Riemannian Geometry

lecture notes

Fall Term 2021/22

Andreas Čap

Institut für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Wien

Email address: Andreas.Cap@univie.ac.at

CHAPTER 1

Fundamentals of Riemannian geometry

After recalling some background, we define Riemannian metrics and Riemannian manifolds. We analyze the basic tensorial operations that become available in the presence of a Riemannian metric. Then we construct the Levi-Civita connection, which is the basic "new" differential operator coming from such a metric.

Background

The purpose of this section is two-fold. On the one hand, we want to relate the general concept of a Riemannian manifold to the geometry of hypersurfaces as known from introductory courses. On the other hand, we recall some facts about tensor fields and introduce abstract index notation.

1.1. Euclidean geometry. The basic object in Euclidean geometry is the n-dimensional *Euclidean space* E^n. One may abstractly start from an affine space of dimension n, but for simplicity, we just take the n-dimensional real vector space $\mathbb{R}^n$ and "forget about the origin". Given two points in this space, there is a well defined vector connecting them, which we denote by $\overrightarrow{xy} \in \mathbb{R}^n$. Identifying E^n with $\mathbb{R}^n$, this can be computed as $\overrightarrow{xy} = y - x$ (which visibly is independent of the location of the origin). On the other hand, given a point $x \in E^n$ and a vector $v \in \mathbb{R}^n$, we can form $x + v \in E^n$. Of course, this satisfies $x + \overrightarrow{xy} = y$ and similar properties. (The abstract definition of an affine space requires the existence of $(x, y) \mapsto \overrightarrow{xy}$ as a map $E^n \times E^n \to \mathbb{R}^n$ and of $+ : E^n \times \mathbb{R}^n \to E^n$ together with some of the basic properties of these operations.)

The second main ingredient to Euclidean geometry is provided by the standard inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$. This allows us to define the Euclidean distance of two points $x, y \in E^n$ by $d(x, y) := \|\overrightarrow{xy}\| = \sqrt{\langle \overrightarrow{xy}, \overrightarrow{xy} \rangle}$.

Let us relate this to differential geometry. Fixing a point $o \in E^n$, the map $x \mapsto \overrightarrow{ox}$ defines a bijection $E^n \to \mathbb{R}^n$. This can be used as a global chart (and any two such charts are compatible) thus making E^n into a smooth manifold. Moreover, one can identify each tangent space T_xE^n with $\mathbb{R}^n$ by mapping $v \in \mathbb{R}^n$ to $c'(0)$, where $c : \mathbb{R} \to E^n$ is the smooth curve defined by $c(t) := x + tv$. Hence we can view the standard inner product on $\mathbb{R}^n$ as defining an inner product on each tangent space of E^n.

The two pictures fit together nicely, as we can see from the appropriate concept of morphisms of Euclidean space, which can be formulated in seemingly entirely different ways:

Proposition 1.1. *For a set-map $f : E^n \to E^n$ the following conditions are equivalent.*

(i) For all points $x, y \in E^n$, we have $d(f(x), f(y)) = d(x, y)$.

(ii) The map f is smooth and for each $x \in E^n$, the tangent map $T_xf : T_xE^n \to T_{f(x)}E^n$ is orthogonal.

(iii) There is an orthogonal linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ such that for all $x, y \in E^n$ we have $f(y) = f(x) + A(\overrightarrow{xy})$.

Proof. The condition in (iii) can be rewritten as $\overrightarrow{f(x)f(y)} = A(\overrightarrow{xy})$ for all $x, y \in E^n$. Since orthogonal linear maps preserve the norms of vectors, we see that (iii)⇒(i). Applying the condition to $y = x + tv$, we get $\overrightarrow{xy} = tv$, so $f(x + tv) = f(x) + tA(v)$. This shows that if f satisfies (iii), then it is smooth and $T_x f = A$ for each $x \in E^n$, so (iii)⇒(ii).

(i)⇒(iii): We claim that a map $F : \mathbb{R}^n \to \mathbb{R}^n$ which satisfies $F(0) = 0$ and which is distance-preserving must be an orthogonal linear map. Since $\|v\| = d(v, 0)$ and $F(0) = 0$, we see that $\|F(v)\| = \|v\|$ for all $v \in \mathbb{R}^n$. Now one of the polarization identities reads as

$$\langle v, w\rangle = \tfrac{1}{2}\left(\|v\|^2 + \|w\|^2 - d(v, w)^2\right),$$

so we conclude that $\langle F(v), F(w)\rangle = \langle v, w\rangle$. In particular, denoting by $\{e_1, \dots, e_n\}$ the (orthonormal) standard basis for $\mathbb{R}^n$, we see that the vectors $F(e_1), \dots, F(e_n)$ also form an orthonormal system and thus an orthonormal basis.

Taking an arbitrary element $v \in \mathbb{R}^n$, we can expand v in the standard basis as $v = \sum_i \langle v, e_i\rangle e_i$. Likewise, we can expand $F(v)$ in the orthonormal basis $\{F(e_i)\}$ as $F(v) = \sum_i \langle F(v), F(e_i)\rangle F(e_i)$. But then $\langle v, e_i\rangle = \langle F(v), F(e_i)\rangle$ implies that $F(\sum_i \lambda_i e_i) = \sum_i \lambda_i F(e_i)$ for all $(\lambda_1, \dots, \lambda_n)$. Hence F is a linear map and knowing this, we have already observed orthogonality.

Starting from a distance-preserving map $f : E^n \to E^n$, we choose a point $o \in E^n$ and define $F : \mathbb{R}^n \to \mathbb{R}^n$ as $F(v) = \overrightarrow{f(o)f(o + v)}$. This evidently satisfies $F(0) = 0$. Moreover, $F(w) - F(v) = \overrightarrow{f(o)f(o + w)} - \overrightarrow{f(o)f(o + v)} = \overrightarrow{f(o + v)f(o + w)}$ and in the same way $\overrightarrow{(o + v)(o + w)} = w - v$, so we see that F is distance-preserving and thus an orthogonal linear map by the claim. By construction, we get $f(x) = f(o) + F(\overrightarrow{ox})$ for all $x \in E^n$. For another point y, we have $\overrightarrow{oy} = \overrightarrow{ox} + \overrightarrow{xy}$ and thus $f(y) = f(o) + F(\overrightarrow{ox}) + F(\overrightarrow{xy}) = f(x) + F(\overrightarrow{xy})$.

(ii)⇒(iii): As in the last step, it suffices to show that a smooth map $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $F(0) = 0$ and for each $v \in \mathbb{R}^n$ the derivative $DF(v) : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal, must itself be an orthogonal linear map.

By assumption, for $X, Y \in \mathbb{R}^n$, we have $\langle DF(v)(X), DF(v)(Y)\rangle = \langle X, Y\rangle$. Taking $w \in \mathbb{R}^n$, we can form $\frac{d}{dt}|_{t=0} DF(v+tw)(X) = D^2F(v)(w, X)$, and this is symmetric in w and X. On the other hand, the map $t \mapsto \langle DF(v+tw)(X), DF(v+tw)(Y)\rangle$ is constant, so differentiating it at $t = 0$, we obtain

$$0 = \langle D^2F(v)(w, X), DF(v)(Y)\rangle + \langle DF(v)(X), D^2F(v)(w, Y)\rangle$$

This means that the tri-linear map $\Phi(X, Y, Z) := \langle D^2F(v)(X, Y), DF(v)(Z)\rangle$ satisfies $\Phi(X, Y, Z) = \Phi(Y, X, Z)$ and $\Phi(X, Z, Y) = -\Phi(X, Y, Z)$. But this implies

$$\begin{aligned}\Phi(X, Y, Z) = -\Phi(X, Z, Y) = &-\Phi(Z, X, Y) = \Phi(Z, Y, X)\\ &= \Phi(Y, Z, X) = -\Phi(Y, X, Z) = -\Phi(X, Y, Z).\end{aligned}$$

Hence we conclude that $\langle D^2F(v)(X, Y), DF(v)(Z)\rangle = 0$ and since the orthogonal map $DF(v)$ is surjective, we see that $D^2F(v) = 0$. But this means that $DF(v) = A$ for some fixed orthogonal linear map $A : \mathbb{R}^n \to \mathbb{R}^n$. This implies that the curve $c(t) = F(tv)$ has derivative $c'(t) = A(v)$ for all t. Hence $F(v) = c(1) = c(0) + \int_0^1 c'(t)dt = 0 + A(v) = A(v)$ for any $v \in \mathbb{R}^n$. □

Definition 1.1. A *Euclidean motion* is a map $f : E^n \to E^n$ which satisfies the equivalent conditions of Proposition 1.1.

The three conditions characterizing Euclidean motions visibly are of very different nature. Condition (i) tells us in a way that the Euclidean distance is the only central ingredient in Euclidean geometry. It is surprising that it is not necessary to assume smoothness initially. Condition (iii) is the most useful one for explicitly describing Euclidean motions and this is often used as the definition. Condition (ii) shows that Euclidean motions are exactly the isometries of E^n in the sense of Riemannian geometry.

1.2. Geometry of curves and surfaces. These classical parts of differential geometry study submanifolds in E^n. To obtain geometric properties, one always requires that things are well behaved (in an appropriate sense) with respect to Euclidean motions. (For example, the curvature of a curve should remain unchanged, while the tangent line should also be moved by the motion.)

In the geometry of surfaces in E^3, one meets a new phenomenon, since there are different kinds of curvatures. This is related to the question whether one can observe the fact that a surface is curved from inside the surface. (In classical language, this is referred to as "inner" or "intrinsic" geometry as opposed to "extrinsic" geometry of surfaces.) The classical examples are provided by a cylinder and a sphere, respectively. While a cylinder is curved from an outside point of view, it can be locally mapped onto an open subset of E^2 in a distance preserving way. In contrast to that, it is not possible to map an open subset of the sphere S^2 onto an open subset of E^2 in such a way that distances are preserved. Here "distance" in the cylinder and in S^2 are defined via the infimum of the arclengths of curves connecting two points (as we will develop the concept on general Riemannian manifolds). This is related to facts like that the sum of the three angles of a (geodesic) triangle on S^2 is always bigger than π and depends on the area of the triangle.

To formalize this concept, one observes that for a smooth submanifold $M \subset E^n$ and a point $x \in M$, the tangent space T_xM can be naturally viewed as a linear subspace of $T_xE^n = \mathbb{R}^n$. Hence one can restrict the standard inner product to the tangent spaces of M, thus defining a smooth $\binom{0}{2}$-tensor field on M. This is called the *first fundamental form*. Roughly speaking, intrinsic quantities are those which depend only on the first fundamental form. To formalize this, one introduces the concept of a local *isometry* between such submanifolds (of the same dimension) as a local diffeomorphism, for which all tangent maps are orthogonal.

If $M \subset E^n$ is a smooth submanifold and $f : E^n \to E^n$ is a Euclidean motion, then $f(M) \subset E^n$ is a smooth submanifold of the same dimension as M, and $f|_M : M \to f(M)$ is an isometry. However, as the example of the cylinder and and the plane shows, there are isometries between submanifolds which do not arise in this way (since the distances of points in $\mathbb{R}^n$ are not preserved). Now the formal definition of a intrinsic quantity is a quantity which is not only invariant under Euclidean motions but also under general isometries.

A fundamental example of an intrinsic quantity is the Gauß curvature for surfaces in E^3. This can be proved directly, but a conceptual approach to understanding this is more involved. This is based on the notion of the covariant derivative which (in view of the original definition of the covariant derivative very surprisingly) turns out to be intrinsic. Then the Gauß curvature for surfaces can be expressed (and is essentially equivalent to) the Riemann curvature, which in turn can be constructed from the covariant derivative and thus is intrinsic.

1.3. Tensor fields and abstract index notation. Let M be a smooth manifold. For a point $x \in M$ one has the tangent space T_xM. One then defines the cotangent space T_x^*M at x to be the dual vector space to the tangent space. A $\binom{\ell}{k}$-tensor field on M then assigns to each point $x \in M$ an element of the tensor product $T_xM \otimes \cdots \otimes T_xM \otimes T_x^*M \otimes \cdots \otimes T_x^*M$ with ℓ factors of the tangent space and k factors of the cotangent space, see Chapter 3 of [**AnaMF**]. The value at x can then be interpreted as a $(k+\ell)$-linear map $(T_xM)^k \times (T_x^*M)^\ell \to \mathbb{R}$, and the assignment should be smooth in the sense that inserting the values of k vector fields and ℓ smooth one-forms into these multilinear maps, one obtains a smooth function on M.

There are two basic point-wise operations with tensor fields, see Section 3.3 of [**AnaMF**]. On the one hand, given an $\binom{\ell}{k}$-tensor field s and a $\binom{\ell'}{k'}$-tensor field t, one can form the tensor product $s \otimes t$, which then is of type $\binom{\ell+\ell'}{k+k'}$. In the picture of multilinear maps, this just feeds the first arguments into the first map and the others into the second map and then multiplies the values. On the other hand, one can form the basic contraction or evaluation map $T_xM \otimes T_x^*M \to \mathbb{R}$, which maps $\xi \otimes \varphi$ to $\varphi(\xi)$. This then leads to a contraction C_r^s mapping $\binom{\ell}{k}$-tensor fields to $\binom{\ell-1}{k-1}$-tensor fields for each r and s with $1 \leq r \leq k$ and $1 \leq s \leq \ell$ specifying which factors in the tensor products should be contracted.

In this last bit it is already visible, that there is some need for notation, since one has to select one of the entries of each type. Abstract index notation as introduced by Roger Penrose offers this possibility. At the same time, this has the advantage that, while the notation makes sense without a choice of local coordinates (and hence there is no need to check that things do not depend on a choice of coordinates) an abstract index expression gives the expression in local coordinates after any such choice.

In abstract index notation, indices are used to indicate the type of tensor fields as well as contractions. A $\binom{\ell}{k}$-tensor field is denoted by some letter with ℓ upper indices and k lower indices. So ξ^i will be a vector field, φ_j a one-form, and A_b^a a $\binom{1}{1}$-tensor field. A tensor product is simply indicated by writing the tensor fields aside of each other, which allows keeping track of the indices. A contraction is indicated by using the same symbol for one upper and one lower index, these indices then are not "free" so they are not to be counted in determining the type. So for example for a $\binom{1}{1}$-tensor field A_b^a there is just one possible contraction which is denoted by A_a^a (or also by A_i^i) and this is a tensor field of type $\binom{0}{0}$, i.e. a smooth function. The space $T_xM \otimes T_x^*M$ can be identified both with $L(T_xM, T_xM)$ and with $L(T_x^*M, T_x^*M)$. Either of these identifications can be obtained by first forming the tensor product with the source space and then applying the unique possible contraction (and the resulting maps are dual to each other). The maps on vector fields and one-forms induced by A_b^a can be written as $A(\xi)^i = A_j^i\xi^j$ respectively as $A(\varphi)_b = A_b^a\varphi_a$. In this picture, the smooth function A_i^i corresponds to the point-wise trace of either of these maps.

Choosing a chart (U, u) for M with local coordinates u^i, one has the corresponding coordinate vector fields $\partial_i = \frac{\partial}{\partial u^i}$ and the dual one-forms du^i. Then one can represent tensor fields by their coefficient functions with respect to the induced bases. For example, a $\binom{1}{1}$-tensor field A can then on U be written as $\sum_{i,j} A_j^i \partial_i \otimes du^j$, and one often omits the sum using Einstein sum convention. Here the A_j^i are smooth functions for each i and j and interpreting A as a field of bilinear maps, one has $A_j^i = A(du^i, \partial_j)$. Given a vector field ξ, we may represent it on U as $\sum_j \xi^j \partial_j$. Therefore, the vector field $A(\ ,\xi)$ can be written as $\sum_j \xi^j A(\ ,\partial_j)$, which in turn is given by $\sum_{i,j} \xi^j A_j^i \partial_i$. Hence the vector field $A(\xi)$ really has coordinate functions $A_j^i\xi^j$ (using Einstein sum convention) and the abstract index expression also gives the expression in local coordinates.

A further ingredient in the calculus with tensor fields is that the identity map (on T_xM or on T_x^*M) defines a canonical element in $T_xM \otimes T_x^*M$. These elements of course fit together to define a canonical $\binom{1}{1}$-tensor field, which in abstract index notation is usually called δ^i_j. Interpreting this as the Kronecker-delta, we again get the coordinate expression in any local coordinate system.

The last important ingredient are symmetrizations and alternations. These can only affect several entries of the same type (vector-field or one-form entries) of a tensor field. Let us consider the simplest situation of a $\binom{0}{2}$-tensor field, whose values are bilinear forms on tangent spaces. If t is such a tensor field, then its symmetrization is defined by $s(\xi, \eta) = \frac{1}{2}(t(\xi, \eta) + t(\eta, \xi))$ while for the alternation, the second summand is subtracted rather than added. So the symmetrization of t can be written as $\frac{1}{2}(t_{ij} + t_{ji})$ and similarly for the alternation. If one has to symmetrize or alternate over more than two entries, one sums over all permutations of the entries, multiplies by the sign of the permutation in the case of the alternation, and divides by the number of permutations. Since this becomes a bit tedious to write out, one denotes a symmetrization over a group of indices by putting them into round brackets and an alternation by putting them into square brackets. The conventions here differ by the division by number of permutations from those used in Section 3.5 of [**AnaMF**]. They are chosen in such a way, that one can efficiently express the fact that a tensor is symmetric respectively alternating. For example a $\binom{0}{k}$-tensor field φ is a k-form if and only if $\varphi_{i_1\dots i_k} = \varphi_{[i_1\dots i_k]}$.

Basic definitions and consequences

1.4. Riemannian metrics and Riemannian manifolds. We will always assume that manifolds are smooth (C^∞) and paracompact, so that partitions of unity are available.

Definition 1.4. (1) A *pseudo-Riemannian metric* on a smooth manifold M is a $\binom{0}{2}$-tensor field g on M such that for each point $x \in M$, the value $g_x : T_xM \times T_xM \to \mathbb{R}$ is a non-degenerate symmetric bilinear form.

(2) A *Riemannian metric* is a pseudo-Riemannian metric such that for each $x \in M$ the value g_x is positive definite and hence defines an inner product on the vector space T_xM.

(3) A *(pseudo-) Riemannian manifold* (M, g) is a smooth manifold M together with a (pseudo-) Riemannian metric g on M.

For a pseudo-Riemannian metric g on M and a point $x \in M$, the bilinear form g_x has a well defined signature (p, q) with $p + q = n = \dim(M)$. By definition, p (respectively q) is the maximal dimension of a linear subspace of T_xM on which the restriction of g_x is positive (respectively negative) definite. From this, it easily follows that the signature is locally constant, and one usually assumes that it is constant on all of M.

The situation with pseudo-Riemannian metrics is a bit unfortunate. On the one hand, they are an interesting topic from a mathematical point of view and they have important applications. In particular, the geometry of pseudo-Riemannian metrics of signature $(1, 3)$ is a fundemental ingredient of general relativity. Moreover, large parts of Riemannian geometry, in particular the study of the Levi-Civita connection and its curvature, generalize to the pseudo-Riemannian case with only minimal changes. On the other hand, some of the fundamental and most intuitive facts about Riemannian metrics, in particular the relation to metrics in the topological sense, do not generalize. Therefore, it is difficult to treat Riemannian and pseudo-Riemannian metrics coherently at the same time, and unfortunately we'll have to focus on the Riemannian case. Still I will try to indicate which parts of the theory generalize without changes.

Proposition 1.4. *(1) For any smooth manifold M, there is a Riemannian metric g on M.*

(2) Let (M, g) be a Riemannian manifold, let (U, u) be a local chart on M. Viewed as a matrix, the local coordinate expression g_{ij} of the tensor field g is symmetric and positive definite and thus invertible. The point-wise inverse matrix defines a smooth $\binom{2}{0}$-tensor field g^{ij} on M such that $g^{ij}g_{jk} = \delta^i_k$.

(3) In the setting of (2) consider the smooth function $\mathrm{vol}_g := \sqrt{\det(g_{ij})}$ on U. Under a change of local coordinates, this function transforms by the absolute value of the determinant of the derivative of the change of coordinates. Hence for any compactly supported smooth function f on M, the product $f \mathrm{vol}_g$ can be integrated over M in a coordinate-independent way.

(4) Let (M, g) be a Riemannian manifold of dimension n. Then for each $x \in M$, there is an open neighborhood U of x in M and there are local vector fields $\xi_1, \dots, \xi_n \in \mathfrak{X}(U)$ such that for each $y \in U$, the vectors $\xi_1(y), \dots, \xi_n(y)$ form an orthonormal basis for T_yM.

Proof. Let (U, u) be a chart on a smooth manifold M. Then for a $\binom{0}{2}$-tensor field g on M, the coordinate expression of g is given by $g_{ij} = g(\partial_i, \partial_j)$. Hence g_x is symmetric if and only if the matrix $(g_{ij}(x))$ is symmetric and g_x is positive definite if and only if the matrix $(g_{ij}(x))$ is positive definite.

(1) The above argument shows that we can find a Riemannian metric on U, for example by taking g_{ij} to be the identity matrix. Now we can choose a covering (U_α, u_α) of M by coordinate charts and a sub-ordinate partition $\{\varphi_i : i \in \mathbb{N}\}$ of unity (see Theorem 1.9 of [**AnaMF**]). For each i, chose $\alpha(i)$ such that $\mathrm{supp}(\varphi_i) \subset U_{\alpha(i)}$, take a Riemannian metric g_i on $U_{\alpha(i)}$ and then put $g := \sum_i \varphi_i g_i$. It follows immediately that this is a symmetric $\binom{0}{2}$-tensor field. Moreover, for a point $x \in M$ and a tangent vector $0 \neq \xi \in T_xM$, we have $g(x)(\xi, \xi) = \sum_i \varphi_i(x) g_i(x)(\xi, \xi)$. Now by construction $g_i(x)(\xi, \xi) > 0$ for all i such that $x \in U_{\alpha(i)}$ and $\varphi_i(x) \geq 0$ for all i, so $0 \leq g(x)(\xi, \xi)$. Moreover, there is at least one i such that $\varphi_i(x) > 0$, which implies $x \in U_{\alpha(i)}$ and hence $g_i(x)(\xi, \xi) > 0$, so $g(x)(\xi, \xi) > 0$, and the proof of (1) is complete.

(2) From above, we know that $(g_{ij}(x))$ is a symmetric, positive definite matrix depending smoothly on x. Hence it is invertible in each point, and we can denote the inverse matrix, which is again symmetric, by $(g^{ij}(x))$. The components of the inverse of a matrix can be computed by determinants via Cramer's rule, so inversion of matrices is a smooth function, so also the g^{ij} depend smoothly on x. Hence $\sum_{ij} g^{ij} \frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial u^j}$ is a well defined $\binom{2}{0}$-tensor field on U. Of course, these tensor fields for different charts agree, thus defining a smooth tensor field on M. The abstract index expression $g^{ij}g_{jk} = \delta^i_k$ just expresses the fact that in local coordinates the matrices are inverse to each other.

(3) Suppose that $U \subset M$ is open and that u_α and u_β are diffeomorphisms from U onto open subsets of $\mathbb{R}^n$. Consider the chart change $u_{\alpha\beta} := u_\alpha \circ u_\beta^{-1} : u_\beta(U) \to u_\alpha(U)$ and its derivative $D(u_{\alpha\beta})$. Writing $D(u_{\alpha\beta})(u_\beta(x)) = A^i_j(x)$ for $x \in U$, we by definition obtain $\frac{\partial}{\partial u^j_\beta} = \sum_i A^i_j(x) \frac{\partial}{\partial u^i_\alpha}$. This implies that the coordinate expressions g^α_{ij} and g^β_{ij} are related by

$$g^\beta_{ij}(x) = \textstyle\sum_{k,\ell} A^k_i(x) A^\ell_j(x) g^\alpha_{k\ell}(x).$$

In terms of matrices, the right hand side can be written as the product with A and its transpose (which is exactly the behavior of the symmetric matrix associated to an inner product under a change of basis). This shows that $\det(g^\beta_{ij}(x)) = \det(A^i_j(x))^2 \det(g^\alpha_{ij}(x))$. Thus the square roots transform by $|\det(A^i_j(x))|$, which is exactly the behavior required for the integral of $f\,\mathrm{vol}_g$ being defined independently of coordinates, compare with Sections 4.1 and 4.2 of [**AnaMF**].

(4) This is the fact that the Gram–Schmidt orthonormalization scheme can be done depending smoothly on a point. Given x, we can find a neighborhood U of x in M and vector fields $\eta_1, \dots, \eta_n \in \mathfrak{X}(U)$ such that the vectors $\eta_1(y), \dots, \eta_n(y)$ form a basis for T_yM for each $y \in U$. (For example, we can use the coordinate vector fields associated to a chart.) Since η_1 is nowhere vanishing on U, $g(\eta_1, \eta_1)$ is a nowhere vanishing smooth function on U, so we can define $\xi_1 := \frac{1}{\sqrt{g(\eta_1,\eta_1)}}\eta_1$. Then by construction $\xi_1(y) \in T_yM$ is a unit vector for each $y \in U$. Next, we define $\tilde{\xi}_2 := \eta_2 - g(\eta_2, \xi_1)\xi_1$, which evidently is a smooth vector field on U such that $g(\tilde{\xi}_2, \xi_1) = 0$. By construction $\eta_2(y)$ and $\xi_1(y)$ are linearly independent for each y, so $\tilde{\xi}_2$ is nowhere vanishing. Thus we can define $\xi_2 := \frac{1}{\sqrt{g(\tilde{\xi}_2,\tilde{\xi}_2)}}\tilde{\xi}_2$, and this is a smooth vector field on u, such that $\xi_1(y)$ and $\xi_2(y)$ form an orthonormal system in T_yM for each $y \in U$. The other ξ_i are constructed similarly. □

Remark 1.4. (1) The simple trick used in the proof of part (1) to glue local Riemannian metrics using a partition of unity depends on the fact that positive definite inner products form a convex set. In fact, the corresponding statement for pseudo-Riemannian metrics is wrong! For example, there are topological obstructions against existence of a pseudo-Riemannian metric of signature $(n-1, 1)$ for even n.

(2) If the manifold M is oriented, then the result in (3) can be stated as the fact that the local coordinate expressions $\sqrt{\det(g_{ij}(x))}dx^1 \wedge \dots \wedge dx^n$ in positively oriented charts fit together and define a nowhere-vanishing differential form of top degree on M. This is called the *volume form* associated to the metric g. In the case of non-orientable manifolds, there is a notion of *densities*, which are the objects that can be integrated independently of coordinates, see Sections 4.1 and 4.2 of [**AnaMF**]. Hence vol_g is also referred to as the *volume density* associated to g. The main moral is that in the presence of a Riemannian metric, one obtains a well defined notion of integration over (compactly supported) smooth functions.

(3) A family $\{\xi_1, \dots, \xi_n\}$ as in part (4) of the Proposition is called a *local orthonormal frame* for M around x. Observe that for $\eta \in \mathfrak{X}(U)$, we get $\eta = \sum_i g(\eta, \xi_i)\xi_i$, so we can write any vector field on U as a linear combination of the ξ_i with smooth coefficients.

1.5. Immediate consequences. Given a Riemannian metric g on a manifold M, one can use the data constructed in Proposition 1.4 to obtain a large number of additional structures. On the level of individual tangent spaces, one may use the point-wise inner product as known from linear algebra, and usually the result will depend smoothly on the point. For example, one can look at the inner product of a tangent vector with itself and at its norm, i.e. at $g_x(X, X)$ respectively $\sqrt{g_x(X, X)}$. If $\xi \in \mathfrak{X}(M)$ is a vector field, then smoothness of the tensor field g implies that $g(\xi, \xi)$ is a smooth function. This function is non-zero unless ξ vanishes in a point. Hence also $\sqrt{g(\xi, \xi)}$ is smooth where ξ is non-zero.

Likewise, for two non-zero tangent vectors ξ and η in a point $x \in M$, one can characterize the angle α between ξ and η by the usual formula $\cos(\alpha) = \frac{g_x(\xi,\eta)}{\sqrt{g_x(\xi,\xi)}\sqrt{g_x(\eta,\eta)}}$. As before, for non-vanishing vector fields, the angle depends smoothly on the point. In particular, given two curves through a point x, one may define the angle between the two curves and, more specifically, one can talk about curves (and more general submanifolds) intersecting orthogonally in a point.

Integrating functions via the volume density vol_g, has several evident applications. From the definition of vol_g it follows that if $f : M \to \mathbb{R}$ is a compactly supported smooth function with non-negative values than $\int_M f \,\mathrm{vol}_g \geq 0$ and $\int_M f \,\mathrm{vol}_g = 0$ in only possible for $f = 0$. Hence one can make the space $C^\infty_c(M, \mathbb{R})$ of smooth functions with compact support into a pre-Hilbert space by defining $\langle f, h\rangle := \int_M fh \,\mathrm{vol}_g$. Hence one can provide the setup for functional analysis by looking at the completion of $C^\infty_c(M, \mathbb{R})$ with respect to the resulting norm, which is the space $L^2(M, \mathbb{R})$ of square integrable functions, and so on.

This can be immediately extended to the space $\mathfrak{X}_c(M)$ of compactly supported vector fields on M. Here one defines a pre-Hilbert structure by $\langle \xi, \eta\rangle := \int_M g(\xi, \eta) \,\mathrm{vol}_g$, or in abstract index notation $\int_M g_{ij}\xi^i\eta^j \,\mathrm{vol}_g$. Again, it is possible to complete this to the space of square-integrable vector fields. Next, we can take the inverse metric g^{ij} as constructed in Proposition 1.4. For each point x, this defines a positive definite inner product on the vector spaces T^*_xM which depends smoothly on the point x. In particular, for two compactly supported one-forms α and β, $g^{ij}\alpha_i\beta_j$ is a smooth function on M, and we can define $\langle \alpha, \beta\rangle := \int_M g^{ij}\alpha_i\beta_j \,\mathrm{vol}_g$. This makes the space $\Omega^1_c(M)$ of one-forms on M with compact support into a pre-Hilbert spaces, which can be completed to the space of square-integrable one-forms.

It is a matter of linear algebra to extend this further. Given inner products on two vector spaces, one obtains an induced inner product on their tensor product. Iterating this, g_x induces inner products on all the spaces $\otimes^k T^*_xM \otimes \otimes^\ell T_xM$ and likewise on the spaces $\Lambda^k T^*_xM$ of alternating k-linear maps $(T_xM)^k \to \mathbb{R}$. All these induced inner products can be characterized in the way that starting from an orthonormal basis of T_xM, also the induced basis of the space in question is orthonormal. Using part (4) of Proposition 1.4, one concludes that there are smooth local orthonormal frames for all these inner products, which implies that they depend smoothly on the point. Integrating point-wise inner products, one can make all spaces of tensor-fields and of differential forms with compact support into pre-Hilbert spaces.

Next, an inner product on a vector space induces an isomorphism with the dual space. Hence given a point x in a Riemannian manifold (M, g) and a tangent vector $\xi \in T_xM$, we obtain a linear functional $T_xM \to \mathbb{R}$ by $\eta \mapsto g_x(\xi, \eta)$. Starting from a vector field $\xi \in \mathfrak{X}(M)$ we can associate to each $x \in M$ the functional $g_x(\xi(x), \_)$. Inserting the values of a smooth vector field η, we obtain the smooth function $g(\xi, \eta)$, so this defines a one-form on M. In abstract index notation, the resulting linear map $\mathfrak{X}(M) \to \Omega^1(M)$ is given by $\xi \mapsto g_{ij}\xi^j$. Similarly, $\alpha \mapsto g^{ij}\alpha_j$ defines a map $\mathfrak{X}(M) \to \Omega^1(M)$, which is inverse to the other one. Thus the metric g induces an isomorphism between vector fields and one-forms.

This readily generalizes to tensor fields of arbitrary type. In view of abstract index notation this is often phrased as "raising and lowering indices using the metric" (and its inverse). For example, given a $\binom{1}{1}$-tensor field $A = A^i_j$, we can use the metric to lower the upper index and form the $\binom{0}{2}$-tensor field $A^k_j g_{ik}$. This corresponds to the bilinear form $(\xi, \eta) \mapsto g(\xi, A(\eta))$. This bilinear form can be decomposed into a symmetric and a skew symmetric part as $A^k_{(j} g_{i)k} + A^k_{[j} g_{i]k}$. One can then convert these parts back to $\binom{1}{1}$-tensor fields to obtain a decomposition of A itself. For example, for the symmetric part, this reads as

$$\tfrac{1}{2} g^{i\ell}(A^k_\ell g_{jk} + A^k_j g_{\ell k}) = \tfrac{1}{2}(g^{i\ell}A^k_\ell g_{jk} + A^k_j \delta^i_k) = \tfrac{1}{2}(A^i_j + g^{ik}A^\ell_k g_{\ell j}).$$

To interpret this result, observe that the for the linear map $B^i_j := g^{ik}A^\ell_k g_{\ell j}$ we can write $g(B(\xi),\eta)$ as

$$g_{aj}B^a_i\xi^i\eta^j = g_{aj}g^{ab}A^c_b g_{ci}\xi^i\eta^j = \delta^b_j A^c_b g_{ci}\xi^i\eta^j = A^c_j g_{ci}\xi^i\eta^j,$$

so by symmetry of g, this coincides with $g(\xi, A(\eta))$. Hence $B_x : T_xM \to T_xM$ is simply the adjoint of A_x with respect to the inner product g_x, and so we have just applied the usual formula for the symmetric part from linear algebra in each point.

1.6. Hodge-$*$ operator, codifferential, and Laplacian. Let us discuss a more complicated but very important construction based on the ideas from Section 1.5. Let (M, g) be an oriented Riemannian manifold of dimension n. Then we can view the volume form vol_g as a nowhere vanishing element of $\Omega^n(M)$, thus identifying for each point $x \in M$ the space $\Lambda^n T^*_x M$ with $\mathbb{R}$. For each point $x \in M$ and each $k = 0, \dots, n$, the wedge product defines a bilinear map $\Lambda^k T^*_x M \times \Lambda^{n-k} T^*_x M \to \Lambda^n T^*_x M$. Linear algebra tells us that this gives rise to a linear isomorphism $\Lambda^{n-k} T^*_x M \to L(\Lambda^k T^*_x M, \Lambda^n T^*_x M)$. Using $\mathrm{vol}_g(x)$ to identify $\Lambda^n T^*_x M$ with $\mathbb{R}$, we can identify the target space with the dual space $(\Lambda^k T^*_x M)^*$. But from above, we know that g_x induces an inner product $\tilde{g}_x$ on $\Lambda^k T^*_x M$ which gives an identification of the dual space with $\Lambda^k T^*_x M$ itself. Otherwise put, for each $\psi \in \Lambda^k T^*_x M$, there is a unique element $*\psi \in \Lambda^{n-k} T^*_x M$ such that for each $\varphi \in \Lambda^k T^*_x M$ we have $\varphi \wedge *\psi = \tilde{g}_x(\varphi, \psi)\,\mathrm{vol}_g(x)$.

PROPOSITION 1.6. *Let (M, g) be a oriented Riemannian manifold of dimension n.*

(1) For each $k = 0, \dots, n$, the point-wise $*$-operation defined above gives rise to a linear isomorphism $* : \Omega^k(M) \to \Omega^{n-k}(M)$ which is characterized by $\alpha \wedge *\beta = \tilde{g}(\alpha, \beta)\,\mathrm{vol}_g$ for all $\alpha, \beta \in \Omega^k(M)$. Moreover, for any $\beta \in \Omega^k(M)$, we get $*(*\beta) = (-1)^{k(n-k)}\beta$.

(2) Let d be the exterior derivative and define $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ as $\delta\beta := (-1)^{nk+n+1} * d * \beta$. Then this satisfies $\delta^2 = \delta \circ \delta = 0$. If M is compact, then δ is adjoint to d with respect to the L^2-inner products on the spaces $\Omega^*(M)$ introduced in 1.5.

(3) Suppose that M is compact. Then the operator $\Delta := \delta d + d\delta : \Omega^k(M) \to \Omega^k(M)$ is self-adjoint with respect to the L^2-inner product from 1.5. Moreover, for $\alpha \in \Omega^k(M)$, we get $\Delta(\alpha) = 0$ if and only if $d\alpha = 0$ and $\delta\alpha = 0$, while for $\beta \in \Omega^{k-1}(M)$, $\Delta(d\beta) = 0$ implies $d\beta = 0$.

PROOF. (1) We first have to show that for a smooth k-form $\beta \in \Omega^k(M)$ the point-wise definition of $*\beta$ gives rise to a smooth form. This is a local question, so we can restrict to an open subset U for which there is a positively oriented local orthonormal frame $\xi_1, \dots, \xi_n$, see Proposition 1.4. Then we define $\alpha^1, \dots, \alpha^n \in \Omega^1(U)$ to be the dual forms, i.e. $\alpha^i(\xi_j) = \delta^i_j$ for all i, j. Then for each $x \in U$ the values $(\alpha^{i_1} \wedge \dots \wedge \alpha^{i_k})(x)$ with $1 \le i_1 < \dots < i_k \le n$ form an orthonormal basis for $\Lambda^k T^*_x M$. Moreover, it is easy to see that $\alpha^1 \wedge \dots \wedge \alpha^n = \mathrm{vol}_g$ on U. But this implies that among the basis elements $(\alpha^{j_1} \wedge \dots \wedge \alpha^{j_{n-k}})(x)$ for $\Lambda^{n-k} T^*_x M$, there is a unique one, for which the wedge product with $(\alpha^{i_1} \wedge \dots \wedge \alpha^{i_k})(x)$ coincides with $\pm\mathrm{vol}_g(x)$, while all other wedge-products are zero. But this exactly means that, with a sign that is independent of x, we have

$$*(\alpha^{i_1} \wedge \dots \wedge \alpha^{i_k})(x) = \pm(\alpha^{j_1} \wedge \dots \wedge \alpha^{j_{n-k}})(x)$$

where $\{j_1, \dots, j_{n-k}\}$ is the complement of $\{i_1, \dots, i_k\}$ in $\{1, \dots, n\}$. Hence for each of the forms $\alpha^{i_1} \wedge \dots \wedge \alpha^{i_k}$ the point-wise $*$ defines a smooth $(n-k)$-form. Since any k form can be written as a linear combination of these with smooth coefficients and $*$ is evidently linear, we conclude that $*\beta$ is smooth for each $\beta \in \Omega^k(M)$.

To prove the second part of (1), we observe that in the defining equation $\alpha \wedge *\beta = \tilde{g}(\alpha, \beta) \operatorname{vol}_g$, the right hand side is symmetric in α and β. Thus we see that $\alpha \wedge *\beta = \beta \wedge *\alpha = (-1)^{k(n-k)} *\alpha \wedge \beta$ for all $\alpha, \beta \in \Omega^k(M)$. Next, for $\alpha \in \Omega^k(M)$ and $\gamma \in \Omega^{n-k}(M)$, we compute

$$\tilde{g}(\alpha, *\gamma) \operatorname{vol}_g = \alpha \wedge * * \gamma = (-1)^{(n-k)k} * \alpha \wedge *\gamma = (-1)^{(n-k)k} \tilde{g}(*\alpha, \gamma) \operatorname{vol}_g .$$

Applying this to $\gamma = *\beta$ for $\beta \in \Omega^k$, we get $\tilde{g}(\alpha, * * \beta) = (-1)^{k(n-k)} \tilde{g}(*\alpha, *\beta)$. But above we have seen that $*$ maps an orthonormal system in $\Lambda^k T^*_x M$ to an orthonormal system in $\Lambda^{n-k} T^*_x M$. Hence it is orthogonal, so in particular $\tilde{g}(*\alpha, *\beta) = \tilde{g}(\alpha, \beta)$, and this implies that last statement in (1).

(2) Up to a sign, $\delta\delta\beta$ equals $*d * *d * \beta$ and since the two middle $*$'s also produce a sign only, $d^2 = 0$ implies $\delta^2 = 0$. On the other hand, observe that the sign in the definition of δ is chosen in such a way that for $\beta \in \Omega^{k+1}(M)$ we have $*\delta\beta = (-1)^{k+1} d*\beta$. Now taking $\alpha \in \Omega^k(M)$, we can form $\alpha \wedge *\beta \in \Omega^{n-1}(M)$ and by Stokes' theorem, we get

$$0 = \int_M d(\alpha \wedge *\beta) = \int_M d\alpha \wedge *\beta + (-1)^k \int_M \alpha \wedge d*\beta = \int_M \tilde{g}(d\alpha, \beta) \operatorname{vol}_g - \int_M \tilde{g}(\alpha, \delta\beta) \operatorname{vol}_g .$$

By definition of the L^2-inner product from 1.5, this simply equals $\langle d\alpha, \beta \rangle - \langle \alpha, \delta\beta \rangle$ and adjointness follows.

(3) This is now a simple direct computation. For $\alpha, \beta \in \Omega^k(M)$, we get using the adjointness from (2):

$$\langle \Delta(\alpha), \beta \rangle = \langle \delta d\alpha, \beta \rangle + \langle d\delta\alpha, \beta \rangle = \langle d\alpha, d\beta \rangle + \langle \delta\alpha, \delta\beta \rangle,$$

and in the same way, one shows that this equals $\langle \alpha, \Delta(\beta) \rangle$. If $\Delta(\alpha) = 0$, then $0 = \langle \Delta(\alpha), \alpha \rangle$ and the above computation shows that $0 = \langle d\alpha, d\alpha \rangle + \langle \delta\alpha, \delta\alpha \rangle$. Since $\langle\ ,\ \rangle$ is a positive definite inner product, this implies $d\alpha = 0$ and $\delta\alpha = 0$.

Applying this to $\alpha = d\beta$, we see that $\Delta(d\beta) = 0$ implies $\delta d\beta = 0$. But this gives $0 = \langle \delta d\beta, \beta \rangle = \langle d\beta, d\beta \rangle$ and hence $d\beta = 0$. □

Definition 1.6. (1) The operator $*$ is called the *Hodge-$*$-operator* associated to the Riemannian metric g.

(2) The operator δ is called the *codifferential* associated to g.

(3) The operator Δ is called the *Laplace–Beltrami operator* associated to g.

Remark 1.6. (1) For the basic adjointness results in part (2) and (3), compactness of M is not really necessary. In general, one may consider both d and δ as operators on differential forms with compact support and then adjointness is still true.

(2) The Laplace–Beltrami operator is of fundamental importance in large areas of differential geometry and of analysis. Differential forms in the kernel of Δ are called *harmonic forms*. In the case of a compact manifold, Δ extends to an essentially self adjoint operator on L^2-forms, so one can do spectral theory and so on. One can also look at the analog of the heat equation on a compact Riemannian manifold, which is of fundamental importance in geometric analysis.

(3) The last part of Proposition 1.6 is the starting point for Hodge theory on compact Riemannian manifolds. As we have proved, for a harmonic k-form α we get $d\alpha = 0$, so one may look at the class of α in the de-Rham cohomology group $H^k(M)$, which by definition is the quotient of the kernel of $d : \Omega^k(M) \to \Omega^{k+1}(M)$ by the image of $d : \Omega^{k-1}(M) \to \Omega^k(M)$. The last statement in the proposition then shows that this maps the space of harmonic k-forms injectively to $H^k(M)$. Using a bit of functional

analysis, one proves that this map is also surjective and thus a linear isomorphism. Hence any cohomology class contains a unique harmonic representative.

1.7. Arclength and the distance function. The next direct way to use a Riemannian metric is related to arclength of curves.

DEFINITION 1.7. Let (M, g) be a Riemannian manifold and let $c : [a, b] \to M$ be a smooth curve defined on a compact interval in $\mathbb{R}$.

Then we define the *arclength* $L(c)$ and the *energy* $E(c)$ of c by

$$L(c) := \int_a^b \sqrt{g_{c(t)}(c'(t), c'(t))}dt$$
$$E(c) := \tfrac{1}{2}\int_a^b g_{c(t)}(c'(t), c'(t))dt.$$

Of course, the factor $\frac{1}{2}$ in the definition of the energy is just a matter of convention. It is motivated by the definition of kinetic energy in physics. There is an obvious concept of reparametrization of a smooth curve, in which one replaces c by $c \circ \varphi$ for a diffeomorphism φ. As we shall see below, the arclength of a curve remains unchanged if the curve is reparametrized. For some applications, this is an advantage, but for other purposes, like for finding distinguished curves, it is a disadvantage and it is better to use the energy.

For technical purposes, it is better to work with curves which are only piece-wise smooth. Here by a *piece-wise smooth curve* $c : [a, b] \to M$ we mean a continuous curve $c : [a, b] \to M$ such that there is a subdivision $a = t_0 < t_1 < \dots < t_{N-1} < t_N = b$ of $[a, b]$ such that for each $i = 0, \dots, N-1$ the restriction of c to $[t_i, t_{i+1}]$ is smooth. Putting $c_i := c|_{[t_i,t_{i+1}]}$ one then defines $L(c) = \sum_{i=0}^{N-1} L(c_i)$ and $E(c) = \sum_{i=0}^{N-1} E(c_i)$. One immediately verifies that this is well defined (i.e. there is no problem with adding additional points to the sub-division around which c is smooth anyway).

PROPOSITION 1.7. *(1) The arclength of smooth curves is invariant under orientation preserving reparametrizations, i.e. if $c : [a, b] \to M$ is a smooth curve and $\varphi : [a', b'] \to [a, b]$ is a diffeomorphism with $\varphi'(t) > 0$ for all t, then $L(c \circ \varphi) = L(c)$.*

(2) For points x, y in a connected Riemannian manifold M define $d_g(x, y)$ as the infimum of the arclengths $L(c)$ of piece-wise smooth curves $c : [a, b] \to M$ with $c(a) = x$ and $c(b) = y$. Then (M, d_g) is a metric space and the topology induced by the metric d_g coincides with the manifold topology on M.

PROOF. (1) This is the same computation as in Euclidean space. By the chain rule, we have $(c \circ \varphi)'(t) = c'(\varphi(t)) \cdot \varphi'(t)$ and thus

$$\sqrt{g((c\circ\varphi)(t))((c\circ\varphi)'(t), (c\circ\varphi)'(t))} = |\varphi'(t)|\sqrt{g(c(\varphi(t)))(c'(\varphi(t)), c'(\varphi(t)))}.$$

By assumption, $\varphi'(t) > 0$, so we may leave out the absolute value and the result follows by the substitution rule for one-dimensional integrals.

(2) If $c : [a, b] \to M$ is a smooth curve, then the function in the integral defining $L(c)$ is continuous and non-negative. Hence $L(c) \geq 0$ and $L(c) = 0$ if and only if the integrand is identically zero and hence c is constant. Since M is assumed to be connected, any two points in M can be connected by at least one piece-wise smooth curve and hence $d_g : M \times M \to \mathbb{R}_{\geq 0}$ is well defined. The fact that $d_g(x, y) = d_g(y, x)$ follows easily since one can run through curves in the opposite direction. The triangle inequality $d_g(x, z) \leq d_g(x, y) + d_g(y, z)$ follows since having given a curve c connecting

x to y and a curve $\tilde{c}$ connecting y to z, one can simply run through them successively to obtain a curve of length $L(c) + L(\tilde{c})$ which connects x to z.

Let us next consider the special case $M = \mathbb{R}^n$, endowed with an arbitrary Riemannian metric g. We compare d_g to the Euclidean distance focusing on (a neighborhood of) the point $0 \in \mathbb{R}^n$. Now $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, and we consider the map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $(x, v) \mapsto \sqrt{g(x)(v, v)}$. This map is clearly continuous and positive unless $v = 0$. Looking at the compact set $\bar{B}_1(0) \times S^{n-1}$ we thus see that there are constants $0 < C_1 < C_2$ such that $C_1 \leq \sqrt{g(x)(v, v)} \leq C_2$ provided that $\|x\| \leq 1$ and $\|v\| = 1$. This in turn implies that for $\|x\| \leq 1$ we have

$$C_1\|v\| \leq \sqrt{g(x)(v, v)} \leq C_2\|v\|$$

Hence for a piece-wise smooth curve c whose image is contained in the closed unit ball, the arclength $L^g(c)$ with respect to g and the Euclidean arclength $L^E(c)$ are related by $C_1L^E(c) \leq L^g(c) \leq C_2L^E(c)$. In particular for $0 < \epsilon < 1$ and $x \in B_\epsilon(0)$, the straight line provides a curve of length $< \epsilon C_2$ connecting 0 to x, so the $B_\epsilon(0)$ is contained in the d_g-ball around 0 of radius ϵC_2.

Conversely, suppose we have given $0 < \epsilon < 1/C_1$ and a curve $c : [a, b] \to \mathbb{R}^n$ with $c(a) = 0$ and $L(c) < \epsilon$. Then we first prove that c cannot leave the unit ball. Indeed, if c leaves the unit ball, we let $t_0 \in [a, b]$ be the infimum of $\{t : \|c(t)\| \geq 1\}$ and look at the curve $\tilde{c} := c|_{[a,t_0]}$. Then $\tilde{c}$ stays inside the closed unit ball and satisfies $L^g(\tilde{c}) < 1/C_1$ and hence $L^E(\tilde{c}) < 1$, which is a contradiction. Hence we conclude that $L^E(c) < \epsilon C_1$ and hence $c(b) \in B_{\epsilon C_1}(0)$. Hence $B_{\epsilon C_1}(0)$ contains the d_g-ball of radius ϵ around 0.

Now returning to a general Riemannian manifold (M, g) and a point $x \in M$, we can choose a chart (U, u) for M with $x \in U$, $u(x) = 0$, and $u(U) = \mathbb{R}^n$. Then u is a homeomorphism, and we can pull back $g|_U$ by u^{-1} to a Riemannian metric on $\mathbb{R}^n$. Of course, for a curve c with values in U, the arclength of c with respect to g coincides with the arclength of $u \circ c$ with respect to the pullback metric. Now from above we conclude that there is an $\epsilon > 0$ such that curves of length $\leq \epsilon$ stay in U. Hence if $y \in M$ is such that $d_g(x, y) = 0$ then $y \in U$. But then the above considerations show that $u(y)$ has Euclidean distance zero to $0 = u(x)$ and hence $y = x$. Hence (M, d_g) is a metric space, and the above argument shows that any Riemannian metric on $\mathbb{R}^n$ produces the usual neighborhoods of $0 \in \mathbb{R}^n$. Since u is a homeomorphism, we see that d_g leads to the usual neighborhoods of x, which completes the proof. □

REMARK 1.7. (1) One may now go ahead as in the Euclidean case, and consider regular parametrizations. For any regularly parametrized curve, one can then obtain a reparametrization by arclength (as usual by solving an ODE). This means the $g(c(t))(c'(t), c'(t)) = 1$ and hence $t = L(c|_{[a,t]})$ for all $t \in [a, b]$.

(2) The relation to metrics in the topological sense is the main point where things go wrong for pseudo-Riemannian metrics. The notion of energy still makes sense in the pseudo-Riemannian setting, but the energy of a non-trivial curve can be zero or negative. (In physical applications, this is a feature, since it allows to distinguish space-like, time-like, and light-like curves.) In particular, there is no direct relation to metric spaces in the pseudo-Riemannian case. Still, energy and arclength and constructions analogous to metric spaces are important tools there.

The Levi-Civita connection

After we have exploited the tensorial operations arising from a Riemannian metric on a smooth manifold, we will next construct and study the fundamental family of "new" differential operators available in the presence of such a metric. While the motivation of this concept from submanifold geometry is not very difficult, things are constructed on an abstract Riemannian manifold in different order.

1.8. Motivation. There are various concepts in submanifold geometry that are related to the covariant derivative. The most intuitive among these probably is the notion of a geodesic: The simplest non-trivial curves in E^n are the affine lines $t \mapsto x+tv$ with $x \in E^n$, $v \in \mathbb{R}^n$ and $t \in \mathbb{R}$. For a general curve $t \mapsto c(t)$, one can view the derivative c' as a map to $\mathbb{R}^n$, so it is no problem to form the second derivative c'', which again is an $\mathbb{R}^n$-valued function. The affine lines in E^n are exactly the curves for which c' is constant or equivalently $c'' = 0$. Now if $M \subset E^n$ is a smooth submanifold, then in general M will not contain any pieces of affine lines. However, there is a nice class of curves in M, which can be thought of as the paths of particles which move freely in M. Namely, for a smooth curve $c : I \to M$, one requires that for each $t \in I$, the second derivative $c''(t)$ is perpendicular to the tangent space $T_{c(t)}M \subset \mathbb{R}^n$. Intuitively, this means that acceleration is only there to keep the curve on the submanifold. These curves are the geodesics of M, and one shows that, given $x \in M$ and $X \in T_xM$, there locally is a unique geodesic $c : I \to M$ with $c(0) = x$ and $c'(0) = X$.

As a slight variation, one can consider the concept of parallel transport. In E^n one can transport a tangent vector $X \in T_xE^n = \mathbb{R}^n$ parallely to all of E^n by looking at the vector field corresponding to the constant function X. To be applicable to submanifolds, one has to modify this concept by only looking at it along a curve. Namely, for a curve $c : I \to E^n$, a vector field along c is a smooth function $X : I \to \mathbb{R}^n$, which we view as associating to t a tangent vector in the point $c(t)$. Then one can simply say that X is parallel along c if the function X is constant. Now this concept can be adapted to a smooth submanifold $M \subset E^n$. Given a smooth curve $c : I \to M$, one defines a *vector field along* c as a smooth map $X : I \to \mathbb{R}^n$ such that $X(t) \in T_{c(t)}M$ for all $t \in \mathbb{R}$. Then one says that X *is parallel along* c if for each $t \in I$ the derivative $X'(t)$ is perpendicular to $T_{c(t)}M$. In local coordinates, this amounts to a system of linear first order ODE. Hence any tangent vector can be transported parallely along a curve, i.e. it can be extended uniquely to a vector field which is parallel along the curve.

Observe that a curve c is a geodesic if and only if $c'(t)$ (which evidently defines a vector field along c) is parallel along c. In this sense, parallel transport is easier to deal with than geodesics are. Simple examples of surfaces in E^3 show that the concept of parallel transport only makes sense along curves. Take the unit sphere S^2 and a tangent vector $X \neq 0$ at the north pole. Then take the great circle in S^2 obtained by intersecting the sphere with the plane orthogonal to X. Then along this great circle the constant vector field on E^3 corresponding to X is tangent to S^2, so it must be parallel along the curve. So transporting X parallely to the south pole along this curve, one obtains X. In contrast to this, if one takes the great circle emanating from the north pole in direction X and transports X parallely along this to the south pole, one obtains $-X$! This is another way to see that the sphere is (intrinsically) curved.

The last step is to absorb these ideas into the definition of the covariant derivative, an analog of a directional derivative for vector fields. Suppose that $M \subset E^n$ is a submanifold and $\eta \in \mathfrak{X}(M)$ is a vector field, which we can view as a smooth function $\eta : M \to \mathbb{R}^n$ such that $\eta(x) \in T_xM \subset \mathbb{R}^n$ for all $x \in M$. Then given a point $x \in M$ and a tangent vector $X \in T_xM$, one forms $X \cdot \eta \in \mathbb{R}^n$ (the directional derivative of the function η in direction X) and projects the result orthogonally into T_xM to obtain an element $\nabla_X\eta(x) \in T_xM$. This depends smoothly on the point in the sense that for

$\xi, \eta \in \mathfrak{X}(M)$, one obtains a smooth vector field $\nabla_\xi \eta$ in this way. There are two crucial properties of this operation. On the one hand, taking $\eta, \zeta \in \mathfrak{X}(M)$ and their point-wise inner product, one gets

$$\xi \cdot \langle \eta, \zeta \rangle = \langle \xi \cdot \eta, \zeta \rangle + \langle \eta, \xi \cdot \zeta \rangle.$$

Since ζ and η lie in the tangent spaces to M, the inner products in the right hand side remain unchanged if one replaces $\xi \cdot \eta$ by $\nabla_\xi \eta$ and $\xi \cdot \zeta$ by $\nabla_\xi \zeta$. Hence we see that ∇ satisfies a Leibniz rule with respect to the first fundamental form.

On the other hand, consider the skew-symmetrization $\nabla_\xi \eta - \nabla_\eta \xi$ of the operation. This can be computed as the orthogonal projection of $\xi \cdot \eta - \eta \cdot \xi$ to the tangent spaces of M. However, it is well known that $\xi \cdot \eta - \eta \cdot \xi = [\xi, \eta]$, the Lie bracket, which is automatically contained in the tangent space. Hence $\nabla_\xi \eta - \nabla_\eta \xi = [\xi, \eta]$, which is referred to as *torsion-freeness* of the covariant derivative. Having the covariant derivative at hand, the fact that a vector field ξ is parallel along c can be written as $0 = \nabla_{c'(t)} \xi$ for all t. (One has to check that this also makes sense for vector fields along c.) So one can recover the more intuitive concepts discussed above.

1.9. Existence and uniqueness of the Levi-Civita connection. It turns out that it is easiest to generalize the covariant derivative to Riemannian manifolds and then derive the other concepts as consequences.

DEFINITION 1.9. Let M be a smooth manifold.

(1) A *linear connection* on TM is an operator $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, which is bilinear over $\mathbb{R}$ and satisfies

$$\nabla_{f\xi} \eta = f \nabla_\xi \eta \qquad \nabla_\xi (f\eta) = (\xi \cdot f)\eta + f \nabla_\xi \eta$$

for all $\xi, \eta \in \mathfrak{X}(M)$ and all $f \in C^\infty(M, \mathbb{R})$.

(2) If ∇ is a linear connection on TM, then the *torsion* of ∇ is the bilinear map $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$T(\xi, \eta) := \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta].$$

The connection ∇ is called *torsion-free* if and only if its torsion vanishes identically.

(3) A linear connection ∇ on TM is said to be *metric* with respect to a Riemannian metric g on M if and only if

$$\xi \cdot g(\eta, \zeta) = g(\nabla_\xi \eta, \zeta) + g(\eta, \nabla_\xi \zeta)$$

for all $\xi, \eta, \zeta \in \mathfrak{X}(M)$.

While this is not really needed for our purposes, observe that the torsion of any linear connection actually defines a $\binom{1}{2}$-tensor field on M. To see this, we just have to prove that $T(\xi, \eta)$ is bilinear over smooth functions. Now if we replace η by $f\eta$ for $f \in C^\infty(M, \mathbb{R})$, then $\nabla_\xi (f\eta) = (\xi \cdot f)\eta + f \nabla_\xi \eta$ and $\nabla_{f\eta} \xi = f \nabla_\eta \xi$ by definition of a linear connection. On the other hand, it is well known that $[\xi, f\eta] = (\xi \cdot f)\eta + f[\xi, \eta]$, which shows that $T(\xi, f\eta) = fT(\xi, \eta)$. Since $T(\eta, \xi) = -T(\xi, \eta)$ is evident, we see that T indeed is a tensor field. This is why the torsion is an important concept.

One of the most fundamental results of Riemannian geometry is the following

THEOREM 1.9. *Let (M, g) be a Riemannian manifold. Then there is a unique torsion-free linear connection on TM, which is metric for g.*

We discuss two proofs for this result, which very different in nature. While the first proof is entirely global, it is slightly mysterious why it works. The second proof requires some local input, but is makes the algebraic background clear. In both proofs we leave some straightforward verifications to the reader.

First proof. The global proof is based on the fact that, assuming the existence of a torsion free metric linear connection, one can derive a formula for it via a nice trick. Take three vector fields ξ, η, and ζ on M. Write out the definition of being metric three times with the vector fields cyclically permuted and taking the negative in one case, we get

$$\begin{aligned}0 &= \xi \cdot g(\eta, \zeta) - g(\nabla_\xi \eta, \zeta) - g(\eta, \nabla_\xi \zeta)\\ 0 &= \eta \cdot g(\zeta, \xi) - g(\nabla_\eta \zeta, \xi) - g(\zeta, \nabla_\eta \xi)\\ 0 &= -\zeta \cdot g(\xi, \eta) + g(\nabla_\zeta \xi, \eta) + g(\xi, \nabla_\zeta \eta).\end{aligned}$$

Adding up these three lines, we of course get zero. We can always exchange arguments in g and then use bilinearity. Via torsion freeness, we can replace $-\nabla_\xi \zeta + \nabla_\zeta \xi$ by $-[\xi, \zeta]$, $-\nabla_\eta \zeta + \nabla_\zeta \eta$ by $-[\eta, \zeta]$, and $-\nabla_\xi \eta - \nabla_\eta \xi$ by $-2\nabla_\xi \eta + [\xi, \eta]$. Bringing the term involving $\nabla_\xi \eta$ to the other side, we arrive at the so-called *Koszul formula*, which expresses $2g(\nabla_\xi \eta, \zeta)$ as

$$\xi \cdot g(\eta, \zeta) + \eta \cdot g(\zeta, \xi) - \zeta \cdot g(\xi, \eta) + g([\xi, \eta], \zeta) - g([\xi, \zeta], \eta) - g([\eta, \zeta], \xi). \tag{1.1}$$

Observe that in the right hand side, only the Lie bracket and the action of vector fields on smooth functions is used. If we have a torsion free metric connection ∇, then this formula allows us to compute, for each $x \in M$, the value $g_x(\nabla_\xi \eta(x), \zeta(x))$. For fixed ξ and η, we can of course realize any element of T_xM as $\zeta(x)$ for an appropriate vector field ζ. Hence $\nabla_\xi \eta(x)$ is uniquely determined by these values, and since this can be done in each point, $\nabla_\xi \eta$ is uniquely determined. Since this works for arbitrary vector fields, the uniqueness part of the theorem follows.

To prove existence, we show that the formula (1.1) can be used to *define* a linear connection ∇. Let us first fix two vector fields ξ and η, and view (1.1) as defining an operator that sends a vector field ζ to a smooth function. This map is linear and one verifies directly that it is even linear over smooth functions, so we have actually defined a one-form on M. From Section 1.5 we know that this can be expressed as $g(\varphi, \zeta)$ for a uniquely determined vector field $\varphi \in \mathfrak{X}(M)$, and we define $\nabla_\xi \eta := \frac{1}{2}\varphi$.

Doing this for all vector fields ξ and η, we obtain an operator $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, which is bilinear since (1.1) is evidently linear in ξ and in η. Next, one verifies that (1.1) is linear over smooth functions in ξ. This means that

$$g(\nabla_{f\xi} \eta, \zeta) = f g(\nabla_\xi \eta, \zeta) = g(f \nabla_\xi \eta, \zeta).$$

As above, this shows that ∇ is linear over smooth functions in the first argument. On the other hand, replacing η by $f\eta$ in (1.1), one obtains the product of (1.1) by f plus $2(\xi \cdot f)g(\eta, \zeta)$. Bringing the function into the metric, we conclude that $\nabla_\xi (f\eta) = (\xi \cdot f)\eta + f\nabla_\xi \eta$. Hence ∇ defines a linear connection on TM.

To prove torsion-freeness, we observe that the first two summands, the last two summands and the third summand in (1.1) are symmetric in ξ and η. Hence we obtain

$$2g(\nabla_\xi \eta - \nabla_\eta \xi, \zeta) = g([\xi, \eta], \zeta) - g([\eta, \xi], \zeta),$$

and torsion freeness follows. On the other hand, the second and third summand, the fourth and fifth summand, and the last summand in (1.1) are skew symmetric in η and ζ. Thus we obtain

$$2(g(\nabla_\xi \eta, \zeta) + g(\eta, \nabla_\xi \zeta)) = 2\xi \cdot g(\eta, \zeta),$$

so ∇ is metric. □

Second Proof. The second proof starts by showing that for any smooth manifold M, there exist linear connections on TM and the space of all such connections can be nicely described. First, in the domain of a chart (U,u), one can represent vector fields as $\xi=\sum_i \xi^i\partial_i$ and $\eta=\sum_j \eta^j\partial_j$ and then define

$$\nabla_\xi\eta := \textstyle\sum_j(\sum_i \xi^i\frac{\partial}{\partial u^i}(\eta^j))\partial_j.$$

One immediately verifies that this defines a linear connection on TU. Now take an atlas $\{(U_\alpha,u_\alpha):\alpha\in I\}$ for M and a subordinate partition $\{\varphi_i : i\in\mathbb{N}\}$ of unity. For each i choose $\alpha(i)$ such that $\operatorname{supp}(\varphi_i)\subset U_{\alpha(i)}$ and consider a linear connection ∇^i on $TU_{\alpha(i)}$ as constructed above. Now taking $\xi,\eta\in\mathfrak{X}(M)$, $\nabla^i_{\varphi_i\xi}\eta$ is defined on $U_{\alpha(i)}$ and vanishes identically outside of the support of φ_i, so it can be extended by zero to a smooth vector field on M. Thus given ξ and η, $\nabla_\xi\eta := \sum_i \nabla^i_{\varphi_i\xi}\eta$ defines a smooth vector field on M. One immediately verifies that this defines a linear connection on TM.

Given two linear connections ∇ and $\hat\nabla$ on TM, one considers their difference, i.e. the map $\Phi:\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ defined by $\Phi(\xi,\eta)=\hat\nabla_\xi\eta-\nabla_\xi\eta$. This expression is bilinear and clearly linear over smooth functions in ξ. But since both connections satisfy the same compatibility condition with respect to multiplication of η by smooth functions, their difference is linear over smooth functions in η, too. Thus, Φ is a smooth $\binom{1}{2}$-tensor field.

Conversely, if ∇ is a linear connection on TM and Φ is a $\binom{1}{2}$-tensor field, then Φ defines a map $\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ which is bilinear over smooth functions, and one immediately verifies that $\hat\nabla_\xi\eta := \nabla_\xi\eta+\Phi(\xi,\eta)$ defines a linear connection on TM. (Technically speaking, we have shown that the space of linear connections on TM is an affine space modeled on the vector space of smooth $\binom{1}{2}$-tensor fields on M.) It is also clear, how such a modification affects the torsion. Denoting by T and $\hat T$ the torsions of ∇ and $\hat\nabla$, we of course get $\hat T(\xi,\eta)=T(\xi,\eta)+(\Phi(\xi,\eta)-\Phi(\eta,\xi))$. In abstract index notation, this reads as $\hat T^i_{jk}=T^i_{jk}+2\Phi^i_{[jk]}$.

Next, suppose that g is a Riemannian metric on M, and that ∇ is some linear connection on TM. Then we can look at the extent to which ∇ fails to be metric for g, i.e. consider the map $\mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\to C^\infty(M,\mathbb{R})$ defined by

$$(\xi,\eta,\zeta)\mapsto A(\xi,\eta,\zeta):=\xi\cdot g(\eta,\zeta)-g(\nabla_\xi\eta,\zeta)-g(\eta,\nabla_\xi\zeta).$$

This mapping evidently is trilinear over $\mathbb{R}$, linear over smooth functions in ξ, and symmetric in η and ζ. But one also verifies readily that it is linear over smooth functions in η and ζ, too, and thus it is given by a $\binom{0}{3}$-tensor field. If we change the connection to $\hat\nabla$ using a $\binom{1}{2}$-tensor field Φ, then the resulting tensor field $\hat A$ evidently satisfies

$$\hat A(\xi,\eta,\zeta)=A(\xi,\eta,\zeta)-g(\Phi(\xi,\eta),\zeta)-g(\eta,\Phi(\xi,\zeta)),$$

or $\hat A_{ijk}=A_{ijk}-g_{\ell k}\Phi^\ell_{ij}-g_{\ell j}\Phi^\ell_{ik}$. Now if we put $\Phi^i_{jk}:=\frac{1}{2}g^{ir}A_{jkr}$ then the resulting change becomes

$$-\tfrac{1}{2}\delta^r_kA_{ijr}-\tfrac{1}{2}\delta^r_jA_{ikr}=-\tfrac{1}{2}A_{ijk}-\tfrac{1}{2}A_{ikj}=-A_{ijk},$$

where in the last step we used that A_{ijk} is symmetric in the last two indices. Hence this change leads to $\hat A=0$, and thus to a linear connection on TM, which is metric for g.

So finally, we can start with a linear connection ∇ on TM which is metric for g. Changing from ∇ to $\hat\nabla$ using Φ, we see from above that $\hat\nabla$ is also metric for g if and only if $0=g_{\ell k}\Phi^\ell_{ij}+g_{\ell j}\Phi^\ell_{ik}$. Otherwise put, the $\binom{0}{3}$-tensor field $\Psi_{ijk}:=\Phi^\ell_{ij}g_{k\ell}$ has to be skew symmetric in j and k. On the other hand, the change of torsion caused by this change of connection is trivial if and only if Φ^ℓ_{ij} is symmetric in i and j, i.e. if and only if Ψ_{ijk} is symmetric in i and j. But we already know from the proof of Proposition

1.1 that these symmetries force Ψ to vanish identically. Hence we see that the map sending a linear connection which is metric for g to its torsion is injective. But both the change of connection and the torsion are point-wise objects. In a point $x \in M$ the changes of metric connections are described by trilinear maps $T_xM \times T_xM \times T_xM \to \mathbb{R}$ which are skew symmetric in the last two entries. Hence this space has dimension $n\frac{n(n-1)}{2}$. On the other hand, the torsion in each point is a skew symmetric bilinear map $T_xM \times T_xM \to T_xM$, so again the space of maps has dimension $n\frac{n(n-1)}{2}$. Hence we conclude that the map between metric connections and torsions is a linear isomorphism, so there is a unique torsion free one. $\square$

1.10. The covariant derivative in local coordinates. We first observe that a linear connection is a local operator and thus can be described in local coordinates.

LEMMA 1.10. *Let M be a smooth manifold and let ∇ be a linear connection on TM. Then the operator ∇ is local in both arguments, i.e. if $U \subset M$ is open and for $\xi, \eta \in \mathfrak{X}(M)$ we either have $\xi|_U = 0$ or $\eta|_U = 0$, then $\nabla_\xi \eta$ vanishes on U.*

Moreover, ∇ is tensorial in the first argument, i.e. if ξ vanishes in some point $x \in M$, then $\nabla_\xi \eta(x) = 0$ for any $\eta \in \mathfrak{X}(M)$.

PROOF. For a point $x \in U$, there is a bump function $\varphi \in C^\infty(M, \mathbb{R})$ such that $\varphi(x) = 1$ and $\operatorname{supp}(\varphi) \subset U$. If $\xi|_U = 0$, then the vector field $\varphi\xi$ vanishes identically, so $0 = \nabla_{\varphi\xi}\eta = \varphi\nabla_\xi\eta$ for any $\eta \in \mathfrak{X}(M)$. Evaluating in x, we get $0 = \varphi(x)\nabla_\xi\eta(x)$ and since $\varphi(x) = 1$, this implies that $\nabla_\xi\eta(x) = 0$.

If $\eta|_U = 0$, then $\varphi\eta = 0$, and we get $0 = \nabla_\xi(\varphi\eta) = (\xi \cdot \varphi)\eta + \varphi\nabla_\xi\eta$. Evaluating in x and using that $\eta(x) = 0$, we again get $\nabla_\xi\eta(x) = 0$.

Now assume that $\xi(x) = 0$ for some point $x \in M$ and choose a chart (U, u) with $x \in U$. Expanding $\xi|_U = \sum_i \xi^i \partial_i$, we conclude from the first part that in computing $\nabla_\xi\eta|_U$ we may replace ξ by this sum. Using the defining properties of ∇, we conclude that $\nabla_\xi\eta|_U = \sum_i \xi^i \nabla_{\partial_i}\eta$. But if $\xi(x) = 0$ then $\xi^i(x) = 0$ for all i and hence $\nabla_\xi\eta(x) = 0$. $\square$

As usual, the lemma implies that $\nabla_\xi\eta|_U$ depends only on $\xi|_U$ and $\eta|_U$ and that $\nabla_\xi\eta(x)$ depends only on $\xi(x)$. The second fact indicates that a linear connection on TM can indeed be thought of as an analog of a directional derivative for vector fields.

This implies that we can compute the action of any linear connection on TM in local coordinates. Consider a local chart (U, u) for M with coordinate vector fields ∂_i and two vector fields $\xi, \eta \in \mathfrak{X}(M)$. Then we can expand the fields as $\xi|_U = \sum_i \xi^i \partial_i$ and $\eta = \sum_j \eta^j \partial_j$ for smooth functions $\xi^i, \eta^j : U \to \mathbb{R}$ and using the lemma, we get

$$\nabla_\xi\eta|_U = \textstyle\sum_{i,j} \nabla_{\xi^i\partial_i}(\eta^j\partial_j) = \sum_{i,j} \xi^i(\partial_i \cdot \eta^j)\partial_j + \sum_{i,j} \xi^i\eta^j\nabla_{\partial_i}\partial_j. \tag{1.2}$$

Since any vector field on U can be expanded in terms of the coordinate vector fields, there are uniquely determined smooth functions $\Gamma^k_{ij} : U \to \mathbb{R}$ for $i, j, k = 1, \dots, n = \dim(M)$ such that

$$\nabla_{\partial_i}\partial_j = \textstyle\sum_k \Gamma^k_{ij}\partial_k.$$

Knowing these functions, one has a complete description of ∇ in local coordinates via

$$\nabla_\xi\eta|_U = \textstyle\sum_{i,j} \xi^i(\partial_i \cdot \eta^j)\partial_j + \sum_{i,j,k} \xi^i\eta^j\Gamma^k_{ij}\partial_k. \tag{1.3}$$

DEFINITION 1.10. The quantities Γ^k_{ij} are called the *connection coefficients* or, in particular in the case of the Levi-Civita connection of a Riemannian metric, the *Christoffel symbols* of the linear connection ∇ with respect to the chart (U, u).

Proposition 1.10. *Let M be a smooth manifold and let ∇ be a linear connection on TM.*

(1) The connection ∇ is torsion-free if and only if its connection coefficients with respect to any chart are symmetric in the lower indices, i.e. $\Gamma_{ij}^k = \Gamma_{ji}^k$ for all i, j, k.

(2) If ∇ is the Levi-Civita connection of a Riemannian metric g on M, then the Christoffel symbols are given explicitly by

$$\Gamma_{ij}^k = \tfrac{1}{2}\textstyle\sum_\ell g^{k\ell}\left(\partial_i \cdot g_{j\ell} + \partial_j \cdot g_{i\ell} - \partial_\ell \cdot g_{ij}\right),$$

where g_{ij} and g^{ij} are the components of the metric and its inverse in local coordinates.

Proof. (1) If ∇ is torsion-free, then $[\partial_i, \partial_j] = 0$ implies $\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i$ and hence symmetry of the connection coefficients. Conversely, the formula (1.3) for ∇ in local coordinates together with the formula for the Lie bracket in local coordinates shows that

$$(\nabla_\xi \eta - \nabla_\eta \xi)|_U = [\xi, \eta]|_U + \textstyle\sum_{i,j,k} \xi^i \eta^j (\Gamma_{ij}^k - \Gamma_{ji}^k)\partial_k.$$

Thus symmetry of the connection coefficients implies torsion-freeness of ∇.

(2) We apply the Koszul formula (1.1) from the first proof of Theorem 1.9 for $\xi = \partial_i$, $\eta = \partial_j$ and $\zeta = \partial_\ell$. Of course, we get $2g(\nabla_{\partial_i}\partial_j, \partial_\ell) = 2\sum_m g_{m\ell}\Gamma_{ij}^m$, so we can recover $2\Gamma_{ij}^k$ from this by multiplying with $g^{k\ell}$ and summing over ℓ. But using that the Lie bracket of two coordinate vector fields always vanishes, we conclude that (1.1) in our case leads to

$$\partial_i \cdot g_{j\ell} + \partial_j \cdot g_{i\ell} - \partial_\ell \cdot g_{ij}.$$

This immediately implies the claim. □

It is easy to compute directly how connection coefficients transform under a change of local coordinates and in particular to see that they do *not* define a tensor field. Nonetheless, it is sometimes useful to interpret them as a tensor field defined on the domain on a coordinate chart. Given a chart (U, u) one thus defines $\Gamma^U : \mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U)$ as $\Gamma^U(\xi, \eta) = \sum_{i,j,k} \xi^i \eta^j \Gamma_{ij}^k \partial_k$. Using this, one can rewrite equation (1.3) for the local coordinate representation of a linear connection as

$$\nabla_\xi \eta|_U = \textstyle\sum_j (\xi \cdot \eta^j)\partial_j + \Gamma^U(\xi, \eta).$$

1.11. Parallel transport. The last formula for the covariant derivative has an important advantage. It shows that in order to compute the value of $\nabla_\xi \eta$ in a point $x \in M$ one only has to know $\eta(x)$ and the derivative of the component functions of η with respect to some chart in direction $\xi(x)$. Given a smooth curve $c : I \to M$ with $c(0) = x$ and $c'(0) = \xi$, these derivatives can be computed as $\xi \cdot \eta^j = \frac{d}{dt}|_{t=0}\eta^j(c(t))$.

Now suppose that we start with a curve $c : I \to M$ and take a vector field $\eta \in \mathfrak{X}(M)$. Then for each $t \in I$, we can look at $\nabla_{c'(t)}\eta(c(t)) \in T_{c(t)}M$. From above we see that this depends only on the restriction of η to the image of c. This allows us to generalize the next part of what was discussed in the motivation in Section 1.9 from embedded submanifolds to general Riemannian manifolds.

Consider an interval $I \subset \mathbb{R}$ and a smooth curve $c : I \to M$ in a manifold M. Then one defines a *vector field along* c as a smooth function $\xi : I \to TM$ such that $\xi(t) \in T_{c(t)}M$ for all $t \in I$. Observe that in the domain of a chart (U, u), we can expand the tangent vectors $\xi(t)$ in terms of the coordinate vector fields ∂_i determined by the chart. Thus we obtain smooth functions ξ^i such that $\xi(t) = \sum_i \xi^i(t)\partial_i(c(t))$. Finally observe that given a vector field ξ along c and a smooth function $f : I \to \mathbb{R}$, one can form $f\xi$ in an obvious way.

Given a linear connection ∇ on TM, the above considerations show that there is a well defined tangent vector $\nabla_{c'(t)}\xi(c(t)) \in T_{c(t)}M$ for each $t \in I$. From the coordinate formula above it follows readily that these fit together to form a smooth vector field $\nabla_{c'}\xi$ along c. The basic properties of this operation are as follows.

Proposition 1.11. *Let (M, g) be a Riemannian manifold and ∇ its Levi-Civita connection.*

(1) The covariant derivative for vector fields along a smooth curve $c : I \to M$ is a linear operator which satisfies the product rule $\nabla_{c'}(f\xi) = f'\xi + f\nabla_{c'}\xi$ for any smooth function $f : I \to \mathbb{R}$. In local coordinates we get

$$\nabla_{c'}\xi(t) = \sum_i (\xi^i)'(t)\partial_i(c(t)) + \Gamma^U(c'(t), \xi(t))(c(t)).$$

(2) For two vector fields ξ and η along c, one has

$$\frac{d}{dt}g(\xi, \eta) = g(\nabla_{c'}\xi, \eta) + g(\xi, \nabla_{c'}\eta),$$

where $g(\xi, \eta)(t) = g(c(t))(\xi(t), \eta(t))$.

(3) Given a point $a \in I$ and a tangent vector $\xi_0 \in T_xM$, where $x = c(a)$, there is a unique vector field $\xi : I \to M$ along c such that $\xi(a) = \xi_0$ and $\nabla_{c'}\xi = 0$.

(4) In the setting of (3) suppose that $[a, b] \subset I$. Then mapping ξ_0 to $\xi(b)$ defines an orthogonal linear isomorphism $T_{c(a)}M \to T_{c(b)}M$.

Proof. (1) Linearity follows immediately from bilinearity of the covariant derivative of vector fields. The formula in local coordinates then follows directly from the considerations in the end of Section 1.10. Since for the components with respect to local coordinates, one clearly has $(f\xi)^i = f\xi^i$, the product rule follows from this coordinate formula.

(2) this follows immediately from the fact that ∇ is metric for g.

(3) From the coordinate formula in (1) is is clear that in local coordinates $\nabla_{c'}\xi = 0$ is a linear system of first order ordinary differential equations on the coordinate functions ξ^i. Hence this admits a unique global solution for any initial value.

(4) Linearity clearly implies that one obtains a linear map $T_{c(a)}M \to T_{c(b)}M$. From (2) it follows that if $\nabla_{c'}\xi = \nabla_{c'}\eta = 0$, then $g(\xi, \eta)$ is constant, which implies orthogonality of the map. □

Definition 1.11. (1) A vector field ξ along c is called *parallel* (along c) if and only if $\nabla_{c'}\xi = 0$.

(2) For $c : [a, b] \to M$, the map $T_{c(a)}M \to T_{c(b)}M$ from part (4) of the proposition is called the *parallel transport along c*.

Parallel transport is closely related to a concept called holonomy. Given a point x in a Riemannian manifold M, one considers piece-wise smooth closed curves starting and ending in x. It is easy to see that parallel transport extends to piece-wise smooth curves without problem, so each such curve gives rise to an orthogonal linear map $T_xM \to T_xM$. It is also easy to see that the resulting linear maps form a subgroup of the orthogonal group $O(T_xM)$ (compositions comes from going through two curves successively, while inversion comes from going in the opposite direction). This is called the *holonomy group* of the metric g in the point x. One further proves that for connected M, the holonomy groups in different points are isomorphic, so one can speak about the holonomy group of M. One of the reasons for the importance of the concept of holonomy is that by a classical result of M. Berger, one can completely classify (in a certain sense) the possible holonomy groups of Riemannian manifolds.

1.12. Geodesics and the exponential map. For a smooth curve $c : I \to M$, the derivative c' of course is a vector field along c. Hence it makes sense to call a curve c a *geodesic* of g, if $\nabla_{c'}c' = 0$ i.e. if c' is parallel along c. We can quickly prove some fundamental results on geodesics:

PROPOSITION 1.12. *Let (M, g) be a Riemannian manifold.*

(1) Given $x \in M$ and $\xi \in T_xM$, there is a unique maximal interval $I \subset \mathbb{R}$ with $0 \in I$ and a unique maximal geodesic $c : I \to M$ with $c(0) = x$ and $c'(0) = \xi$.

(2) Given $x \in M$, there is an open neighborhood U of zero in T_xM such that for each $\xi \in U$, the interval I from (1) contains $[0, 1]$ and mapping ξ to $c(1)$ defines a smooth map $\exp_x : U \to M$.

(3) The map $\exp_x$ from (2) satisfies $\exp_x(0) = x$ and $T_0 \exp_x = \mathrm{id}_{T_xM}$ so choosing U small enough, $\exp_x$ is a diffeomorphism from U onto an open neighborhood of x in M.

(4) Let $\pi : TM \to M$ be the natural projection. There is an open neighborhood V of the zero-section in TM such that for each $\xi \in V$, $\exp_{\pi(\xi)}(\xi) \in M$ is defined. Calling the latter element $\exp(\xi)$, one obtains a smooth map $\exp : V \to M$. Choosing V small enough, $(\pi, \exp) : V \to M \times M$ is a diffeomorphism onto an open neighborhood of the diagonal in $M \times M$.

PROOF. From part (1) of Proposition 1.11, we see that in local coordinates, the equation $0 = \nabla_{c'}c'$ reads as $(c^i)''(t) = -\Gamma^i_{jk}(c(t))(c^j)'(t)(c^k)'(t)$, so this is a (non-linear) system of second order ODEs, which admits unique local solutions for fixed initial values for c and c'. From this, (1) follows by piecing together unique local solutions to maximal solutions.

(2) The fact that solutions of ODEs depend smoothly on the initial data implies that there is an $\epsilon > 0$ such that for each unit vector $\xi \in T_xM$, the maximal interval on which the solution from (1) is defined contains $(-\epsilon, \epsilon)$. Now suppose that $I \subset \mathbb{R}$ is an interval containing zero and $c : I \to M$ is a geodesic. Fix a real number s and consider $\tilde{c}(t) := c(st)$. Then $\tilde{c}'(t) = sc'(st)$, and one easily concludes that $\tilde{c}$ is a geodesic with $\tilde{c}(0) = c(0)$ and $\tilde{c}'(0) = sc'(0)$. Together with the above, this shows that $\exp_x$ is defined and smooth on the ball of radius ϵ and thus on an open neighborhood of 0.

(3) Since the constant curve $c(t) = x$ is a geodesic with $c(0) = x$ and $c'(0) = 0$, we see that $\exp_x(0) = x$. Moreover, the considerations in the proof of part (2) show that the geodesic $c : I \to M$ with $c(0) = x$ and $c'(0) = \xi$ can be written as $t \mapsto \exp_x(t\xi)$ for t close enough to zero. But this shows that

$$T_0 \exp_x \cdot\xi = \tfrac{d}{dt}|_{t=0} \exp_x(t\xi) = \xi,$$

so $T_0 \exp_x = \mathrm{id}(T_xM)$. Hence $\exp_x$ is a local diffeomorphism around 0.

(4) The fact that exp is well defined on an open neighborhood of the zero section in TM follows from smooth dependence of solutions of ODEs on the initial conditions as before. Hence we can consider $(\pi, \exp) : V \to M \times M$. This maps $0_x \in T_xM$ to (x, x), so the diagonal is in the image. Next we claim that $T_{0_x}(\pi, \exp) : T_{0_x}TM \to T_xM \times T_xM$ is injective and thus a linear isomorphism for dimensional reasons. The first component of this map is $T_x\pi$, so this is surjective and thus has a kernel of dimension $n = \dim(M)$. On the other hand, one can view T_xM naturally as a subspace of $T_{0_x}TM$ (via the derivatives of the curves $t \mapsto t\xi$). This is contained in $\ker(T_x\pi)$ and hence has to coincide with this subspace for dimensional reasons. But by construction, the second component of $T_{0_x} \exp$ coincides on this subspace with $T_0 \exp_x$, so the claim follows. Hence we know that $(\pi, \exp)$ is a local diffeomorphism around 0_x for each $x \in M$. A moment of thought

shows that this map is injective and hence a diffeomorphism on sufficiently small open neighborhoods of the zero section. □

An important consequence is that for each $x \in M$ on can use the inverse of $\exp_x$ as a local chart around x. Choosing an orthonormal basis of T_xM, one can identify the target space of this chart with $\mathbb{R}^n$ (endowed with the standard inner product), and thus get local coordinates around x. These are called *normal coordinates centered at x.*

1.13. Curvature. The last topic we discuss in this chapter is the curvature tensor of a Riemannian metric.

PROPOSITION 1.13. *Let (M, g) be a Riemannian manifold and consider the trilinear map $\mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by*

$$R(\xi, \eta)(\zeta) := \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi,\eta]} \zeta.$$

(1) This is given by the action of a $\binom{1}{3}$-tensor field, which in abstract index notation is denoted by $R_{ij}{}^k{}_\ell$ via $R(\xi, \eta)(\zeta)^k = R_{ij}{}^k{}_\ell \xi^i \eta^j \zeta^\ell$.

(2) The tensor field R can be viewed as a two-form with values in skew-symmetric endomorphisms of the tangent bundle, i.e. $g(R(\xi, \eta)(\zeta_1), \zeta_2)$ is skew symmetric both in ξ and η and in ζ_1 and ζ_2, respectively $R_{ij}{}^k{}_\ell = R_{[ij]}{}^k{}_\ell$ and $R_{ij}{}^a{}_\ell g_{ka} = R_{ij}{}^a{}_{[\ell} g_{k]a}$.

(3) In view of the last symmetry, R_x can be viewed as a bilinear form on $\Lambda^2 T_xM$, and as such a form it is symmetric, i.e. $g(R(\xi, \eta)(\zeta_1), \zeta_2) = g(R(\zeta_1, \zeta_2)(\xi), \eta)$, respectively $R_{ij}{}^a{}_k g_{\ell a} = R_{k\ell}{}^a{}_i g_{ja}$.

(4) Finally, R satisfies the first Bianchi-identity

$$0 = R(\xi, \eta)(\zeta) + R(\zeta, \xi)(\eta) + R(\eta, \zeta)(\xi),$$

respectively $R_{[ij}{}^k{}_{\ell]} = 0$.

PROOF. (1) We have to show that the map we have defined is linear over smooth functions in all three entries, but since it is obviously skew-symmetric in ξ and η, it suffices to verify this linearity in η and ζ. Now the second term in the defining formula for R evidently is linear over smooth functions in η, while for the first term, we compute for $f \in C^\infty(M, \mathbb{R})$:

$$\nabla_\xi \nabla_{f\eta} \zeta = \nabla_\xi (f \nabla_\eta \zeta) = f \nabla_\xi \nabla_\eta \zeta + (\xi \cdot f) \nabla_\eta \zeta.$$

But on the other hand, $[\xi, f\eta] = f[\xi, \eta] + (\xi \cdot f)\eta$, which after inserting into the covariant derivative cancels the other contribution.

To verify linearity over smooth functions in ζ, we take $f \in C^\infty(M, \mathbb{R})$ and compute

$$\begin{aligned}\nabla_\xi \nabla_\eta f\zeta &= \nabla_\xi (f \nabla_\eta \zeta + (\eta \cdot f)\zeta) \\ &= f \nabla_\xi \nabla_\eta \zeta + ((\xi \cdot f)\nabla \eta \zeta + (\eta \cdot f)\nabla_\xi \zeta) + (\xi \cdot \eta \cdot f)\zeta\end{aligned}$$

The middle sum is symmetric in ξ and η and thus cancels with the corresponding term coming from $-\nabla_\eta \nabla_\xi f\zeta$. On the other hand,

$$\nabla_{[\xi,\eta]} f\zeta = f \nabla_{[\xi,\eta]} \zeta + ([\xi, \eta] \cdot f)\zeta.$$

By definition of the Lie bracket $[\xi, \eta] \cdot f = \xi \cdot \eta \cdot f - \eta \cdot \xi \cdot f$, so linearity over smooth functions in ζ follows.

(2) We have already observed that $R(\xi, \eta)(\zeta)$ is skew symmetric in ξ and η, so $R_{ij}{}^k{}_\ell = R_{[ij]}{}^k{}_\ell$. On the other hand, $R_{ij}{}^a{}_\ell g_{ka}$ is just the $\binom{0}{4}$-tensor field defined by

$(\xi,\eta,\zeta_1,\zeta_2)\mapsto g(R(\xi,\eta)(\zeta_1),\zeta_2)$, and we have to prove that this is skew symmetric in ζ_1 and ζ_2. Now we can compute directly as follows

$$\begin{aligned}\xi\cdot\eta\cdot g(\zeta_1,\zeta_2)&=\xi\cdot(g(\nabla_\eta\zeta_1,\zeta_2)+g(\zeta_1,\nabla_\eta\zeta_2))\\&=g(\nabla_\xi\nabla_\eta\zeta_1,\zeta_2)+g(\nabla_\eta\zeta_1,\nabla_\xi\zeta_2)+g(\nabla_\xi\zeta_1,\nabla_\eta\zeta_2)+g(\zeta_1,\nabla_\xi\nabla_\eta\zeta_2).\end{aligned}$$

Observe that the middle two terms in the last expression are symmetric in ξ and η, hence they will vanish if we subtract the same term with ξ and η exchanged. But then if we further subtract

$$[\xi,\eta]\cdot g(\zeta_1,\zeta_2)=g(\nabla_{[\xi,\eta]}\zeta_1,\zeta_2)+g(\zeta_1,\nabla_{[\xi,\eta]}\zeta_2),$$

then the left hand side will vanish by definition of the Lie bracket, while on the right hand side we get

$$g(R(\xi,\eta)(\zeta_1),\zeta_2)+g(\zeta_1,R(\xi,\eta)(\zeta_2)).$$

(4) Expanding $R(\xi,\eta)(\zeta)+R(\zeta,\xi)(\eta)+R(\eta,\zeta)(\xi)$ according to the definition of R, the first term $\nabla_\xi\nabla_\eta\zeta$ from the first summand adds up with the second term $-\nabla_\xi\nabla_\zeta\eta$ from the second summand to $\nabla_\xi[\eta,\zeta]$ by torsion freeness. Again by torsion freeness, this adds up with the last term $-\nabla_{[\eta,\zeta]}\xi$ from the last summand to $[\xi,[\eta,\zeta]]$. This can be similarly done for the other terms to see that

$$R(\xi,\eta)(\zeta)+R(\zeta,\xi)(\eta)+R(\eta,\zeta)(\xi)=[\xi,[\eta,\zeta]]+[\zeta,[\xi,\eta]]+[\eta,[\zeta,\xi]],$$

which vanishes by the Jacobi identity for the Lie bracket of vector fields. Since $R(\xi,\eta)(\zeta)$ is skew symmetric in ξ and η, its complete alternation coincides with $1/3$ times the sum over all cyclic permutations of the arguments.

(3) This identity is a formal consequence of the other ones. Writing $S_{ijk\ell}:=R_{ij}{}^a{}_k g_{a\ell}$, we know skew symmetry in (i,j) and in (k,ℓ) from (2) and $0=S_{ijk\ell}+S_{kij\ell}+S_{jki\ell}$ from (4). Using this, we compute

$$\begin{aligned}S_{ijk\ell}&=-S_{kij\ell}-S_{jki\ell}=S_{ki\ell j}+S_{jk\ell i}=-S_{\ell kij}-S_{i\ell kj}-S_{\ell jki}-S_{k\ell ji}\\&=2S_{k\ell ij}+S_{i\ell jk}+S_{\ell jik}=2S_{k\ell ij}-S_{ji\ell k}.\end{aligned}$$

Since the last term equals $-S_{ijk\ell}$ the claimed symmetry $S_{ijk\ell}=S_{k\ell ij}$ follows. □

The Riemann curvature tensor is the fundamental invariant of a Riemannian metric. As the discussion of the symmetries shows, it is a rather complicated object, and extracting parts of the curvature, which are more easily handled is an important problem in Riemannian geometry. We will discuss some aspects of this in the next chapter.

1.14. Remarks on isometries. Let us apply the concepts discussed so far to obtain some basic facts on isometries, which are the appropriate concept of morphisms in the category of Riemannian manifolds. This discussion also shows that the concepts we have developed so far actually are naturally associated to Riemannian manifolds.

Definition 1.14. Let (M,g) and (N,h) be Riemannian manifolds of dimension n. An *isometry* between M and N is a smooth map $\Phi:M\to N$ such that for each $x\in M$ the tangent map $T_x\Phi:T_xM\to T_{\Phi(x)}N$ is orthogonal with respect to the inner products g_x and $h_{\Phi(x)}$.

Observe that by definition $T_x\Phi$ always has to be a linear isomorphism, so Φ is a local diffeomorphism. In particular, one may always pull back arbitrary tensor fields along isometries. Moreover, since Riemannian metrics can be restricted to open subsets, there is an obvious concept of a *local isometry*. For simplicity, one often restricts to the case of isometries which are diffeomorphisms.

For an isometric diffeomorphism $\Phi : (M, g) \to (N, h)$ we have induced linear isomorphisms $\Phi^* : C^\infty(N, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ and likewise for all kinds of geometric objects. From the constructions in Proposition 1.4 it is clear that Φ^* maps the inverse metric to h to the inverse metric to g and is also compatible with the volume forms. In particular, we see that the maps Φ^* are always orthogonal for the L^2-inner products we have constructed in 1.5 and hence extend to isomorphisms of the Hilbert space completions. Likewise, the pullback along Φ is compatible with the Hodge-$*$ operation, since the maps induced by Φ are compatible with the induced inner products on the spaces $\Lambda^k T_x^* M$ and $\Lambda^k T_{\Phi(x)}^* N$. Hence Φ^* is also compatible with the codifferential and the Laplace–Beltrami operator on forms.

Next, for a smooth curve $c : [a, b] \to M$, $\Phi \circ c$ is a smooth curve in N, and since $(\Phi \circ c)'(t) = T_{c(t)}\Phi \cdot c'(t)$, orthogonality of the tangent maps of Φ implies that c and $\Phi \circ c$ have the same arclength. Denoting by d_g and d_h the metrics on M and N as defined in 1.7 and assuming that Φ is a diffeomorphism, we conclude that $d_h(\Phi(x), \Phi(y)) = d_g(x, y)$ for all $x, y \in M$. This means that Φ is an isometry between the metric spaces (M, d_g) and (N, d_h).

Proposition 1.14. *Let (M, g) and (N, h) be Riemannian manifolds of the same dimension n with Levi-Civita connections ∇^M and ∇^N and Riemann curvature tensors R^M and R^N, and let $\Phi : M \to N$ be an isometry.*

(1) For $\xi, \eta \in \mathfrak{X}(M)$ we have $\Phi^*(\nabla^N_\xi \eta) = \nabla^M_{\Phi^*\xi} \Phi^* \eta$.

(2) Φ is compatible with the curvature tensors, i.e. $\Phi^* R^N = R^M$.

(3) Φ is compatible with the covariant derivative of smooth vector fields along smooth curves. Thus it is compatible with the parallel transport along smooth curves and maps geodesics in M to geodesics in N.

Proof. (1) This is a local question, so we may replace M and N by U and $\Phi(U)$ where Φ restricts to a diffeomorphism on U. Then consider the operation $\mathfrak{X}(\Phi(U)) \times \mathfrak{X}(\Phi(U)) \to \mathfrak{X}(\Phi(U))$ defined by $(\xi, \eta) \mapsto (\Phi^{-1})^*(\nabla^M_{\Phi^*\xi} \Phi^* \eta)$. This is evidently bilinear and since pullbacks are linear over smooth functions it follows readily that it is linear over smooth functions in the first variable. On the other hand, one uses $\Phi^*(f\eta) = (f \circ \Phi)\Phi^*\eta$ and $(\Phi^*\xi) \cdot (f \circ \Phi) = (\xi \cdot f) \circ \Phi$ to conclude that this satisfies a Leibniz rule in second variable, so we have constructed a linear connection on $T\Phi(U)$. Next, alternating this operation, we just have to use $[\Phi^*\xi, \Phi^*\eta] = \Phi^*([\xi, \eta])$ to conclude that this linear connection is torsion free.

Finally, since the tangent maps of Φ are all orthogonal, we conclude that

$$h_{\Phi(x)}(\xi(\Phi(x)), \eta(\Phi(x))) = g_x(\Phi^*\xi(x), \Phi^*\eta(x)),$$

so $h(\xi, \eta) \circ \Phi = g(\Phi^*\xi, \Phi^*\eta)$. Thus for a third vector field $\zeta \in \mathfrak{X}(U)$, we can write $(\zeta \cdot h(\xi, \eta)) \circ \Phi$ as $(\Phi^*\zeta) \cdot g(\Phi^*\xi, \Phi^*\eta)$. Now apply compatibility of ∇^M with the metric and rewrite $g(\nabla^M_{\Phi^*\zeta}\Phi^*\xi, \Phi^*\eta)$ as $h((\Phi^{-1})^*(\nabla^M_{\Phi^*\zeta}\Phi^*\xi), \eta) \circ \Phi$ and likewise for the other summand. Since Φ is a diffeomorphism, we can forget about the composition with Φ and conclude that the connection we have defined is compatible with h and hence has to coincide with ∇^N by Theorem 1.9. From this, the result follows by applying Φ^*.

(2) Since Φ is a local diffeomorphism, we can realize all tangent vectors in a point x as the values of vector fields of the form $\Phi^*\xi$ for $\xi \in \mathfrak{X}(N)$. But the formula in (1) together with the definition of curvature shows that $R^M(\Phi^*\xi, \Phi^*\eta)(\Phi^*\zeta) = \Phi^*(R^N(\xi, \eta)(\zeta))$, which implies the claim.

(3) For a smooth curve c in M, $\Phi \circ c$ is a smooth curve in N and for a vector field ξ along c, $\Phi_*\xi(t) := T_{c(t)}\Phi \cdot \xi(t)$ is a vector field along $\Phi \circ c$. Now the result in (1)

easily implies that $\Phi_*\nabla_{c'}\xi = \nabla_{(\Phi\circ c)'}\Phi_*\xi$. This immediately implies compatibility with the parallel transport and since by definition $(\Phi \circ c)' = \Phi_* c'$, the claim on geodesics follows, too. □

This implies that isometries are rather rare in various senses. For example, consider Euclidean space E^n. By definition, the coordinate vector fields ∂_i on E^n satisfy $\nabla_\xi \partial_i = 0$ for any vector field ξ on E^n. But this easily shows that $R(\xi, \eta)(\partial_i) = 0$, so on E^n the Riemann curvature vanishes identically. From part (2) of the proposition, we thus conclude that if (M, g) is a Riemannian manifold and $x \in M$ is a point, there may be an isometry from an open neighborhood U of x in M to E^n only if the Riemann curvature R^M vanishes identically on U.

If we endow $\mathbb{R}^n$ with an arbitrary Riemannian metric g, then the Riemann curvature tensor of g can be considered as a smooth function to (a subspace of) $\otimes^3\mathbb{R}^{n*} \otimes \mathbb{R}^n$. Now even taking into account all the symmetries of the curvature tensor, the target space is high dimensional. Suppose it is possible to choose the metric g in such a way that the curvature tensor defines an injective function. Then part (2) of the proposition implies that the only isometry between open subsets of $\mathbb{R}^n$ endowed with the restriction of this metric, is the identity map, so there are no non-trivial local isometries. Indeed, one can make this precise and show that the space of Riemannian metrics on a smooth manifold M in dimension $n \geq 3$ contains an open dense subset (in a suitable topology) consisting of metric which do not admit any non-trivial local isometries.

Finally, Proposition 1.1 shows that any isometry of Euclidean space is a Euclidean motion. The proof of (ii)⇒(iii) we have given actually applies more generally to show that for connected open subsets U and V of E^n any isometry $f : U \to V$ is the restriction of a Euclidean motion. So even for this simplest example of a Riemannian manifold, isometries are rather rare (they form a finite dimensional manifold). Now we can prove an analog of this result for arbitrary Riemannian manifolds.

Corollary 1.14. *Let (M, g) and (N, h) be Riemannian manifolds of dimension n such that M is connected, and let $x \in M$ be a point. Then an isometry $\Phi : M \to N$ is uniquely determined by $\Phi(x)$ and $T_x\Phi$.*

Proof. From part (3) of the proposition, we know that an isometry Φ maps geodesics to geodesics. Hence if $c : I \to M$ is a geodesic with $c(0) = x$ and $c'(0) = \xi$, then $\Phi \circ c$ is a geodesic through $\Phi(x)$ with initial direction $T_x\Phi \cdot \xi$. In terms of the exponential mapping this means that $\Phi \circ \exp_x = \exp_{\Phi(x)} \circ T_x\Phi$ holds on the domain of definition of $\exp_x$. By Proposition 1.12, $\exp_x$ restricts to a diffeomorphism from some open neighborhood of zero onto a neighborhood of x in M. But then the restriction of Φ to this neighborhood is uniquely determined by $\Phi(x)$ and $T_x\Phi$.

Given two isometries $\Phi, \Psi : M \to N$, this shows that the set

$$\{x \in M : \Phi(x) = \Psi(x) \text{ and } T_x\Phi = T_x\Psi\}.$$

is open in M. On the other hand, its complement is evidently open, hence if non-empty, this set coincides with M, since M is connected. □

CHAPTER 2

Some more advanced topics

Having the core notions of Riemannian geometry at hand, we briefly discuss "how things go on from here" in several directions. There is some dependence between the different topics we discuss, but this is not too strong. Hence to a large extent the individual parts of this chapter can be studied independently of each other.

Moving frames – Examples

We start by discussing the fundamentals of E. Cartan's moving frame method. This gives a systematic way for computing the Levi-Civita connection and the Riemann curvature tensor of a Riemannian manifold in terms of local orthonormal frames and coframes. The method builds on the calculus of differential forms.

2.1. Local orthonormal frames and coframes. One of the basic difficulties in Riemannian geometry is that it is impossible to choose local coordinates which are well adapted to a Riemannian metric. This is basically due to the fact that the Riemann curvature tensor constructed in 1.13 is a *local invariant* of a Riemannian metric, which tells us that Riemannian metrics in general do not locally look the same. For example, suppose that one has a local chart (U, u) on a Riemannian manifold such that the corresponding coordinate vector fields ∂_i form an orthonormal basis of T_xM for each $x \in U$. Then (compare with Proposition 2.7 below) u is an isometry to the subset $u(U) \subset \mathbb{R}^n$ with the restriction of the usual metric on $\mathbb{R}^n$. As observed in 1.13, such an isometry can only exist if the Riemann curvature vanishes identically on U.

A possible replacement for adapted coordinates are local orthonormal frames, which we have met in 1.4. Given a Riemannian manifold (M, g) of dimension n and an open subset $U \subset M$, a local orthonormal frame for U is a family $\{s_1, \dots, s_n\}$ of vector fields defined on U such that $g(s_i, s_j) = \delta_{ij}$ on U. This means that for each $x \in U$, the tangent vectors $s_1(x), \dots, s_n(x) \in T_xM$ form an orthonormal basis for T_xM (with respect to g_x). In Proposition 1.4 we have proved that local orthonormal frames always exist. Since there is a better calculus for differential forms available than for vector fields, it is better to use the dual concept defined as follows.

Definition 2.1. Let (M, g) be a Riemannian manifold of dimension n and let $U \subset M$ be an open subset. A *local orthonormal coframe* on U is a family $\{\sigma^1, \dots, \sigma^n\}$ of one-forms defined on U such that $g|_U = \sum_{i=1}^n \sigma^i \otimes \sigma^i$.

Lemma 2.1. *Let (M, g) be a Riemannian manifold of dimension n and let $U \subset M$ be an open subset. A family $\{\sigma^1, \dots \sigma^n\}$ of elements of $\Omega^1(U)$ is a local orthonormal coframe if and only if for each $x \in U$ the elements $\sigma^1(x), \dots, \sigma^n(x)$ form a basis for T_x^*M, for which the dual basis of T_xM is orthonormal. In particular, local orthonormal coframes always exist.*

Proof. This is just a linear algebra statement. Starting with a local orthonormal coframe, we get $g_x = \sum_i \sigma^i(x) \otimes \sigma^i(x)$, so non-degeneracy of g_x implies that for each $\xi \in T_xM$, there is at least one i such that $\sigma^i(x)(\xi) \neq 0$. This implies that the $\sigma^i(x)$ are linearly independent and thus form a basis of T_x^*M. Denoting the dual basis by s_i we conclude that $g_x(s_i,s_j)=\sum_k \sigma^k(x)(s_i)\sigma^k(x)(s_j)=\delta_{ij}$ so the dual basis is orthonormal.

Conversely, suppose that $\sigma^1,\dots,\sigma^n$ is a family of one-forms satisfying the condition on the values in x. Then g_x and $\sum_i \sigma^i(x)\otimes\sigma^i(x)$ agree whenever one inserts two elements of the basis dual to $\{\sigma^1(x),\dots,\sigma^n(x)\}$ and hence on all pairs of vectors.

In particular, we see that we can obtain a local orthonormal coframe by forming the dual basis to a local orthonormal frame in each point, so existence follows from Proposition 1.4. □

From now on, we will usually work in a local orthonormal coframe $\{\sigma^1,\dots,\sigma^n\}$ with dual orthonormal frame $\{s_1,\dots,s_n\}$, so $\sigma^i(s_j)=\delta^i_j$. This simply means that any vector field ξ in the domain of the frames can be written as $\xi=\sum_i\sigma^i(\xi)s_i$. Likewise, a one-form can, in the domain of the frames, be written as $\varphi=\sum_j\varphi(s_j)\sigma^j$, and similarly for more complicated tensor fields.

It is actually possible to develop the fundamentals of Riemannian geometry in the language of local orthonormal coframes. One defines objects in terms of such a coframe and then proves that different coframes lead to the same object. In particular, texts taking this approach contain lots of computations on how various quantities behave under a change of frame. In the approach we take, such computations are not needed, since we only compute quantities which we already know to be well defined in terms of a local coframe.

2.2. Connection and curvature in a moving frame. Consider a local orthonormal coframe $\{\sigma^1,\dots,\sigma^n\}$ for a Riemannian manifold (M,g) defined on $U\subset M$ with dual frame $\{s_1,\dots,s_n\}$. To describe the Levi-Civita connection in the frame, we observe that for each $\xi\in\mathfrak{X}(U)$ and each $i=1,\dots,n$, $\nabla_\xi s_i$ is a smooth vector field on U, so we can write it as $\sum_j\omega_i^j(\xi)s_j$ for smooth functions $\omega_i^j(\xi)$, $i=1,\dots,n$, which depend on ξ. But by definition for a smooth function $f\in C^\infty(U,\mathbb{R})$, we have $\nabla_{f\xi}s_i=f\nabla_\xi s_i$, and hence $\omega_i^j(f\xi)=f\omega_i^j(\xi)$ for all i,j. Thus each ω_i^j actually is a smooth one-form on U, and it is natural to view (ω_i^j) as a matrix of one-forms on U. This is called the *matrix of connection forms* associated to the coframe $\{\sigma^i\}$.

It is even easier to describe the Riemann curvature tensor in a local frame. Namely, given vector fields $\xi,\eta\in\mathfrak{X}(U)$, we expand $R(\xi,\eta)(s_i)=\sum_j\Omega_i^j(\xi,\eta)s_j$. The fact that R is a tensor immediately implies that Ω_i^j actually is a two-form on U for each i and j. Hence we also view (Ω_i^j) as a matrix of two-forms, called the *matrix of curvature forms* associated to the coframe $\{\sigma^i\}$.

Proposition 2.2. *(1) The matrix (ω_j^i) of connection forms associated to a local orthonormal coframe $\{\sigma^i\}$ is skew symmetric, i.e. $\omega_j^i=-\omega_i^j$ and for each $i=1,\dots,n$ it satisfies the equation*

$$0=d\sigma^i+\textstyle\sum_j\omega_j^i\wedge\sigma^j.$$

These two properties uniquely determine (ω_i^j).

(2) The corresponding matrix (Ω_j^i) of curvature forms is also skew symmetric and it is given by

$$\Omega_j^i=d\omega_j^i+\textstyle\sum_k\omega_k^i\wedge\omega_j^k.$$

Proof. (1) By definition, we have

$$\omega_i^j(\xi)=\sigma^j(\nabla_\xi s_i)=g(\nabla_\xi s_i,s_j).$$

But since $g(s_i, s_j)$ is always constant, compatibility of ∇ with g implies that $0 = g(\nabla_\xi s_i, s_j) + g(s_i, \nabla_\xi s_j)$ and thus $\omega_i^j(\xi) = -\omega_j^i(\xi)$, so skew symmetry follows.

For a vector field $\eta \in \mathfrak{X}(U)$, we have noted in 2.1 that $\eta = \sum_j \sigma^j(\eta)s_j$. Hence we compute

$$\nabla_\xi \eta = \textstyle\sum_j \nabla_\xi(\sigma^j(\eta)s_j) = \sum_j(\xi \cdot \sigma^j(\eta))s_j + \sum_{j,k} \sigma^j(\eta)\omega_j^k(\xi)s_k.$$

Otherwise put, we get

$$\sigma^i(\nabla_\xi \eta) = \xi \cdot \sigma^i(\eta) + \textstyle\sum_j \sigma^j(\eta)\omega_j^i(\xi).$$

Now subtract the analogous term with ξ and η exchanged and further subtract $\sigma^i([\xi, \eta])$ from both sides. Then in the left hand side, we get zero by torsion freeness of ∇. In the right hand side, we can use the definition of the exterior derivative to conclude that

$$0 = d\sigma^i(\xi, \eta) + \textstyle\sum_j \left(\omega_j^i(\xi)\sigma^j(\eta) - \omega_j^i(\eta)\sigma^j(\xi)\right),$$

and the last term just represents $\sum_j(\omega_j^i \wedge \sigma^j)(\xi, \eta)$.

To prove the statement on uniqueness, we consider the difference of two skew symmetric matrices of one-forms, which both satisfy the equations. Then this is a matrix (τ_i^j) of one-forms such that $\tau_j^i = -\tau_i^j$ and such that $\sum_j \tau_j^i \wedge \sigma^j = 0$ for each $i = 1, \dots, n$. Now evaluate the last expression on (s_k, s_ℓ) to get $0 = \tau_\ell^i(s_k) - \tau_k^i(s_\ell)$. Hence if we put $\Phi_{ijk} := \tau_j^i(s_k)$, we get $\Phi_{ijk} = -\Phi_{ijk}$ and $\Phi_{ijk} = \Phi_{ikj}$ and we know from the proof of Proposition 1.1 that this implies $\Phi_{ijk} = 0$ and hence $\tau_j^i = 0$ for all i and j.

(2) By definition,

$$\Omega_i^j(\xi, \eta) = \sigma^j(R(\xi, \eta)(s_i)) = g(R(\xi, \eta)(s_i), s_j),$$

so skew symmetry follows from part (2) of Proposition 1.13. From the defining equation $\nabla_\eta s_i = \sum_k \omega_i^k(\eta)s_k$, we conclude that

$$\nabla_\xi \nabla_\eta s_i = \textstyle\sum_k(\xi \cdot \omega_i^k(\eta))s_k + \sum_{k,\ell} \omega_i^k(\eta)\omega_k^\ell(\xi)s_\ell,$$

and hence

$$\sigma^j(\nabla_\xi \nabla_\eta s_i) = \xi \cdot \omega_i^j(\eta) + \textstyle\sum_k \omega_k^j(\xi)\omega_i^k(\eta).$$

To obtain $\Omega_j^i(\xi, \eta)$ we have to subtract the corresponding term with ξ and η exchanged and further subtract $\sigma^j(\nabla_{[\xi,\eta]}s_i) = \omega_i^j([\xi, \eta])$. Now the result follows immediately from the definition of the exterior derivative and of the wedge product. □

2.3. Examples. (1) **Flat space**: In Euclidean space E^n, we take one of the global charts from 1.1 to identify E^n with $\mathbb{R}^n$. Then the corresponding coordinate vector fields ∂_i form a global orthonormal frame. The dual coframe is simply given by $\sigma^i = dx^i$ for $i = 1, \dots, n$. Since $d\sigma^i = 0$ for all i, we conclude that both the matrix (ω_j^i) of connection forms and the matrix (Ω_j^i) of curvature forms vanish identically in this frame.

Vanishing of the curvature forms reflects reflects the fact that the Riemann curvature of E^n vanishes identically, so this is a property of Euclidean space. Vanishing of the connections forms is not a property of Euclidean space but of the particularly nice frame that we have chosen. For more general frames the connection forms and thus the computation showing that the curvature forms vanish become much more complicated.

(2) **The sphere**: Let us consider the unit sphere $S^n := \{x \in \mathbb{R}^{n+1} : \langle x, x\rangle = 1\}$ with the Riemannian metric induced from $\mathbb{R}^{n+1}$. To get simple formulae, we use a particularly nice chart, the *stereographic projection*. Let $N = e_{n+1} \in S^n$ be the north pole, put $U := S^n \setminus \{N\}$ and define $u : U \to \mathbb{R}^n$ by

$$u(x) = u(x^1, \dots, x^{n+1}) = \tfrac{1}{1-x^{n+1}}(x^1, \dots, x^n)$$

(To interpret this geometrically, one views $\mathbb{R}^n$ as the affine hyperplane through $-N$ which is orthogonal to N and one maps each point $x \in S^n$ to the intersection of the ray from N through x with that affine hyperplane.) One immediately verifies that the map

$$(u^1, \dots, u^n) \mapsto \tfrac{1}{\langle u,u\rangle+1}(2u^1, \dots, 2u^n, \langle u,u\rangle - 1)$$

is inverse to u. The ith partial derivative of this mapping is given by

$$\tfrac{-2u^i}{(1+\langle u,u\rangle)^2}(2u, \langle u,u\rangle - 1) + \tfrac{1}{1+\langle u,u\rangle}(2e_i, 2u^i),$$

which shows that we can write $\frac{\partial}{\partial u^i} \circ u^{-1}$ as

$$\tfrac{-2u^i}{(1+\langle u,u\rangle)^2}\left(\sum_{j=1}^n 2u^j \tfrac{\partial}{\partial x^j} + (\langle u,u\rangle - 1)\tfrac{\partial}{\partial x^{n+1}}\right) + \tfrac{2}{1+\langle u,u\rangle}\left(\tfrac{\partial}{\partial x^i} + u^i \tfrac{\partial}{\partial x^{n+1}}\right).$$

Now we can compute the inner products of these vector fields using that the fields $\frac{\partial}{\partial x^j}$ are orthonormal. The bracket in the first summand is independent of i and inserting it twice into the metric, one gets $4\langle u,u\rangle + (\langle u,u\rangle - 1)^2 = (1+\langle u,u\rangle)^2$. So the contributions to $g(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^k}) \circ u^{-1}$ is given by $\frac{4u^iu^k}{(1+\langle u,u\rangle)^2}$. Likewise from the second terms, one obtains a contribution of $\frac{4}{(1+\langle u,u\rangle)^2}(\delta_{ik} + u^iu^k)$. Finally, the terms mixing the two summands give a contribution of $\frac{-8u^iu^k}{(1+\langle u,u\rangle)^2}$. Altogether, we see that

$$g(\tfrac{\partial}{\partial u^i}, \tfrac{\partial}{\partial u^k}) \circ u^{-1} = \tfrac{4}{(1+\langle u,u\rangle)^2}\delta_{ik}.$$

Now putting $f(x) := \frac{1}{2}(1 + \langle u(x), u(x)\rangle)$, i.e. $f = \frac{1}{2}(1 + \sum_i (u^i)^2)$ we see that $\{f\frac{\partial}{\partial u^i}\}$ is a local orthonormal frame and hence the one-forms $\sigma^i := \frac{1}{f}du^i$ form a local orthonormal coframe.

Consequently, $d\sigma^i = -\frac{1}{f^2}df \wedge du^i$ and since $df = \sum_j u^j du^j$ this can be written as $\sum_j \frac{u^j}{f^2} du^i \wedge du^j = \sum_j u^j \sigma^i \wedge \sigma^j$. This can be written as $-\sum_j \omega^i_j \wedge \sigma^j$ with

$$\omega^i_j := u^i\sigma^j - u^j\sigma^i = \tfrac{u^i}{f}du^j - \tfrac{u^j}{f}du^i. \tag{2.1}$$

This evidently satisfies $\omega^j_i = -\omega^i_j$ and thus gives the matrix of connection forms associated to our coframe.

Applying the exterior derivative to (2.1) immediately gives

$$d\omega^i_j = -\tfrac{u^i}{f^2}df \wedge du^j + \tfrac{u^j}{f^2}df \wedge du^i + \tfrac{2}{f}du^i \wedge du^j.$$

On the other hand, using $df = \sum_k u^k du^k$, we compute

$$\sum_k (\tfrac{u^i}{f}du^k - \tfrac{u^k}{f}du^i) \wedge (\tfrac{u^k}{f}du^j - \tfrac{u^j}{f}du^k) = \tfrac{u^i}{f^2}df \wedge du^j - \tfrac{u^j}{f^2}df \wedge du^i - \tfrac{\sum(u^k)^2}{f^2}du^i \wedge du^j.$$

Hence we directly get $\Omega^i_j = \frac{1}{f^2}du^i \wedge du^j = \sigma^i \wedge \sigma^j$. To understand the form of the curvature more explicitly, we look at the elements s_a of the orthonormal frame. By definition of the matrix of curvature forms, we have $R(\xi,\eta)(s_j) = \sum_i \Omega^i_j(\xi,\eta)s_i$ and hence $g(R(\xi,\eta)(s_j), s_i) = \Omega^i_j(\xi,\eta)$. Thus we can compute $g(R(s_a,s_b)(s_c), s_d)$ as

$$\Omega^d_c(s_a,s_b) = \sigma^d(s_a)\sigma^c(s_b) - \sigma^c(s_a)\sigma^d(s_a) = g(s_a,s_d)g(s_b,s_c) - g(s_a,s_c)g(s_b,s_d).$$

Since this is a tensorial expression, it holds for arbitrary vector fields instead of the elements of the frame, which shows that, in abstract index notation, we have $R_{ij}{}^a{}_\ell g_{ka} = g_{ik}g_{j\ell} - g_{i\ell}g_{jk}$ respectively $R_{ij}{}^k{}_\ell = \delta^k_i g_{j\ell} - \delta^k_j g_{i\ell}$. This is the simplest way to construct a tensor with curvature symmetries out of the metric. We will later say that the sphere has constant (positive) sectional curvature. As in the case of Euclidean space these computations get significantly more involved in more general frames.

(3) **Hyperbolic space**: Although this example is quite different from the sphere, the computations will quickly become very similar. We consider the open unit ball $\{x \in \mathbb{R}^n : \langle x, x\rangle < 1\}$ and define a metric there as $g := \frac{4}{(1-\langle x,x\rangle)^2} g_0$, where g_0 is the restriction of the flat metric. (As we define it here, this may seem rather artificial, but it arises from several other pictures in a natural way.) Putting $f(x) := \frac{1}{2}(1 - \langle x, x\rangle)$ we see that the vector fields $f\partial_i$ form an orthonormal frame, and the corresponding orthonormal coframe is obtained by putting $\sigma^i := \frac{1}{f} dx^i$. The only difference compared to the case of the sphere now is that $df = -\sum x^i dx^i$, so there is a sign change compared to the case of the sphere. This sign change carries over to $d\sigma^i$ and hence to ω^i_j, so this time we get $\omega^i_j = -x^i\sigma^j + x^j\sigma^i = -\frac{x^i}{f} dx^j + \frac{x^j}{f} dx^i$. As in the case of the sphere, one directly verifies that this leads to $\Omega^i_j = -\sigma^i \wedge \sigma^j$, so again there is a sign change compared to the sphere. As in the case of the sphere, one then verifies that $R_{ij}{}^a{}_\ell g_{ka} = -g_{ik}g_{j\ell} + g_{i\ell}g_{jk}$ respectively $R_{ij}{}^k{}_\ell = -\delta^k_i g_{j\ell} + \delta^k_j g_{i\ell}$. We will say that hyperbolic space has constant negative sectional curvature.

Geodesics, distance and completeness

One of the fundamental facts in Euclidean geometry is the fact that a line segment provides the shortest path connecting two points. Since the analogs of straight lines in general Riemannian manifolds are the geodesics, it is a natural question whether any two points can be connected by a geodesic and whether this is a (or even the) shortest curve connecting the two points.

The geodesics of a Riemannian metric also lead to a natural notion of completeness for Riemannian manifolds. It turns out that completeness is closely related to the interpretation of geodesics as shortest curves. Using this relation, this concept of completeness turns out to be equivalent to completeness in the sense of metric spaces. This result is called the Hopf–Rinow theorem, and it is one of the cornerstones of Riemannian geometry.

2.4. The first variational formula. We start with an elementary characterization of geodesics which is a first step towards identifying them as "shortest curves". As we have noted in 1.7, the arclength of a curve is invariant under reparametrizations, which make it less suitable for the purpose of characterizing curves, so we use the energy instead. We study the behavior of the energy under a variation of curves. Given a smooth curve $c : [a, b] \to M$, such a variation is a smooth mapping $\gamma : [a, b] \times (-\epsilon, \epsilon) \to M$ such that $\gamma(t, 0) = c(t)$. Evidently, we can view such a variation as a smooth family $\{c_s : [a, b] \to M : |s| < \epsilon\}$ of curves by putting $c_s(t) := \gamma(t, s)$. The "direction" of such a variation can be described by $r(t) := \frac{\partial}{\partial s}|_{s=0}\gamma(t, s)$. This evidently is a vector field along c called the *variational vector field* determined by γ. A particularly interesting case is provided by variations fixing the endpoints, where one in addition requires that $\gamma(a, s) = c(a)$ and $\gamma(b, s) = c(b)$ for all s. The infinitesimal version of this condition of course is $r(a) = r(b) = 0$.

Given a variation γ of c, we can consider the resulting variation of energy, i.e. look at $E(s) := \frac{1}{2}\int_a^b g(\gamma(t, s))(\gamma'(t, s), \gamma'(t, s))dt$, where we write $\gamma'(t, s)$ for $\frac{\partial}{\partial t}\gamma(t, s)$. Evidently, this is a smooth function $(-\epsilon, \epsilon) \to \mathbb{R}$, so we can try to compute the infinitesimal variation $\frac{d}{ds}|_{s=0}E(s)$ of energy. The result is very appealing:

PROPOSITION 2.4 (First variational formula). *Let γ be a smooth variation of $c : [a, b] \to M$ with variation vector field r. Then the infinitesimal variation of energy is* *given by*

$$\tfrac{d}{ds}|_{s=0}E(s) = -\int_a^b g(c(t))(\nabla_{c'}c'(t), r(t))dt + g(c(b))(c'(b), r(b)) - g(c(a))(c'(a), r(a)).$$

In particular, a smooth curve c is a critical point for the energy under all variations with fixed endpoints if and only if c is a geodesic.

Proof. The formula on $[a, b]$ clearly follows from the analogous formula on small sub-intervals of $[a, b]$. Thus, we may restrict to the case that γ has values in the domain U of a chart (U, u) for M. In this chart, we can use local coordinate expressions and in terms of the corresponding component functions, we get

$$E(s) = \tfrac{1}{2}\int_a^b \textstyle\sum_{i,j} g_{ij}(\gamma(t,s))(\gamma'(t,s))^i(\gamma'(t,s))^j dt.$$

Differentiating this with respect to s at $s = 0$, we can exchange the derivative with the integral and then use the product rule and the chain rule. Observe that $\frac{\partial}{\partial s}|_{s=0}\gamma(t,s) = r(t)$ and since partial derivative commute, we get that $\frac{\partial}{\partial s}|_{s=0}\gamma'(t,s) = r'(t)$. Using this, we see that $\frac{d}{ds}|_{s=0}E(s)$ is given by

$$\tfrac{1}{2}\int_a^b \textstyle\sum_{i,j,\ell} \partial_\ell g_{ij}(c(t))(r(t))^\ell(c'(t))^i(c'(t))^j dt + \displaystyle\int_a^b \textstyle\sum_{i,j} g_{ij}(c(t))(r'(t))^i(c'(t))^j dt. \tag{2.2}$$

Now in the second summand, we can integrate by parts to remove the derivative from r. First, this gives boundary terms, which evidently reduce to $g(c(b))(r(b), c'(b)) - g(c(a))(r(a), c'(a))$. To this, we have to add an integral term, for which we obtain in the same way as above

$$-\int_a^b (r(t))^i \left(\textstyle\sum_{i,j,\ell} \partial_\ell g_{ij}(c(t))(c'(t))^\ell(c'(t))^j + \sum_{i,j} g_{ij}(c(t))(c''(t))^j\right) dt.$$

The second summand can be expressed as $-\int_a^b g(c(t))(r(t), c''(t))dt$. Moreover, the formula for Γ^k_{ij} in part (2) of Proposition 1.10 exactly says that the first summand adds up with the first summand in (2.2) to $-\int_a^b g(\Gamma^U(c(t))(c'(t), c'(t)), r(t))$. Since Proposition 1.11 shows that $\nabla_{c'}c'(t) = c''(t) + \Gamma^U(c(t))(c'(t), c'(t))$, this completes the proof. □

The computation in the proof actually allows an elementary approach to the construction of the Levi-Civita connection. Motivated by the computation, one shows that, in the domain of a chart, one can write $2(Dg(x)(\xi))(\xi, \eta) - (Dg(x)(\eta))(\xi, \xi)$ as $g(x)(Q_x(\xi), \eta)$ for a quadratic form Q_x. This then determines a symmetric bilinear form Γ_x such that $Q_x(\xi) = \Gamma_x(\xi, \xi)$. Then one can use these forms to define a covariant derivative in charts and verify directly that the definitions in different charts coincide, so one obtains a globally defined covariant derivative.

2.5. Minimizing curves. Given a point x in a Riemannian manifold (M, g) we have seen in Proposition 1.12 that there is an open neighborhood of zero in T_xM on which the exponential map $\exp_x$ restricts to a diffeomorphism onto an open neighborhood of x in M. In particular, there is a number $\epsilon > 0$ such that $\exp_x$ restricts to a diffeomorphism from the ball of radius ϵ (with respect to g_x) in T_xM onto a neighborhood U of x in M. We will phrase this by saying that U is *the geodesic ball of radius ϵ around x*. So geodesic balls exist for sufficiently small radii but not necessarily for large radii. Now any point y in a geodesic ball U can be written as $\exp(X)$ for some X in that ball, and hence $t \mapsto \exp_x(tX)$ defines a geodesic $c : [0,1] \to M$ such that $c(0) = x$ and $c(1) = y$. So any point in a geodesic ball can be joined to x by a geodesic.

On the other hand, for $0 < \delta < \epsilon$, we can consider the sphere of radius δ in T_xM. Its image under $\exp_x$ is called the *geodesic sphere* $S_\delta(x)$ of radius δ around x.

Lemma 2.5 (Gauß). *Let x be a point in a Riemannian manifold (M, g) and let $\epsilon > 0$ be chosen in such a way that the geodesic ball U of radius ϵ around x exists. Then for each $0 < \delta < \epsilon$, the geodesic sphere $S_\delta(x)$ is a smooth submanifold in M and the geodesics through x intersect this submanifold orthogonally.*

Proof. Since any sphere in T_xM is a submanifold in any ball containing it, and $S_\delta(x)$ is the image of one of these spheres under a diffeomorphism, it is a submanifold, too. Now take any smooth curve $v(s)$ in the sphere of radius δ in T_xM and for $t \in [0,1]$ define $\gamma(t,s) := \exp_x(tv(s))$. This is a smooth variation of the curve $c(t) = \exp_x(tv(0))$ which is a geodesic. But it is also true that for each fixed s, the curve $c_s(t) = \exp_x(tv(s))$ is a geodesic. Thus $g(c_s(t))(c_s'(t), c_s'(t))$ is constant and its value at $t = 0$ of course is $g_x(v(s), v(s)) = \delta^2$. In particular, the energy of this variation is constant in s, so $0 = \frac{d}{ds}|_{s=0}E(s)$.

But we can also compute this infinitesimal variation using the first variational formula, and since c is a geodesic, only the boundary terms survive in this formula. Moreover, $\frac{\partial}{\partial s}\gamma(t,s) := (T_{tv(s)})\exp_x(t\frac{d}{ds}v(s))$, so the variation vector field r satisfies $r(0) = 0$ and $r(1) = T_{v(0)}\exp_x \cdot v'(0)$. Thus the first variational formula simply tells us that $0 = g(\exp_x(v(0)))(c'(1), \xi)$ for any tangent vector ξ which can be written as $T_{v(0)}\exp_x \cdot v'(0)$. By construction, any vector tangent to $S_\delta(x)$ can be written in this form, so the whole tangent space of the geodesic sphere is orthogonal to the tangent vector $c'(1)$ of the geodesic c. □

Now by a *minimizing curve*, we mean a piece-wise smooth curve $c : [a,b] \to M$ which is a shortest connection between its endpoints, i.e. satisfies $d(c(a), c(b)) = L(c)$. We can next prove that for nearby points, minimizing curves exist and are geodesics (up to parametrization).

Proposition 2.5. *Let (M, g) a Riemannian manifold, $x \in M$ a point and $\epsilon > 0$ such that the a geodesic ball U around x of radius ϵ exists, and put $B_\epsilon(0) := \{\xi \in T_xM : g_x(\xi,\xi) < \epsilon^2\}$.*

(1) Let $u : [a,b] \to (0,\epsilon)$ and $v : [a,b] \to T_xM$ be smooth functions such that $g_x(v(t), v(t)) = 1$ for all t and put $c(t) := \exp_x(u(t)v(t))$. Then the arc length of c satisfies $L(c) \geq |u(b) - u(a)|$ and equality holds if and only if u is monotonous and v is constant.

(2) For $y = \exp_x(\xi) \in U$, the geodesic $t \mapsto \exp_x(t\xi)$ is a minimizing curve joining x to y, and up to reparametrizations it is the unique such curve.

Proof. (1) By construction, we get $c'(t) = T_{u(t)v(t)}\exp_x(u'(t)v(t) + u(t)v'(t))$. Along the line spanned by $v(t)$, the vector $T\exp_x(v(t))$ is the speed vector of a geodesic, whence we conclude that $g(T\exp_x(u'(t)v(t)), T\exp_x(u'(t)v(t))) = |u'(t)|^2$. On the other hand, $T_{u(t)v(t)}\exp_x \cdot (u(t)v'(t))$ is tangent to $S_{u(t)}(x)$ and hence orthogonal to $T_{u(t)v(t)}\exp_x \cdot (u'(t)v(t))$ by Lemma 2.5. Hence we get $g(c'(t), c'(t)) \geq |u'(t)|^2$ with equality only if $v'(t) = 0$.

Hence we obtain $L(c) \geq \int_a^b |u'(t)|dt \geq |\int_a^b u'(t)dt| = |u(b) - u(a)|$ as claimed. The first inequality becomes an equality if and only if $v'(t) = 0$ for all t i.e. iff v is constant. The second inequality becomes an equality if and only if $u'(t)$ has constant sign and hence u is monotonous.

(2) By assumption, $y \in S_\rho(x)$ for some $\rho < \epsilon$. Of course have $d(x, y) \leq \rho$, since the geodesic joining x to y has length ρ. From (1) we conclude that a curve joining x to y which stays in $S_\rho \cup \exp_x(B_\rho(0))$ has length at least ρ, since outside of x, any such curve can be written in the form used in (1). But any curve leaving this set has to have larger length, since the part up to the first intersection with $S_\rho(x)$ already has length $\geq \rho$. This shows that $d(x, y) = \rho$, so the geodesic is a minimizing curve.

Conversely, a minimizing curve connecting x to y must stay in $S_\rho \cup \exp_x(B_\rho(0))$. Now it follows immediately from the definition that the restriction of a minimizing curve to a smaller interval is still minimizing. Applying the equality part of (1) outside of x shows that a minimizing curve there must be of the form $\exp_x(u(t)v)$ for a monotonous function u, and hence a reparametrization of the geodesic $\exp_x(tv)$. □

We can further use this to conclude that short pieces of minimizing curves always are geodesics.

Corollary 2.5. *Let $c : [a, b] \to M$ be a piece-wise smooth minimizing curve. Then for each $t \in (a, b)$, there are $a' < t < b'$ such that $c|_{(a',b')}$ is a reparametrization of a geodesic. In particular, c can be parametrized smoothly.*

Proof. Given t, we claim that we can find $a' < t$ and $\epsilon > 0$ such that there is a geodesic ball U of radius ϵ around $c(a')$ such that $c(t) \in U$. Having shown this, the fact that U is open implies that there is a $b' > t$ such that $c([a', b']) \subset U$. As we have noticed above already, $c|_{[a',b']}$ is also minimizing, so the result follows from the last part of Proposition 2.5.

To prove the claim, recall the by part (4) of Proposition 1.12, there is an open neighborhood V of the zero section in TM on which $(\pi, \exp)$ restricts to a diffeomorphism. This implies that we can find an open neighborhood W of $c(t)$ in M and a number $\epsilon > 0$ such that $\tilde{V} := \{\xi : \pi(\xi) \in W, |\xi| < \epsilon\} \subset V$, where the norm of ξ is taken with respect to $g_{\pi(\xi)}$. Continuity of c then implies that we can choose $a' < t$ such that $c(a') \in W$ and $(c(a'), c(t)) \in (\pi, \exp)(\tilde{V})$, which shows that a' and ϵ have the required properties. □

2.6. Completeness and the Hopf–Rinow theorem. In our discussion of geodesics in 1.12, we have proved existence of local solutions to the geodesic equation. The natural completeness condition coming from geodesics is that all these solutions are defined for all times.

Definition 2.6. A Riemannian metric g on a smooth manifold M is called (geodesically) *complete* if for any $x \in M$ and $\xi \in T_xM$, there exists a geodesic $c : \mathbb{R} \to M$ such that $c(0) = x$ and $c'(0) = \xi$. In this case, (M, g) is called a (geodesically) complete Riemannian manifold.

The Hopf–Rinow theorem shows that the notion of geodesic completeness is equivalent to completeness of the metric space (M, d_g) and at the same time proves an important property of complete Riemannian manifolds.

Theorem 2.6 (Hopf-Rinow). *Let (M, g) be a connected smooth Riemannian manifold and let d_g be the distance function associated to g as in Proposition 1.7. Then the following conditions are equivalent*

(i) The metric g is geodesically complete.

(ii) (M, d_g) is a complete metric space, i.e. any Cauchy sequence converges.

(iii) (M, d_g) has the Heine–Borel property, i.e. bounded closed subsets are compact.

(iv) There exists a point $x \in M$ such that $\exp_x$ is defined on all of T_xM.

Moreover, these equivalent conditions imply *(v) For any two points $x, y \in M$, there is a minimizing geodesic connecting x to y.*

Proof. It is clear that (i) implies (iv), and the fact that (iii) implies (ii) is a general result for metric spaces. (A Cauchy sequence is a bounded set, so (iii) implies that its closure is compact. Hence there is a convergent subsequence, which already implies that the initial Cauchy sequence converges.)

(ii)$\Rightarrow$(i): Assume that (ii) holds and that c is a geodesic in M, whose maximal interval (a, b) of definition is finite. Without loss of generality, we may assume that $g(c'(t), c'(t))$ (which is constant since c is a geodesic) is equal to one. This implies that for all $s, t \in (a, b)$ we have $d_g(c(s), c(t)) \leq |t - s|$. It suffices to show that the domain of definition of c can be extended on one side. Thus assume that $b < \infty$, choose a sequence t_i converging to b and consider the sequence $(c(t_i))$ in (M, d_g). By construction, this is a Cauchy sequence, so there is a point $x \in M$ such that $c(t_i)$ converges to x. As in the proof of Corollary 2.5 we can find an index i and a number $\epsilon > 0$ such that the geodesic ball of radius ϵ around $c(t_i)$ exists and contains x. Then $\gamma(s) := \exp_{c(t_i)}(sc'(t_i))$ is a well defined geodesic for $|s| < \epsilon$. But $\gamma(0) = c(t_i)$ and $\gamma'(0) = c'(t_i)$ so $\gamma(s) = c(t_i + s)$ as long as $t_i + s \in (a, b)$. But by assumption $b < t_i + \epsilon$, so we obtain an extension of the domain of definition to $(a, t_i + \epsilon)$, which is a contradiction.

We next claim that if for a point $x \in M$, $\exp_x$ is defined on all of T_xM, then for any point $y \in M$, there is a minimizing geodesic connecting x to y. Put $r = d_g(x, y)$, let U be a geodesic ball around x of some radius $\epsilon > 0$ and fix $\delta < \epsilon$. Then the geodesic sphere $S_\delta(x) \subset M$ is the image of a compact submanifold of $B_\epsilon(0)$ under a diffeomorphism and hence compact. Thus there is a point $z \in S_\delta(x)$ at which the continuous function $d_g(\ , y)$ attains its minimum. From Proposition 2.5 we know that any point in $S_\delta(x)$ has distance δ from x. Together with the fact that any piece-wise smooth curve from x to y has to intersect $S_\delta(x)$, this easily implies that $d_g(z, y) = r - \delta$.

Now there is a unique unit vector $\xi \in T_xM$ such that $z = \exp_x(\delta\xi)$ and we consider the geodesic $c(t) := \exp_x(t\xi)$ emanating from x in direction ξ. By construction, this satisfies $g_{c(t)}(c'(t), c'(t)) = 1$, so it is parametrized by arclength. Now we define $A := \{t \in [\delta, r] : d_g(c(t), y) = r - t\}$, and we want to show that $r \in A$, which implies that $c(r) = y$, and hence the claim. As observed above, $d_g(z, y) = r - \delta$, so $\delta \in A$ and A is non-empty. Moreover, $A \subset [\delta, r]$ is the subset on which two continuous functions agree, so it is closed.

Therefore, putting $s_0 := \sup(A)$, we get $s_0 \in A$. If $s_0 < r$, then we can find a $0 < \delta' < \epsilon$ such that $s_0 + \delta' < r$ and the geodesic ball of radius ϵ around $c(s_0)$ exists. As above, the geodesic sphere $S_{\delta'}(c(s_0))$ contains a point z' which has minimal distance to y, and $d_g(z', y) = r - s_0 - \delta'$. But this implies that $d_g(z', x) \geq s_0 + \delta'$. As above, we can write $z' = \exp_{c(s_0)}(\delta'\xi')$ for a unit vector $\xi' \in T_{c(s_0)}M$, and we denote by $\tilde{c}$ the corresponding unit speed geodesic emanating from $c(s_0)$. This shows that first going from x to $c(s_0)$ via c and then going to z' via $\tilde{c}$ is a minimizing curve connecting x to z'. By Corollary 2.5 this has to coincide with a geodesic on a neighborhood of s_0, which is only possible if $\xi' = c'(s_0)$. But this implies that $s_0 + \delta' \in A$, which is a contradiction. Thus the proof of the claim is complete.

Using this claim, we can now prove that (iv) implies (iii), which completes the proof of the equivalences. Indeed, if $K \subset M$ is bounded then there is a constant C such that $d_g(x, y) \leq C$ for all $y \in K$, where x is the point occurring in (iv). But by the claim, this implies that K is contained in the image of the closed ball of radius C in T_xM under $\exp_x$, which is compact by continuity of $\exp_x$. Hence if K is closed, it is compact, too.

Having the equivalence at hand, we see that if (iv) is satisfied for one point $x \in M$, it implies (i), which in turn says that (iv) is satisfied for any point of M. Hence (v) follows from the claim. □

COROLLARY 2.6. *(1) Any compact Riemannian manifold is complete.*

(2) If M is a closed submanifold of $\mathbb{R}^n$ for some n, and one endows M with the Riemannian metric g induced from the inner product of $\mathbb{R}^n$, then (M,g) is complete.

(3) If (M,g) is a complete Riemannian manifold, then for each $x \in M$, the exponential map defines a surjection $\exp_x : T_xM \to M$.

PROOF. (1) Follows from the well known fact that compact metric spaces are automatically complete.

For (2), observe that for a smooth curve in M connecting two points x and y, the arclength is always at least the Euclidean distance between x and y. But this shows that any subset in M which is bounded with respect to d_g is also bounded with respect to Euclidean distance, so closed subsets with this property are automatically compact.

(3) immediately follows from condition (v) in the Hopf–Rinow theorem. □

It turns out that hyperbolic space as discussed in part (3) of 2.3 is a complete Riemannian manifold. This example nicely illustrates two general phenomena. Starting from the unit ball in $\mathbb{R}^n$ with the restriction g_0 of the flat metric (which evidently is not complete), we have obtained the hyperbolic metric as a so-called *conformal rescaling*, i.e. $g = fg_0$ for a positive smooth function f. Rescaling a metric conformally does change the notion of length, but it does not change the notions of angles, so in particular, one obtains the same concept of orthogonality. Now the first general phenomenon mentioned above is that given an arbitrary Riemannian manifold (M,g_0), one can always find a positive smooth function $f : M \to \mathbb{R}$ such that $g := fg_0$ defines a complete Riemannian metric on M. Intuitively, one can think about this as "moving the missing points to infinity".

The second phenomenon is a kind of converse of this. By the Hopf–Rinow theorem, for a non-compact, complete Riemannian manifold (M,g), M must be unbounded with respect to the distance function d_g. In the case of hyperbolic space, we can also start with the hyperbolic metric g and view g_0 as a conformal rescaling of g, in which the manifold becomes bounded. Again this works in general, so any Riemannian metric can be rescaled to one leading to a bounded distance on M (which then has to be incomplete unless M is compact).

Covariant derivative of tensor fields

The covariant derivative and parallel transport can be extended to tensor fields, basically by requiring certain naturality properties. This for example allows us to form the covariant derivative of the curvature. Moreover, it leads to an interpretation in which we can iterate covariant derivatives and thus construct higher order differential operators.

2.7. Basic notions. The extension of the covariant derivative is determined by requiring certain naturality properties. These properties are analogous to those satisfied by the Lie derivative with respect to a vector field, see Section 3.4 of [**AnaMF**]. On the one hand, for smooth functions, one already has an appropriate operation given by the usual action of vector fields on smooth functions. Let us denote by $\mathcal{T}^\ell_k(M)$ the space of smooth $\binom{\ell}{k}$-tensor fields on a smooth manifold M. Then we want to use the Levi-Civita connection to define operators $\nabla : \mathfrak{X}(M) \times \mathcal{T}^\ell_k(M) \to \mathcal{T}^\ell_k(M)$ with properties analogous to the covariant derivative. In particular, ∇ should be linear over smooth functions in the $\mathfrak{X}(M)$ component.

It turns out that the only thing to require in addition is a compatibility with tensor products and with contractions. This then pins down the whole operation completely.

PROPOSITION 2.7. *Suppose that ∇ is a linear connection on the tangent bundle of a smooth manifold M. Then this extends uniquely to a family of bilinear operators $\nabla : \mathfrak{X}(M) \times \mathcal{T}^k_\ell(M) \to \mathcal{T}^k_\ell(M)$ which are linear over smooth functions in the first variable, commute with contractions, and satisfy $\nabla_\xi(s \otimes t) = (\nabla_\xi s) \otimes t + s \otimes \nabla_\xi t$ as well as $\nabla_\xi f = \xi \cdot f$ for $f \in \mathcal{T}^0_0(M) = C^\infty(M, \mathbb{R})$.*

PROOF. Let us first look at the case of $\mathcal{T}^0_1(M) = \Omega^1(M)$. Given $\xi, \eta \in \mathfrak{X}(M)$ and $\varphi \in \Omega^1(M)$ we can write the smooth function $\varphi(\eta)$ as the result of the unique possible contraction applied to $\varphi \otimes \eta \in \mathcal{T}^1_1(M)$. If an extension with the required properties exists, then the contraction of $(\nabla_\xi \varphi) \otimes \eta + \varphi \otimes (\nabla_\xi \eta)$ has to coincide with $\xi \cdot \varphi(\eta)$. Thus we try defining $\nabla_\xi \varphi$ as a map $\mathfrak{X}(M) \to C^\infty(M, \mathbb{R})$ by

$$\nabla_\xi \varphi(\eta) := \xi \cdot \varphi(\eta) - \varphi(\nabla_\xi \eta). \tag{2.3}$$

This map is immediately seen to be linear over smooth functions in η, so we have defined $\nabla_\xi \varphi \in \Omega^1(M)$. Moreover, the definition readily implies that $\nabla_{f\xi} \varphi = f \nabla_\xi \varphi$ and that

$$(\nabla_\xi (f\varphi))(\eta) = f \nabla_\xi \varphi(\eta) + (\xi \cdot f)\varphi(\eta)$$

and hence $\nabla_\xi f\varphi = f \nabla_\xi \varphi + (\xi \cdot f)\varphi$.

Having this at hand, the general definition of the covariant derivative is motivated in the same way. Given $t \in \mathcal{T}^\ell_k$ and $\xi \in \mathfrak{X}(M)$, we define $\nabla_\xi t$ as a $(k + \ell)$-linear map $\mathfrak{X}(M)^k \times \Omega^1(M)^\ell \to C^\infty(M, \mathbb{R})$ by

$$\begin{aligned} (\nabla_\xi t)(\eta_1, \dots, \eta_k, \varphi^1, \dots, \varphi^\ell) &:= \xi \cdot t(\eta_1, \dots, \eta_k, \varphi^1, \dots, \varphi^\ell) \\ &- \textstyle\sum_{i=1}^k t(\eta_1, \dots, \nabla_\xi \eta_i, \dots, \eta_k, \varphi^1, \dots, \varphi^\ell) \\ &- \textstyle\sum_{j=1}^\ell t(\eta_1, \dots, \eta_k, \varphi^1, \dots, \nabla_\xi \varphi^j, \dots, \varphi^\ell). \end{aligned} \tag{2.4}$$

Similarly as above, one verifies directly that this map is linear over smooth functions in each η_i and each φ_j, so we have defined $\nabla_\xi t \in \mathcal{T}^\ell_k(M)$. We also see directly from the formula that $\nabla_{f\xi} t = f \nabla_\xi t$. As in the case of one-forms, this formula is forced from the properties we want to achieve, since $t(\eta_1, \dots, \varphi_\ell)$ can be obtained via a sequence of contractions from $t \otimes \eta_1 \otimes \dots \otimes \varphi_\ell$. This shows the the required properties pin down the covariant derivative completely.

So it remains to prove the compatibility with tensor products and with contractions in general. Concerning tensor products, we take $t \in \mathcal{T}^\ell_k(M)$ and $s \in \mathcal{T}^{\ell'}_{k'}(M)$ and $\xi \in \mathfrak{X}(M)$ and expand the defining equation for $\nabla_\xi (t \otimes s)(\eta_1, \dots, \eta_{k+k'}, \varphi^1, \dots, \varphi^{\ell+\ell'})$ as in (2.4). By definition $(t \otimes s)(\eta_1, \dots, \eta_{k+k'}, \varphi^1, \dots, \varphi^{\ell+\ell'})$ is given by

$$t(\eta_1, \dots, \eta_k, \varphi^1, \dots, \varphi^\ell) s(\eta_{k+1}, \dots, \eta_{k+k'}, \varphi^{\ell+1}, \dots, \varphi^{\ell+\ell'}).$$

Applying ξ to this product of smooth functions, we apply the Leibniz rule. The first term in the result adds up with those terms in which the covariant derivatives hits one of the first k η's or one of the first ℓ φ's to

$$(\nabla_\xi t)(\eta_1, \dots, \eta_k, \varphi^1, \dots, \varphi^\ell) s(\eta_{k+1}, \dots, \eta_{k+k'}, \varphi^{\ell+1}, \dots, \varphi^{\ell+\ell'}).$$

This is exactly the action of $(\nabla_\xi t) \otimes s$ on the given vector fields an one-forms. In the same way, the remaining terms add up to the action of $t \otimes \nabla_\xi s$, so the compatibility with tensor products is proved.

Let us next look at the basic contraction, which can be viewed as a tensorial operator $C : \mathcal{T}^1_1(M) \to C^\infty(M, \mathbb{R})$. Given $\eta \in \mathfrak{X}(M)$ and $\varphi \in \Omega^1(M)$, we get $\eta \otimes \varphi \in \mathcal{T}^1_1(M)$ and $C(\eta \otimes \varphi) = \varphi(\eta)$. The definition of ∇ on $\Omega^1(M)$ together with compatibility with the tensor product shows that

$$C(\nabla_\xi(\eta \otimes \varphi)) = \xi \cdot \varphi(\eta) = \nabla_\xi(C(\eta \otimes \varphi)).$$

The definition in (2.4) also implies that the covariant derivative on tensor fields is a local operator. But locally any element of $\mathcal{T}^1_1(M)$ can be written as a finite sum of such tensor products, so compatibility of ∇ with C follows.

Now let us consider a general contraction $\mathcal{T}^\ell_k(M) \to \mathcal{T}^{\ell-1}_{k-1}(M)$, say the one contracting the ith upper index into the jth lower one. On a tensor field of the form $t \otimes \psi \otimes s$ with $t \in \mathcal{T}^{i-1}_{j-1}(M)$, $\psi \in \mathcal{T}^1_1(M)$ and $s \in \mathcal{T}^{\ell-i}_{k-j}(M)$, this contraction is given by $C(\psi)t \otimes s$. For $\xi \in \mathfrak{X}(M)$ we then conclude that the contraction of $\nabla_\xi(t \otimes \psi \otimes s)$ is given by

$$C(\psi)(\nabla_\xi t) \otimes s + C(\nabla_\xi \psi)t \otimes s + C(\psi)t \otimes \nabla_\xi s.$$

Since we have verified $C(\nabla_\xi \psi) = \xi \cdot C(\psi)$ already, we see that this coincides with $\nabla_\xi(C(\psi)t \otimes s)$. Locally, any element of $\mathcal{T}^\ell_k(M)$ can be written as a finite sum of such tensor products, so compatibility of the contraction with the covariant derivative holds in general. Since general contractions can be obtained by iterating contractions of a single pair of indices, the proof is complete. □

Remark 2.7. (1) For a smooth function f and a tensor field t, $f \otimes t$ is just the product ft, so $\nabla_\xi(ft) = (\xi \cdot f)t + f\nabla_\xi t$ holds in general as a consequence of the compatibility with tensor products.

(2) Given a tensor field $g \in T^0_2(M)$, the formula for the covariant derivative from the proof reads as

$$(\nabla_\xi g)(\eta, \zeta) = \xi \cdot g(\eta, \zeta) - g(\nabla_\xi \eta, \zeta) - g(\eta, \nabla_\xi \zeta).$$

Hence the condition that a linear connection ∇ on TM is metric with respect to a Riemannian metric g on M reads as $\nabla_\xi g = 0$ for the induced connection and any vector field ξ.

2.8. Parallel tensor fields. From the formula (2.4) for the covariant derivative in the proof of Proposition 2.7, we can easily derive a description in local coordinates. In the domain of a chart (U, u), a tensor field $t \in \mathcal{T}^\ell_k(M)$ is determined by the functions $t^{i_1\dots i_\ell}_{j_1\dots j_k}$ which can be obtained as

$$t^{i_1,\dots,i_\ell}_{j_1\dots j_k} = t(\partial_{j_1}, \dots, \partial_{j_k}, du^{i_1}, \dots, du^{i_\ell}).$$

Writing $\xi \in \mathfrak{X}(M)$ as $\sum_i \xi^i \partial_i$ in the domain of the chart, we by definition get $\nabla_\xi \partial_j = \sum_{i,a} \xi^i \Gamma^a_{ij} \partial_a$. Likewise, we can expand $\nabla_\xi du^i = \sum_j (\nabla_\xi du^i)(\partial_j) du^j$, which easily leads to $\nabla_\xi du^i = \sum_{j,a} \xi^j \Gamma^i_{ja} du^a$. Together, these observations immediately imply that

$$\begin{aligned}(\nabla_\xi t)^{i_1\dots i_\ell}_{j_1\dots j_k} = \xi \cdot t^{i_1\dots i_\ell}_{j_1\dots j_k} - \textstyle\sum_{i,a} \xi^i \Gamma^a_{ij_1} t^{i_1\dots i_\ell}_{a j_2\dots j_k} - \dots - \textstyle\sum_{i,a} \xi^i \Gamma^a_{ij_k} t^{i_1\dots i_\ell}_{j_1\dots j_{k-1} a} \\ - \textstyle\sum_{j,a} \xi^j \Gamma^{i_1}_{ja} t^{a i_2\dots i_\ell}_{j_1\dots j_k} - \dots - \textstyle\sum_{j,a} \xi^j \Gamma^{i_\ell}_{ja} t^{i_1\dots i_{\ell-1} a}_{j_1\dots j_k}.\end{aligned}$$

As in the case of vector fields, this implies that to compute $\nabla_\xi t(x)$, it suffices to know t along the flow line of ξ through x. Consequently, we can mimic the developments in 1.11 in the case of tensor fields. Given a smooth curve $c : I \to M$, we define $\binom{\ell}{k}$-tensor fields along c and then obtain a well defined linear operator $t \mapsto \nabla_{c'} t$ on the space of such tensor fields. In particular, there is the concept of a tensor field being parallel along a curve. Since in local coordinates being parallel is again a system of linear first order ODE, for $a \in I$ and $x = c(a) \in M$, we can uniquely extend any element $t_0 \in \otimes^\ell T_xM \otimes \otimes^k T_x^*M$ to a $\binom{\ell}{k}$-tensor field along c which is parallel along c. For $[a,b] \subset I$, this gives rise to a well defined parallel transport of tensors along c. From the construction, one easily verifies that this is exactly the map which gets functorially induced by the parallel transport of vector fields.

For the Levi-Civita connection of a Riemannian manifold (M,g) we have noted above that the induced connection on $\mathcal{T}^0_2(M)$ has the property that $\nabla_\xi g = 0$ for any ξ. A tensor field with this property is called *parallel* since it is parallel along any smooth curve. Surprisingly, parallel tensor fields of any type on a Riemannian manifold can be described provided that on knows the holonomy of the metric as introduced in 1.11. Given a point $x \in M$, we have introduced there the holonomy group $\mathrm{Hol}_x(M)$ of M at x, which is a subgroup of the orthogonal group $O(T_xM)$. Observe that any linear automorphism of T_xM induces a linear automorphism of each of the tensor powers $\otimes^\ell T_xM \otimes \otimes^k T_x^*M$. Hence any element of the holonomy group acts on the values of tensor fields of any type at x.

Proposition 2.8. *Let (M,g) be a connected Riemannian manifold and let $x \in M$ be a point.*

(1) A parallel tensor field $t \in \mathcal{T}^\ell_k(M)$ is uniquely determined by its value $t(x) \in \otimes^\ell T_xM \otimes \otimes^k T_x^*M$.

(2) Given an element $t_0 \in \otimes^\ell T_xM \otimes \otimes^k T_x^*M$, there is a parallel tensor field $t \in \mathcal{T}^\ell_k(M)$ such that $t(x) = t_0$ if and only if t_0 is mapped to itself by any element of the holonomy group $\mathrm{Hol}_x(M)$ of M at x.

Proof. (1) If $t \in \mathcal{T}^\ell_k(M)$ is parallel, it is parallel along each smooth curve. Given a point y in M, connectedness of M implies that there is a smooth curve $c : [a,b] \to M$ such that $c(a) = x$ and $c(b) = y$. But then we must have $t(y) = \mathrm{Pt}_c(t(x))$.

(2) The necessity of the condition follows readily since t is parallel along each smooth curve. To prove sufficiency, one observes that the fact that t_0 is preserved by any element of $\mathrm{Hol}_x(M)$ is equivalent to the fact that for two curves c and $\tilde{c}$ connecting x to some point $y \in M$, we get $\mathrm{Pt}_c(t_0) = \mathrm{Pt}_{\tilde{c}}(t_0)$. This is because transporting t_0 to y parallely along c and transporting the result back to x parallely along $\tilde{c}$ is the parallel transport along the pice-wise smooth closed curve obtained by first running through c and then backwards through $\tilde{c}$. Hence this is given by the action of an element of the holonomy group.

Knowing this, we can extend t_0 to a tensor field t by defining $t(y)$ as $\mathrm{Pt}_c(t_0)$ for any pice-wise smooth curve c connecting x to y. It is easy to see that the result is smooth and it is parallel along any smooth curve by construction. □

Note that the statement that g is parallel fits nicely into the picture, since any element of $\mathrm{Hol}_x(M)$ is orthogonal with respect to g_x and this exactly means that the induced map on $\otimes^2 T_x^*M$ preserves g_x.

2.9. Natural differential operators. We can interpret the covariant derivative as a linear differential operator (even in the case of vector fields). In this picture the covariant derivative can be iterated, thus providing the possibility to construct operators of higher order.

The first observation we need is that for a tensor field $t \in \mathcal{T}^\ell_k(M)$ we can consider the $(k+\ell+1)$-linear map $\nabla t : \mathfrak{X}(M)^{k+1} \times \Omega^1(M)^\ell \to C^\infty(M,\mathbb{R})$ defined by

$$(\nabla t)(\eta_0, \dots, \eta_k, \varphi_1, \dots, \varphi_\ell) := (\nabla_{\eta_0} t)(\eta_1, \dots, \eta_k, \varphi_1, \dots, \varphi_\ell).$$

From Proposition 2.7 we know that $\nabla_{\eta_0}t$ is a tensor field, so this is linear over smooth functions in all entries but η_0. But in Proposition 2.7 we have also seen that $\nabla_{f\eta_0}t = f\nabla_{\eta_0}t$, so $\nabla t \in \mathcal{T}^\ell_{k+1}(M)$. But then it is clear that we can form $\nabla^2 t = \nabla(\nabla t) \in \mathcal{T}^\ell_{k+2}$, and more generally, $\nabla^r t$ for any integer r.

In these terms, there is a natural interpretation of the curvature. Namely, for $\zeta \in \mathfrak{X}(M)$, we can consider $\nabla^2\zeta \in \mathcal{T}^1_2(M)$. To compute this, we have to observe that $\nabla\zeta \in \mathcal{T}^1_1(M)$ is, as a bilinear map $\mathfrak{X}(M) \times \Omega^1(M) \to C^\infty(M,\mathbb{R})$ given by $(\nabla\zeta)(\eta,\varphi) = \varphi(\nabla_\eta\zeta)$. Consequently, we get

$$(\nabla^2\zeta)(\xi,\eta,\varphi) = (\nabla_\xi(\nabla\zeta))(\eta,\varphi) = \xi\cdot(\varphi(\nabla_\eta\zeta)) - \varphi(\nabla_{\nabla_\xi\eta}\zeta) - (\nabla_\xi\varphi)(\nabla_\eta\zeta).$$

The first and last term add up to $\varphi(\nabla_\xi\nabla_\eta\zeta)$, which implies that, as a bilinear map $\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$, we obtain

$$(\nabla^2\zeta)(\xi,\eta) = \nabla_\xi\nabla_\eta\zeta - \nabla_{\nabla_\xi\eta}\zeta.$$

In view of torsion-freeness, this implies that

$$R(\xi,\eta)(\zeta) = (\nabla^2\zeta)(\xi,\eta) - (\nabla^2\zeta)(\eta,\xi),$$

which interprets the curvature as the alternation of the square of a covariant derivative.

In the context of abstract index notation, one can use an expression like $\nabla_j t^{i_1\dots i_\ell}_{j_1\dots j_k}$ to denote ∇t for a tensor field $t = t^{i_1\dots i_\ell}_{j_1\dots j_k} \in T^\ell_k(M)$. This has to be handled with care, since one has to decide which terms are really differentiated. The usual convention is that if there are no brackets, then a covariant derivative acts on all terms to its right. Thus $\nabla_j\xi^i\varphi_k$ represents $\nabla(\xi\otimes\varphi)$ and the compatibility of the covariant derivative with the tensor product can be written as $\nabla_j\xi^i\varphi_k = (\nabla_j\xi^i)\varphi_k + \xi^i\nabla_j\varphi_k$. Alternatively, the first of these summands can be written as $\varphi_k\nabla_j\xi^i$.

In these terms, one can now easily describe some operators. For example, the fact that $\nabla_\xi f = \xi\cdot f$ for a smooth function f immediately implies that $\nabla f = df$. Likewise, for a one-form $\varphi = \varphi_i$, we have by definition

$$(\nabla\varphi)(\xi,\eta) = (\nabla_\xi\varphi)(\eta) = \xi\cdot\varphi(\eta) - \varphi(\nabla_\xi\eta).$$

Torsion-freeness of ∇ together with the global formula for the exterior derivative implies that

$$d\varphi(\xi,\eta) = (\nabla\varphi)(\xi,\eta) - (\nabla\varphi)(\eta,\xi),$$

so in abstract index notation the exterior derivative can be written as $\varphi_i \mapsto 2\nabla_{[i}\varphi_{j]}$. One can also verify that for a one-form $\varphi = \varphi_i$ the codifferential is given by $\delta\varphi = g^{ij}\nabla_i\varphi_j$. Together with the observation on the exterior derivative of functions from above, this shows that for a smooth function f, the Laplacian is given by $\Delta f = g^{ij}\nabla_i\nabla_j f$.

As an example of a natural differential operator, let us study the so-called *Killing operator* on one-forms. This is the operator mapping $\Omega^1(M)$ to the space of symmetric $\binom{0}{2}$-tensor fields, defined by $\varphi_i \mapsto \nabla_{(i}\varphi_{j)}$. One calls one-forms which lie in the kernel of this operator *Killing one-forms* and the vector fields dual to these (i.e. given by $\xi^i = g^{ij}\varphi_j$) are called *Killing vector fields.*

Proposition 2.9. *Let (M,g) be a Riemannian manifold. Then we have*
(1) For $\varphi = \varphi_i \in \Omega^1(M)$ the following conditions are equivalent

(i) *φ is a Killing one-form*
(ii) $\nabla\varphi = \frac{1}{2}d\varphi$
(iii) *Each local flow of the dual vector field $\xi^i = g^{ij}\varphi_j$ is a local isometry for g.*

(2) If M is connected, then φ is uniquely determined by the values $\varphi(x)$ and $\nabla\varphi(x)$ for any point $x \in M$. In particular, the space of Killing one-forms has dimension at most $\frac{n(n+1)}{2}$, where $n = \dim(M)$.

PROOF. (1) By definition φ is a is a Killing one-form if and only if $\nabla_i\varphi_j$ has trivial symmetrization and thus is skew symmetric. Since we have observed already that $d\varphi = 2\nabla_{[i}\varphi_{j]}$ we see that (i) and (ii) are equivalent. Next, we show that the condition in (iii) is equivalent to $\mathcal{L}_\xi g = 0$, where $\mathcal{L}_\xi$ denotes the Lie derivative. Indeed, by definition, one has $\mathcal{L}_\xi g(x) = \frac{d}{dt}|_{t=0}(\mathrm{Fl}^\xi_t)^*g(x)$, see Section 3.4 of [**AnaMF**]. This shows that $\mathcal{L}_\xi g = 0$ if the local flows of ξ are isometries. Conversly, one can argue similarly to the proof of Lemma 2.11 of [**AnaMF**] to show that $\frac{d}{dt}|_{t=t_0}(\mathrm{Fl}^\xi_t)^*g(x) = ((\mathrm{Fl}^\xi_{t_0})^*\mathcal{L}_\xi g)(x)$ whenever the flow through x is defined up to $t = t_0$. Hence if $\mathcal{L}_\xi g = 0$, then $(\mathrm{Fl}^\xi_t)^*g(x)$ is constant in t whenever the flow is defined which completes the proof of the claim.

The explict formula for the Lie derivative on tensor fields from Section 3.4 of [**AnaMF**] then reads as

$$(\mathcal{L}_\xi g)(\eta,\zeta) = \xi \cdot g(\eta,\zeta) - g([\xi,\eta],\zeta) - g(\eta,[\xi,\zeta]).$$

By torsion-freeness of the Levi-Civita connection, we can write $[\xi,\eta] = \nabla_\xi\eta - \nabla_\eta\xi$, and likewise for the other bracket. But then the fact that ∇ is metric shows that we end up with

$$(\mathcal{L}_\xi g)(\eta,\zeta) = g(\nabla_\eta\xi,\zeta) + g(\eta,\nabla_\zeta\xi),$$

and the right hand side is the symmetrization of $g_{ai}\nabla_j\xi^a = \nabla_j\varphi_i$. Hence we see that (i) is equivalent to (iii).

(2) Suppose that φ_i is a Killing one-form, and put $\mu_{ij} = \frac{1}{2}d\varphi = \nabla_i\varphi_j$. Now a nice trick allows us to compute $\nabla_i\mu_{jk}$ (as a consequence of the equation satisfied by φ). Namely, by construction, we have $d\mu = 0$. Similarly to the case of one-forms discussed above, one verifies that $d\mu$ can be computed as a multiple of the complete alternation of $\nabla_i\mu_{jk}$. Hence we conclude that

$$\nabla_i\mu_{jk} = -\nabla_k\mu_{ij} - \nabla_j\mu_{ki} = \nabla_k\nabla_j\varphi_i - \nabla_j\nabla_k\varphi_i.$$

Similarly to the case of vector fields, one now verifies that the commutator of covariant derivatives can be expressed via the curvature. More precisely, one verifies that

$$\nabla_k\nabla_j\varphi_i - \nabla_j\nabla_k\varphi_i = R_{jk}{}^{\ell}{}_i\varphi_\ell.$$

Thus we conclude that for the pair $\binom{\varphi_i}{\mu_{jk}}$ we can compute the component-wise covariant derivative in terms of the values of the components and the (known) curvature of g. Along a smooth curve, this gives a first order ODE on the pair $\binom{\varphi_i}{\mu_{jk}}$, so the values along the curve are determined from the value of the pair in one point. Since M is connected, this implies the claim. □

Decomposing and interpreting curvature

From the discussion of curvature symmetries in 1.13, it is already visible that the Riemann curvature tensor is a rather complicated object. Therefore, constructing simpler objects out of the Riemann tensor is important for many applications.

2.10. Flat manifolds. Before we start decomposing the Riemann curvature tensor, we discuss the geometric meaning of vanishing of the curvature. Observe that by Lemma 1.10, the covariant derivative is a local operator. The definition of curvature in 1.13 then implies that the curvature is a local invariant of a Riemannian manifold, i.e. the restriction of the curvature to an open subset U depends only on the restriction of the metric to U. Hence vanishing of the curvature is a local condition, so we can only hope for local characterizations of this property.

PROPOSITION 2.10. *Let (M, g) be a Riemannian manifold of dimension n. Then for a point $x \in M$, the following conditions are equivalent*

(1) The Riemann curvature tensor vanishes on an open neighborhood of x.

(2) There is a chart (U, u) for M with $x \in U$ such that the coordinate vector fields ∂_i form an orthonormal basis of each tangent space.

(3) There are vector fields $\{\xi_1, \dots, \xi_n\}$ defined on a neighborhood of x which are all parallel, i.e. such that $\nabla_\eta \xi_i = 0$ for any $\eta \in \mathfrak{X}(M)$ and any $i = 1, \dots, n$ and such that $\{\xi_1(x), \dots, \xi_n(x)\}$ is a basis for T_xM.

(4) There is an isometric diffeomorphism from an open neighborhood of x in M onto an open subset of Euclidean space.

PROOF. (3)$\Rightarrow$(2): Let us first orthonormalize the basis $\{\xi_1(x), \dots, \xi_n(x)\}$ and write the corresponding orthonormal basis as $\eta_j(x) = \sum_i a^i_j \xi_i(x)$ with $a^i_j \in \mathbb{R}$. Putting $\eta_j = \sum_i a^i_j \xi_i$ we of course get $\nabla_\eta \eta_j = 0$ for each j and any $\eta \in \mathfrak{X}(M)$ since the a^i_j are constant. Since ∇ is metric, this implies that for all i, j, the functions $g(\eta_i, \eta_j)$ are constant, so the η_i are orthonormal wherever they are defined.

On the other hand, we in particular get $\nabla_{\eta_i} \eta_j = 0$ for all i and j, which by torsion freeness implies $[\eta_i, \eta_j] = 0$ for all i and j. By Corollary 2.11 of [**AnaMF**], there is a chart (U, u) around x, such that $\partial_i = \eta_i$ for all i, so (2) holds.

(2)$\Rightarrow$(4): By definition, the chart map u is a diffeomorphism from the open neighborhood U of x onto an open subset of $\mathbb{R}^n$. Moreover for $y \in U$, the tangent map $T_y u$ maps the orthonormal basis $\{\partial_i(y)\}$ to the standard basis of $\mathbb{R}^n$. Hence $T_y u$ is orthogonal, so u is an isometry.

(4)$\Rightarrow$(1): This is clear since by Proposition 1.14, any isometry is compatible with the Riemann curvature tensors, and the curvature vanishes on $\mathbb{R}^n$.

(1)$\Rightarrow$(3): Since this is a local question, it suffices to do this locally around $0 \in \mathbb{R}^n$ for an arbitrary Riemannian metric g on $\mathbb{R}^n$ with vanishing curvature. We denote by $x^1, \dots, x^n$ the standard coordinates and by ∂_i the corresponding coordinate vector fields. Choose an orthonormal basis $\xi_1(0), \dots, \xi_n(0)$ of $T_0\mathbb{R}^n$ and extend each of these tangent vectors to a vector field ξ_i on $\mathbb{R}^n$ as follows. To get $\xi_i(x^1, \dots, x^n)$, first translate $\xi_i(0)$ parallely along the line $t \mapsto (t, 0, \dots, 0)$ to the point $(x^1, 0, \dots, 0)$ then translate parallely along $t \mapsto (x^1, t, 0, \dots, 0)$ to $(x^1, x^2, 0, \dots, 0)$ and so on. So we have to prove that the resulting vector fields ξ_i are all parallel.

Now by construction ξ_i is parallel along each of the lines $t \mapsto (y^1, \dots, y^{n-1}, t)$, so $\nabla_{\partial_n} \xi_i = 0$. The same argument shows that $\nabla_{\partial_{n-1}} \xi_i$ vanishes on the subspace of all points with last coordinate equal to 0. But vanishing of the curvature together with $[\partial_{n-1}, \partial_n] = 0$ implies that $\nabla_{\partial_n} \nabla_{\partial_{n-1}} \xi_i = \nabla_{\partial_{n-1}} \nabla_{\partial_n} \xi = 0$. Hence $\nabla_{\partial_{n-1}} \xi_i$ is parallel along each of the lines $t \mapsto (y^1, \dots, y^{n-1}, t)$ and vanishes for $t = 0$, so it vanishes identically. Next $\nabla_{\partial_{n-2}} \xi_i$ vanishes in all points for which the last two coordinates are zero, and using vanishing of the curvature one first shows that this extends to all points with vanishing last coordinate and then to all of $\mathbb{R}^n$. Iteratively, we get $\nabla_{\partial_j} \xi_i = 0$ for all i and j, so the ξ_i are indeed parallel. □

2.11. Sectional curvature and space forms. The concept of sectional curvature is on the one hand motivated by the relation between Gauß curvature and the Riemann curvature tensor for surfaces in $\mathbb{R}^3$. On the other hand, as we have noted in Proposition 1.13, the Riemann tensor can be interpreted as a bilinear form on $\Lambda^2 TM$, so one can look at the values of this form on the wedge product of two tangent vectors.

In view of the symmetries of the Riemann tensor, it does not make sense to insert four copies of a tangent vector into R. However, given two tangent vectors ξ and η in T_xM, there is an essentially unique way to insert them into the curvature. A slight variation of this idea with nicer properties is the following.

DEFINITION 2.11. (1) Let (M, g) be a Riemannian manifold with curvature tensor R. Then for a point $x \in M$ and two linearly independent tangent vectors $X, Y \in T_xM$, one defines the *sectional curvature* by

$$K(x)(X, Y) := \frac{g_x(R_x(X, Y)(Y), X)}{g_x(X, X)g_x(Y, Y) - g_x(X, Y)^2} \in \mathbb{R}. \tag{2.5}$$

(2) One says that g has *constant sectional curvature* $a \in \mathbb{R}$ if and only if $K(x)(X, Y) = a$ for all x, X and Y.

Observe first that by the Cauchy-Schwarz inequality for the inner product g_x, the denominator in (2.5) is non-zero since X and Y are linearly independent. The motivation for this denominator is that $K(x)(X, Y)$ depends only on the plane in T_xM spanned by the two vectors. Indeed, replacing X and Y by $aX + bY$ and $cX + dY$, the skew symmetry properties of R show that the numerator in (2.5) gets multiplied by $(ad - bc)^2$. On the other hand, the square of the area of the parallelogram spanned by X and Y can be computed as $|X|^2|Y|^2 \sin^2(\alpha) = |X|^2|Y|^2(1 - \cos^2(\alpha))$, where α is the angle between the two vectors, and this is exactly the denominator in (2.5).

This also implies that it is sufficient to consider $K(x)(X, Y)$ for orthonormal tangent vectors, and $K(x)(X, Y) = g_x(R_x(X, Y)(Y), X)$ in this case. Now the explicit formula for the curvatures of the sphere and of hyperbolic space from 2.3 show that the sphere has constant sectional curvature $+1$, while hyperbolic space has constant sectional curvature -1. Of course, $\mathbb{R}^n$ has constant sectional curvature 0. These three basic examples are called the *space forms* and in some sense they are the simplest Riemannian manifolds.

Other constant values of sectional curvature are not terribly interesting, since one may always rescale the metric by a positive constant. Since the Levi-Civita connection of g is also metric for any constant positive multiple of g, it follows that such a constant rescaling does not change the Levi-Civita connection and hence also the Riemann tensor remains unchanged. However, passing from the Riemann tensor to sectional curvature involves the metric which implies that passing from g to ag means passing from K to $\frac{1}{a}K$.

It can be shown in general that any manifold of constant sectional curvature 1 (respectively -1) is locally isometric to S^n (respectively $\mathcal{H}^n$), while manifolds of constant sectional curvature 0 are flat and thus locally isometric to $\mathbb{R}^n$ by Proposition 2.10. Finally, it turns out that if in each point, the sectional curvature has the same value for all planes in the tangent space, then the metric automatically has constant sectional curvature.

2.12. The covariant derivative of the curvature. Aparat from constant sectional curvature as discussed in 2.11, there is a second idea to define a concept of "constant curvature" for a Riemannian manifold. Namely, we can consider the Riemann curvature tensor R as a $\binom{1}{3}$-tensor field and form its covariant derivative ∇R, which then is a tensor field of type $\binom{1}{4}$. Before we study vanishing of this tensor field, we prove the so-called *second Bianchi identity* (or differential Bianchi identity) which is the last main symmetry property of the curvature tensor.

PROPOSITION 2.12 (Second Bianchi identity). *Let (M, g) be a Riemannian manifold with Riemann curvature tensor R. Then the covariant derivative of the Riemann tensor satisfies*

$$0 = (\nabla_\xi R)(\eta, \zeta) + (\nabla_\zeta R)(\xi, \eta) + (\nabla_\eta R)(\zeta, \xi)$$

for all $\xi, \eta, \zeta \in \mathfrak{X}(M)$. In abstract index notation, this reads as $0 = \nabla_{[i} R_{jk]}{}^{\ell}{}_m$.

PROOF. This is most easliy verified in terms of the expression of the curvature in a local orthonormal frame, but also in this setting quite a bit of computation is needed. Apply the exterior derivative to the defining equation

$$\Omega^i_j = d\omega^i_j + \textstyle\sum_k \omega^i_k \wedge \omega^k_j$$

for the curvature two-forms and reinsert for terms involving $d\omega$'s. This gives

$$d\Omega^i_j = \textstyle\sum_k \Omega^i_k \wedge \omega^k_j - \sum_k \omega^i_k \wedge \Omega^k_j - \sum_{k,\ell} \omega^i_\ell \wedge \omega^\ell_k \wedge \omega^k_j + \sum_{k,\ell} \omega^i_k \wedge \omega^k_\ell \wedge \omega^\ell_j,$$

and clearly the last two sums cancel. This is already the second Bianchi identity in the moving frame from, and we have to interpret it in terms of covariant derivatives. Recall from 2.2 that $\Omega^i_j(\eta, \zeta) = g(R(\eta, \zeta)(s_j), s_i)$, where the s_i are the elements of the given orthonormal frame. Differentiating this smooth function with $\xi \in \mathfrak{X}(M)$, we obtain

$$\xi \cdot \Omega^i_j(\eta, \zeta) = g(\nabla_\xi R(\eta, \zeta)(s_j), s_i) + g(R(\eta, \zeta)(s_j), \nabla_\xi s_i).$$

The naturality properties of the covariant derivative from Propostion 2.7 imply that

$$\nabla_\xi R(\eta, \zeta)(s_j) = ((\nabla_\xi R)(\eta, \zeta))(s_j) + R(\nabla_\xi \eta, \zeta)(s_j) + R(\eta, \nabla_\xi \zeta)(s_j) + R(\eta, \zeta)(\nabla_\xi s_j).$$

Now by definition $\nabla_\xi s_j = \sum_k \omega^k_j(\xi) s_k$ and likewise we can insert for $\nabla_\xi s_i$ above. Inserting all that above, we obtain

$$\begin{aligned} \xi \cdot \Omega^i_j(\eta, \zeta) = {} & g(((\nabla_\xi R)(\eta, \zeta))(s_j), s_i) + \Omega^i_j(\nabla_\xi \eta, \zeta) - \Omega^i_j(\nabla_\xi \zeta, \eta) \\ & + \textstyle\sum_k \omega^k_j(\xi) \Omega^i_k(\eta, \zeta) + \sum_k \omega^k_i(\xi) \Omega^k_j(\eta, \zeta). \end{aligned}$$

Summing this up over all cyclic permutations of ξ, η and ζ the second and third terms in the right hand side add up to

$$\Omega^i_j([\xi, \eta], \zeta) + \Omega^i_j([\eta, \zeta], \xi) + \Omega^i_j([\zeta, \xi], \eta),$$

and bringing this to the other side, we obtain $d\Omega^i_j(\xi, \eta, \zeta)$ on the left hand side. On the other hand, summing the last two terms in the right hand side and using $\omega^k_i = -\omega^i_k$, one gets

$$(\textstyle\sum_k \omega^k_j \wedge \Omega^i_k - \sum \omega^i_k \wedge \Omega^k_j)(\xi, \eta, \zeta).$$

Hence we conclude that the sum over all cyclic permutations of ξ, η and ζ of

$$g(((\nabla_\xi R)(\eta, \zeta))(s_j), s_i)$$

vanishes for all i and j, which implies the claim. □

Now let us study the condition of parallel curvature for Riemannian manifolds. It turns out that this is related to so called symmetries. Here by a symmetry in a point x of a smooth manifold M one means a local diffeomorphism $\sigma = \sigma_x$ defined on a neighborhood of x such that $\sigma(x) = x$ and $T_x\sigma = -\operatorname{id}_{T_xM}$. Note that in case that (M, g) is a Riemannian manifold and σ is an isometry for g, these conditions determine σ locally around x. Indeed, in this case, we must have $\sigma(\exp_x(\xi)) = \exp_x(-\xi)$ for all $\xi \in T_xM$ such that the left hand side is defined, compare with 1.14. Conversely, we can clearly define a local symmetry at x by $\exp_x \circ - (\exp_x)^{-1}$ on any geodesic ball around x. This is called the *geodetic reflection* in x, but it is not a (local) isometry in general.

DEFINITION 2.12. Let (M, g) be a connected Riemannian manifold.

(1) (M, g) is called a *locally symmetric space* if and only if for each point $x \in M$, the geodetic reflection defines an isometry on some open neighborhood of x.

(2) (M, g) is called a *symmetric space* if and only if for each point $x \in M$, the geodetic reflection in x extends to a globally defined isometry of M.

It turns out that (M, g) is a locally symmetric space if and only if the Riemann curvature tensor R of g is parallel, i.e. iff $\nabla R = 0$. The necessity of this condition is easy to see. If the geodestic reflection defines an isometry σ_x on an open neighborhood U of x, then $(\sigma_x)^*(\nabla R) = \nabla R$, see Proposition 1.14. But the action of $(\sigma_x)^*(\nabla R)(x)$ on three tangent vectors in T_xM is given by hitting the tangent vectors with $T_x\sigma_x = -\operatorname{id}$, so $(\sigma_x)^*(\nabla R)(x) = -\nabla R(x)$. The sufficiency is much more complicated to prove.

Second, it turns out that the difference between locally symmetric spaces and symmetric spaces comes from topology. Indeed, one can prove that a simply connected locally symmetric space automatically is a symmetric space. In particular, given a locally symmetric space (M, g) one can form the universal covering space $\widetilde{M}$. This is a simply connected space endowed with a covering map $p : \widetilde{M} \to M$. This covering map is a local homeomorphism, so one can use charts of M to make $\widetilde{M}$ into a smooth manifold in such a way that p becomes a local diffeomorphism. Further, one can pull back the tensor field g on M to $\widetilde{M}$ to obtain a Riemannian metric $\tilde{g}$ on $\widetilde{M}$ and then p becomes a local isometry. By construction, $(\widetilde{M}, \tilde{g})$ is a locally symmetric space and thus a symmetric space by simple connectedness.

To analyze symmetric spaces, one first proves that they are *homogenous*, i.e. for two points x and y in a symmetric space (M, g), there always is an isometry $f : M \to M$ such that $f(x) = y$. This follows easily from the same fact in the case that x and y can be connected by a geodesic, which is obvious since x is mapped to y by reflecting in the middle point between x and y on that geodesic. Now there is a general result ("Meyers–Steerod Theorem") that says that the group of isometries of a Riemannian manifold is always a Lie group. The Lie algebra of this group turns out to be isomorphic to the space of those vector fields dual to Killing one-forms (as in Proposition 2.9) which are complete. Thus a homogeneous Riemannian manifold is realized as a homogeneous space of its isometry group. One can then study the condition of being symmetric in terms of Lie theory, which leads to a complete classification of symmetric spaces. Locally symmetric spaces are then obtained by further quotenting by discrete subgroups of the isometry group, and a lot is known about such subgroups.

Apart from the fact that they provide many interesting examples of Riemannian manifolds (indculding spheres, hyperbolic spaces, and Grassmann manifolds) they also play an important role in holonomy theory. In fact, in the classification of holonomy groups mentioned in 1.11 and 2.8, one always has to distinguish between the case of locally symmetric spaces and manifold for which the curvature tensor is not parallel. For the locally symmetric case, the classification of symmetric spaces in terms of Lie theory also gives a classification of holonomy groups, in the other case, the possible holonomy groups are classified by a classical theorem of M. Berger.

2.13. Decomposing the curvature tensor. An idea to obtain simpler objects from the Riemann curvature tensor is to try taking traces. Due to the symmetries of the curvature tensor, there is initially only one trace (up to sign) which has the potential to be non-zero. Writing the curvature tensor as $R_{ij}{}^k{}_\ell$ a contraction is defined by either choosing one of the lower indices and contracting k into it or by choosing two of the lower indices and contrating them with the inverse metric. Now the skew symmetry results from part (2) of Proposition 1.13 on the one hand imply that $g^{ij}R_{ij}{}^k{}_\ell = 0$ and $R_{ij}{}^k{}_k = 0$ as well as the fact that the remaining contractions (k into i or j, or ℓ with i or j with the inverse metric) all agree up to sign.

DEFINITION 2.13. Let (M, g) be a smooth Riemannian manifold of dimension $n \geq 3$ with Riemann curvature tensor $R_{ij}{}^k{}_\ell$.

(1) The *Ricci curvature* of g is the $\binom{0}{2}$-tensor field Ric defined by $\mathrm{Ric}_{ij} := R_{ki}{}^k{}_j$.

(2) The *scalar curvature* of g is the smooth function R on M defined by $R := g^{ij}\,\mathrm{Ric}_{ij}$.

(3) The *Schouten tensor* of g is defined by $\mathsf{P}_{ij} := \frac{1}{n-2}(\mathrm{Ric}_{ij} - \frac{1}{2(n-1)}Rg_{ij})$.

(4) The *Weyl curvature* of g is the $\binom{1}{3}$-tensorfeld W defined by

$$W_{ij}{}^k{}_\ell := R_{ij}{}^k{}_\ell - \left(2\delta^k_{[i}\mathsf{P}_{j]\ell} - 2g_{\ell[i}\mathsf{P}_{j]a}g^{ak}\right).$$

(5) The metric g is called *Ricci flat* if $\mathrm{Ric}_{ij} = 0$.

(6) The metric g is called an *Einstein metric* if its Ricci curvature (or equivalently its Schouten tensor) is proportional to the metric, i.e. if $\mathrm{Ric}_{ij} = \frac{1}{n}Rg_{ij}$.

Let us next verify the basic properites of these quantitites.

PROPOSITION 2.13. *For any Riemannian manifold (M, g) the following hold.*

(1) The Ricci curvature and the Schouten tensor are both symmetric and they satisfy $\mathrm{Ric}_{ij} = (n-2)\mathsf{P}_{ij} + \mathsf{P}g_{ij}$, where $\mathsf{P} = g^{ij}\mathsf{P}_{ij} = \frac{1}{2(n-1)}R$ is the trace of the Schouten tensor.

(2) The Weyl curvature has all symmetries of the Riemann curvature tensor as in parts (2) – (4) of Proposition 1.13 and in addition is totally tracefree, i.e. we have

$$W_{ij}{}^k{}_\ell = W_{[ij]}{}^k{}_\ell \quad W_{ij}{}^a{}_\ell g_{ka} = W_{ij}{}^a{}_{[\ell}g_{k]a} \quad W_{ij}{}^a{}_\ell g_{ka} = W_{\ell k}{}^a{}_i g_{ja} \quad W_{[ij}{}^k{}_{\ell]} = 0 \quad W_{ki}{}^k{}_j = 0$$

PROOF. (1) By defintion, the Ricci curvature can be written as $\mathrm{Ric}_{ij} = g^{k\ell}R_{ki}{}^a_j g_{\ell a}$. From Proposition 1.13, we know that $R_{ki}{}^a_j g_{\ell a} = R_{j\ell}{}^a_k g_{ia} = R_{\ell j}{}^a_i g_{ka}$ and applying $g^{k\ell}$ to the last expression, we by definition get Ric_{ji}. Symmtry of the Schouten tensor then follows by definition.

From the definition of the Schouten tensor, it follows readily that $\mathrm{Ric}_{ij} = (n-2)\mathsf{P}_{ij} + \frac{1}{2(n-1)}Rg_{ij}$. Contracting this equations with g^{ij}, we see that $R = (n-2)\mathsf{P} + \frac{n}{2(n-1)}R$, and hence $\frac{n-2}{2(n-1)}R = (n-2)\mathsf{P}$, which implies the claim.

(2) Lowering the index k in the definition of the Weyl curvature, we see that $W_{ij}{}^a{}_\ell g_{ka}$ is obtained from $R_{ij}{}^a{}_\ell g_{ka}$ by subtracting

$$2g_{k[i}\mathsf{P}_{j]\ell} - 2g_{\ell[i}\mathsf{P}_{j]k}.$$

From this form it is evident that this term is skew symmetric in i and j as well as in k and ℓ. Morover, if we expand the alternations, symmetry of g and P implies that each of the resulting terms is symmetric in two of the three indices i, j and ℓ. Therefore, the complete alternation of this expression over these three indices vanishes, so $R_{[ij}{}^k{}_{\ell]} = 0$ implies $W_{[ij}{}^k{}_{\ell]} = 0$. In the proof of Proposition 1.13, we have seen that the symmetries derived so far imply that $W_{ij}{}^a{}_\ell g_{ka} = W_{\ell k}{}^a{}_i g_{ja}$, so it remains to prove that $W_{kj}{}^k{}_\ell = 0$. To do this, we expand the alternations in the definition of the Weyl curvature to obtain

$$W_{ij}{}^k{}_\ell = R_{ij}{}^k{}_\ell - \delta^k_i\mathsf{P}_{j\ell} + \delta^k_j\mathsf{P}_{i\ell} + g_{\ell i}\mathsf{P}_{ja}g^{ak} - g_{\ell j}\mathsf{P}_{ia}g^{ak}.$$

Contracting the indices i and k, we get

$$W_{kj}{}^k{}_\ell = \mathrm{Ric}_{j\ell} - n\mathsf{P}_{j\ell} + \mathsf{P}_{j\ell} + \mathsf{P}_{j\ell} - g_{j\ell}\mathsf{P},$$

which equals $\mathrm{Ric}_{j\ell} - ((n-2)\mathsf{P}_{j\ell} + \mathsf{P} g_{j\ell}) = 0$. □

In view of this result we can reinterpret the definiton of the Weyl curvature as a the decomposition

$$R_{ij}{}^k{}_\ell = W_{ij}{}^k{}_\ell + \left(2\delta^k_{[i}\mathsf{P}_{j]\ell} - 2g_{\ell[i}\mathsf{P}_{j]a}g^{ak}\right)$$

of the Riemann curvature into a trace-free part and a trace part. This trace part can be equivalently be described by Ric_{ij} or by P_{ij}, and it again splits into a trace-free part and trace part as $\mathrm{Ric}_{ij} = (\mathrm{Ric}_{ij} - \frac{1}{n}Rg_{ij}) + \frac{1}{n}Rg_{ij}$ and similarly for P_{ij}. By definition, the metric is Einstein if and only if the tracefree part of the Ricci curvature vanishes identically.

Forming a constraction of the second Bianchi identity from Proposition 2.12, one sees that $\nabla_i R = \frac{1}{2}g^{jk}\nabla_j \mathrm{Ric}_{ik}$. In the case of an Einstein metric, the right hand side becomes $\frac{1}{2n}\nabla_i R$, so we conclude that for an Einstein metric, the scalar curvature is constant. This constant value is referred to as the Einstein constant of the metric, it is mainly of interest whether this is postive, negative or zero ("Ricci-flat metrics").

Example 2.13. Consider the metric on the sphere S^n from example (2) of 2.3, so $R_{ij}{}^k{}_\ell = \delta^k_i g_{j\ell} - \delta^k_j g_{i\ell}$. This gives $\mathrm{Ric}_{j\ell} = (n-1)g_{j\ell}$ and $R = n(n-1)$, which implies that the metric on the sphere is Einstein with positive scalar curvature.

Inserting this into the definitions, we obtain $\mathsf{P}_{ij} = \frac{1}{n-2}((n-1)g_{ij} + \frac{n}{2}g_{ij}) = \frac{1}{2}g_{ij}$. Inserting into the definition shows that the Weyl curvature of the sphere vanishes. So also from our current point of view, these are the simplest possible curvature tensors.

Likewise, hyperbolic space has vanishing Weyl curvature, and is Einstein with negative scalar curvature $R = -n(n-1)$.

The part of the curvature tensor which is most easily to interpret is the Weyl curvature. This is related to the concept of conformal rescaling that we have met in 2.6. There we said that two metrics g and $\hat{g}$ on a manifold M are conformal to each other if and only if there is a positive smooth function f on M such that $\hat{g} = fg$. This defines an equivalence relation on the set of Riemannian metrics on M. It turns out that conformal metrics have the same Weyl curvature, so one says that the Weyl curvature is a *conformal invariant*. It further turns out that the Weyl curvature vanishes identically if and only if the metric is (locally) conformally flat, i.e. if each point in M admits an open neighborhood on which the metric is conformal to a flat metric as characterized in Proposition 2.10.

This gives a simple explanation why the Weyl curvatures of the sphere and of hyperbolic space vanish. For hyperbolic space, we have defined the hyperbolic metric as a conformal rescaling of the flat metric on the ball. Likewise, the computation in 2.3 shows that in the chart defined by stereographic projection, the metric on a sphere is a conformal rescaling of the flat metric on $\mathbb{R}^n$, so again this is evidently conformally flat.

2.14. Curvature and normal coordinates. We complete this part by a short discussion of the relation between normal coordinates and the curvature tensor. This is useful for understanding "how well" normal coordinates are adapted to the Riemannian manifold in a point. On the other hand, it provides explanations for the meanings of the values in a point of several curvature quantities.

Recall from 1.12 that normal coordinates centered at x are obtained by using the inverse of $\exp_x$ as a chart and an orthonormal basis of T_xM to identify this space with $\mathbb{R}^n$. In the resulting local coordinates, the point x corresponds to $0 \in \mathbb{R}^n$ and we consider the local coordinate expression g_{ij} of the metric in these coordinates. Now first of all, since $T_0 \exp_x = \mathrm{id}_{T_xM}$, we see that $g_{ij}(0) = \delta_{ij}$. Second, we know that the radial lines in normal coordinates correspond to geodesics. This means that if X is a linear combination of the coordinate vector fields ∂_i with constant coefficients, then $\nabla_X X(0) = 0$. This implies that $\Gamma^U(X, X)$ vanishes in the point 0, and since Γ^U is symmetric, polarization implies that Γ^U vanishes in 0. Hence all the Christoffel symbols Γ^k_{ij} vanish at the origin. From the definition in 1.11, we conclude that this implies that $\partial_i \cdot g_{j\ell} + \partial_j \cdot g_{i\ell} - \partial_\ell \cdot g_{ij}$ vanishes in 0 for all indices i, j and ℓ. Adding the same term with i and ℓ exchanged, we see that $2\partial_j \cdot g_{i\ell}$ vanishes in 0, so all partial derivatives of the components g_{ij} vanish at the origin.

This says that the flat metric in normal coordinates approximates g in x to first order, but this is already as good as things can get. We can see this by deriving the coordinate expression for the curvature tensor, which shows that the values of its components in 0 can be computed from the Christoffel symbols and their partial derivatives in 0. Hence they depend only on the partial derivatives of the g_{ij} up to second order, so we cannot have vanishing second order partials in 0 unless the curvature vanishes in x.

Lemma 2.14. *In arbitray local coordinates, the Riemann curvature tensor is in terms of the Christoffel symbols given by*

$$R(\partial_i, \partial_j)(\partial_\ell) = \sum_k \Bigg(\partial_i(\Gamma^k_{j\ell}) - \partial_j(\Gamma^k_{i\ell}) + \sum_a \left(\Gamma^a_{j\ell}\Gamma^k_{ia} - \Gamma^a_{i\ell}\Gamma^k_{ja}\right)\Bigg)\partial_k.$$

Proof. By definition of the Christoffel symbols, $\nabla_{\partial_j}\partial_\ell = \sum_k \Gamma^k_{j\ell}\partial_k$, and hence

$$\nabla_{\partial_i}\nabla_{\partial_j}\partial_\ell = \sum_k \left((\partial_i \cdot \Gamma^k_{j\ell})\partial_k - \Gamma^k_{j\ell}\nabla_{\partial_i}\partial_k\right).$$

Expanding the covariant derivative in terms of Christoffel symbols, and using that $[\partial_i, \partial_j] = 0$, the claimed formula then follows from the definition of curvature. □

In the special case of normal coordinates, we see that the components of the curvature tensor are given by $R_{ij}{}^k{}_\ell(0) = \partial_i \cdot \Gamma^k_{j\ell}(0) - \partial_j \cdot \Gamma^k_{i\ell}(0)$. In fact it turns out that the relation between the curvature and the second derivatives of the functions g_{ij} is much simpler than one would expect. To formulate this, we consider the functions $R_{ijk\ell} := g(R(\partial_i, \partial_j)(\partial_k), \partial_\ell)$. The values of these functions in a point are exactly the first non-trivial Taylor coefficients of the functions g_{ij}:

Theorem 2.14. *The Taylor expansion of the components g_{ij} of the metric in normal coordinates $(u^1, \dots, u^n)$ centered in x in the point $u = 0$ is given by*

$$g_{ij}(u) = \delta_{ij} + \tfrac{1}{3}R_{ik\ell j}(x)u^k u^\ell + O(|u|^3).$$

The proof of this and the following consequences is beyond the scope of this course. Having derived this Taylor development, one can construct various expansions which lead to the values of various curvature quantities at x as Taylor coefficients. We list these expansions without detailed proofs.

Let us start with sectional curvature as discussed in 2.11. Here we have to specify a two-dimensional subspace $E \subset T_xM$, and the sectional curvature associated to this plane is given by inserting an orthonormal basis $\{X, Y\}$ of E into the formula from Definition 2.11. We denote the resulting value by $K(x)(E)$. To interpret this, we take a small radius $r > 0$ and let $C_r \subset M$ be the image under $\exp_x$ of the circle of Radius r in $E \subset T_xM$. Let $L(r)$ denote the arclength of this smooth closed curve in M. Then it turns out that

$$L(r) = 2\pi r - \tfrac{\pi}{3}K(x)(E)r^3 + O(r^4).$$

In particular, for positive sectional curvature, the circles are shorter than their Euclidean counterparts while for negative sectional curvature they are longer.

Next, the Ricci curvature in x measures the infinitesimal growth of the volume density $\sqrt{\det(g_{ij}(u))}$. More precisely, one has

$$\sqrt{\det(g_{ij}(u))} = 1 - \tfrac{1}{6}\operatorname{Ric}_{ij}(x)u^i u^j + O(|u|^3).$$

So positive definite Ricci curvature (as in the case of the sphere) means that the volume element gets smaller when leaving the origin.

Finally, scalar curvature $R(x)$ can be interpreted in terms of the growth of volumes of geodesic balls and spheres. Let us denote by ω_n the volume of the unit ball in $\mathbb{R}^n$. Then for sufficiently small r, we let $B_r(x)$ denote the image under $\exp_x$ of the ball of radius r in T_xM, while by $S_r(x)$ we denote the geodesic sphere of radius r. Then it turns out that the volume of $B_r(x)$ and the area of $S_r(x)$ grow as

$$\begin{aligned}\operatorname{Vol}(B_r(x)) &= \omega_n r^n \left(1 - \tfrac{1}{6(n+2)}R(x)r^2 + O(r^3)\right)\\ \operatorname{Vol}(S_r(x)) &= n\omega_n r^{n-1} - \tfrac{1}{6}R(x)\omega_n r^{n+1} + O(r^{n+2}).\end{aligned}$$

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