The ambient metric (mathematical aspects)

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- In my talk, I will survey the basic properties of the ambient metric, its relation to other conformally invariant objects, and some applications.
- I will also discuss the motivation for the ambient metric construction which comes from complex analysis and CR geometry.

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Basic notions

Two (Pseudo–)Riemannian metrics g_{ij} and $\hat{g}_{ij}$ on a smooth manifold M are *conformally equivalent* if there is a smooth function $\varphi : M \to \mathbb{R}$ such that $\hat{g}_{ij} = e^{2\varphi} g_{ij}$.

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It is very easy to construct (scalar) invariants associated to a Riemannian manifold (M, g): Take the Riemann curvature $R_{ij}{}^{k}{}_{\ell}$ and its iterated covariant derivatives $\nabla_{a_1} \dots \nabla_{a_r} R_{ij}{}^{k}{}_{\ell}$ multiply several such objects up and then define $I(g)$ to be a complete contraction of such an expression.

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It is a natural idea to look at particularly robust Riemannian invariants, which behave nicely under conformal changes. The simplest of those are the conformal invariants:

Definition

A *conformal invariant* of weight w is a Riemannian invariant I such that for $\hat{g} = e^{2\varphi} g$ one obtains $I(\hat{g}) = e^{w\varphi} I(g)$.

Similarly, it is very easy to construct (linear) differential operators acting on tensor or spinor bundles, which are intrinsic to a Riemannian manifold (M, g): One just takes iterated covariant derivatives, the curvature and its covariant derivatives and then forms tensorial operations like contractions to define a differential operator D^g depending on g. Again, one can look at such operators which are well behaved with respect to conformal changes:

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Definition

A Riemannian invariant linear differential operator D^g is called *conformally covariant* of weight (a, b) if and only if for $\hat{g} = e^{2\varphi} g$ one gets

$$D^{\hat{g}}(e^{a\varphi} s) = e^{b\varphi} D^g(s).$$

Examples

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The simplicity of these examples is misleading. Although conformal geometry was studied intensively, only very few other conformal invariants were know classically.
Likewise, a naive approach to conformally covariant operators is very effective for order 1 or 2, but gets quickly out of hand in higher orders. A conformally covariant modification of Δ^2 in dimension 4 was first constructed in 1983 (Paneitz operator).

To overcome these difficulties, it is desirable to work in a conformally invariant way throughout. The basic ideas for this have been developed in the 1920s and 30s, but only in the last decades things have turned into fairly effective calculi. All these approaches are based on the *homogeneous model* of conformal structures.

To overcome these difficulties, it is desirable to work in a conformally invariant way throughout. The basic ideas for this have been developed in the 1920s and 30s, but only in the last decades things have turned into fairly effective calculi. All these approaches are based on the *homogeneous model* of conformal structures.

The sphere S^n can be realized as the space of null–lines in $\mathbb{R}^{n+1,1}$ and inherits a canonical conformal structure from the Lorentzian inner product. This leads to a transitive action of $G := SO(n+1,1)$ on S^n by conformal isometries, and an identification $S^n \cong G/P$, where $P \subset G$ is the stabilizer of an isotropic line (Poincaré conformal group). The group P is an extension of the conformal group $CO(n)$ by $\mathbb{R}^n$, so $CO(n)$ is naturally a quotient of P.

Projectivizing the interior of the light cone, one obtains hyperbolic space $\mathcal{H}^{n+1}$ and the natural action realizes G as the isometry group of $\mathcal{H}^{n+1}$. The sphere, viewed as the projectivized light cone, thus naturally shows up as the conformal infinity of $\mathcal{H}^{n+1}$ in this picture (with the group G of conformal isometries).

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The basic object associated to a conformal structure is the canonical Cartan connection introduced by E. Cartan in the 1920s. On the homogeneous model, one observes the $G \to G/P \cong S^n$ is a principal P–bundle, and the Maurer–Cartan form defines a trivialization of the tangent bundle $TG \cong G \times \mathfrak{g}$ with nice properties, where $\mathfrak{g} = \mathfrak{so}(n+1,1)$. The Cartan connection generalizes this description to arbitrary conformal structures:

Theorem (E. Cartan)

Let $(M, [g])$ be a conformal manifold. Then one can naturally extend the conformal frame bundle to a principal P–bundle $\mathcal{P} \to M$ which can be canonically endowed with a Cartan connection $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$. The pair $(\mathcal{P}, \omega)$ is uniquely determined up to isomorphism.

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Already in the 1930s, T. Thomas developed an equivalent approach via a canonical vector bundles endowed with natural linear connections. This has been rediscovered and developed under the name *tractor bundles* during the last years. The basic example is the *standard tractor bundle* $\mathcal{T} \to M$ which comes with a bundle metric of signature $(n+1, 1)$, a canonical line subbundle $\mathcal{T}^1 \subset \mathcal{T}$ with isotropic fibers, and a natural linear connection. It can be obtained as $\mathcal{T} = \mathcal{P} \times_P \mathbb{R}^{n+1,1}$. On the homogeneous model, it corresponds to the trivial bundle $S^n \times \mathbb{R}^{n+1,1}$ with the tautological subbundle.

Structure

The ambient metric construction can also be motivated from the homogeneous model. It is easy to generalize the point of view of S^n as a projectivized light cone: One simply attaches to each point of M the line formed by the metrics in the conformal class to define a cone. The ambient metric construction then tries to obtain an analog of the surrounding space $\mathbb{R}^{n+1,1}$ and the flat metric on this space.

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There is a deeper background however, (which also came first historically) in which the whole construction becomes more natural. This comes from the complex analog of the whole setup, which is related to complex analysis and CR geometry.

Let $\Omega \subset \mathbb{C}^{n+1}$ be a (suitably convex) bounded domain with smooth boundary $M = \partial\Omega$. Then for each $x \in M$ the tangent space $T_xM \subset \mathbb{C}^{n+1}$ has real dimension $2n+1$, so the maximal complex subspace $H_xM \subset T_xM$ must have complex dimension n. The family H_x of complex subspaces is called a *CR–structure* on M, and many analytic properties of Ω are reflected in this geometric structure.

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The homogeneous model

Consider $\mathbb{C}^{n+1,1}$ with a Hermitian form of Lorentzian signature. Then the space of lines in the light cone respectively in its interior can be identified with the unit sphere S^{2n+1} respectively the unit ball B^{n+1} in $\mathbb{C}^{n+1}$. The resulting actions of $G = SU(n+1,1)$ identify G with the group of CR–automorphisms of S^{2n+1} respectively of holomorphic automorphisms of B^n.

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In the 1970s Ch. Fefferman did groundbreaking work on such domains. Motivated by considerations from complex analysis, he came up with the following construction:

Given $\Omega \subset \mathbb{C}^{n+1}$ with $M = \partial\Omega$ consider the domain $\Omega_\# = (\mathbb{C} \setminus 0) \times \Omega \subset (\mathbb{C} \setminus 0) \times \mathbb{C}^{n+1}$ and its boundary $M_\# = (\mathbb{C} \setminus 0) \times M$. Consider a defining function r for M (i.e. $M = r^{-1}(0)$ and dr is nonzero on M), and define $r_\#(z_0, z) := |z_0|^2 r(z)$. Then this is a defining function for $M_\#$ and it can be used as the potential for a Lorentzian Kähler metric $g_\#$ defined locally around $M_\#$.

The Ricci curvature of $g_\#$ can be easily computed as a (highly non–linear) expression in the partial derivatives of r up to second order. In particular, $g_\#$ is Ricci flat (and hence Calabi–Yau) locally around $M_\#$ if and only if r is a solution of a complex Monge–Ampère equation.

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In general, this Monge–Ampère equation does not admit smooth solutions (and the solvability questions were later on sorted out completely by Cheng–Yau), but Fefferman found an ingenious algorithm to algorithmically construct approximate solutions and proved that they are uniquely determined. It is then rather easy to show that an appropriate jet of the metric $g_\#$ constructed from such an approximate solution is intrinsic to the domain Ω respectively the CR structure on M.

Remark

There is a stronger connection to conformal structures. The restriction of $g_{\#}$ to $M_{\#} = (\mathbb{C} \setminus 0) \times M$ turns out to be degenerate with the degenerate directions given by the real rays in $(\mathbb{C} \setminus 0)$. Factoring by these real rays, one obtains an induced conformal structure on $\tilde{M} = M \times S^1$. The resulting conformal manifold is called the *Fefferman space* of M. On the one hand, this allows the use of conformal geometry for the study of CR–structures (which was the original reason for Fefferman's interest in the topic). On the other hand, it gives rise to a nice subclass of conformal structures.

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The relation between conformally invariant objects and calculi on $\tilde{M}$ and CR–invariant objects and calculi is well understood and explicitly described by now. Also there are nice characterizations of Fefferman spaces (up to local isomorphism).

Structure

We have seen that we obtain an analog of the cone over a conformal manifold $(M, [g])$ by attaching to each point $x \in M$ the set of values g_x of all metrics in the conformal class. This defines a smooth manifold $\mathcal{G}$ with a projection $p : \mathcal{G} \to M$.

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Now one considers $\mathcal{G} \times \mathbb{R}$ with the extended action of $\mathbb{R}_+$ and $\mathbb{R}_+$–invariant neighborhoods $\tilde{\mathcal{G}}$ of $\mathcal{G} = \mathcal{G} \times 0$ in there, together with Lorentzian metrics $\tilde{g}$ on $\tilde{\mathcal{G}}$ which are homogeneous of degree two and restrict to g_0 on $\mathcal{G}$.

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- Now one makes a power–series ansatz (in ρ) for $\tilde{g}$ and tries to determine the terms in this series from Ricci flatness of $\tilde{g}$ iteratively.

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It turns out that the result depends heavily on the parity of n.

Theorem (Fefferman–Graham)

1. If n is odd, then all terms in the power series can be determined uniquely. In particular, one can find a pair $(\tilde{\mathcal{G}}, \tilde{g})$ for which $\tilde{g}$ is Ricci flat to infinite order along $\mathcal{G}$.

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2. If n is even, then there is a formal obstruction to finding a power series solution in order $n/2$. One can find a pair $(\tilde{\mathcal{G}}, \tilde{g})$ for which $\tilde{g}$ is Ricci flat to order $n/2 - 1$ along $\mathcal{G}$. Even if the obstruction vanishes, there is a formal indeterminacy at this order.

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3. In both cases, the solution is unique up to diffeomorphism which are defined locally around $\mathcal{G} \times 0$ and fix this subset and addition of terms which vanish to infinite order respectively to order $n/2$ along $\mathcal{G} \times 0$.

Remarks

(1) Since $\tilde{g}$ is assumed to be homogeneous, one may factor by the $\mathbb{R}_+$–action to obtain picture of a *Poincaré metric* g_+ on $M \times [0, \epsilon)$. This is a Riemannian Einstein metric, which has the given conformal class as conformal infinity. One may simply translate between results in the ambient picture and this picture.

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(3) There are a few global results on Poincaré metrics. For example, any conformal class on S^n close to the standard one, is induced by a unique complete Poincaré– Einstein metric close to the hyperbolic metric on B^{n+1} (Graham–Lee, Biquard). These results are of completely different nature than the ones of Fefferman–Graham.

Relation to tractors

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- $T\tilde{\mathcal{G}}|_{\mathcal{G}}$ descends to a vector bundle $\mathcal{T} \to M$, $\tilde{g}$ descends to a bundle metric on $\mathcal{T}$, and the vertical subbundle induces an isotropic line subbundle $\mathcal{T}^1 \subset \mathcal{T}$.

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Theorem (Č.-Gover)

If $\tilde{g}$ has vanishing Ricci curvature along $\mathcal{G}$, then this is isomorphic to the standard tractor bundle and its canonical connection. If $\tilde{g}$ is Ricci flat to higher order, one can compute $\tilde{R}$ and its covariant derivative from tractor data.

Immediate applications

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- The obstruction in even dimensions can be viewed as a tracefree symmetric two–tensor $\mathcal{O}_{ij}$ on M of certain weight. This is a conformal invariant, which turns out to be very important.
- If a conformal class contains an Einstein metric, then $\mathcal{O}_{ij} = 0$. (Starting from an Einstein representative as above, the equation for Ricci flatness of $\tilde{g}$ reduces to an ODE.)

GJMS operators

- Choosing a representative g of the conformal class and a smooth function f on M, one can canonically lift f to a function $\mathcal{G}$ which is homogeneous of degree $w \in \mathbb{R}$. (Changing to $\hat{g} = e^{2\varphi}g$ then leads to $\hat{f} = e^{w\varphi}f$).

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- Then extend this lift to a smooth function $\tilde{f}$ on $\tilde{\mathcal{G}}$ and apply $\tilde{\Delta}^k$ (for k small enough to be unambiguous), and $\tilde{\Delta}^k(\tilde{f})$ is homogeneous of degree $w - 2k$.

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- Then extend this lift to a smooth function $\tilde{f}$ on $\tilde{\mathcal{G}}$ and apply $\tilde{\Delta}^k$ (for k small enough to be unambiguous), and $\tilde{\Delta}^k(\tilde{f})$ is homogeneous of degree $w - 2k$.
- One proves that for fixed k, w can be uniquely chosen in such a way that $\tilde{\Delta}^k(\tilde{f})$ depends only on f and not on the extension $\tilde{\varphi}$. Hence it induces a differential operator P_{2k}, which is conformally covariant of weight $(w, w - 2k)$. In the Poincaré metric picture, the P_{2k} are deeply related to scattering theory.

Branson's Q–curvature

- For $\dim(M) = 2n$, fix a representative metric g and build coordinates (t, x, ρ) on $\tilde{\mathcal{G}}$ as before (with a small additional normalization). Then consider the function $\log(t)$ on $\tilde{\mathcal{G}}$ and define function Q^g on M by $Q^g(x) := -\tilde{\Delta}^n(|\log(t)|)(1, x, 0)$.

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- Then Q^g is a smooth function on M which defines a Riemannian invariant, called *Branson's Q–curvature*. This has a very nice (linear) conformal transformation law. Namely for $\hat{g} = e^{2\varphi} g$ one gets $Q^{\hat{g}} = Q^g + P_{2n}(\varphi)$.

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- Together with self–adjointness of P_{2n} this implies that for compact M integrating Q^g with respect to the volume form of g defines a global conformal invariant.

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