrak{h}$ with $H_i:=E_{i,i}-E_{n+1+i,n+1+i}$: We declare a linear functional $\lambda:\mathfrak{h}\to\mathbb{R}$ to be positive if there exists $i\in\{1,..,n\}$ such that $\lambda(H_k)=0$ for all $k<i$ and such that $\lambda(H_i)>0$. Then we obtain

a total ordering on $\mathfrak{h}^*$ by defining $\lambda \leq \mu$ if and only if $\mu - \lambda$ is positive. With respect to this choice of positivity, we obtain the set of positive roots $\Delta^+ = \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\} \cup \{\varepsilon_i : 1 \leq i \leq n\}$ and the simple subsystem of roots $\Delta^0 = \{\alpha_1, ..., \alpha_n\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, ..., n-1$ and $\alpha_n = \varepsilon_n$. So we have $\Delta_r^+ = \Delta^+ \cap \Delta_r = \Delta^+|_{\mathfrak{h}}$ and $\Delta_r^0 = \Delta^0 \cap \Delta_r = \Delta^0|_{\mathfrak{h}}$.
Now let $\mathfrak{p}$ be the parabolic subalgebra of $\mathfrak{g}$ corresponding to $\Sigma = \{\alpha_n\}$:



As we know we can realize $\mathfrak{p}$ as stabilizer of a highest weight line in some irreducible representation of $\mathfrak{g}$. The irreducible representation of $\mathfrak{g}_{\mathbb{C}}$ with highest weight $\lambda = 2\omega_{\alpha_n}$, where ω_{α_n} is the fundamental weight corresponding to the simple root α_n, is the representation $\bigwedge^n \mathbb{C}^{2n+1}$, where $\mathbb{C}^{2n+1}$ is the standard representation of $\mathfrak{g}_{\mathbb{C}}$. The eigenspace of the highest weight is generated by $e_1 \wedge ... \wedge e_n$. One can easily show that the stabilizer of this eigenspace is just the stabilizer of the isotropic subspace $\mathbb{C}^n$ of $(\mathbb{C}^{2n+1}, Q)$. As we remarked in section 2.2. the complex parabolic subalgebra corresponding to α_n, which is just $\mathfrak{p}_{\mathbb{C}}$, must coincides with this stabilizer. Since $\mathfrak{g}$ is the stabilizer of $\mathbb{R}^{2n+1}$ in $\mathfrak{g}_{\mathbb{C}}$, we conclude that $\mathfrak{p}$ is the stabilizer of $\mathbb{R}^n$.
The parabolic subgroup P of G corresponding to $\mathfrak{p}$ is then the stabilizer of $\mathbb{R}^n$ with respect to the standard representation $\mathbb{R}^{2n+1}$ of G.
From above we know that in order to determine the second largest possible dimension of all automorphism groups of regular parabolic geometries of type (G, P) over connected manifolds, we have to study proper graded subalgebras of $\mathfrak{g}$, where $\mathfrak{g}$ is endowed with the grading corresponding to $\mathfrak{p}$.
The Σ-height of the maximal root $\varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\alpha_2 + ... + 2\alpha_n$ is 2. So $\mathfrak{p}$ corresponds to a $|2|$-grading and this is given by

$$\mathfrak{g} = \begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix}$$

with block sizes $n, 1$ and n as before. Hence we get the linear isomorphisms $\mathfrak{g}_2 \simeq \bigwedge^2 \mathbb{R}^n$, $\mathfrak{g}_{-2} \simeq \bigwedge^2 (\mathbb{R}^n)^*$, $\mathfrak{g}_1 \simeq \mathbb{R}^n$, $\mathfrak{g}_{-1} \simeq (\mathbb{R}^n)^*$ and $\mathfrak{g}_0 \simeq \mathfrak{gl}(n, \mathbb{R})$.
We now want to interpret the restrictions of the Lie bracket to the grading components. The restriction of the Lie bracket to $\mathfrak{g}_2 \times \mathfrak{g}_{-1}$ can be seen as the standard representation of $\mathfrak{o}(n)$ on $\mathbb{R}^n$, since for

$$X = \begin{pmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}_2 \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ -w^t & 0 & 0 \\ 0 & w & 0 \end{pmatrix} \in \mathfrak{g}_{-1}$$

we have $[X,Y] = \begin{pmatrix} 0 & Bw & 0 \\ 0 & 0 & -(Bw)^t \\ 0 & 0 & 0 \end{pmatrix}$.

The same holds for the restriction to $\mathfrak{g}_{-2} \times \mathfrak{g}_1$.

The bracket $\mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1$ is given by the standard representation of $\mathfrak{gl}(n,\mathbb{R})$ on $\mathbb{R}^n$, since for

$$X = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -A^t \end{pmatrix} \in \mathfrak{g}_0 \quad \text{and} \quad Y = \begin{pmatrix} 0 & v & 0 \\ 0 & 0 & -v^t \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}_1$$

we have $[X,Y] = \begin{pmatrix} 0 & Av & 0 \\ 0 & 0 & -(Av)^t \\ 0 & 0 & 0 \end{pmatrix}$.

From proposition 2.1. we know that the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is then just dual to the $\mathfrak{g}_0$-module $\mathfrak{g}_1$.

For the bracket $[\ ,\ ] : \mathfrak{g}_{-1} \times \mathfrak{g}_1 \to \mathfrak{g}_0$, we have

$$\left[\begin{pmatrix} 0 & 0 & 0 \\ -w^t & 0 & 0 \\ 0 & w & 0 \end{pmatrix}, \begin{pmatrix} 0 & v & 0 \\ 0 & 0 & -v^t \\ 0 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} AB & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -(AB)^t \end{pmatrix}$$

where $A = \begin{pmatrix} v_1 & \dots & v_1 \\ \vdots & \vdots & \vdots \\ v_n & \dots & v_n \end{pmatrix}$ and $B = \begin{pmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_n \end{pmatrix}$.

So this bracket is just given by the tensor product of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ and obviously $\mathfrak{g}_0 \simeq \mathfrak{g}_{-1} \otimes \mathfrak{g}_1$.

We notice that by proposition 2.1. and corollary 2.1. (or one can read it off the above) we have $[\mathfrak{g}_{i+1}, \mathfrak{g}_{-1}] = \mathfrak{g}_i$ for $i = -1, -2$ and $[\mathfrak{g}_{i-1}, \mathfrak{g}_1] = \mathfrak{g}_i$ for $i = 1, 2$. Since $\mathfrak{g} = \mathfrak{so}(n+1,n)$ is simple, also $[\mathfrak{g}_1, \mathfrak{g}_{-1}] = \mathfrak{g}_0$ must be true.

Now we want to determine the largest possible dimension of proper graded subalgebras of $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

PROPOSITION 3.1. *Let $\mathfrak{so}(n+1,n) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be the $|2|$-grading on $\mathfrak{so}(n+1,n)$ from above and $\mathfrak{b}$ a proper graded subalgebra. Then we have:*
(1) The dimension of $\mathfrak{b}$ is at most $2n^2 - n + 1$.
(2) If $\mathfrak{b}$ contains $\mathfrak{g}_{-1}$ and $\dim(\mathfrak{b}_1) = i < n$, the dimension of $\mathfrak{b}$ is at most $\frac{n(n-1)}{2} + n + n^2 - (n-i)i + i + \frac{i(i-1)}{2} \leq 2n^2 - n + 1$.

PROOF. We will prove (1) and (2) together. Therefore we set $\mathfrak{g} = \mathfrak{so}(n+1,n)$ and $d_j := \dim(\mathfrak{b}_j)$ for $j = -2, ..., 2$.

First we suppose that $\mathfrak{b}$ contains $\mathfrak{g}_{-1}$.

From $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = [\mathfrak{b}_{-1}, \mathfrak{b}_{-1}] \subseteq \mathfrak{b}_{-2}$ follows that $\mathfrak{g}_-$ is contained in $\mathfrak{b}$. In this case we must have $\mathfrak{b}_1 \neq \mathfrak{g}_1$, since otherwise from

$\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1] = [\mathfrak{b}_1, \mathfrak{b}_1] \subseteq \mathfrak{b}_2$ and $\mathfrak{g}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1] = [\mathfrak{b}_{-1}, \mathfrak{b}_1] \subseteq \mathfrak{b}_0$ it would follow that $\mathfrak{b} = \mathfrak{g}$. So $d_1 \leq n-1$.
Assume that the dimension of $\mathfrak{b}_1$ is $i \leq n-1$. Without loss of generality we can assume that $\mathfrak{b}_1$ is the i-dimensional subspace generated by $e_1, ..., e_i \in \mathbb{R}^n$: Indeed, since G_0 acts on $\mathfrak{g}_1$ by the standard representation, we can find for any i dimensional subspace V of $\mathfrak{g}_1$ a $g \in G_0$ such that the image of V under the grading preserving isomorphism $Ad(g) : \mathfrak{g} \to \mathfrak{g}$ is the subspace generated by $e_1, ..., e_i$.
We consider the bracket on $\mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1$ given by the standard representation of $\mathfrak{gl}(n, \mathbb{R})$. Since this representation is irreducible and $[\mathfrak{b}_0, \mathfrak{b}_1] \subseteq \mathfrak{b}_1$, it follows that $\mathfrak{b}_0 \neq \mathfrak{g}_0$ for $i \neq 0$.
Suppose that $i \neq 0$. Then the stabilizer of $\mathfrak{b}_1$ in $\mathfrak{gl}(n, \mathbb{R})$ is the subspace consisting of the matrices, where $a_{k,l} = 0$ for $k = i+1, ..., n$ and $l = 1, ..., i$. Therefore the dimension of $\mathfrak{b}_0$ can be at most $n^2 - (n-i)i$ in order to stabilize $\mathfrak{b}_1$. Since $(n-i)i$ is smallest, when $i = 1$ or $i = n-1$, we have $d_0 \leq n^2 - n + 1$ for $d_1 \neq 0$. Now we consider the bracket on $\mathfrak{g}_2 \times \mathfrak{g}_{-1} \to \mathfrak{g}_1$ given by the standard representation of $\mathfrak{o}(n)$. The subspace $[\mathfrak{b}_2, \mathfrak{b}_{-1}] = [\mathfrak{b}_2, \mathfrak{g}_{-1}]$ must be contained in $\mathfrak{b}_1 = \mathbb{R}^i$, so $\mathfrak{b}_2$ consists of matrices B in $\mathfrak{o}(n)$, where $b_{l,k} = 0$ for $l = i+1, ..., n$ and $k = 1, ..., n$. Hence

$$d_2 \leq \frac{n(n-1)}{2} - (n-i)i - \frac{(n-i)(n-i-1)}{2} = \frac{(i-1)i}{2} \leq$$
$$\leq \frac{(n-1)(n-2)}{2} = \frac{n^2 - 3n + 2}{2}.$$

For $d_1 = i = 0$ we obtain that $d_2 = 0$, since for $X \neq 0$ in $\mathfrak{b}_2$ we have $ad(X) : \mathfrak{g}_{-1} = \mathfrak{b}_{-1} \to \mathfrak{g}_1$ is nonzero. So in this case the dimension of $\mathfrak{b}$ can be at most $\frac{n(n-1)}{2} + n + n^2 = \frac{3n^2+n}{2} \leq 2n^2 - n + 1$.
Finally we conclude that the dimension of any proper graded subalgebra $\mathfrak{b}$ that contains $\mathfrak{g}_{-1}$ is at most

$$\frac{n(n-1)}{2} + n + (n^2 - (n-1)) + (n-1) + \frac{n^2 - 3n + 2}{2} = 2n^2 - n + 1.$$

In fact we obtain that

$$\dim(\mathfrak{b}) \leq \frac{n(n-1)}{2} + n + (n^2 - (n-i)i) + i + \frac{i(i-1)}{2} \leq 2n^2 - n + 1,$$

when $d_1 = i$.
Now let $\mathfrak{b}$ be any proper graded subalgebra of $\mathfrak{g}$. When $d_{-1} = n$, we are in the above case. So the dimension of $\mathfrak{b}$ is at most $2n^2 - n + 1$. When $d_1 = n$, we obtain $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \mathfrak{g}_2 \subseteq \mathfrak{b}$ by using $[\mathfrak{b}_1, \mathfrak{b}_1] = [\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$. Since $\mathfrak{g}_-$ and $\mathfrak{p}_+$ behave completely symmetrically, we conclude that also in this case the dimension of $\mathfrak{b}$ is at most $2n^2 - n + 1$.
It remains to look at the case, where d_1 and d_{-1} are strictly smaller than n.
We claim that $d_{-1} + d_0 + d_1 \leq n^2 + 1$, when $d_1, d_{-1} < n$.

To prove this we assume that $d_1 = i < n$ and $i \neq 0$. Then we already know that $d_0 \leq n^2 - (n-i)i$ and if $d_{-1} \leq n-i$, we have $d_{-1} + d_0 + d_1 \leq n^2 - (n-i)i + i + (n-i) = n + n^2 - (n-i)i \leq n + n^2 - (n-(n-1))(n-1) = n^2+1$. Assume now that $n > k = d_{-1} > n-i$. By the duality of the $\mathfrak{g}_0$-modules $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$, it follows that $d_0 \leq n^2 - (n-i)i - (k-(n-i))(n-k)$. Hence

$$\begin{aligned} d_{-1} + d_0 + d_1 &\leq k + n^2 - (n-i)i - (k-(n-i))(n-k) + i \\ &\leq k + n^2 - (n-i)i + i - (k-(n-i)) = \\ &= n^2 - (n-i)i + i + n - i = n^2 - (n-i)i + n \\ &\leq n^2 - (n-1) + n = n^2 + 1. \end{aligned}$$

If $d_1 = 0$ and $n > d_{-1} = i$, or the other way around, we get $d_{-1} + d_0 + d_1 \leq n^2 - (n-i)i + i \leq n^2 - (n-(n-1))(n-1) + n - 1 = n^2$. Putting all this together we obtain the claim. So we have

$$\dim(\mathfrak{b}) \leq n(n-1) + n^2 + 1 = 2n^2 - n + 1.$$

□

A graded subalgebra of dimension $2n^2 - n + 1$ is for example

$$\mathfrak{b} := \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1 \oplus \mathfrak{b}_2$$

where $\mathfrak{b}_1 = \mathbb{R}^{n-1}$, $\mathfrak{b}_2 = \bigwedge^2 \mathbb{R}^{n-1}$ and $\mathfrak{b}_0 = \{X \in \mathfrak{gl}(n, \mathbb{R}) : x_{n,i} = 0 \quad \text{for} \quad 1 \leq i \leq n-1\}$.
By section 2.3. and the above propsition, we obtain the following theorem:

Theorem 3.1. *Let M be a connected manifold and $(\mathcal{G} \to M, \omega)$ be a regular parabolic geometry of type $(SO(n+1,n), P)$ with P as above. If the dimension of the automorphism group $Aut(\mathcal{G}, \omega)$ of $(\mathcal{G} \to M, \omega)$ is strictly smaller than $\dim(SO(n+1,n)) = 2n^2 + n$, then*

$$\dim(Aut(\mathcal{G}, \omega)) \leq 2n^2 - n + 1.$$

Now we want to show that the graded subalgebras

$$\mathfrak{b}^k = \bigwedge^2 (\mathbb{R}^n)^* \oplus (\mathbb{R}^n)^* \oplus \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \oplus \mathbb{R}^k \oplus \bigwedge^2 \mathbb{R}^k$$

of $\mathfrak{g}$ for $1 \leq k \leq n-1$ can be realized as the Lie algebras of automorphism groups of regular parabolic geometries of type $(G = SO(n+1,n), P)$.
We consider the homogeneous model $(G \to G/P, \omega_{MC})$.
Recall that a subsapce W of $\mathbb{R}^{2n+1}$ is *totally isotropic* with respect to Q if W is contained in its own orthogonal complement, i.e. if $Q(v,w) = 0$ for all $v, w \in W$.
G acts on the set of all n-dimensional totally isotropic subspaces of $(\mathbb{R}^{2n+1}, Q)$, since G is the subgroup of $SL(2n+1, \mathbb{R})$ preserving Q. Moreover this action is transitive: Let V be a n-dimensional totally

isotropic subspace of $\mathbb{R}^{2n+1}$ and choose a basis $\{v_1, ..., v_n\}$. Extend this basis to a basis of $\mathbb{R}^{2n+1}$ such that the matrix A having as columns these basis vectors is in G. This is possible by the theorem of Witt. So A maps the standard basis of the totally isotropic subspace $\mathbb{R}^n$ to the basis $\{v_1, ..., v_n\}$ of the totally isotropic subspace V.
We know that the stabilizer of the totally isotropic subspace $\mathbb{R}^n$ is exactly P. Therefore we can identify G/P with the set of all n-dimensional totally isotropic subspaces of $(\mathbb{R}^{2n+1}, Q)$.
Since the curvature of the homogeneous model vanishes identically, $(G \to G/P, \omega_{MC})$ is in particular a regular parabolic geometry. The homogeneous space G/P is connected, since P contains elements of both connected components of G.
Hence $Aut(G, \omega_{MC}) = \{\lambda_g : G \to G \text{ for } g \in G\} \cong G$ by section 1.2. For any open subset $U \subset G/P$ the restriction $(p^{-1}(U) \to U, \omega_{MC}|_{p^{-1}(U)})$ of the parabolic geometry is again a parabolic geometry of type (G, P) with vanishing curvature. If in addition U is connected we know by the theorem of Liouville (theorem 1.3), that the automorphism group $Aut(p^{-1}(U), \omega_{MC}|_{p^{-1}(U)}) = \{\lambda_g : G \to G : \lambda_g(p^{-1}(U)) \subset p^{-1}(U)\}$.
We claim that there is a connected open subset U_k such that the Lie algebra of $Aut(p^{-1}(U_k), \omega_{MC}|_{p^{-1}(U_k)})$ is isomorphic to $\mathfrak{b}^k$.
In fact, identify G/P with the set of all n-dimensional totally isotropic subspaces of $(\mathbb{R}^{2n+1}, Q)$ and let W_k be the totally isotropic subspace generated by $e_{n+k+2}, ..., e_{2n+1}$. Then define U_k to be the open subset

$$U_k := \{Z \text{ } n\text{-dimensional totally isotropic subspace} : Z \cap W_k = \{0\}\}$$

We set

$$H_k := \{g \in G : \lambda_g(p^{-1}(U_k)) \subset p^{-1}(U_k)\} = \{g \in G : \overline{\lambda}_g(U_k) \subset U_k\}.$$

Now we claim that H_k must coincide with the set $\{g \in G : g{\cdot}W_k = W_k\}$. Indeed, for $g \in G$ with $g.W_k = W_k$ and any $Z \in U_k$ we get $gZ \cap W_k = gZ \cap g.W_k = \{0\}$. Conversely, let $g \in G$ such that $gZ \cap W_k = \{0\}$ for all n-dimensional totally isotropic subspaces $Z \in U_k$. Suppose there exists an element $w \in W_k$ with $gw \neq W_k$. Since $Q(gw, gw) = Q(w, w) = 0$, we conclude that we can extend gw to an n-dimensional totally isotropic subspace Z intersecting W_k only in zero.
So $g^{-1}Z \cap W_k \neq \{0\}$, a contradiction.
So we have

$$\begin{aligned} H_k &= \{g \in G : g \cdot W_k = W_k\} = \\ &= \{g \in G : \; g_{ij} = 0 \text{ for } 1 \leq i \leq n+k+1 \text{ and } n+k+2 \leq j \leq 2n+1\} \\ &= \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}. \end{aligned}$$

Its Lie algebra is therefore given by

$$\begin{aligned}\mathfrak{h}_k: &= \{X \in \mathfrak{g} : X \cdot W_k = W_k\} = \\ &= \{X \in \mathfrak{g} :\ X_{ij} = 0 \text{ for } 1 \leq i \leq n+k+1 \text{ and } n+k+2 \leq j \leq 2n+1\} \\ &= \mathfrak{b}^k.\end{aligned}$$

Finally we conclude that the Lie algebra of $Aut(p^{-1}(U_k), \omega_{MC}|_{p^{-1}(U_k)})$ is isomorphic to $\mathfrak{b}^k$ as claimed.

3.2. Some background on composition algebras

Now we collect some facts about composition algebras, which we will need in the next sections to give a description of certain simple Lie algebras. More details and proofs can be found in [**9**].

DEFINITION. *A composition algebra C over a field k is a k-algebra (not necessarily associative) over k with identity element e such that there exists a non-degenerate quadratic form $N : C \to k$, called the norm, which satisfies $N(xy) = N(x)N(y)$ for all $x, y \in C$. We call a quadratic form N non-degenerate, if its associated symmetric bilinear form $B_N(x, y) := N(x+y) - N(x) - N(y)$ is non-degenerate.*
N is defined to be isotropic, if there exists a nonzero element $x \in C$ such that $N(x) = 0$. If not, N is called anisotropic.

An immediate consequence of the definition is that $N(e) = 1$ and that $N(x) = \frac{1}{2}B_N(x, x)$, if $char(k) \neq 2$.
Let (C, N) be a composition algebra over a field k, then a short computation shows that

$$B_N(x_1y, x_2y) = B_N(x_1, x_2)N(y) \text{ and } B_N(xy_1, xy_2) = N(x)B_N(y_1, y_2)$$

for all $x, y, x_1, x_2, y_1, y_2 \in C$.
Replacing y by $y_1 + y_2$ in the first equation we obtain

$$B_N(x_1y_1, x_2y_2) + B_N(x_1y_2, x_2y_1) = B_N(x_1, x_2)B_N(y_1, y_2)$$

for all $x_1, x_2, y_1, y_2 \in C$.
Hence we get

$$\begin{gathered}B_N(x^2 - B_N(x,e)x + N(x)e, y) = \\ = B_N(x^2, y) - B_N(x,e)B_N(x,y) + N(x)B_N(e,y) = \\ = B_N(x^2, y) - B_N(x,e)B_N(x,y) + B_N(x,xy) = 0\end{gathered}$$

for all $x, y \in C$. So by the non-degeneracy of B_N any $x \in C$ satisfies the following equation:

$$x^2 - B_N(x,e)x + N(x)e = 0 \quad (*)$$

Replacing x by $x + y$ in $(*)$ we conclude that

$$xy + yx - B_N(x,e)y - B_N(y,e)x + B_N(x,y)e = 0 \quad (**)$$

for all $x, y \in C$.
We remark that C is *power associative*, i.e the subalgebra $k[x]$ generated by x is associative for every $x \in C$.
On any composition algebra C we have the *conjugation* $^{-} : C \to C$, which is given by the negative of the reflection in the hyperplane orthogonal to e: $\overline{x} = B_N(x, e)e - x$. We call $\overline{x}$ the *conjugate* of x. The conjugation is an involutive linear automorphism on C. Hence C splits into the direct sum of its 1-eigenspace and its -1-eigenspace. The 1-eigenspace is generated by e and the -1-eigenspace is given by $e^{\perp}$ and is called the set of imaginary elements and will be denoted by $Im(C)$. Hence any element $x \in C$ can be written as $x = x_1 + x_{-1}$ according to the eigenspace decomposition. x_1 is called the real part of x, denoted by $Re(x) := x_1$ and x_{-1} is called the imaginary part of x, denoted by $Im(x) := x_{-1}$.
Using $(**)$ yields to

$$B_N(x, y) = \overline{x}y + \overline{y}x$$

Moreover by straightforward computations we have the following rules:

$$x\overline{x} = \overline{x}x = N(x)e$$

$$\overline{xy} = \bar{y}\bar{x}$$

$$\overline{\overline{x}} = x$$

$$\overline{x + y} = \overline{x} + \overline{y}$$

$$N(x) = N(\overline{x})$$

$$B_N(x, y) = B_N(\overline{x}, \overline{y})$$

One can easily show that an element $x \in C$ has an inverse if and only if $N(x) \neq 0$ and then $x^{-1} = N(x)^{-1}\overline{x}$.
The only possible dimensions of composition algebras are 1 (if the $char(k) \neq 2$), 2, 4 and 8, see **[9]**. In dimension 1 and 2 the algebras are commutative and associative, in dimension 4 they are associative, but not commutative and in dimension 8 they are neither commutative nor associative.
We can distinguish between to types of composition algebras: those whose norm is isotropic and those whose norm is anistropic. In the first case the composition algebras have zero divisors and are called a *split composition algebras.* One can prove that there is up to isomorphism exactly one split composition algebra over any field k in each of the dimensions 2, 4 and 8.
In the second case every non zero element has an inverse and the composition algebra is called *composition division algebra.*
We are particulary interested in composition algebras over $\mathbb{R}$. In dimension 1 we have the real numbers. In dimension 2, 4 and 8 we have up to isomorphism exactly two different composition algebras, corresponding to whether the norm is positive definite or isotropic. In the

first case we obtain in dimension 2 the complex numbers, $\mathbb{C}$, in dimension 4 the (Hamilton) quaternions, $\mathbb{H}$, and in dimension 8 the (Cayley) octonions, $\mathbb{O}$. In the second case we obtain in dimension 2 the so called split complex numbers, $\mathbb{C}_s$, in dimension 4 the split quaternions, $\mathbb{H}_s$, and in dimension 8 the so called split octonions, $\mathbb{O}_s$.
Since the dimension of a maximal totally isotropic subspace of a composition algebra with isotropic norm is half the dimension of the algebra, we see that the associated indefinte bilinear form is of signature $(1,1)$, $(2,2)$ and $(4,4)$ respectively.
Now we will focus on the quaternions, the split quaternions and the split octonions, since we will use them in the next section to describe certain simple Lie algebras.
The quaternions $\mathbb{H}$ are defined as the four dimensional vector space over $\mathbb{R}$ with basis 1, i, j, k, endowed with the bilinear multiplication, which is given by the conditions that 1 is a unity element and $i^2=-1$, $j^2=-1$ and $ij=-ji=k$.
The conjugation $^- : \mathbb{H} \to \mathbb{H}$ is given by

$$x = 1x_1 + ix_2 + jx_3 + kx_4 \mapsto \overline{x} = 1x_1 - ix_2 - jx_3 - kx_4.$$

The norm is then defined as $N(x) = \overline{x}x$.
$(\mathbb{H}, N)$ is the unique four-dimensional composition algebra over $\mathbb{R}$ with positive definite norm.
The split quaternions $\mathbb{H}_s$ can be realized as the 2×2 matrices over $\mathbb{R}$ with the determinant as norm. The conjugation is then given by

$$x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \mapsto \overline{x} = \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix}$$

Observe that the imaginary elements $Im(\mathbb{H}_s)$ are just the trace free matrices.
Obviously $(\mathbb{H}_s, \det)$ is the unique four dimensional composition algebra over $\mathbb{R}$ with isotropic norm.
The split octonions $\mathbb{O}_s$ can be realized as the vector space of matrices of the form $\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix}$ with $\xi, \eta \in \mathbb{R}$ and $x, y \in \mathbb{R}^3$, where the multiplication is defined by

$$\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} \begin{pmatrix} \xi' & x' \\ y' & \eta' \end{pmatrix} = \begin{pmatrix} \xi\xi' + \langle x, y' \rangle & \xi x' + \eta' x + y \wedge y' \\ \eta y' + \xi' y + x \wedge x' & \eta\eta' + \langle y, x' \rangle \end{pmatrix}$$

for $\langle\ ,\ \rangle$ the standard inner product on $\mathbb{R}^3$ and $\wedge$ the cross product.
The norm is

$$N(\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix}) = \xi\eta - \langle x, y \rangle$$

and the associated bilinear form of signature $(4,4)$ is then

$$B_N(\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix}, \begin{pmatrix} \xi' & x' \\ y' & \eta' \end{pmatrix}) = \xi\eta' + \xi'\eta - \langle x, y' \rangle - \langle x', y \rangle$$

In this case the imaginary elements are

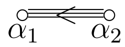
$$Im(\mathbb{O}_s) = \{\begin{pmatrix} \xi & x \\ y & -\xi \end{pmatrix} : x, y \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^3\}.$$

3.3. The case G_2

Now we want to determine the second largest possible dimension of automorphism groups of regular parabolic geometry of type (G, P), where G is a real connected simple Lie group with Lie algebra the split real form of the complex simple Lie algebra $\mathfrak{g}_2$ and P some parabolic subgroup, which we will specify in the sequel.
The abstract root system G_2 of rank 2 can be realized in the following way: Denote by $(\ ,\ )$ the standard inner product on $\mathbb{R}^3$. Let V be the subspace of $\mathbb{R}^3$, which is orthogonal to $e_1 + e_2 + e_3$ and let I be the $\mathbb{Z}$-span of e_1, e_2 and e_3. Set $I' = I \cap V$. Then $\Phi := \{\lambda \in I' : (\lambda, \lambda) = 2 \text{ or } 6\} = \pm\{e_1 - e_2, e_1 - e_3, e_2 - e_3, 2e_1 - e_2 - e_3, 2e_2 - e_1 - e_3, 2e_3 - e_1 - e_2\}$ is a root system of type G_2. As a base choose $\Phi^0 = \{\alpha_1, \alpha_2\}$, where $\alpha_1 = e_1 - e_2$ and $\alpha_2 = -2e_1 + e_2 + e_3$.
For $\lambda \in \Phi$ we define $\lambda^\vee \in V^*$ to be the unique linear functional such that the reflection s_λ on the hyperplane $\lambda^\perp = \{\beta \in V : (\beta, \lambda) = 0\}$ is given by $s_\lambda(\mu) = \mu - \langle \mu, \lambda^\vee \rangle \lambda$, where $\langle \mu, \lambda^\vee \rangle = \lambda^\vee(\mu)$. One can show that $\lambda^\vee = \frac{2(\lambda,\ \ )}{(\lambda,\lambda)}$.
The Cartan matrix is therefore given by

$$\begin{pmatrix} \langle \alpha_1, \alpha_1^\vee \rangle & \langle \alpha_1, \alpha_2^\vee \rangle \\ \langle \alpha_2, \alpha_1^\vee \rangle & \langle \alpha_2, \alpha_2^\vee \rangle \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

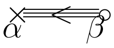
and so the Dynkin diagram corresponding to Φ is



From the structure theory of complex semisimple Lie algebras, we know that there is a complex simple Lie algebra (unique up to isomorphism) of dimension 14, whose root system is isomorphic to the abstract root system G_2. By the theorem of Serre (see 18.3 in [**4**]) we can construct such a Lie algebra $\mathfrak{g}$ by defining $\mathfrak{g}$ as the complex Lie algebra generated by the six elements $(H_{\alpha_1}, H_{\alpha_2}, E_{\alpha_1}, E_{\alpha_2}, F_{\alpha_1}, F_{\alpha_2})$ and the relations:
a) $[H_{\alpha_1}, H_{\alpha_2}] = 0$
b) $[E_{\alpha_i}, F_{\alpha_i}] = H_{\alpha_i}$ for $i = 1, 2$, $[E_{\alpha_i}, F_{\alpha_j}] = 0$ for $i \neq j$
c) $[H_{\alpha_i}, E_{\alpha_j}] = \langle \alpha_j, \alpha_i^\vee \rangle E_{\alpha_j}$ and $[H_{\alpha_i}, F_{\alpha_j}] = -\langle \alpha_j, \alpha_i^\vee \rangle F_{\alpha_j}$ for $i, j \in \{1, 2\}$
d) $ad(E_{\alpha_i})^{-\langle \alpha_j, \alpha_i^\vee \rangle + 1}(E_{\alpha_j}) = 0$ for $i \neq j$.
e) $ad(F_{\alpha_i})^{-\langle \alpha_j, \alpha_i^\vee \rangle + 1}(F_{\alpha_j}) = 0$ for $i \neq j$.

It follows that the subalgebra generated by $H_\alpha := H_{\alpha_1}$ and $H_\beta := H_{\alpha_2}$, which we will denote by $\mathfrak{h}$, is a Cartan subalgebra of $\mathfrak{g}$ and that the linear functionals α and β on $\mathfrak{h}$, given by $\alpha(H_{\alpha_i}) = \langle \alpha_i, \alpha_1^\vee \rangle$ and

$\beta(H_{\alpha_i}) = \langle \alpha_i, \alpha_2^\vee \rangle$ for $i = 1, 2$, form a simple subsystem of the root system Δ of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Δ coincides with the set $\pm\{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}$. We define generators for the root spaces: $E_\alpha := E_{\alpha_1}$ and $E_\beta := E_{\alpha_2}$ are generators of the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ respectively. $F_\alpha := F_{\alpha_1}$ and $F_\beta := F_{\alpha_2}$ are generators of the root spaces $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-\beta}$ respectively. Further set $E_{i\alpha+\beta} := ad^i(E_\alpha)(E_\beta)$ and $F_{i\alpha+\beta} := ad^i(F_\alpha)(F_\beta)$ for $i = 1, 2, 3$, these are generators of the root spaces $\mathfrak{g}_{i\alpha+\beta}$ and $\mathfrak{g}_{-i\alpha-\beta}$. Finally, we define $E_{3\alpha+2\beta} := [E_{3\alpha+\beta}, E_\beta]$ and $F_{3\alpha+2\beta} := [F_{3\alpha+\beta}, F_\beta]$, generators of the root spaces $\mathfrak{g}_{3\alpha+2\beta}$ and $\mathfrak{g}_{-3\alpha-2\beta}$.
We are interested in the grading on $\mathfrak{g}$, which corresponds to the parabolic subalgebra



Since the maximal root is $3\alpha + 2\beta$, this has to be a $|3|$-grading and it is given by $\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$, where $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta}$, $\mathfrak{g}_{\pm 1} = \mathfrak{g}_{\pm\alpha} \oplus \mathfrak{g}_{\pm(\alpha+\beta)}$, $\mathfrak{g}_{\pm 2} = \mathfrak{g}_{\pm(2\alpha+\beta)}$ and $\mathfrak{g}_{\pm 3} = \mathfrak{g}_{\pm(3\alpha+\beta)} \oplus \mathfrak{g}_{\pm(3\alpha+2\beta)}$. Since all root spaces are 1-dimensional, we have $\dim(\mathfrak{g}_{\pm 1}) = 2$, $\dim(\mathfrak{g}_{\pm 2}) = 1$, $\dim(\mathfrak{g}_{\pm 3}) = 2$ and $\dim(\mathfrak{g}_0) = 4$.
Now we have all ingredients in hand to prove the following proposition:

Proposition 3.2. *Let $\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ be the $|3|$-grading corresponding to $\Sigma = \{\alpha\}$. Then the dimension of a proper graded subalgebra of $\mathfrak{g}$ is at most 9.*
If $\mathfrak{g}^s$ is the split real form of $\mathfrak{g}$ which is given by

$$\mathfrak{g}^s := \mathfrak{h}_0 \oplus \bigoplus_{\lambda \in \Delta^+} \mathbb{R}E_\lambda \oplus \mathbb{R}F_\lambda$$

then the restricted roots with respect to $\mathfrak{h}_0$ are just the restrictions to $\mathfrak{h}_0$ of the roots in Δ. So the grading on $\mathfrak{g}^s$ corresponding to $\Sigma = \{\alpha\}$ induces the grading on $(\mathfrak{g}^s)_\mathbb{C} = \mathfrak{g}$ from above and the dimension of a proper graded subalgebra of $\mathfrak{g}^s$ is as well at most 9.

Proof. Let $\mathfrak{b} = \mathfrak{b}_{-3} \oplus \mathfrak{b}_{-2} \oplus \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1 \oplus \mathfrak{b}_2 \oplus \mathfrak{b}_3$ be a proper graded subalgebra of $\mathfrak{g}$ and set $d_i := \dim(\mathfrak{b}_i)$ for $i = -3, ..., 3$.
First we assume that $\mathfrak{b}$ contains $\mathfrak{g}_{-1}$. In this case we have $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = [\mathfrak{b}_{-1}, \mathfrak{b}_{-1}] \subseteq \mathfrak{b}_{-2}$ and so $\mathfrak{b}_{-2} = \mathfrak{g}_{-2}$. Using this we obtain that $\mathfrak{g}_{-3} = [\mathfrak{g}_{-2}, \mathfrak{g}_{-1}] = [\mathfrak{b}_{-2}, \mathfrak{b}_{-1}] \subseteq \mathfrak{b}_{-3}$ and hence also $\mathfrak{b}_{-3} = \mathfrak{g}_{-3}$. So $\mathfrak{g}_-$ is contained in $\mathfrak{b}$.
We conclude that $\mathfrak{b}_1 \neq \mathfrak{g}_1$, since otherwise from $\mathfrak{g}_0 = [\mathfrak{g}_1, \mathfrak{g}_{-1}] = [\mathfrak{b}_1, \mathfrak{b}_{-1}] \subseteq \mathfrak{b}_0$, $\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1] = [\mathfrak{b}_1, \mathfrak{b}_1] \subseteq \mathfrak{b}_2$ and $\mathfrak{g}_3 = [\mathfrak{g}_2, \mathfrak{g}_1] = [\mathfrak{b}_2, \mathfrak{b}_1] \subseteq \mathfrak{b}_3$ it would follow that $\mathfrak{g} = \mathfrak{b}$. So $d_1 \leq 1$.
Now we consider the bracket on $\mathfrak{g}_2 \times \mathfrak{g}_{-1} \to \mathfrak{g}_1$. We observe that $d_2 = 0$: $\mathfrak{g}_{-1}$ is generated by F_α and $F_{\alpha+\beta}$. If $\mathfrak{b}_2$ is nonzero, so $\mathfrak{b}_2 = \mathfrak{g}_2$, and we have

$$[E_{2\alpha+\beta}, aF_\alpha + bF_{\alpha+\beta}] = a[E_{2\alpha+\beta}, F_\alpha] + b[E_{2\alpha+\beta}, F_{\alpha+\beta}]$$

for $a, b \in \mathbb{C}$ therefore one gets that $ad(E_{2\alpha+\beta}) : \mathfrak{g}_{-1} = \mathfrak{b}_{-1} \to \mathfrak{g}_1$ is surjective by using that the nonzero elements $[E_{2\alpha+\beta}, F_\alpha]$ and $[E_{2\alpha+\beta}, [F_\alpha, F_\beta]]$ generate the root spaces $\mathfrak{g}_{\alpha+\beta}$ and $\mathfrak{g}_\alpha$ respectively. So $\mathfrak{b}_1 = \mathfrak{g}_1$, a contradiction.
Next, we look at the bracket $\mathfrak{g}_3 \times \mathfrak{g}_{-1} \to \mathfrak{g}_2$. If there is a nonzero element $X := aE_{3\alpha+\beta} + bE_{3\alpha+2\beta}$ in $\mathfrak{b}_3$, we obtain

$$[aE_{3\alpha+\beta} + bE_{3\alpha+2\beta}, xF_\alpha + yF_{\alpha+\beta}] =$$

$$= ax[E_{3\alpha+\beta}, F_\alpha] + by[E_{3\alpha+2\beta}, F_{\alpha+\beta}] = (akx + bly)E_{2\alpha+\beta}$$

for some $k, l \in \mathbb{C} \setminus \{0\}$. Since a or b has to be $\neq 0$, $ad(X) : \mathfrak{b}_{-1} \to \mathfrak{g}_2$ is surjective, contradicting $d_2 = 0$. Therefore also $d_3 = 0$.
When $d_1 = 1$, we must have $d_0 \leq 3$ in order to have $[\mathfrak{b}_0, \mathfrak{b}_1] \subset \mathfrak{b}_1$. In fact, let $X = rE_\alpha + sE_{\alpha+\beta}$ be a generator of $\mathfrak{b}_1$ and $Y = a_1H_\alpha + a_2H_\beta + a_3E_\beta + a_4F_\beta$ an element in $\mathfrak{g}_0$. Then we get

$$\begin{aligned}
[Y, X] &= a_1r[H_\alpha, E_\alpha] + a_2r[H_\beta, E_\alpha] + a_3r[E_\beta, E_\alpha] + \\
&\quad + a_1s[H_\alpha, E_{\alpha+\beta}] + a_2s[H_\beta, E_{\alpha+\beta}] + a_4s[F_\beta, E_{\alpha+\beta}] = \\
&= (a_1r\alpha(H_\alpha) + a_2r\alpha(H_\beta) + a_4sc)E_\alpha + \\
&\quad + (a_3r + a_1s(\alpha+\beta)(H_\alpha) + a_2s(\alpha+\beta)(H_\beta))E_{\alpha+\beta}
\end{aligned}$$

for some $c \in \mathbb{C} \setminus \{0\}$, using that $\alpha - \beta$ and $\alpha + 2\beta$ are no roots. The equation

$$\begin{pmatrix} r\alpha(H_\alpha) & r\alpha(H_\beta) & 0 & sc \\ s(\alpha+\beta)(H_\alpha) & s(\alpha+\beta)(H_\beta) & r & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} k \\ m \end{pmatrix}$$

has a solution for any $k, m \in \mathbb{C}$, Indeed, as α and $\alpha + \beta$ are non zero on H_α and on H_β and r or s must be $\neq 0$, it follows that the rank of the matrix has to be two. Hence $ad(X) : \mathfrak{g}_0 \to \mathfrak{g}_1$ is surjective and so $d_0 \leq 3$.
Putting all together, we obtain $\dim(\mathfrak{b}) \leq 9$.
Now suppose that $\mathfrak{b}$ is any graded proper subalgebra.
If $d_1 = 2$, we obtain that $\dim(\mathfrak{b}) \leq 9$, since this case behaves completely symmetrically to the case above.
So it only remains to consider the case, where $d_1 \leq 1$ and $d_{-1} \leq 1$. We can restrict ourselves to the following three cases:
1) $d_1 = 0$ and $d_{-1} = 0$
2) $d_1 = 0$ and $d_{-1} = 1$
3) $d_1 = 1$ and $d_{-1} = 1$
since the case $d_1 = 1$ and $d_{-1} = 0$ behaves symmetrically to case 2).
We start with the first case: Consider the bracket on $\mathfrak{g}_3 \times \mathfrak{g}_{-2} \to \mathfrak{g}_1$ and on $\mathfrak{g}_{-3} \times \mathfrak{g}_2 \to \mathfrak{g}_{-1}$. We have

$$[aE_{3\alpha+\beta} + bE_{3\alpha+\beta}, F_{2\alpha+\beta}] = ackE_\alpha + bclE_{\alpha+\beta}$$

and

$$[aF_{3\alpha+\beta} + bF_{3\alpha+\beta}, E_{2\alpha+\beta}] = ack'F_\alpha + bcl'F_{\alpha+\beta}$$

for all $a, b, c \in \mathbb{C}$ and some nonzero elements k, k', l, l' in $\mathbb{C}$. So we see that $[\mathfrak{g}_3, \mathfrak{g}_{-2}] = \mathfrak{g}_1$ and $[\mathfrak{g}_{-3}, \mathfrak{g}_2] = \mathfrak{g}_{-1}$. For $[\mathfrak{b}_3, \mathfrak{b}_{-2}] = 0$, we must have $d_3 = 0$ or $d_{-2} = 0$, and similarly $[\mathfrak{b}_{-3}, \mathfrak{b}_2] = 0$, implies $d_{-3} = 0$ or $d_2 = 0$. So in any case we obtain $\dim(\mathfrak{b}) \leq 8$.
Next consider the second case: From $d_1 = 0$, it follows again that $d_3 = 0$ or $d_{-2} = 0$. From $d_{-1} = 1$ we analogously obtain $d_0 \leq 3$, as for $d_1 = 1$ above. In order to have $[\mathfrak{b}_{-3}, \mathfrak{b}_2] \subset \mathfrak{b}_{-1}$, $d_{-3} = 1$ or $d_2 = 0$. So we conclude that $\dim(\mathfrak{b}) \leq 8$.
Now consider the last case: We already know that $d_1 = d_{-1} = 1$ implies $d_0 \leq 3$. Similarly, in order to have $[\mathfrak{b}_{\pm 3}, \mathfrak{b}_{\mp 2}] \subset \mathfrak{b}_{\pm 1}$, the conditions $d_3 = 1$ or $d_{-2} = 0$ and $d_{-3} = 1$ or $d_2 = 0$ must be satisfied. Hence $\dim(\mathfrak{b}) \leq 9$.
Finally, putting all this together, we conclude that the dimension of a proper graded subalgebra of $\mathfrak{g}$ is at most 9. □

Examples of graded subalgebras of dimension 9 are:

(1) $\mathfrak{b} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ or $\mathfrak{b} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$.
(2) $\mathfrak{b} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$
with $\dim(\mathfrak{b}_0) = 3$ and $\dim(\mathfrak{b}_1) = 1$ or
$\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$
with $\dim(\mathfrak{b}_0) = 3$ and $\dim(\mathfrak{b}_{-1}) = 1$.
(3) $\mathfrak{b} = \mathfrak{b}_{-3} \oplus \mathfrak{b}_{-2} \oplus \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1 \oplus \mathfrak{b}_2 \oplus \mathfrak{b}_3$
with $\mathfrak{b}_0 = \mathfrak{h} \oplus \mathfrak{g}_{-\beta}$, $\mathfrak{b}_1 = \mathfrak{g}_\alpha$, $\mathfrak{b}_{-1} = \mathfrak{g}_{-\alpha-\beta}$, $\mathfrak{b}_{\pm 2} = \mathfrak{g}_{\pm(2\alpha+\beta)}$, $\mathfrak{b}_3 = \mathfrak{g}_{3\alpha+\beta}$ and $\mathfrak{b}_{-3} = \mathfrak{g}_{-3\alpha-2\beta}$.

In fact these are all types of graded subalgebras of dimension 9:
By the proof of the theorem we only have to regard the two cases:
a) $d_1 = 2$ or $d_{-1} = 2$
b) $d_{-1} = 1$ and $d_1 = 1$.
In case a) the first part of the proof shows that the possible types of graded subalgebras of dimension 9 are given by the examples 1) and 2). According to the proof of the theorem we could have three possible types in case b):
1 1 1 3 1 1 1
2 0 1 3 1 0 2
2 1 1 3 1 0 1
Observe that only the first type can actually occur:
For type two, we would obtain that $d_3 = d_{-3} = 0$ in order to have $[\mathfrak{b}_{\pm 3}, \mathfrak{b}_{\mp 1}] = 0$ and for type three we would obtain $d_3 = 0$ in order to have $[\mathfrak{b}_3, \mathfrak{b}_{-1}] = 0$.
As a consequence of proposition 3.2. this yields the following theorem:

Theorem 3.2. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ (resp. $\mathfrak{g}^s$) endowed with the $|3|$-grading from above and P the corresponding parabolic subgroup. If the dimension of the automorphism group $Aut(G,\omega)$ of a regular parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G,P) over a connected manifold M is strictly smaller than $dim(G) = 14$, then we have*

$$\dim(Aut(\mathcal{G},\omega)) \leq 9.$$

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ (resp. $\mathfrak{g}^s$) and P be the parabolic subgroup corresponding to $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ (resp. $\mathfrak{p} = \mathfrak{g}_0^s \oplus \mathfrak{g}_1^s \oplus \mathfrak{g}_2^s \oplus \mathfrak{g}_3^s$). We consider the homogeneous model $(p : G \to G/P, \omega_{MC})$, which is a regular parabolic geometry of type (G,P) over a connected manifold. Set $o = p(e)$, where e is the neutral element in G and define $U \subset G/P$ to be the open subset $U := \{G/P \setminus o\}$. Then we get that $\{g \in G : \overline{\lambda}_g(U) \subseteq U\} = \{g \in G : \lambda_g(P) \subseteq P\} = P$. Therefore the Lie algebra of $Aut(p^{-1}(U), \omega_{MC}|_{p^{-1}(U)})$ is isomorphic to the 9-dimensional Lie subalgebra $\mathfrak{p}$.
To realize the other types of 9-dimensional proper graded subalgebras from above as Lie algebras of automorphism groups of some parabolic geometries of type (G,P), we will need a more concrete description of the homogeneous model.
We will do this in the real case. By G we denote a connected Lie group with Lie algebra the split real form $\mathfrak{g} := \mathfrak{g}^s$. Consider the split octonions

$$\mathbb{O}_s = \left\{ \begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} : \xi, \eta \in \mathbb{R} \text{ and } x, y \in \mathbb{R}^3 \right\}$$

with norm

$$N(\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix}) = \xi\eta - \langle x, y \rangle.$$

G can be realized as the group of algebra automorphism of $\mathbb{O}_s$, which we denote by $Aut(\mathbb{O}_s)$, see **[9]**. One can show that an algebra automorphism of a composition algebra automatically preserves the norm N respectively the associated bilinear form B_N. So $G = Aut(\mathbb{O}_s)$ is contained in the stabilizer of $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in the orthogonal group $O(N) := O(B_N)$. In fact, one can see that $Aut(\mathbb{O}_s)$ is connected and hence it already lies in $SO(N) := SO(B_N)$.
We know from section 3.2. that $B_N : \mathbb{O}_s \times \mathbb{O}_s \to \mathbb{R}$ given by

$$B_N(\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix}, \begin{pmatrix} \xi' & x' \\ y' & \eta' \end{pmatrix}) = \xi\eta' + \xi'\eta - \langle x, y' \rangle - \langle x', y \rangle$$

is of signature $(4,4)$. Now we can restrict B_N to

$$Im(\mathbb{O}_s) = \left\{ \begin{pmatrix} \xi & x \\ y & -\xi \end{pmatrix} \right\}$$

and obtain a bilinear form Q on $Im(\mathbb{O}_s)$ of signature $(4,3)$. Choose the following basis of $Im(\mathbb{O}_s)$:

$$X_1 = \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix},\ X_2 = \begin{pmatrix} 0 & e_2 \\ 0 & 0 \end{pmatrix},\ X_3 = \begin{pmatrix} 0 & e_3 \\ 0 & 0 \end{pmatrix},\ X_4 = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$X_5 = \begin{pmatrix} 0 & 0 \\ -e_1 & 0 \end{pmatrix},\ X_6 = \begin{pmatrix} 0 & 0 \\ -e_2 & 0 \end{pmatrix} \text{ and } X_7 = \begin{pmatrix} 0 & 0 \\ -e_3 & 0 \end{pmatrix}.$$

With respect to this basis Q is given by

$$J := \begin{pmatrix} 0 & & I_3 \\ & 1 & \\ I_3 & & 0 \end{pmatrix}.$$

Moreover, there is an isomorphism between the stabilizer of E in $SO(B_N)$ and the $SO(Q) = SO(4,3)$ defined by $\varphi \mapsto \varphi|_{Im(\mathbb{O}_s)}$.
Therefore $Aut(\mathbb{O}_s) \subset SO(4,3)$ and the Lie algebra $\mathfrak{g}$ of $Aut(\mathbb{O}_s)$, given by the derivations of the algebra $\mathbb{O}_s$, can be seen as subalgebra of

$$\mathfrak{so}(4,3) = \left\{ \begin{pmatrix} A & v & B \\ -w^t & 0 & -v^t \\ C & w & -A^t \end{pmatrix} : A \in \mathfrak{gl}(3,\mathbb{R}), C, B \in \mathfrak{o}(3) \text{ and } v, w \in \mathbb{R}^3 \right\}.$$

The Lie algebra $\mathfrak{g}$ consists exactly of those matrices in $\mathfrak{so}(4,3)$ that act as derivations on $Im(\mathbb{O}_s)$. It turns out that

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & v & B \\ -w^t & 0 & -v^t \\ C & w & -A^t \end{pmatrix} : A \in \mathfrak{sl}(3,\mathbb{R}), Be_i = -\frac{1}{\sqrt{2}}(w \wedge e_i), Ce_i = \frac{1}{\sqrt{2}}(v \wedge e_i) \right\}.$$

The subalgebra $\mathfrak{h}$ of diagonal matrices in $\mathfrak{g}$ is a Cartan subalgebra, where the corresponding roots are given by

$$\Delta = \pm\{\varepsilon_i : 1 \le i \le 3\} \cup \pm\{\varepsilon_i - \varepsilon_j : 1 \le i < j \le 3\}.$$

Choosing the ordering on $\mathfrak{h}^*$ induced from the basis $H_1 = E_{1,1} + E_{7,7} - E_{3,3} - E_{5,5}$ and $H_2 = E_{2,2} + E_{7,7} - E_{3,3} - E_{6,6}$, we obtain as positive roots $\Delta^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$, where $\alpha := \varepsilon_2$ and $\beta := \varepsilon_1 - \varepsilon_2$ are the simple roots. For the root spaces we obtain

$$\mathfrak{g} = \begin{pmatrix} \mathfrak{h} & \mathfrak{g}_{\beta} & \mathfrak{g}_{3\alpha+2\beta} & \mathfrak{g}_{\alpha+\beta} & 0 & \mathfrak{g}_{2\alpha+\beta} & \mathfrak{g}_{-\alpha} \\ \mathfrak{g}_{-\beta} & \mathfrak{h} & \mathfrak{g}_{3\alpha+\beta} & \mathfrak{g}_{\alpha} & \mathfrak{g}_{2\alpha+\beta} & 0 & \mathfrak{g}_{-\alpha-\beta} \\ \mathfrak{g}_{-3\alpha-2\beta} & \mathfrak{g}_{-3\alpha-\beta} & \mathfrak{h} & \mathfrak{g}_{-2\alpha-\beta} & \mathfrak{g}_{-\alpha} & \mathfrak{g}_{-\alpha-\beta} & 0 \\ \mathfrak{g}_{-\alpha-\beta} & \mathfrak{g}_{-\alpha} & \mathfrak{g}_{2\alpha+\beta} & 0 & \mathfrak{g}_{\alpha+\beta} & \mathfrak{g}_{\alpha} & \mathfrak{g}_{-2\alpha-\beta} \\ 0 & \mathfrak{g}_{-2\alpha-\beta} & \mathfrak{g}_{\alpha} & \mathfrak{g}_{-\alpha-\beta} & \mathfrak{h} & \mathfrak{g}_{-\beta} & \mathfrak{g}_{-3\alpha-2\beta} \\ \mathfrak{g}_{-2\alpha-\beta} & 0 & \mathfrak{g}_{\alpha+\beta} & \mathfrak{g}_{-\alpha} & \mathfrak{g}_{\beta} & \mathfrak{h} & \mathfrak{g}_{-3\alpha-\beta} \\ \mathfrak{g}_{\alpha} & \mathfrak{g}_{\alpha+\beta} & 0 & \mathfrak{g}_{2\alpha+\beta} & \mathfrak{g}_{3\alpha+2\beta} & \mathfrak{g}_{3\alpha+\beta} & \mathfrak{h} \end{pmatrix}.$$

Consider the $|3|$-grading on $\mathfrak{g}$ form above. Then $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ is the stabilizer of the line generated by e_7 with respect of the standard representation of $\mathfrak{g}$ on $\mathbb{R}^7$. Otherwise put, $\mathfrak{p}$ is the stabilizer of the highest weight line through X_7 of the standard representation of $\mathfrak{g} =$

$Der(\mathbb{O}_s)$ on $Im(\mathbb{O}_s)$. This is exactly the fundamental representation with respect to the fundamental weight $\omega_\alpha = 2\alpha + \beta$.
Hence the parabolic subgroup P corresponding to $\mathfrak{p}$ is the stabilizer of the line through X_7 with respect of the standard representation of $G = Aut(\mathbb{O}_s)$ on $Im(\mathbb{O}_s)$.
The representation of G on $Im(\mathbb{O}_s)$ induces an action of G on the projective space of all isotropic lines in $Im(\mathbb{O}_s)$, i. e. the quotient $C/\mathbb{R}^+$, where $C = \{X \in Im(\mathbb{O}_s) : X \neq 0 \text{ and } Q(X,X) = 0\}$. The line through X_7 lies in $C/\mathbb{R}^+$. Since P is the stabilizer of $[X_7]$, G/P can be identified with the orbit $G[X_7]$, which itself coincides with $C/\mathbb{R}^+$: Indeed, since the orbit $G[X_7]$ is of maximal possible dimension, namely $5 = \dim(C/\mathbb{R}^+)$, it has to be open. But it has also to be closed, since the quotient of a semisimple Lie group by a parabolic subgroup is always compact (G/P is diffeomorphic to $K/K \cap P$, where K is a maximal compact subgroup of G, see **[5]** chapter VI). Hence we obtain $G/P \simeq G[X_7] = C/\mathbb{R}^+$ by the connectedness of $C/\mathbb{R}^+$ (cf. **[7]**).
Now consider the 9-dimensional graded subalgebras of $\mathfrak{g}$:

$$\mathfrak{b}^1 = \mathfrak{g}_- \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$$

where $\mathfrak{b}_0 = \mathfrak{h} \oplus \mathfrak{g}_{-\beta}$, $\mathfrak{b}_1 = \mathfrak{g}_\alpha$ and $\mathfrak{b}_2 = \mathfrak{b}_3 = \{0\}$
and

$$\mathfrak{b}^2 = \mathfrak{b}_{-3} \oplus \mathfrak{b}_{-2} \oplus \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1 \oplus \mathfrak{b}_2 \oplus \mathfrak{b}_3$$

where $\mathfrak{b}_0 = \mathfrak{h} \oplus \mathfrak{g}_{-\beta}$, $\mathfrak{b}_1 = \mathfrak{g}_\alpha$, $\mathfrak{b}_{-1} = \mathfrak{g}_{-\alpha-\beta}$, $\mathfrak{b}_{\pm 2} = \mathfrak{g}_{\pm(2\alpha+\beta)}$, $\mathfrak{b}_3 = \mathfrak{g}_{3\alpha+\beta}$ and $\mathfrak{b}_{-3} = \mathfrak{g}_{-3\alpha-2\beta}$.
Let B^1 be the connected subgroup of G with Lie algebra $\mathfrak{b}^1$ and consider the orbit $B^1[X_7] =: O$. We can identify O with the quotient $B^1/(B^1 \cap P)$ and see that O is an orbit of maximal dimension, since $\dim(\mathfrak{b}^1/\mathfrak{p} \cap \mathfrak{b}^1) = 5$. So O has to be open. We set

$$K := \{g \in G : \lambda_g(p^{-1}(O)) \subset p^{-1}(O)\} = \{g \in G : \overline{\lambda}_g(O) \subset O\}.$$

Observe that $B^1 \subset K$ and $K \neq G$. The Lie algebra $\mathfrak{k}$ of K coincides with the Lie algebra $\mathfrak{aut}(p^{-1}(O), \omega_{MC}|_{p^{-1}(O)})$ of $Aut(p^{-1}(O), \omega_{MC}|_{p^{-1}(O)})$. Since $\mathfrak{b}^1 \subset \mathfrak{aut}(p^{-1}(O), \omega_{MC}|_{p^{-1}(O)})$ and $\mathfrak{aut}(p^{-1}(O), \omega_{MC}|_{p^{-1}(O)}) \neq \mathfrak{g}$, we conclude that $\mathfrak{b}^1 = \mathfrak{aut}(p^{-1}(O), \omega_{MC}|_{p^{-1}(O)})$, since $\mathfrak{b}^1$ is a proper graded subalgebra of maximal dimension in $\mathfrak{g}$.
One can show by the same arguments that $\mathfrak{b}^2$ can be realized as Lie algebra of $Aut(p^{-1}(\widetilde{O}), \omega_{MC}|_{p^{-1}(\widetilde{O})})$, where $\widetilde{O} = B^2[X_6]$.

3.4. The case $Sp(n+1,1)$

We define $\mathbb{H}^n = \mathbb{H} \times ... \times \mathbb{H}$ to be the right module over $\mathbb{H}$, where the addition and the scalar multiplication is componentwise.
A *hermitian form* on $\mathbb{H}^n$ is a map $Q : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$, which satisfies:
(1) Q is linear in the second variable:

$$Q(v, w_1 + w_2 a) = Q(v, w_1) + Q(v, w_2) a$$

for all $v, w_1, w_2 \in \mathbb{H}^n$ and $a \in \mathbb{H}$.
(2) Q is conjugate linear in the first variable:

$$Q(v_1 + v_2 a, w) = Q(v_1, w) + \overline{a} Q(v_2, w)$$

for all $v_1, v_2, w \in \mathbb{H}^n$ and $a \in \mathbb{H}$.
(3) $Q(v, w) = \overline{Q(w, v)}$ for all $v, w \in \mathbb{H}^n$
We call Q non-degenerate, if $Q(v, w) = 0$ for all $w \in \mathbb{H}^n$ implies $v = 0$.
By $\langle\ ,\ \rangle : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$ we denote the *standard hermitian form* defined by $\langle v, w \rangle = \sum_i \overline{v_i} w_i$.
Observe that for $n = 1$ we have $N(x) = \langle x, x \rangle$ and that
$B_N(x, y) = \langle x, y \rangle + \overline{\langle x, y \rangle} = 2Re(\langle x, y \rangle)$.

We denote by $M(n, \mathbb{H})$ the set of $n \times n$ matrices over $\mathbb{H}$.
Then we have a natural embedding ι of $M(n, \mathbb{H})$ into $M(2n, \mathbb{C})$: For $X = A + iB + jC + kD$ with $A, B, C, D \in M(n, \mathbb{R})$

$$\iota(X) = \begin{pmatrix} z_1(X) & -\overline{z_2(X)} \\ z_2(X) & \overline{z_1(X)} \end{pmatrix},$$

where $z_1(X) = A + iB$, $z_2(X) = C - iD$. Identifying $\mathbb{H}^n$ with $\mathbb{C}^{2n}$ via the $\mathbb{C}$-isomorphism

$$v \mapsto \begin{pmatrix} z_1(v) \\ z_2(v) \end{pmatrix}$$

we obtain that left multiplication by X on $\mathbb{H}^n$ corresponds to left multiplication by $\iota(X)$ on $\mathbb{C}^{2n}$.
This embedding enables us to define a determinant on $M(n, \mathbb{H})$. For an element $X \in M(n, \mathbb{H})$ we set $\det(X) := \det(\iota(X))$. One can show that on $M(n, \mathbb{H})$ det has values in $\mathbb{R}^+$.
We set

$$GL(n, \mathbb{H}) := \{X \in M(n, \mathbb{H}) : \det(X) \neq 0\}$$

and

$$SL(n, \mathbb{H}) := \{X \in M(n, \mathbb{H}) : \det(X) = 1\}.$$

These are not complex but real Lie subgroups of $GL(2n, \mathbb{C})$ and $SL(2n, \mathbb{C})$ respectively. The Lie algebras of the real Lie groups $GL(n, \mathbb{H})$ and $SL(n, \mathbb{H})$ are $\mathfrak{gl}(n, \mathbb{H})$ the set of all $n \times n$ matrices over $\mathbb{H}$ viewed as Lie algebra and

$$\mathfrak{sl}(n, \mathbb{H}) := \{X \in \mathfrak{gl}(n, \mathbb{H}) : Re(tr(X)) = 0\}$$

respectively.
By $Sp(n + 1, 1)$ we denote the closed subgroup of $SL(n + 2, \mathbb{H})$ which preserves the hermitian form $Q(v, w) = \overline{v}^t I_{n+1,1} w$, where

$$I_{n+1,1} = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix} \in M(n + 2, \mathbb{H})$$

Observe that for $X \in GL(n+2,\mathbb{H})$ $Q(Xv,Xw) = Q(v,w)$ for all $v,w \in \mathbb{H}^n$ if and only if $X^* I_{n+1,1} X = I_{n+1,1}$, where $X^* := \overline{(X^t)}$. So we obtain

$$Sp(n+1,1) = \{X \in GL(n+2,\mathbb{H}) : X^* I_{n+1,1} X = I_{n+1,1}\}$$

since $1 = \det(I_{n+1,1})^2 = \det(X^*)\det(X) = \det(X)^2$ and $\det(X) \in \mathbb{R}^+$. Therefore we see that the Lie algebra $\mathfrak{sp}(n+1,1)$ of $Sp(n+1,1)$ is given by

$$\mathfrak{sp}(n+1,1) := \{X \in \mathfrak{gl}(n+2,\mathbb{H}) : X^* I_{n+1,1} + I_{n+1,1} X = 0\} =$$

$$= \left\{ \begin{pmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n+1} & x_{1,n+2} \\ -\overline{x}_{1,2} & x_{2,2} & \dots & x_{2,n+1} & x_{2,n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\overline{x}_{1,n+1} & -\overline{x}_{2,n+1} & \dots & x_{n+1,n+1} & x_{n+1,n+2} \\ \overline{x}_{1,n+2} & \overline{x}_{2,n+2} & \dots & \overline{x}_{n+1,n+2} & x_{n+2,n+2} \end{pmatrix} : x_{i,i} \in Im(\mathbb{H}) \right\}$$

of dimension $2(n+2)^2 + n + 2$.
This is a real form of $\mathfrak{sp}(2(n+2),\mathbb{C})$ and we denote by σ the corresponding conjugation. Let $\theta : \mathfrak{sp}(n+1,1) \to \mathfrak{sp}(n+1,1)$ be the Cartan involution given by $\theta(X) = -X^*$. Then we denote the corresponding Cartan decomposition by $\mathfrak{sp}(n+1,1) = \mathfrak{k} \oplus \mathfrak{q}$, where $\mathfrak{k}$ is the 1-eigenspace and $\mathfrak{q}$ the -1-eigenspace.
In order to find a θ-stable maximally non-compact Cartan subalgebra, we choose a maximal abelian subspace in $\mathfrak{q}$. Such a subspace is for example

$$\mathfrak{a} := \left\{ \begin{pmatrix} 0 & \dots & 0 & a \\ 0 & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots \\ a & 0 & \dots & 0 \end{pmatrix} : a \in \mathbb{R} \right\}$$

We set

$$\mathfrak{m} := Z_{\mathfrak{k}}(\mathfrak{a}) = \left\{ \begin{pmatrix} x_{1,1} & 0 & \dots & 0 & 0 \\ 0 & x_{2,2} & \dots & x_{2,n+1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -\overline{x}_{2,n+1} & \dots & x_{n+1,n+1} & 0 \\ 0 & \dots & \dots & 0 & x_{1,1} \end{pmatrix} : x_{i,i} \in Im(\mathbb{H}) \right\}$$

and fix a maximal abelian subalgebra

$$\mathfrak{t} := \left\{ \begin{pmatrix} t_1 & & & \\ & \ddots & & \\ & & t_{n+1} & \\ & & & t_1 \end{pmatrix} : t_i \in i\mathbb{R} \right\}$$

of $\mathfrak{m}$.
Then we know that

$$\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{t} = \left\{ \begin{pmatrix} ix_1 & \dots & 0 & a \\ 0 & \ddots & \vdots & 0 \\ \vdots & \vdots & ix_{n+1} & \vdots \\ a & 0 & \dots & ix_1 \end{pmatrix} : a, x_i \in \mathbb{R} \text{ for } 1 \leq i \leq n+1 \right\}$$

is a θ-stable maximally non-compact Cartan subalgebra of $\mathfrak{sp}(n+1,1)$. Let H be an element of the subspace $\mathfrak{a} \oplus i\mathfrak{t}$ of $\mathfrak{h}_{\mathbb{C}}$, on which all roots are real. The set of roots of $\mathfrak{sp}(n+2,\mathbb{C})$ with respect to $\mathfrak{h}_{\mathbb{C}}$ is

$$\Delta(\mathfrak{sp}(n+2,\mathbb{C}), \mathfrak{h}_{\mathbb{C}}) = \{\pm\alpha_i \pm \alpha_j : 1 \leq i < j \leq n+2\} \cup \{\pm 2\alpha_i : 1 \leq i \leq n+2\},$$

where

$$\alpha_i(H) = \begin{cases} x_1 + a & \text{if } i = 1 \\ x_1 - a & \text{if } i = 2 \\ 2x_{i-1} & \text{if } 3 \leq i \leq n+2 \end{cases}$$

Now we choose a basis of the subspace $\mathfrak{a} \oplus i\mathfrak{t} \subseteq \mathfrak{h}_{\mathbb{C}}$ given by $H_1 = E_{1,n+2} + E_{n+2,1}$, $H_2 = E_{1,1} + E_{n+2,n+2}$ and $H_i = E_{i-1,i-1}$ for $3 \leq i \leq n+2$. We define $\alpha \in \Delta$ to be positive if and only if there is an index j such that $\alpha(H_i) = 0$ for all $i < j$ and $\alpha(H_j) > 0$. This is an admissible positive subsystem, since for $\alpha \in \Delta^+ \setminus \Delta_c$ we have $\sigma(H_1) = H_1$ and $\overline{\alpha(H_1)} = \alpha(H_1) > 0$ and therefore $\sigma^*\alpha \in \Delta^+$. It follows that

$$\begin{aligned} \Delta^+ &= \{\alpha_1 \pm \alpha_j : 2 \leq j \leq n+2\} \cup \{-\alpha_2 \pm \alpha_j : 3 \leq j \leq n+2\} \\ &\cup \{\alpha_i \pm \alpha_j : 3 \leq i < j \leq n+2\} \cup \{2\alpha_i : i \neq 2\} \cup \{-2\alpha_2\} \end{aligned}$$

and

$$\Delta^0 = \{\alpha_1 + \alpha_2\} \cup \{-\alpha_2 - \alpha_3\} \cup \{\alpha_i - \alpha_{i+1} : 3 \leq i \leq n+1\} \cup \{2\alpha_{n+2}\}.$$

The restricted roots Δ_r are $\{\pm\lambda\} \cup \{\pm 2\lambda\}$, where for $A \in \mathfrak{a}$ $\lambda(A) = a$. The set of simple restricted roots consists only of one element, $\Delta_r^0 = \{\lambda\}$. The restricted root spaces are given as follows:

$$\mathfrak{g}_1 = \mathfrak{g}_\lambda = \left\{ \begin{pmatrix} 0 & x_{1,2} & \dots & x_{1,n+1} & 0 \\ -\overline{x}_{1,2} & 0 & \dots & 0 & \overline{x}_{1,2} \\ \vdots & & \ddots & & \vdots \\ -\overline{x}_{1,n+1} & 0 & \dots & 0 & \overline{x}_{1,n+1} \\ 0 & \overline{x}_{1,2} & \dots & x_{1,n+1} & 0 \end{pmatrix} : x_{1,i} \in \mathbb{H} \text{ for } 2 \leq i \leq n+1 \right\}$$

$$\mathfrak{g}_{-1} = \mathfrak{g}_{-\lambda} = \left\{ \begin{pmatrix} 0 & x_{1,2} & \dots & x_{1,n+1} & 0 \\ -\overline{x}_{1,2} & 0 & \dots & 0 & -\overline{x}_{1,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\overline{x}_{1,n+1} & 0 & \dots & 0 & -\overline{x}_{1,n+1} \\ 0 & -x_{1,2} & \dots & -x_{1,n+1} & 0 \end{pmatrix} : x_{1,i} \in \mathbb{H} \text{ for } 2 \leq i \leq n+1 \right\}$$

$\dim_{\mathbb{R}}(\mathfrak{g}_{\pm 1}) = 4n$

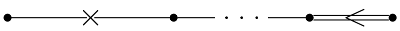
$$\mathfrak{g}_2 = \mathfrak{g}_{2\lambda} = \left\{ \begin{pmatrix} x & 0 & \dots & 0 & -x \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ x & 0 & \dots & 0 & -x \end{pmatrix} : x \in Im(\mathbb{H}) \right\}$$

$$\mathfrak{g}_{-2} = \mathfrak{g}_{-2\lambda} = \left\{ \begin{pmatrix} x & 0 & \dots & 0 & x \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ -x & 0 & \dots & 0 & -x \end{pmatrix} : x \in Im(\mathbb{H}) \right\}$$

$\dim_{\mathbb{R}}(\mathfrak{g}_{\pm 2}) = 3$

$$\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m} = \left\{ \begin{pmatrix} x & & a \\ & M & \\ a & & x \end{pmatrix} : a \in \mathbb{R}, x \in Im(\mathbb{H}) \text{ and } M = -M^* \right\}$$

$\dim_{\mathbb{R}}(\mathfrak{g}_0) = 2n^2 + n + 4$
Now we consider the parabolic subalgebra $\mathfrak{p}$ corresponding to $\Sigma = \Delta_r^0 = \{\lambda\}$



or equivalently to the $|2|$-grading $\mathfrak{sp}(n+1,1) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_{\pm 2} = \mathfrak{g}_{\pm 2\lambda}$, $\mathfrak{g}_{\pm 1} = \mathfrak{g}_{\pm\lambda}$ and $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$. So $\mathfrak{p}$ is given by $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_\lambda \oplus \mathfrak{g}_{2\lambda}$.
Now we want to interpret the restrictions of the Lie bracket on the grading components. First we look at the restriction to $\mathfrak{g}_2 \times \mathfrak{g}_{-1} \to \mathfrak{g}_1$, which is given by scalar multiplication, since for $X \in \mathfrak{g}_2$ and $Y \in \mathfrak{g}_{-1}$ we have

$$[X, Y] = [\begin{pmatrix} x & 0 & -x \\ 0 & 0 & 0 \\ x & 0 & -x \end{pmatrix}, \begin{pmatrix} 0 & -y^* & 0 \\ y & 0 & y \\ 0 & y^* & 0 \end{pmatrix}] = \begin{pmatrix} 0 & 2(yx)^* & 0 \\ -2(yx) & 0 & 2(yx) \\ 0 & 2(yx)^* & 0 \end{pmatrix}$$

for $x \in Im(\mathbb{H})$ and $y \in \mathbb{H}^n$. The same holds obviously for the bracket on $\mathfrak{g}_{-2} \times \mathfrak{g}_1 \to \mathfrak{g}_{-1}$.
Next we consider the brackets $[\ ,\ ] : \mathfrak{g}_{\pm 1} \times \mathfrak{g}_{\pm 1} \to \mathfrak{g}_{\pm 2}$. These are given by two times the imaginary part of the standard hermitian form $\langle\ ,\ \rangle$

on $\mathbb{H}^n$, since for $v, w \in \mathbb{H}^n$ we obtain

$$[\begin{pmatrix} 0 & -v^* & 0 \\ v & 0 & \pm v \\ 0 & \pm v^* & 0 \end{pmatrix}, \begin{pmatrix} 0 & -w^* & 0 \\ w & 0 & \pm w \\ 0 & \pm w^* & 0 \end{pmatrix}] =$$

$$= \begin{pmatrix} \langle w, v\rangle - \overline{\langle w, v\rangle} & 0 & \pm(\langle w, v\rangle - \overline{\langle w, v\rangle}) \\ 0 & & 0 \\ \mp(\langle w, v\rangle - \overline{\langle w, v\rangle}) & 0 & -(\langle w, v\rangle - \overline{\langle w, v\rangle}) \end{pmatrix}$$

The brackets on $\mathfrak{g}_0 \times \mathfrak{g}_{\pm 1} \to \mathfrak{g}_{\pm 1}$ are given by

$$[\begin{pmatrix} x & & a \\ & M & \\ a & & x \end{pmatrix}, \begin{pmatrix} 0 & -v^* & 0 \\ v & 0 & \pm v \\ 0 & \pm v^* & 0 \end{pmatrix}] =$$

$$\begin{pmatrix} 0 & -(Mv - v(x \pm a))^* & 0 \\ Mv - v(x \pm a) & 0 & \pm(Mv - v(x \pm a)) \\ 0 & \pm(Mv - v(x \pm a))^* & 0 \end{pmatrix}$$

where $a \in \mathbb{R}$, $x \in Im(\mathbb{H})$ and M a skew-hermitian matrix in $\mathfrak{gl}(n, \mathbb{H})$. Let P be the parabolic subgroup of $Sp(n+1, 1)$ corresponding to $\mathfrak{p}$. In order to study possible dimensions of automorphism groups of regular parabolic geometries of type $(Sp(n+1, 1), P)$, we will now look at proper graded subalgebras of $\mathfrak{sp}(n+1, 1)$. Therefore we will need the following lemma:

LEMMA 3.1. *Let $Q : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$ be a non-degenerate hermitian form and V a real subspace of $\mathbb{H}^n$ of dimension $> n$. Then there exist $v, w \in V$ such that $Im(Q(v, w)) \neq 0$. Moreover, if W is a real subspace such that $Im(Q(v, w)) = 0$ for all $v, w \in W$, then $W \otimes \mathbb{H} \to \mathbb{H}^n$ is injective.*

PROOF. First we will prove the lemma for a hermitian form on $\mathbb{H}$. So we assume that $Q : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is any non-degenerate hermitian form on $\mathbb{H}$ and that V is a real subspace of dimension > 1. Since Q has to be a multiple of the standard hermitian form $\langle\ ,\ \rangle$, we can suppose without loss of generality that $Q = \langle\ ,\ \rangle$.
Suppose $Im(\langle v, w\rangle) = 0$ for all $v, w \in V$. In particular, for $0 \neq v \in V$ we have $Im(\langle v, w\rangle) = 0$ for all $w \in V$. When $v = a + ib + jc + kd$ and $w = x_1 + ix_2 + jx_3 + kx_4$, we obtain $Im(\langle v, w\rangle) = 0$ if and only if

$$\begin{pmatrix} -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Using that $v \neq 0$ one can easily see that the rank of the matrix is 3 and therefore the space of solutions is one dimensional and obviously generated by v. Hence the dimension of V has to be 1, a contradiction. Now we prove the general claim. Again without loss of generality we

can assume that Q is the standard hermitian form on $\mathbb{H}^n$, since Q is given modulo conjugation by a matrix of the form $I_{r,s}$ for some $r, s \in \{0, 1, ..., n\}$ satisfying $r + s = n$. Let $V \subset \mathbb{H}^n$ be a real subspace of dimension $> n$ and define $\pi_i : \mathbb{H}^n \to \mathbb{H}$ to be the i-th projection of $\mathbb{H}^n$.
Since $\dim_{\mathbb{R}}(V) > n$, there must exist j with $\dim_{\mathbb{R}}(\pi_j(V)) \geq 2$. From above we know that we can find $v_j, w_j \in \pi_j(V)$ with $Im(\overline{v}_j w_j) \neq 0$. Choose $v_k = w_k \in \pi_k(V)$ for $k \neq j$.
Then $Im(\langle v, w\rangle) \neq 0$ for $v := \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ and $w := \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ in V.
For the second assertion observe that the proof shows in particular that for a real subspace $W \subset \mathbb{H}^n$ satisfying $Im(\langle v, w\rangle) = 0$ for all $v, w \in W$, $\dim_{\mathbb{R}}(W \cap Z) \leq 1$ for all one-dimensional quaternionic subspaces $Z \subset \mathbb{H}^n$. But this means exactly that the map $W \otimes \mathbb{H} \to \mathbb{H}^n$ induced by scalar multiplication is injective. $\square$

Now we can prove the following proposition:

PROPOSITION 3.3. *Let* $\mathfrak{sp}(n+1, 1) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ *the* $|2|$*-grading from above.*
If $\mathfrak{b} = \mathfrak{b}_{-2} \oplus \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1 \oplus \mathfrak{b}_2$ *is a proper graded subalgebra that contains* $\mathfrak{g}_{-1}$ *or* $\mathfrak{g}_1$*, then* $\dim(\mathfrak{b}) \leq 2n^2 + 5n + 7$

PROOF. We set $\mathfrak{g} := \mathfrak{sp}(n+1, 1)$ and $d_i = \dim_{\mathbb{R}}(\mathfrak{b}_i)$ for $i = -2, ..., 2$. Without loss of generality we can assume that $\mathfrak{b}_{-1} = \mathfrak{g}_{-1}$. From $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = [\mathfrak{b}_{-1}, \mathfrak{b}_{-1}] \subseteq \mathfrak{b}_{-2}$ follows that $\mathfrak{g}_-$ is contained in $\mathfrak{b}$. We must also have $\mathfrak{b}_1 \neq \mathfrak{g}_1$, since otherwise $\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1] = [\mathfrak{b}_1, \mathfrak{b}_1] \subseteq \mathfrak{b}_2$ and $\mathfrak{g}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1] = [\mathfrak{b}_{-1}, \mathfrak{b}_1] \subseteq \mathfrak{b}_0$ would imply $\mathfrak{b} = \mathfrak{g}$. Hence $d_1 < 4n$.
Considering the bracket on $\mathfrak{g}_2 \times \mathfrak{g}_{-1}$ given by scalar multiplication, we see that $d_2 = 0$, since otherwise we would obtain a contradiction to $\mathfrak{b}_1 \neq \mathfrak{g}_1$ using the fact that for any nonzero element X in $\mathfrak{b}_2$ $ad(X) : \mathfrak{g}_{-1} = \mathfrak{b}_{-1} \to \mathfrak{g}_1$ is surjective.
Now we look at the bracket $\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_2$ given by the imaginary part of the standard hermitian form. In order to have $[\mathfrak{b}_1, \mathfrak{b}_1] = 0$, we conclude $d_1 \leq n$ using lemma 3.1.
Finally we consider the bracket $\mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1$, which is given by $Mv + v(a - x)$ for $v \in \mathbb{H}^n$, $a \in \mathbb{R}, x \in Im(\mathbb{H})$ and M a skew-hermitian matrix. Assume that $d_1 = k \leq n$ and $k \neq 0$. By the second assertion of lemma 3.1 we can assume without loss of generality that $\mathfrak{b}_1 = \mathbb{R}^k \subseteq \mathbb{H}^k \subseteq \mathbb{H}^n$. In order to have $[\mathfrak{b}_0, \mathfrak{b}_1] \subseteq \mathfrak{b}_1$ we must have $x = 0$ and

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix},$$

where $M_1 \in \mathfrak{o}(k)$ and M_2 skew-hermitian of order $n-k$. So the dimension of $\mathfrak{b}_0$ equals

$$\dim_{\mathbb{R}}(\mathfrak{b}_0) = \frac{k(k-1)}{2} + 2(n-k)(n-k-1) + 3(n-k) + 1$$

Therefore,

$$\dim_{\mathbb{R}}(\mathfrak{b}_0 \oplus \mathfrak{b}_1) = \frac{5k^2 - (8n+1)k + 4n^2 + 2n + 2}{2}$$

is a quadratic polynomial in k having positive leading coefficient. Hence, $\dim_{\mathbb{R}}(\mathfrak{b}_0 \oplus \mathfrak{b}_1)$ can only be maximal at $k = 1$ or $k = n-1$. In fact, it turns out that it takes its maximum at $k = 1$, so

$$\dim_{\mathbb{R}}(\mathfrak{b}_0 \oplus \mathfrak{b}_1) \leq 2n^2 - 3n + 3$$

for $1 \leq d_1 \leq n-1$.
For $d_1 = 0$ we obtain $\dim_{\mathbb{R}}(\mathfrak{b}) \leq \dim_{\mathbb{R}}(\mathfrak{g}_- \oplus \mathfrak{g}_0) = 2n^2 + 5n + 7$.
So $\dim_{\mathbb{R}}(\mathfrak{b}) \leq 2n^2+5n+7$ for any proper graded subalgebra $\mathfrak{b}$ containing $\mathfrak{g}_{-1}$. $\square$

As a consequence we obtain

THEOREM 3.3. *Let $(\mathcal{G} \to M, \omega)$ be a regular parabolic geometry of type $(Sp(n+1,1), P)$ over a connected manifold M. If $\dim(Aut(\mathcal{G},\omega)) < \dim(Sp(n+1,1)) = 2(n+2)^2 + n + 2$ and $\mathfrak{aut}(\mathcal{G},\omega)$ contains $\mathfrak{g}_{-1}$ or $\mathfrak{g}_1$, then*

$$\dim(Aut(\mathcal{G},\omega)) \leq 2n^2 + 5n + 7$$

The only proper graded subalgebras that contain $\mathfrak{g}_{-1}$ or $\mathfrak{g}_1$ of dimension $2n^2 + 5n + 7$ are $\mathfrak{g}_- \oplus \mathfrak{g}_0$ and $\mathfrak{p}$ respectively. We can realize these as Lie algebras of automorphism groups of parabolic geometries as follows:
Since $Sp(n+1,1)$ is connected, the homogeneous model is a regular parabolic geometry over a connected manifold and we obtain

$$Aut(Sp(n+1,1,\omega_{MC})) = \{\lambda_g : Sp(n+1,1) \to Sp(n+1,1) : g \in Sp(n+1,1)\}.$$

Set $o = p(e)$, where e is the neutral element in $Sp(n+1,1)$ and define $U \subset Sp(n+1,1)/P$ to be the open subset $U := Sp(n+1,1)/P \setminus o$. Then we get that

$$\{g \in Sp(n+1,1) : \overline{\lambda}_g(U) \subset U\} = \{g \in G : \lambda_g(P) \subset P\} = P.$$

Therefore the Lie algebra of $Aut(p^{-1}(U), \omega_{MC}|_{p^{-1}(U)})$ is isomorphic to $\mathfrak{p}$. To obtain the subalgebra $\mathfrak{g}_- \oplus \mathfrak{g}_0$ one has just to take out of $Sp(n+1,1)/P$ a different point.

3.5. The case $Sp(6,\mathbb{R})$

Let G be $Sp(6,\mathbb{R})$, the closed subgroup of $SL(6,\mathbb{R})$ that preserves the non-degenerate skew-symmetric bilinear form $Q : \mathbb{R}^6 \times \mathbb{R}^6 \to \mathbb{R}$ given by $Q(v,w) = v^t K w$, where

$$K = \begin{pmatrix} 0 & & & & & -1 \\ & & & & 1 & \\ & & & -1 & & \\ & & 1 & & & \\ & -1 & & & & \\ 1 & & & & & 0 \end{pmatrix}$$

Since for $X \in SL(6,\mathbb{R})$ we have that $Q(Xv, Xw) = Q(v,w)$ for all $v, w \in \mathbb{R}^6$ if and only if $X^t K X = K$, we obtain

$$G = Sp(6,\mathbb{R}) = \{X \in SL(6,\mathbb{R}) : X^t K X = K\}$$

So the Lie algebra of G is

$$\mathfrak{sp}(6,\mathbb{R}) := \{X \in \mathfrak{gl}(6,\mathbb{R}) : X^t K + K X = 0\} =$$

$$= \left\{ \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} & -x_{1,5} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & -x_{2,4} & x_{1,4} \\ x_{4,1} & x_{4,2} & x_{4,3} & -x_{3,3} & x_{2,3} & -x_{1,3} \\ x_{5,1} & x_{5,2} & -x_{4,2} & x_{3,2} & -x_{2,2} & x_{1,2} \\ x_{6,1} & -x_{5,1} & x_{4,1} & -x_{3,1} & x_{2,1} & -x_{1,1} \end{pmatrix} : x_{i,j} \in \mathbb{R} \right\}$$

This is a real form of $\mathfrak{sp}(6,\mathbb{C})$ of dimension 21. In section 3.2. we mentioned that the split quaternions can be realized as the algebra of 2×2-matrices over $\mathbb{R}$. Using this we can identify $\mathfrak{sp}(6,\mathbb{R})$ with

$$\left\{ \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & -\overline{A}_{1,2} \\ A_{3,1} & -\overline{A}_{2,1} & -\overline{A}_{1,1} \end{pmatrix} : A_{1,3}, A_{2,2}, A_{3,1} \in Im(\mathbb{H}_s) \text{ and } A_{1,1}, A_{1,2}, A_{2,1} \in \mathbb{H}_s \right\}$$

Let $\theta : \mathfrak{sp}(6,\mathbb{R}) \to \mathfrak{sp}(6,\mathbb{R})$ be the Cartan involution given by $\theta(X) = -X^t$.

$$\text{Then } \mathfrak{h} := \left\{ \begin{pmatrix} a & 0 & & & & \\ 0 & b & & & & \\ & & c & 0 & & \\ & & 0 & -c & & \\ & & & & -b & 0 \\ & & & & 0 & -a \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

is a θ-stable Cartan subalgebra.
The roots Δ of $\mathfrak{sp}(6,\mathbb{C})$ with respect to $\mathfrak{h}_{\mathbb{C}}$ are given by

$$\Delta = \{\pm(\varepsilon_i + \varepsilon_j) : 1 \leq i < j \leq 3\}$$
$$\cup\{\pm(\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq 3)\} \cup \{\pm 2\varepsilon_i : 1 \leq i \leq 3\}.$$

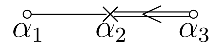
So we see that the restrictions of the roots to $\mathfrak{h}$ are real and therefore $\mathfrak{sp}(6,\mathbb{R})$ is a split real form of $\mathfrak{sp}(6,\mathbb{C})$. Hence the Satake diagram of $\mathfrak{sp}(6,\mathbb{R})$ and the Dynkin diagram of $\mathfrak{sp}(6,\mathbb{C})$ coincide.
Let $\{H_1, H_2, H_3\}$ be the basis of $\mathfrak{h}$, where $H_j = E_{j,j} - E_{7-j,7-j}$ for $j = 1,2,3$. Fixing on $\mathfrak{h}^*$ the ordering induced from this basis, we obtain

$$\Delta^+ = \{\varepsilon_i + \varepsilon_j : 1 \leq i < j \leq 3\} \cup \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq 3\} \cup \{2\varepsilon_i : 1 \leq i \leq 3\}$$

and

$$\Delta^0 = \{\alpha_1 := \varepsilon_1 - \varepsilon_2, \alpha_2 := \varepsilon_2 - \varepsilon_3, \alpha_3 := 2\varepsilon_3\}.$$

Now consider the standard parabolic subalgebra $\mathfrak{p}$ corresponding to the set $\Sigma = \{\alpha_2\}$:



The Σ-height of the maximal root $2\varepsilon_1 = 2\alpha_1 + 2\alpha_2 + \alpha_3$ is 2, so $\mathfrak{p}$ corresponds to a $|2|$-grading. This grading is given by

$$\mathfrak{sp}(6,\mathbb{R}) = \begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix}$$

and we obtain the linear isomorphisms $\mathfrak{g}_{\pm 1} \simeq \mathbb{H}_s$ and $\mathfrak{g}_{\pm 2} \simeq Im(\mathbb{H}_s) = \mathfrak{sl}(2,\mathbb{R})$.

We saw that $\mathfrak{sp}(6,\mathbb{R})$ and $\mathfrak{sp}(2,1)$ are both real forms of $\mathfrak{sp}(6,\mathbb{C})$. The $|2|$-grading on $\mathfrak{sp}(6,\mathbb{R})$ is the same as the $|2|$-grading on $\mathfrak{sp}(2,1)$ of the last section, but we will see that we have 15 as the maximal dimension of proper graded subalgebras of $\mathfrak{sp}(6,\mathbb{R})$ containing $\mathfrak{g}_{-1}$, not 14 as in the case $\mathfrak{sp}(2,1)$, since $\mathbb{H}_s$ is not a division ring.
Let V be a finite dimensional vector space over $\mathbb{R}$ endowed with a non-degenerate symmetric bilinear form. We will denote by $CSO(V)$ the subgroup of $GL(V)$ generated by $SO(V)$ and $\mathbb{R}^+$. Its Lie algebra $\mathfrak{cso}(V)$ is then the subalgebra of $\mathfrak{gl}(V)$ generated by $\mathbb{R}$ and $\mathfrak{so}(V)$.
Consider $\mathbb{H}_s$ as a real vector space endowed with the bilinear form $B_N(X,Y) = \overline{X}Y + \overline{Y}X$ of signature $(2,2)$. Note the the restriction of this form to $Im(\mathbb{H}_s)$ is of signature $(1,2)$.
One can easily see that the maps

$$Ad : G_0 \to GL(\mathfrak{g}_{\pm 1})$$

and

$$Ad : G_0 \to GL(\mathfrak{g}_{\pm 2})$$

have images in $CSO(\mathfrak{g}_{\pm 1})$ and $CSO(\mathfrak{g}_{\pm 2})$ respectively. In fact, these images already conincide with $CSO(\mathfrak{g}_{\pm 1})$ resp. $CSO(\mathfrak{g}_{\pm 2})$.
Now we will interpret the Lie brackets on the grading components of $\mathfrak{sp}(6,\mathbb{R})$. We will obtain results similar to the case $\mathfrak{sp}(2,1)$.
For the bracket on $\mathfrak{g}_2 \times \mathfrak{g}_{-1} \to \mathfrak{g}_1$ we obtain

$$[\begin{pmatrix} 0 & 0 & X \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ A & 0 & 0 \\ 0 & -\overline{A} & 0 \end{pmatrix}] = \begin{pmatrix} 0 & \overline{AX} & 0 \\ 0 & 0 & -AX \\ 0 & 0 & 0 \end{pmatrix}$$

for $X \in \mathfrak{sl}(2,\mathbb{R})$ and $A \in \mathfrak{gl}(2,\mathbb{R})$. Hence it is given by multiplication of A and X.
Similary, for the bracket on $\mathfrak{g}_{-2} \times \mathfrak{g}_1 \to \mathfrak{g}_{-1}$, we obtain

$$[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & A & 0 \\ 0 & 0 & -\overline{A} \\ 0 & 0 & 0 \end{pmatrix}] = \begin{pmatrix} 0 & 0 & 0 \\ -\overline{XA} & 0 & 0 \\ 0 & XA & 0 \end{pmatrix}$$

for $X \in \mathfrak{sl}(2,\mathbb{R})$ and $A \in \mathfrak{gl}(2,\mathbb{R})$.
The brackets on $\mathfrak{g}_{\pm 1} \times \mathfrak{g}_{\pm 1} \to \mathfrak{g}_{\pm 2}$ are two times the imaginary part of the standard hermitian form $\langle X, Y\rangle := \overline{X}Y$ on $\mathbb{H}_s$, since

$$[\begin{pmatrix} 0 & \overline{X} & 0 \\ 0 & 0 & -X \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \overline{Y} & 0 \\ 0 & 0 & -Y \\ 0 & 0 & 0 \end{pmatrix}] = \begin{pmatrix} 0 & 0 & \overline{Y}X - \overline{\overline{Y}X} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and analogously for $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1}$.
Now we consider the brackets on $\mathfrak{g}_0 \times \mathfrak{g}_{\pm 1} \to \mathfrak{g}_{\pm 1}$.

For $A := \begin{pmatrix} A_{1,1} & 0 & 0 \\ 0 & A_{2,2} & 0 \\ 0 & 0 & -\overline{A}_{1,1} \end{pmatrix} \in \mathfrak{g}_0$, $B := \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & -\overline{X} & 0 \end{pmatrix} \in \mathfrak{g}_{-1}$ and

$$C := \begin{pmatrix} 0 & Y & 0 \\ 0 & 0 & -\overline{Y} \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}_1$$

we obtain

$$[A,B] = \begin{pmatrix} 0 & 0 & 0 \\ A_{2,2}X - XA_{1,1} & 0 & 0 \\ 0 & -\overline{(A_{2,2}X - XA_{1,1})} & 0 \end{pmatrix}$$

and

$$[A,C] = \begin{pmatrix} 0 & A_{1,1}Y - YA_{2,2} & 0 \\ 0 & 0 & -\overline{(A_{1,1}Y - YA_{2,2})} \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore we obtain the isomorphisms $\mathfrak{g}_0 \simeq \mathfrak{cso}(\mathfrak{g}_{\pm 1})$.

For the bracket on $\mathfrak{g}_0 \times \mathfrak{g}_2 \to \mathfrak{g}_2$ we obtain that

$$[\begin{pmatrix} A_{1,1} & 0 & 0 \\ 0 & A_{2,2} & 0 \\ 0 & 0 & -\overline{A}_{1,1} \end{pmatrix}, \begin{pmatrix} 0 & 0 & M \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}] = \begin{pmatrix} 0 & 0 & A_{1,1}M - \overline{A_{1,1}M} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So this bracket is given by $2Im(A_{1,1}M)$. Similarly, for the bracket on $\mathfrak{g}_0 \times \mathfrak{g}_{-2}$ we have $-2Im(MA_{1,1})$. Therefore we obtain surjections $\mathfrak{g}_0 \twoheadrightarrow \mathfrak{cso}(\mathfrak{g}_{\pm 2})$.
Regarding the bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_1 \to \mathfrak{g}_0$, we get

$$[\begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & -\overline{X} & 0 \end{pmatrix}, \begin{pmatrix} 0 & A & 0 \\ 0 & 0 & -\overline{A} \\ 0 & 0 & 0 \end{pmatrix}] = \begin{pmatrix} -AX & 0 & 0 \\ 0 & XA - \overline{XA} & 0 \\ 0 & 0 & \overline{AX} \end{pmatrix}$$

and finally the bracket on $\mathfrak{g}_{-2} \times \mathfrak{g}_2 \to \mathfrak{g}_0$ turns out to be

$$[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}] = \begin{pmatrix} -YX & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & XY \end{pmatrix}$$

Now we can prove the following proposition:

PROPOSITION 3.4. *Let $\mathfrak{sp}(6,\mathbb{R}) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be the $|2|$-grading from above. If $\mathfrak{b} = \mathfrak{b}_{-2} \oplus \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1 \oplus \mathfrak{b}_2$ is a graded subalgebra that contains $\mathfrak{g}_{-1}$, then the dimension of $\mathfrak{b}$ is at most* 15.

PROOF. We set $\mathfrak{g} := \mathfrak{sp}(6,\mathbb{R})$ and $d_i := \dim(\mathfrak{b}_i)$ for $i = -2, ..., 2$. Identify $\mathfrak{g}_{\pm 1}$ with $\mathbb{H}_s$, $\mathfrak{g}_{\pm 2}$ with $Im(\mathbb{H}_s)$ and $\mathfrak{g}_0$ with $\mathbb{H}_s \times Im(\mathbb{H}_s)$.
From $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = [\mathfrak{b}_{-1}, \mathfrak{b}_{-1}] \subseteq \mathfrak{b}_{-2}$ it follows that $\mathfrak{g}_-$ is contained in $\mathfrak{b}$. We must also have $\mathfrak{b}_1 \neq \mathfrak{g}_1$, since otherwise $\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1] = [\mathfrak{b}_1, \mathfrak{b}_1] \subseteq \mathfrak{b}_2$ and $\mathfrak{g}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1] = [\mathfrak{b}_{-1}, \mathfrak{b}_1] \subseteq \mathfrak{b}_0$ would imply $\mathfrak{b} = \mathfrak{g}$. Therefore $d_1 < 4$.
If we consider the bracket on $\mathfrak{g}_2 \times \mathfrak{g}_{-1} \to \mathfrak{g}_1$ given by multiplication, we see that $d_2 \leq 1$: Suppose that $d_2 \geq 2$. Any subspace of the space of trace free real 2×2 matrices of dimension ≥ 2 contains an invertible element and so we can find an invertible element in $\mathfrak{b}_2$. Since for any invertible element X $ad(X) : \mathfrak{g}_{-1} = \mathfrak{b}_{-1} \to \mathfrak{g}_1$ is surjective, we get a contradiction to $\mathfrak{b}_1 \neq \mathfrak{g}_1$. So $\mathfrak{b}_2$ is either 0 or a one dimensional subspace of non invertible elements of $\mathfrak{g}_2 = Im(\mathbb{H}_s)$.
In order to have $[\mathfrak{b}_1, \mathfrak{b}_1] \subseteq \mathfrak{b}_2$, we must have $d_1 \leq 2$: Assume that $d_1 = 3$ and consider the bracket on $\mathfrak{g}_1 \times \mathfrak{g}_1$ given by the imaginary part of the standard hermitian form on $\mathbb{H}_s$. Since $d_1 = 3$ and since a totally isotropic subspace is at most 2-dimensional, we can find an element $X \in \mathfrak{b}_1$ such that $\langle X, X \rangle \neq 0$. By the non-degeneracy of $\langle\ ,\ \rangle$, we obtain an $\mathbb{H}_s$-isomorphism given by $\langle X, . \rangle : \mathbb{H}_s \to \mathbb{H}_s$. Hence $\dim([\mathfrak{b}_1, \mathfrak{b}_1]) > 2$, a contradiction.
We proceed further by a case by case distinction.
First case: Suppose $d_2 = 0$.

Then we can have $d_1 = 0, 1$ or 2 and all cases can be actually realized. So we assume first that in addition to $d_2 = 0$ $d_1 = 2$.
Now consider the bracket on $\mathfrak{g}_0 \times \mathfrak{g}_1$ given by $AX - XB$ for $(A, B) \in \mathfrak{g}_0 = \mathbb{H}_s \times Im(\mathbb{H}_s)$ and $X \in \mathfrak{g}_1 = \mathbb{H}$. It can be seen as an isomorphism $\mathfrak{g}_0 \to \mathfrak{cso}(\mathfrak{g}_1)$. Observe that $\mathfrak{b}_1 \subset \mathbb{H}_s$ has to be totally isotropic, since otherwise $d_2 \geq 1$. Since $[\mathfrak{b}_0, \mathfrak{b}_1] \subseteq \mathfrak{b}_1$, $\mathfrak{b}_0 \subseteq \mathfrak{g}_0$ is contained in the stabilizer of the totally isotropic plane $\mathfrak{b}_1$ in $\mathfrak{cso}(\mathfrak{g}_1)$. Therefore $\dim(\mathfrak{b}_0) \leq 6$ and $\dim(\mathfrak{b}) \leq 15$.
Now suppose that $d_2 = 0$ and $d_1 = 1$. Then $\mathfrak{b}_0$ lies in the stabilizer of the isotropic line $\mathfrak{b}_1$ and so $\dim(\mathfrak{b}_0) \leq 5$. Hence $\dim(\mathfrak{b}) \leq 13$.
Second case: Suppose $d_2 = 1$.
Since $Ad : G_0 \to CSO(\mathfrak{g}_2)$ is surjective, we can suppose without loss of generality that $\mathfrak{b}_2$ is the isotropic line generated by $L := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.
The image of $\mathfrak{b}_{-1} = \mathfrak{g}_{-1}$ under $ad(L)$ is the two dimensional subspace of $\mathfrak{g}_1$ of elements of the form

$$ad(L)(\mathfrak{b}_{-1}) = \left\{ \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \right\}.$$

Since $\mathfrak{b}_1$ must contain this subspace and $d_1 \leq 2$, we get $ad(L)(\mathfrak{b}_{-1}) = \mathfrak{b}_1$.
By straight forward computations we obtain that

$$\mathfrak{b}_0 \subseteq \left\{ \begin{pmatrix} a_1 & 0 \\ a_3 & a_4 \end{pmatrix} \right\} \times \left\{ \begin{pmatrix} b_1 & 0 \\ b_3 & -b_1 \end{pmatrix} \right\}$$

in order to have $[\mathfrak{b}_0, \mathfrak{b}_1] \subseteq \mathfrak{b}_1$ and $[\mathfrak{b}_0, \mathfrak{b}_2] \subseteq \mathfrak{b}_2$. So, $d_0 \leq 5$ and $\dim(\mathfrak{b}) \leq 15$
Putting all this together, we finally conclude that $\dim(\mathfrak{b}) \leq 3 + 4 + 8 = 15$. $\square$

Proper graded subalgebras of $\mathfrak{g}$ of dimension 15 are for example:

$\mathfrak{b} = \mathfrak{g}_- \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$
where $\mathfrak{b}_0 = \mathbb{H}_s \times \left\{ \begin{pmatrix} b_1 & 0 \\ b_3 & -b_1 \end{pmatrix} \right\}$, $\mathfrak{b}_1 = \left\{ \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \right\}$ and $\mathfrak{b}_2 = \{0\}$.

or

$\mathfrak{b} = \mathfrak{g}_- \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1 \oplus \mathfrak{b}_2$

where $\mathfrak{b}_0 = \left\{ \begin{pmatrix} -a_2 + a_3 + a_4 & a_2 \\ a_3 & a_4 \end{pmatrix} \times \begin{pmatrix} b_1 & b_3 - 2b_1 \\ b_3 & -b_1 \end{pmatrix} \right\}$, $\mathfrak{b}_1 = \left\{ \begin{pmatrix} x & -x \\ y & -y \end{pmatrix} \right\}$
and $\mathfrak{b}_2 = \left\{ \begin{pmatrix} a & -a \\ a & -a \end{pmatrix} \right\}$

As a consequence of proposition 3.4. we obtain the following theorem:

THEOREM 3.4. *Let P be the parabolic subgroup of $Sp(6,\mathbb{R})$ corresponding to $\mathfrak{p}$ from above and let $(\mathcal{G} \to M, \omega)$ a regular parabolic geometry of type $(Sp(6,\mathbb{R}), P)$ over a connected manifold M.*
If $\dim(Aut(\mathcal{G},\omega)) < \dim(Sp(6,\mathbb{R})) = 21$ *and* $\mathfrak{aut}(\mathcal{G},\omega)$ *contains* $\mathfrak{g}_{-1}$ *or* $\mathfrak{g}_1$*, then*

$$\dim(Aut(\mathcal{G},\omega)) \leq 15$$

To realize the 15-dimensional graded subalgebra

$$\mathfrak{b} = \left\{ \begin{pmatrix} a_1 & a_2 & x & 0 & 0 & 0 \\ a_3 & a_4 & y & 0 & 0 & 0 \\ x_1 & x_2 & b_1 & 0 & 0 & 0 \\ x_3 & x_4 & b_3 & -b_1 & y & -x \\ c_1 & c_2 & -x_4 & x_2 & -a_4 & a_2 \\ c_3 & -c_1 & x_3 & -x_1 & a_3 & -a_1 \end{pmatrix} \right\}$$

as Lie algebra of the automorphism group of some parabolic geometry of type $(G = Sp(6,\mathbb{R}), P)$ we give a description of the homogeneous space G/P.
First we recall that G is connected and therefore the homogeneous model is a regular parabolic geometry over a connected manifold. Moreover $Aut(G, \omega_{MC}) = \{\lambda_g : G \to G : g \in G\}$. The parabolic subalgebra $\mathfrak{p}$ is obviously the stabilizer of the totally isotropic subspace $\mathbb{R}^2$ of $(\mathbb{R}^6, Q)$. So we conclude that P is the stabilizer of $\mathbb{R}^2$ with respect to the standard representation $\mathbb{R}^6$.
G acts naturally on all 2-dimensional totally isotropic subspaces of $\mathbb{R}^6$ and this action is transitive by the theorem of Witt. Since P is exactly the isotropy group of $\mathbb{R}^2$, we can identify G/P with the set of all all 2-dimensional totally isotropic subspaces of $\mathbb{R}^6$.
Now consider the orbit $B[\mathbb{R}^2] =: O$, where B is the connected subgroup of G with Lie algebra $\mathfrak{b}$. Identifyng O with $B/(B \cap P)$ we see that this orbit is of maximal dimension, since $\dim(\mathfrak{b}/(\mathfrak{b} \cap \mathfrak{p})) = 7$. Hence O is open and we conclude that $Aut(p^{-1}(O), \omega_{MC}|_{p^{-1}(O)})$ has Lie algebra $\mathfrak{b}$ using the same arguments as in section 3.3.

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