Riemannian Geometry

rough lecture notes

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Preface

CHAPTER 1

Fundamentals of Riemannian geometry

After recalling some background, we define Riemannian metrics and Riemannian manifolds. We analyze the basic tensorial operations that become available in the presence of a Riemannian metric. Then we construct the Levi-Civita connection, which is the basic "new" differential operator coming from such a metric.

Background

The purpose of this section is two–fold. On the one hand, we want to relate the general concept of a Riemannian manifold to the geometry of hypersurfaces as known from introductory courses. On the other hand, we recall some facts about tensor fields and introduce abstract index notation.

1.1. Euclidean Geometry. The basic object in Euclidean geometry is the n–dimensional *Euclidean space* E^n. One may abstractly start from an affine space of dimension n, but for simplicity, we just take the n–dimensional real vector space $\mathbb{R}^n$ and "forget about the origin". Given two points in this space, there is a well defined vector connecting them, which we denote by $\overrightarrow{xy} \in \mathbb{R}^n$. Identifying E^n with $\mathbb{R}^n$, this can be computed as $\overrightarrow{xy} = y - x$ (which visibly is independent of the location of the origin). On the other hand, given a point $x \in E^n$ and a vector $v \in \mathbb{R}^n$, we can form $x + v \in E^n$. Of course, this satisfies $x + \overrightarrow{xy} = y$ and similar properties. (The abstract definition requires the existence of $(x, y) \mapsto \overrightarrow{xy}$ as a map $E^n \times E^n \to \mathbb{R}^n$ and of $+ : E^n \times \mathbb{R}^n \to E^n$ together with some of the basic properties of these operations.)

The second main ingredient to Euclidean geometry is provided by the standard inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$. This allows us to define the Euclidean distance of two points $x, y \in E^n$ by $d(x, y) := \|\overrightarrow{xy}\| = \sqrt{\langle \overrightarrow{xy}, \overrightarrow{xy} \rangle}$.

Let us relate this to differential geometry. Fixing a point $o \in E^n$, the map $x \mapsto \overrightarrow{ox}$ defines a bijection $E^n \to \mathbb{R}^n$. This can be used as a global chart (and any two such charts are compatible) thus making E^n into a smooth manifold. Moreover, one can identify each tangent space T_xE^n with $\mathbb{R}^n$ by mapping $v \in \mathbb{R}^n$ to $c'(0)$, where $c : \mathbb{R} \to E^n$ is the smooth curve defined by $c(t) := x + tv$. Hence we can view the standard inner product on $\mathbb{R}^n$ as defining an inner product on each tangent space of E^n.

The two pictures fit together nicely, as we can see from the appropriate concept of morphisms of Euclidean space, which can be formulated in seemingly entirely different ways:

Proposition 1.1. *For a set–map $f : E^n \to E^n$ the following conditions are equivalent.*

(i) For all points $x, y \in E^n$, we have $d(f(x), f(y)) = d(x, y)$.

(ii) The map f is smooth and for each $x \in E^n$, the tangent map $T_xf : T_xE^n \to T_{f(x)}E^n$ is orthogonal.

(iii) There is an orthogonal linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ such that for all $x, y \in E^n$ we have $f(y) = f(x) + A(\overrightarrow{xy})$.

Proof. The condition in (iii) can be rewritten as $\overrightarrow{f(x)f(y)} = A(\overrightarrow{xy})$ for all $x, y \in E^n$. Since orthogonal linear maps preserve the norms of vectors, we see that (iii)⇒(i). Applying the condition to $y = x + tv$, we get $\overrightarrow{xy} = tv$, so $f(x + tv) = f(x) + tA(v)$. This shows that if f satisfies (iii), then it is smooth and $T_x f = A$ for each $x \in E^n$, so (iii)⇒(ii).

(i)⇒(iii): We claim that a map $F : \mathbb{R}^n \to \mathbb{R}^n$ which satisfies $F(0) = 0$ and which is distance–preserving must be an orthogonal linear map. Since $\|v\| = d(v, 0)$ and $F(0) = 0$, we see that $\|F(v)\| = \|v\|$ for all $v \in \mathbb{R}^n$. Now one of the polarization identities reads as

$$\langle v, w\rangle = \tfrac{1}{2}\left(\|v\|^2 + \|w\|^2 - d(v, w)^2\right),$$

so we conclude that $\langle F(v), F(w)\rangle = \langle v, w\rangle$. In particular, denoting by $\{e_1, \dots, e_n\}$ the (orthonormal) standard basis for $\mathbb{R}^n$, we see that the vectors $F(e_1), \dots, F(e_n)$ also form an orthonormal system and thus an orthonormal basis.

Taking an arbitrary element $v \in \mathbb{R}^n$, we can expand v in the standard basis as $v = \sum_i \langle v, e_i\rangle e_i$. Likewise, we can expand $F(v)$ in the orthonormal basis $\{F(e_i)\}$ as $F(v) = \sum_i \langle F(v), F(e_i)\rangle F(e_i)$. But then $\langle v, e_i\rangle = \langle F(v), F(e_i)\rangle$ implies that $F(\sum_i \lambda_i e_i) = \sum_i \lambda_i F(e_i)$ for all $(\lambda_1, \dots, \lambda_n)$. Hence F is a linear map and knowing this, we have already observed orthogonality.

Starting from a distance–preserving map $f : E^n \to E^n$, we choose a point $o \in E^n$ and define $F : \mathbb{R}^n \to \mathbb{R}^n$ as $F(v) = \overrightarrow{f(o)f(o + v)}$. This evidently satisfies $F(0) = 0$. Moreover, $F(w) - F(v) = \overrightarrow{f(o)f(o + w)} - \overrightarrow{f(o)f(o + v)} = \overrightarrow{f(o + v)f(o + w)}$ and in the same way $\overrightarrow{(o + v)(o + w)} = w - v$, so we see that F is distance–preserving and thus an orthogonal linear map by the claim. By construction, we get $f(x) = f(o) + F(\overrightarrow{ox})$ for all $x \in E^n$. For another point y, we have $\overrightarrow{oy} = \overrightarrow{ox} + \overrightarrow{xy}$ and thus $f(y) = f(o) + F(\overrightarrow{ox}) + F(\overrightarrow{xy}) = f(x) + F(\overrightarrow{xy})$.

(ii)⇒(iii): As in the last step, it suffices to show that a smooth map $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $F(0) = 0$ and for each $v \in \mathbb{R}^n$ the derivative $DF(v) : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal, must itself be an orthogonal linear map.

By assumption, for $X, Y \in \mathbb{R}^n$, we have $\langle DF(v)(X), DF(v)(Y)\rangle = \langle X, Y\rangle$. Taking $w \in \mathbb{R}^n$, we can form $\frac{d}{dt}|_{t=0} DF(v + tw)(X) = D^2F(v)(w, X)$, and this is symmetric in w and X. On the other hand, the map $t \mapsto \langle DF(v + tw)(X), DF(v + tw)(Y)\rangle$ is constant, so differentiating it at $t = 0$, we obtain

$$0 = \langle D^2F(v)(w, X), DF(v)(Y)\rangle + \langle DF(v)(X), D^2F(v)(w, Y)\rangle$$

This means that the tri–linear map $\Phi(X, Y, Z) := \langle D^2F(v)(X, Y), DF(v)(Z)\rangle$ satisfies $\Phi(X, Y, Z) = \Phi(Y, X, Z)$ and $\Phi(X, Z, Y) = -\Phi(X, Y, Z)$. But this implies

$$\begin{aligned}\Phi(X, Y, Z) = -\Phi(X, Z, Y) = -\,\Phi(Z, X, Y) = \Phi(Z, Y, X)&\\ =\Phi(Y, Z, X) = -\Phi(Y, X, Z) = -\Phi(X, Y, Z).&\end{aligned}$$

Hence we conclude that $\langle D^2F(v)(X, Y), DF(v)(Z)\rangle = 0$ and since the orthogonal map $DF(v)$ is surjective, we see that $D^2F(v) = 0$. But this means that $DF(v) = A$ for some fixed orthogonal linear map $A : \mathbb{R}^n \to \mathbb{R}^n$. This implies that the curve $c(t) = F(tv)$ has derivative $c'(t) = A(v)$ for all t. Hence $F(v) = c(1) = c(0) + \int_0^1 c'(t)dt = 0 + A(v) = A(v)$ for any $v \in \mathbb{R}^n$. □

Definition 1.1. A *Euclidean motion* is a map $f : E^n \to E^n$ which satisfies the equivalent conditions of this Proposition.

The three conditions characterizing Euclidean motions visibly are of very different nature. Condition (i) tells us in a way that the Euclidean distance is the only central ingredient in Euclidean geometry. It is surprising that it is not necessary to assume smoothness initially. Condition (iii) is the most useful one for explicitly describing Euclidean motions and this is often used as the definition. Condition (ii) shows that Euclidean motions are exactly the isometries of E^n in the sense of Riemannian geometry.

1.2. Geometry of curves and surfaces. These classical parts of differential geometry study submanifolds in E^n. To obtain geometric properties, one always requires that things are well behaved (in an appropriate sense) with respect to Euclidean motions. (For example, the curvature of a curve should remain unchanged, while the tangent line should also be moved by the motion.)

In the geometry of surfaces in E^3, one meets a new phenomenon, since there are different kinds of curvatures. This is related to the question whether one can observe the fact that a surface is curved from inside the surface. (In classical language, this was referred to as "inner" or "intrinsic" geometry as opposed to "extrinsic" geometry of surfaces.) The classical examples are provided by a cylinder and a sphere respectively. While a cylinder is curved from an outside point of view, it can be locally mapped onto an open subset of E^2 in a distance preserving way. In contrast to that, it is not possible to map an open subset of the sphere S^2 onto an open subset of E^2 in such a way that distances are preserved. Here "distance" in the cylinder and in S^2 are defined via the infimum of the arclengths of curves connecting two points (as we will develop the concept on general Riemannian manifolds). This is related to facts like that the sum of the three angles of a (geodesic) triangle on S^2 is always bigger than π and depends on the area of the triangle.

To formalize this concept, one observes that for a smooth submanifold $M \subset E^n$ and a point $x \in M$, the tangent space T_xM can be naturally viewed as a linear subspace of $T_xE^n = \mathbb{R}^n$. Hence one can restrict the standard inner product to the tangent spaces of M, thus defining a smooth $\binom{0}{2}$–tensor field on M. This is called the *first fundamental form*. Roughly speaking, intrinsic quantities are those which depend only on the first fundamental form. To formalize this, one introduces the concept of a local *isometry* between such submanifolds (of the same dimension) as a local diffeomorphism, for which all tangent maps are orthogonal.

If $M \subset E^n$ is a smooth submanifold and $f : E^n \to E^n$ is a Euclidean motion, then $f(M) \subset E^n$ is a smooth submanifold of the same dimension as M, and $f|_M : M \to f(M)$ is an isometry. However, as the example of the cylinder and and the plane shows, there are isometries between submanifolds which do not arise in this way (since the distances of points in $\mathbb{R}^n$ are not preserved). Now the formal definition of a intrinsic quantity is a quantity which is not only invariant under Euclidean motions but also under general isometries.

The fundamental intrinsic quantity is the Gauß curvature for surfaces in E^3. This can be proved directly, but a conceptual approach to understanding this is more involved. This is based on the notion of the covariant derivative which (in view of the original definition of the covariant derivative very surprisingly) turns out to be intrinsic. Then the Gauß curvature for surfaces can be expressed (and is essentially equivalent to) the Riemann curvature, which in turn can be constructed from the covariant derivative and thus is intrinsic.

1.3. Tensor fields and abstract index notation. Let M be a smooth manifold. For a point $x \in M$ one has the tangent space T_xM. One then defines the cotangent space T_x^*M at x to be the dual vector space to the tangent space. A $\binom{\ell}{k}$–tensor field on M then assigns to each point $x \in M$ an element of the tensor product $T_xM \otimes \dots \otimes T_xM \otimes T_x^*M \otimes \dots \otimes T_x^*M$ with ℓ factors of the tangent space and k factors of the cotangent space. The value at x can then be interpreted as a $(k+\ell)$–linear map $(T_xM)^k \times (T_x^*M)^\ell \to \mathbb{R}$, and the assignment should be smooth in the sense that inserting the values of k vector fields and ℓ smooth one–forms into these multilinear maps, one obtains a smooth function on M.

There are two basic point–wise operations with tensor fields. On the one hand, given an $\binom{\ell}{k}$–tensor field s and a $\binom{\ell'}{k'}$–tensor field t, one can form the tensor product $s \otimes t$, which then is of type $\binom{\ell+\ell'}{k+k'}$. In the picture of multilinear maps, this just feeds the first arguments into the first map and the others into the second map and then multiplies the values. On the other hand, one can form the contraction or evaluation map $T_xM \otimes T_x^*M \to \mathbb{R}$, which maps $\xi \otimes \varphi$ to $\varphi(\xi)$. This can be extended to contracting one covariant entry of a $\binom{\ell}{k}$–tensor field with one contravariant entry to obtain an $\binom{\ell-1}{k-1}$–tensor field.

In this last bit it is already visible, that there is some need for notation, since one has to select one of the entries of each type. Abstract index notation as introduced by Roger Penrose offers this possibility. At the same time, this has the advantage that, while the notation makes sense without a choice of local coordinates (and hence there is no need to check that things do not depend on a choice of coordinates) an abstract index expression gives the expression in local coordinates after any such choice.

In abstract index notation, indices are used to indicate the type of tensor fields as well as contractions. A $\binom{\ell}{k}$–tensor field is denoted by some letter with ℓ upper indices and k lower indices. So ξ^i will be a vector field, φ_j a one–form, and A^a_b a $\binom{1}{1}$–tensor field. A tensor product is simply indicated by writing the tensor fields aside of each other, which allows keeping track of the indices. A contraction is indicated by using the same symbol for one upper and one lower index, these indices then are not "free" so they are not to be counted in determining the type. So for example for a $\binom{1}{1}$–tensor field A^a_b there is just one possible contraction which is denoted by A^a_a (or also by A^i_i) and this is a tensor field of type $\binom{0}{0}$, i.e. a smooth function. The space $T_xM \otimes T_x^*M$ can be identified both with $L(T_xM, T_xM)$ and with $L(T_x^*M, T_x^*M)$. Either of these identifications can be obtained by first forming the tensor product with the source space and then applying the unique possible contraction (and the resulting maps are dual to each other). The maps on vector fields and one–forms induced by A^a_b can be written as $A(\xi)^i = A^i_j \xi^j$ respectively as $A(\varphi)_b = A^a_b \varphi_a$. In this picture, the smooth function A^i_i corresponds to the point–wise trace of either of these maps.

Choosing a chart (U, u) for M with local coordinates u^i, one has the corresponding coordinate vector fields $\partial_i = \frac{\partial}{\partial u^i}$ and the dual one–forms du^i. Then one can represent tensor fields by their coefficient functions with respect to the induced bases. For example, a $\binom{1}{1}$–tensor field A can then on U be written as $\sum_{i,j} A^i_j \partial_i \otimes du^j$, and one often omits the sum using Einstein sum convention. Here the A^i_j are smooth functions for each i and j and interpreting A as a field of bilinear maps, one has $A^i_j = A(du^i, \partial_j)$. Given a vector field ξ, we may represent it on U as $\sum_j \xi^j \partial_j$. Therefore, the vector field $A(\ ,\xi)$ can be written as $\sum_j \xi^j A(\ ,\partial_j)$, which in turn is given by $\sum_{i,j} \xi^j A^i_j \partial_i$. Hence the vector field $A(\xi)$ really has coordinate functions $A^i_j \xi^j$ (using Einstein sum convention) and the abstract index expression also gives the expression in local coordinates.

A further ingredient in the calculus with tensor fields is that the identity map (on T_xM or on T_x^*M) defines a canonical element in $T_xM \otimes T_x^*M$. These elements of course fit together to define a canonical $\binom{1}{1}$–tensor field, which in abstract index notation is usually called δ^i_j. Interpreting this as the Kronecker–delta, we again get the coordinate expression in any local coordinate system.

The last important ingredient are symmetrizations and alternations. These can only affect several entries of the same type (covariant or contravariant) of a tensor field. Let us consider the simplest situation of a $\binom{0}{2}$–tensor field, whose values are bilinear forms on tangent spaces. If t is such a tensor field, then its symmetrization is defined by $s(\xi,\eta) = \frac{1}{2}(t(\xi,\eta)+t(\eta,\xi))$ while for the alternation, the second summand is subtracted rather than added. So the symmetrization of t can be written as $\frac{1}{2}(t_{ij}+t_{ji})$ and similarly for the alternation. If one has to symmetrize or alternate over more than two entries, one sums over all permutations of the entries, multiplies by the sign of the permutation in the case of the alternation, and divides by the number of permutations. Since this becomes a bit tedious to write out, one denotes a symmetrization over a group of indices by putting them into round brackets and an alternation by putting them into square brackets. The conventions are chosen in such a way, that one can efficiently express the fact that a tensor is symmetric respectively alternating. For example a $\binom{0}{k}$–tensor field φ is a k–form if and only if $\varphi_{i_1\dots i_k} = \varphi_{[i_1\dots i_k]}$.

Basic definitions and consequences

1.4. Riemannian metrics and Riemannian manifolds. We will always assume that manifolds are smooth (C^∞) and paracompact, so that partitions of unity are available.

Definition 1.4. (1) A *pseudo–Riemannian metric* on a smooth manifold M is a $\binom{0}{2}$–tensor field g on M such that for each point $x \in M$, the value $g_x : T_xM \times T_xM \to \mathbb{R}$ is a non–degenerate symmetric bilinear form.

(2) A *Riemannian metric* is a pseudo–Riemannian metric such that for each $x \in M$ the value g_x is positive definite and hence defines an inner product on the vector space T_xM.

(3) A *(pseudo–) Riemannian manifold* (M, g) is a smooth manifold M together with a (pseudo–) Riemannian metric g on M.

For a pseudo–Riemannian metric g on M and a point $x \in M$, the bilinear form g_x has a well defined signature (p, q) with $p + q = n = \dim(M)$. By definition, p (respectively q) is the maximal dimension of a linear subspace of T_xM on which the restriction of g_x is positive (respectively negative) definite. From this, it easily follows that the signature is locally constant, and one usually assumes that it is constant on all of M.

The situation with pseudo–Riemannian metrics is a bit unfortunate. On the one hand, they are an interesting topic from a mathematical point of view and they have important applications. In particular, the geometry of pseudo–Riemannian metrics of signature $(1, 3)$ forms a large part of general relativity. Moreover, large parts of Riemannian geometry, in particular the study of the Levi–Civita connection and its curvature, generalize to the pseudo–Riemannian case with only minimal changes. On the other hand, some of the fundamental and most intuitive facts about Riemannian metrics, in particular the relation to metrics in the topological sense, do not generalize. Therefore, it is difficult to treat Riemannian and pseudo–Riemannian metrics coherently at the same time, and unfortunately we'll have to focus on the Riemannian case. Still I will try to indicate which parts of the theory generalize without changes.

Proposition 1.4. *(1) For any smooth manifold M, there is a Riemannian metric g on M.*

(2) Let (M,g) be a Riemannian manifold, let (U,u) be a local chart on M. Viewed as a matrix, the local coordinate expression g_{ij} of the tensor field g is symmetric and positive definite and thus invertible. The point–wise inverse matrix defines a smooth $\binom{2}{0}$–tensor field g^{ij} on M such that $g^{ij}g_{jk}=\delta^i_k$.

(3) In the setting of (2) consider the smooth function $\mathrm{vol}_g := \sqrt{\det(g_{ij})}$ on U. Under a change of local coordinates, this function transforms by the absolute value of the determinant of the derivative of the change of coordinates. Hence for any smooth function f on M, the product $f\,\mathrm{vol}_g$ can be integrated over compact subsets of M in a coordinate–independent way.

(4) Let (M,g) be a Riemannian manifold of dimension n. Then for each $x\in M$, there is an open neighborhood U of x in M and there are local vector fields $\xi_1,\dots,\xi_n\in\mathfrak{X}(U)$ such that for each $y\in U$, the vectors $\xi_1(y),\dots,\xi_n(y)$ form an orthonormal basis for T_yM.

Proof. Let (U,u) be a chart on a smooth manifold M. Then for a $\binom{0}{2}$–tensor field g on M, the coordinate expression of g is given by $g_{ij}=g(\partial_i,\partial_j)$. Hence g_x is symmetric if and only if the matrix $(g_{ij}(x))$ is symmetric and g_x is positive definite if and only if the matrix $(g_{ij}(x))$ is positive definite.

(1) The above argument shows that we can find a Riemannian metric on U, for example by taking g_{ij} to be the identity matrix. Now we can choose a covering (U_α,u_α) of M by coordinate charts and a sub–ordinate partition $\{\varphi_\alpha\}$ of unity. For each α take a Riemannian metric g_α on U_α and then put $g:=\sum_\alpha\varphi_\alpha g_\alpha$. It follows immediately that this is a symmetric $\binom{0}{2}$–tensor field. Moreover, for a point $x\in M$ and a tangent vector $0\neq\xi\in T_xM$, we have $g(x)(\xi,\xi)=\sum_\alpha\varphi_\alpha(x)g_\alpha(x)(\xi,\xi)$. Now by construction $g_\alpha(x)(\xi,\xi)\geq 0$ for all α such that $x\in U_\alpha$ and $\varphi_\alpha(x)\geq 0$ for all α, so $0\leq g(x)(\xi,\xi)$. Moreover, there is at least one α such that $\varphi_\alpha(x)>0$, which implies $x\in U_\alpha$ and hence $g_\alpha(x)(\xi,\xi)>0$, so $g(x)(\xi,\xi)>0$, and the proof of (1) is complete.

(2) From above, we know that $(g_{ij}(x))$ is a symmetric, positive definite matrix depending smoothly on x. Hence it is invertible in each point, and we can denote the inverse matrix, which is again symmetric, by $(g^{ij}(x))$. The components of the inverse of a matrix can be computed by determinants via Cramer's rule, so inversion of matrices is a smooth function, so also the g^{ij} depend smoothly on x. Hence $\sum_{ij}g^{ij}du^i\otimes du^j$ is a well defined $\binom{2}{0}$–tensor field on U. Of course, these tensor fields for different charts agree, thus defining a smooth tensor field on M. The abstract index expression $g^{ij}g_{jk}=\delta^i_k$ just expresses the fact that in local coordinates the matrices are inverse to each other.

(3) Suppose that $U\subset M$ is open and that u_α and u_β are diffeomorphisms from U onto open subsets of $\mathbb{R}^n$. Consider the chart change $u_{\alpha\beta}:=u_\beta\circ u_\alpha^{-1}:u_\alpha(U)\to u_\beta(U)$ and its derivative $D(u_{\alpha\beta})$. Writing $D(u_{\alpha\beta})(u_\alpha(x))=A^i_j(x)$ for $x\in U$, we by definition obtain $\frac{\partial}{\partial u^j_\alpha}=\sum_i A^i_j(x)\frac{\partial}{\partial u^i_\beta}$. This implies that the coordinate expressions g^α_{ij} and g^β_{ij} are related by

$$g^\alpha_{ij}(x)=\sum_{k,\ell}A^k_i(x)A^\ell_j(x)g^\beta_{k\ell}(x).$$

In terms of matrices, the right hand side can be written as the product with A and its transpose (which is exactly the behavior of the symmetric matrix associated to an inner product under a change of basis). This shows that $\det(g^\alpha_{ij}(x))=\det(A^i_j(x))^2\det(g^\beta_{ij}(x))$.

Thus the square roots transform by $|\det(A^i_j(x))|$, which is exactly the behavior required for the integral of $f\,\mathrm{vol}_g$ being defined independently of coordinates.

(4) This is the fact that the Gram–Schmidt orthonormalization scheme can be done depending smoothly on a point. Given x, we can find a neighborhood U of x in M and vector fields $\eta_1,\dots,\eta_n\in\mathfrak{X}(U)$ such that the vectors $\eta_1(y),\dots,\eta_n(y)$ form a basis for T_yM for each $y\in U$. (For example, we can use the coordinate vector fields associated to a chart.) Since η_1 is nowhere vanishing on U, $g(\eta_1,\eta_1)$ is a nowhere vanishing smooth function on U, so we can define $\xi_1:=\frac{1}{\sqrt{g(\eta_1,\eta_1)}}\eta_1$. Then by construction $\xi_1(y)\in T_yM$ is a unit vector for each $y\in U$. Next, we define $\tilde{\xi}_2:=\eta_2-g(\eta_2,\xi_1)\xi_1$, which evidently is a smooth vector field on U such that $g(\tilde{\xi}_2,\xi_1)=0$. By construction $\eta_2(y)$ and $\xi_1(y)$ are linearly independent for each y, so $\tilde{\xi}_2$ is nowhere vanishing. Thus we can define $\xi_2:=\frac{1}{\sqrt{g(\tilde{\xi}_2,\tilde{\xi}_2)}}\tilde{\xi}_2$, and this is a smooth vector field on u, such that $\xi_1(y)$ and $\xi_2(y)$ form an orthonormal system in T_yM for each $y\in U$. The other ξ_i are constructed similarly. □

Remark 1.4. (1) The simple trick used in the proof of part (1) to glue local Riemannian metrics using a partition of unity depends on the fact that positive definite inner products form a convex set. In fact, the corresponding statement for pseudo–Riemannian metrics is wrong! For example, there are topological obstructions against existence of a pseudo–Riemannian metric of signature $(n-1,1)$ for even n.

(2) If the manifold M is oriented, then the result in (3) can be stated as the fact that the local coordinate expressions $\sqrt{\det(g_{ij}(x))}dx^1\wedge\dots\wedge dx^n$ in the charts of an oriented atlas fit together and define a global differential form of top degree on M. This is called the *volume form* associated to the metric g. In the case of non–orientable manifolds, there is a notion of *densities*, which are the objects that can be integrated independently of coordinates, see Section 10 of [**Mi**]. Hence vol_g is also referred to as the *volume density* associated to g. The main moral is that in the presence of a Riemannian metric, one obtains a well defined notion of integration over smooth functions.

(3) A family $\{\xi_1,\dots,\xi_n\}$ as in part (4) of the Proposition is called a *local orthonormal frame* for M around x. Observe that then any vector field on U can be written as a linear combination of the ξ_i with smooth coefficients.

1.5. Immediate consequences. Given a Riemannian metric g on a manifold M, one can use the data constructed in Proposition 1.4 to obtain a large number of additional structures. On the level of individual tangent spaces, one may use the point–wise inner product as known from linear algebra, and usually the result will depend smoothly on the point. For example, one can look at the inner product of a tangent vector with itself and at its norm, i.e. at $g_x(\xi,\xi)$ respectively $\sqrt{g_x(\xi,\xi)}$. If $\xi\in\mathfrak{X}(M)$ is a vector field, then smoothness of the tensor field g implies that $g(\xi,\xi)$ is a smooth function. This function is non–zero unless ξ vanishes in a point. Hence also $\sqrt{g(\xi,\xi)}$ is smooth where ξ is non–zero.

Likewise, for two non–zero tangent vectors ξ and η in a point $x\in M$, one can characterize the angle α between ξ and η by the usual formula $\cos(\alpha)=\frac{g_x(\xi,\eta)}{\sqrt{g_x(\xi,\xi)}\sqrt{g_x(\eta,\eta)}}$. As before, for non–vanishing vector fields, the angle depends smoothly on the point. In particular, given two curves through a point x, one may define the angle between the two curves and, more specifically, one can talk about curves (and more general submanifolds) intersecting orthogonally in a point.

Integrating functions via the volume density vol_g, has several evident applications. From the definition of vol_g it follows that if $f : M \to \mathbb{R}$ is a compactly supported smooth function with non–negative values than $\int_M f\,\mathrm{vol}_g \geq 0$ and $\int_M f\,\mathrm{vol}_g = 0$ in only possible for $f = 0$. Hence one can make the space $C^\infty_c(M, \mathbb{R})$ of smooth functions with compact support into a pre–Hilbert space by defining $\langle f, h\rangle := \int_M fh\,\mathrm{vol}_g$. Hence one can provide the setup for functional analysis by looking at the completion of $C^\infty_c(M, \mathbb{R})$ with respect to the resulting norm, which is the space $L^2(M, \mathbb{R})$ of square integrable functions, and so on.

This can be immediately extended to the space $\mathfrak{X}_c(M)$ of compactly supported vector fields on M. Here one defines a pre–Hilbert structure by $\langle \xi, \eta\rangle := \int_M g(\xi, \eta)\,\mathrm{vol}_g$, or in abstract index notation $\int_M g_{ij}\xi^i\eta^j\,\mathrm{vol}_g$. Again, it is possible to complete this to the space of square–integrable vector fields. Next, we can take the inverse metric g^{ij} as constructed in Proposition 1.4. For each point x, this defines a positive definite inner product on the vector spaces T^*_xM which depends smoothly on the point x. In particular, for two one–forms α and β, $g^{ij}\alpha_i\beta_j$ is a smooth function on M, and we can define $\langle \alpha, \beta\rangle := \int_M g^{ij}\alpha_i\beta_j\,\mathrm{vol}_g$. This makes the space $\Omega^1(M)$ of one–forms on M into a pre–Hilbert spaces, which can be completed to the space of square–integrable one–forms.

It is a matter of linear algebra to extend this further. Given inner products on two vector spaces, one obtains an induced inner product on their tensor product. Iterating this, g_x induces inner products on all the spaces $\otimes^k T^*_xM \otimes \otimes^\ell T_xM$ and likewise on the spaces $\Lambda^k T^*_xM$ of alternating k–linear maps $(T_xM)^k \to \mathbb{R}$. All these induced inner products can be characterized in the way that starting from an orthonormal basis of T_xM, also the induced basis of the space in question is orthonormal. Using part (4) of Proposition 1.4, one concludes that there are smooth local orthonormal frames for all these inner products, which implies that they depend smoothly on the point. Integrating point–wise inner products, one can make all spaces of tensor–fields and of differential forms into pre–Hilbert spaces.

Next, an inner product on a vector space induces an isomorphism with the dual space. Hence given a point x in a Riemannian manifold (M, g) and a tangent vector $\xi \in T_xM$, we obtain a linear functional $T_xM \to \mathbb{R}$ by $\eta \mapsto g_x(\xi, \eta)$. Starting from a vector field $\xi \in \mathfrak{X}(M)$ we can associate to each $x \in M$ the functional $g_x(\xi(x), \_)$. Inserting the values of a smooth vector field η, we obtain the smooth function $g(\xi, \eta)$, so this defines a one–form on M. In abstract index notation, the resulting linear map $\mathfrak{X}(M) \to \Omega^1(M)$ is given by $\xi \mapsto g_{ij}\xi^j$. Similarly, $\alpha \mapsto g^{ij}\alpha_j$ defines a map $\mathfrak{X}(M) \to \Omega^1(M)$, which is inverse to the other one. Thus the metric g induces an isomorphism between vector fields and one–forms.

This readily generalizes to tensor fields of arbitrary type. In view of abstract index notation this is often phrased as "raising and lowering indices using the metric" (and its inverse). For example, given a $\binom{1}{1}$–tensor field $A = A^i_j$, we can use the metric to lower the upper index and form the $\binom{0}{2}$–tensor field $A^k_j g_{ik}$. This corresponds to the bilinear form $(\xi, \eta) \mapsto g(\xi, A(\eta))$. This bilinear form can be decomposed into a symmetric and a skew symmetric part as $A^k_{(j}g_{i)k} + A^k_{[j}g_{i]k}$. One can then convert these parts back to $\binom{1}{1}$–tensor fields to obtain a decomposition of A itself. For example, for the symmetric part, this reads as

$$\tfrac{1}{2}g^{i\ell}(A^k_\ell g_{jk} + A^k_j g_{\ell k}) = \tfrac{1}{2}(g^{i\ell}A^k_\ell g_{jk} + A^k_j\delta^i_k) = \tfrac{1}{2}(A^i_j + g^{ik}A^\ell_k g_{\ell j}).$$

To interpret this result, observe that the for the linear map $B^i_j := g^{ik}A^\ell_k g_{\ell j}$ we can write $g(B(\xi),\eta)$ as

$$g_{aj}B^a_i\xi^i\eta^j = g_{aj}g^{ab}A^c_b g_{ci}\xi^i\eta^j = \delta^b_j A^c_b g_{ci}\xi^i\eta^j = A^c_j g_{ci}\xi^i\eta^j,$$

so by symmetry of g, this coincides with $g(\xi, A(\eta))$. Hence $B_x : T_xM \to T_xM$ is simply the adjoint of A_x with respect to the inner product g_x, and so we have just applied the usual formula for the symmetric part from linear algebra in each point.

1.6. Hodge–$*$ operator, codifferential, and Laplacian. Let us discuss a more complicated but very important construction based on the ideas from Section 1.5. Let (M, g) be an oriented Riemannian manifold of dimension n. Then we can view vol_g as a nowhere vanishing element of $\Omega^n(M)$, thus identifying for each point $x \in M$ the space $\Lambda^n T^*_xM$ with $\mathbb{R}$. For each point $x \in M$ and each $k = 0,\dots,n$, the wedge product defines a bilinear map $\Lambda^k T^*_xM \times \Lambda^{n-k}T^*_xM \to \Lambda^n T^*_xM$. Linear algebra tells us that this gives rise to a linear isomorphism $\Lambda^{n-k}T^*_xM \to L(\Lambda^k T^*_xM, \Lambda^n T^*_xM)$. Using $\mathrm{vol}_g(x)$ to identify $\Lambda^n T^*_xM$ with $\mathbb{R}$, we can identify the target space with the dual space $(\Lambda^k T^*_xM)^*$. But from above, we know that g_x induces an inner product $\tilde g_x$ on $\Lambda^k T^*_xM$ which gives an identification of the dual space with $\Lambda^k T^*_xM$ itself. Otherwise put, for each $\beta \in \Lambda^k T^*_xM$, there is a unique element $*\beta \in \Lambda^{n-k}T^*_xM$ such that for each $\alpha \in \Lambda^k T^*_xM$ we have $\alpha \wedge *\beta = \tilde g_x(\alpha,\beta)\,\mathrm{vol}_g(x)$.

Proposition 1.6. *Let (M, g) be a oriented Riemannian manifold of dimension n.*

(1) For each $k = 0,\dots,n$, the point–wise $*$–operation defined above gives rise to a linear isomorphism $* : \Omega^k(M) \to \Omega^{n-k}(M)$ which is characterized by $\alpha \wedge *\beta = \tilde g(\alpha,\beta)\,\mathrm{vol}_g$ for all $\alpha,\beta \in \Omega^k(M)$. Moreover, for any $\beta \in \Omega^k(M)$, we get $*(*\beta) = (-1)^{k(n-k)}\beta$.

(2) Let d be the exterior derivative and define $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ as $\delta\beta := (-1)^{nk+n+1} * d * \beta$. Then this satisfies $\delta^2 = \delta\circ\delta = 0$. If M is compact, then δ is adjoint to d with respect to the L^2–inner products on the spaces $\Omega^*(M)$ introduced in 1.5.

(3) Suppose that M is compact. Then the operator $\Delta := \delta d + d\delta : \Omega^k(M) \to \Omega^k(M)$ is self–adjoint with respect to the L^2 inner product from 1.5. Moreover, for $\alpha \in \Omega^k(M)$, we get $\Delta(\alpha) = 0$ if and only if $d\alpha = 0$ and $\delta\alpha = 0$, while for $\beta \in \Omega^{k-1}(M)$, $\Delta(d\beta) = 0$ implies $d\beta = 0$.

Proof. (1) We first have to show that for a smooth k–form $\beta \in \Omega^k(M)$ the point–wise definition of $*\beta$ gives rise to a smooth form. This is a local question, so we can restrict to an open subset U for which there is a local orthonormal frame $\xi_1,\dots,\xi_n$, see Proposition 1.4. Then we define $\alpha^1,\dots,\alpha^n \in \Omega^1(U)$ to be the dual forms, i.e. $\alpha^i(\xi_j) = \delta^i_j$ for all i, j. Then for each $x \in U$ the values $(\alpha^{i_1}\wedge\dots\wedge\alpha^{i_k})(x)$ with $1 \le i_1 < \dots < i_k \le n$ form an orthonormal basis for $\Lambda^k T^*_xM$. Moreover, it is easy to see that $\alpha^1\wedge\dots\wedge\alpha^n = \mathrm{vol}_g$ on U. But this implies that among the basis elements $(\alpha^{j_1}\wedge\dots\wedge\alpha^{j_{n-k}})(x)$ for $\Lambda^{n-k}T^*_xM$, there is a unique one, for which the wedge product with $(\alpha^{i_1}\wedge\dots\wedge\alpha^{i_k})(x)$ coincides with $\pm\,\mathrm{vol}_g(x)$, while all other wedge–products are zero. But this exactly means that, up to a sign (which is independent of x), we have

$$*(\alpha^{i_1}\wedge\dots\wedge\alpha^{i_k})(x) = (\alpha^{j_1}\wedge\dots\wedge\alpha^{j_{n-k}})(x)$$

where $\{j_1,\dots,j_{n-k}\}$ is the complement of $\{i_1,\dots,i_k\}$ in $\{1,\dots,n\}$. Hence for each of the forms $\alpha^{i_1}\wedge\dots\wedge\alpha^{i_k}$ the point–wise $*$ defines a smooth $(n-k)$–form. Since any k form can be written as a linear combination of these with smooth coefficients and $*$ is evidently linear, we conclude that $*\beta$ is smooth for each $\beta \in \Omega^k(M)$.

To prove the second part of (1), we observe that in the defining equation $\alpha \wedge *\beta = \tilde{g}(\alpha, \beta) \operatorname{vol}_g$, the right hand side is symmetric in α and β. Thus we see that $\alpha \wedge *\beta = \beta \wedge *\alpha = (-1)^{k(n-k)} *\alpha \wedge \beta$ for all $\alpha, \beta \in \Omega^k(M)$. Next, for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{n-k}(M)$, we compute

$$\tilde{g}(\alpha, *\beta) \operatorname{vol}_g = \alpha \wedge * * \beta = (-1)^{(n-k)k} * \alpha \wedge *\beta = (-1)^{(n-k)k} \tilde{g}(*\alpha, \beta) \operatorname{vol}_g .$$

Finally, we take $\alpha, \beta \in \Omega^k$ and compute $\tilde{g}(\alpha, **\beta) = (-1)^{k(n-k)} \tilde{g}(*\alpha, *\beta)$. But above we have seen that $*$ maps an orthonormal system in $\Lambda^k T_x^* M$ to an orthonormal system in $\Lambda^{n-k} T_x^* M$. Hence it is orthogonal, so in particular $\tilde{g}(*\alpha, *\beta) = \tilde{g}(\alpha, \beta)$, and this implies that last statement in (1).

(2) Up to a sign, $\delta\delta\beta$ equals $*d * *d * \beta$ and since the two middle $*$'s also produce a sign only, $d^2 = 0$ implies $\delta^2 = 0$. On the other hand, observe that the sign in the definition of δ is chosen in such a way that for $\beta \in \Omega^{k+1}(M)$ we have $*\delta\beta = (-1)^{k+1} d*\beta$. Now taking $\alpha \in \Omega^k(M)$, we can form $\alpha \wedge *\beta \in \Omega^{n-1}(M)$ and by Stokes' theorem, we get

$$0 = \int_M d(\alpha \wedge *\beta) = \int_M d\alpha \wedge *\beta + (-1)^k \int_M \alpha \wedge d*\beta = \int_M \tilde{g}(d\alpha, \beta) \operatorname{vol}_g - \int_M \tilde{g}(\alpha, \delta\beta) \operatorname{vol}_g .$$

By definition of the L^2–inner product from 1.5, this simply equals $\langle d\alpha, \beta \rangle - \langle \alpha, \delta\beta \rangle$ and adjointness follows.

(3) This is now a simple direct computation. For $\alpha, \beta \in \Omega^k(M)$, we get using the adjointness from (2):

$$\langle \Delta(\alpha), \beta \rangle = \langle \delta d\alpha, \beta \rangle + \langle d\delta\alpha, \beta \rangle = \langle d\alpha, d\beta \rangle + \langle \delta\alpha, \delta\beta \rangle,$$

and in the same way, one shows that this equals $\langle \alpha, \Delta(\beta) \rangle$. If $\Delta(\alpha) = 0$, then $0 = \langle \Delta(\alpha), \alpha \rangle$ and the above computation shows that $0 = \langle d\alpha, d\alpha \rangle + \langle \delta\alpha, \delta\alpha \rangle$. Since $\langle \ , \ \rangle$ is a positive definite inner product, this implies $d\alpha = 0$ and $\delta\alpha = 0$.

Applying this to $\alpha = d\beta$, we see that $\Delta(d\beta) = 0$ implies $\delta d\beta = 0$. But this gives $0 = \langle \delta d\beta, \beta \rangle = \langle d\beta, d\beta \rangle$ and hence $d\beta = 0$. □

Definition 1.6. (1) The operator $*$ is called the *Hodge–$*$–operator* associated to the Riemannian metric g.

(2) The operator δ is called the *codifferential* associated to g.

(3) The operator Δ is called the *Laplace–Beltrami operator* associated to g.

Remark 1.6. (1) For the basic adjointness results in part (2) and (3), compactness of M is not really necessary. In general, one may consider both d and δ as operators on differential forms with compact support and then adjointness is still true.

(2) The Laplace–Beltrami operator is of fundamental importance in large areas of differential geometry and of analysis. Differential forms in the kernel of Δ are called *harmonic forms*. In the case of a compact manifold, Δ extends to an essentially self adjoint operator on L^2–forms, so one can do spectral theory and so on. One can also look at the analog of the heat equation on a compact Riemannian manifold, which is of fundamental importance in geometric analysis.

(3) The last part of the Proposition is the starting point for Hodge–theory on compact Riemannian manifolds. As we have proved, for a harmonic k–form α we get $d\alpha = 0$, so one may look at the class of α in the de–Rham cohomology group $H^k(M)$, which by definition is the quotient of the kernel of $d : \Omega^k(M) \to \Omega^{k+1}(M)$ by the image of $d : \Omega^{k-1}(M) \to \Omega^k(M)$. The last statement in the proposition then shows that this maps the space of harmonic k–forms injectively to $H^k(M)$. Using a bit of functional

analysis, one proves that this map is also surjective and thus a linear isomorphism. Hence any cohomology class contains a unique harmonic representative.

1.7. Arclength and the distance function. The next direct way to use a Riemannian metric is related to arclength of curves.

Definition 1.7. Let (M, g) be a Riemannian manifold and let $c : [a, b] \to M$ be a smooth curve defined on a compact interval in $\mathbb{R}$.

Then we define the *arclength* $L(c)$ and the *energy* $E(c)$ of c by

$$L(c) := \int_a^b \sqrt{g_{c(t)}(c'(t), c'(t))}dt$$
$$E(c) := \tfrac{1}{2} \int_a^b g_{c(t)}(c'(t), c'(t))dt.$$

Of course, the factor $\frac{1}{2}$ in the definition of the energy is just a matter of convention. It is motivated by the definition of kinetic energy in physics. There is an obvious concept of reparametrization of a smooth curve, in which one replaces c by $c \circ \varphi$ for a diffeomorphism φ. As we shall see below, the arclength of a curve remains unchanged if the curve is reparametrized. For some applications, this is an advantage, but for other purposes, like for finding distinguished curves, it is a disadvantage and it is better to use the energy.

For technical purposes, it is better to work with curves which are only piece–wise smooth. Here by a *piece–wise smooth curve* $c : [a, b] \to M$ we mean a continuous curve $c : [a, b] \to M$ such that there is a subdivision $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$ of $[a, b]$ such that for each $i = 0, \dots, N-1$ the restriction of c to $[t_i, t_{i+1}]$ is smooth. Putting $c_i := c|_{[t_i,t_{i+1}]}$ one then defines $L(c) = \sum_{i=0}^{N-1} L(c_i)$ and $E(c) = \sum_{i=0}^{N-1} E(c_i)$. One immediately verifies that this is well defined (i.e. there is no problem with adding additional points to the sub–division around which c is smooth anyway).

Proposition 1.7. *(1) The arclength of smooth curves is invariant under orientation preserving reparametrizations, i.e. if $c : [a, b] \to M$ is a smooth curve and $\varphi : [a', b'] \to [a, b]$ is a diffeomorphism with $\varphi'(t) > 0$ for all t, then $L(c \circ \varphi) = L(c)$.*

(2) For points x, y in a connected Riemannian manifold M define $d_g(x, y)$ as the infimum of the arclengths $L(c)$ of piece–wise smooth curves $c : [a, b] \to M$ with $c(a) = x$ and $c(b) = y$. Then (M, d_g) is a metric space and the topology induced by the metric d_g coincides with the manifold topology on M.

Proof. (1) This is the same computation as in Euclidean space. By the chain rule, we have $(c \circ \varphi)'(t) = c'(\varphi(t)) \cdot \varphi'(t)$ and thus

$$\sqrt{g((c \circ \varphi)(t))((c \circ \varphi)'(t), (c \circ \varphi)'(t))} = |\varphi'(t)|\sqrt{g(c(\varphi(t)))(c'(\varphi(t)), c'(\varphi(t)))}.$$

By assumption, $\varphi'(t) > 0$, so we may leave out the absolute value and the result follows by the substitution rule for one–dimensional integrals.

(2) If $c : [a, b] \to M$ is a smooth curve, then the function in the integral defining $L(c)$ is continuous and non–negative. Hence $L(c) \geq 0$ and $L(c) = 0$ if and only if the integrand is identically zero and hence c is constant. Since M is assumed to be connected, any two points in M can be connected by at least one piece–wise smooth curve and hence $d_g : M \times M \to \mathbb{R}_{\geq 0}$ is well defined. The fact that $d_g(x, y) = d_g(y, x)$ follows easily since one can run through curves in the opposite direction. The triangle inequality $d_g(x, z) \leq d_g(x, y) + d_g(y, z)$ follows since having given a curve c connecting

x to y and a curve $\tilde{c}$ connecting y to z, one can simply run through them successively to obtain a curve of length $L(c) + L(\tilde{c})$ which connects x to z.

Let us next consider the special case $M = \mathbb{R}^n$, endowed with an arbitrary Riemannian metric g. We compare d_g to the Euclidean distance focusing on (a neighborhood of) the point $0 \in \mathbb{R}^n$. Now $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, and we consider the map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $(x, v) \mapsto \sqrt{g(x)(v, v)}$. This map is clearly continuous and positive unless $v = 0$. Looking at the compact set $\bar{B}_1(0) \times S^{n-1}$ we thus see that there are constants $0 < C_1 < C_2$ such that $C_1 \leq \sqrt{g(x)(v, v)} \leq C_2$ provided that $\|x\| \leq 1$ and $\|v\| = 1$. This in turn implies that for $\|x\| \leq 1$ we have

$$C_1\|v\| \leq \sqrt{g(x)(v, v)} \leq C_2\|v\|$$

Hence for a piece–wise smooth curve c whose image is contained in the closed unit ball, the arclength $L^g(c)$ with respect to c and the Euclidean arclength $L^E(c)$ are related by $C_1 L^E(c) \leq L^g(c) \leq C_2 L^E(c)$. In particular for $0 < \epsilon < 1$ and $x \in B_\epsilon(0)$, the straight line provides a curve of length $< \epsilon C_2$ connecting 0 to x, so the $B_\epsilon(0)$ is contained in the d_g–ball around 0 of radius ϵC_2.

Conversely, suppose we have given $0 < \epsilon < 1/C_1$ and a curve $c : [a, b] \to \mathbb{R}^n$ with $c(a) = 0$ and $L(c) < \epsilon$. Then we first prove that c cannot leave the unit ball. Indeed, if c leaves the unit ball, we let $t_0 \in [a, b]$ be the infimum of $\{t : \|c(t)\| \geq 1\}$ and look at the curve $\tilde{c} := c|_{[a,t_0]}$. Then $\tilde{c}$ stays inside the closed unit ball and satisfies $L^g(\tilde{c}) < 1/C_1$ and hence $L^E(\tilde{c}) < 1$, which is a contradiction. Hence we conclude that $L^E(c) < \epsilon C_1$ and hence $c(b) \in B_{\epsilon C_1}(0)$. Hence $B_{\epsilon C_1}(0)$ contains the d_g–ball of radius ϵ around 0.

Now returning to a general Riemannian manifold (M, g) and a point $x \in M$, we can choose a chart (U, u) for M with $x \in U$, $u(x) = 0$, and $u(U) = \mathbb{R}^n$. Then u is a homeomorphism, and we can pull back $g|_U$ by u^{-1} to a Riemannian metric on $\mathbb{R}^n$. Of course, for a curve c with values in U, the arclength of c with respect to g coincides with the arclength of $u \circ c$ with respect to the pullback metric. Now from above we conclude that there is an $\epsilon > 0$ such that curves of length $\leq \epsilon$ stay in U. Hence if $y \in M$ is such that $d_g(x, y) = 0$ then $y \in U$. But then the above considerations show that $u(y)$ has Euclidean distance zero to $0 = u(x)$ and hence $y = x$. Hence (M, d_g) is a metric space, and the above argument shows that any Riemannian metric on $\mathbb{R}^n$ produces the usual neighborhoods of $0 \in \mathbb{R}^n$. Since u is a homeomorphism, we see that d_g leads to the usual neighborhoods of x, which completes the proof. □

Remark 1.7. (1) One may now go ahead as in the Euclidean case, and consider regular parametrizations. For any regularly parametrized curve, one can then obtain a reparametrization by arclength (as usual by solving an ODE). This means the $g(c(t))(c'(t), c'(t)) = 1$ and hence $t = L(c|_{[a,t]})$ for all $t \in [a, b]$.

(2) The relation to metrics in the topological sense is the main point where things go wrong for pseudo–Riemannian metrics. The notion of energy still makes sense in the pseudo–Riemannian setting, but the energy of a non–trivial curve can be zero or negative. (In physical applications, this is a feature, since it allows to distinguish space–like, time–like, and light–like curves.) There is no well defined notion of arclength and no nice relation to metric spaces in the pseudo–Riemannian case.

The Levi–Civita connection

After we have exploited the tensorial operations arising from a Riemannian metric on a smooth manifold, we will next construct and study the fundamental family of "new" differential operators available in the presence of such a metric. While the motivation of this concept from submanifold geometry is not very difficult, things are constructed on an abstract Riemannian manifold in different order.

1.8. Motivation. The most intuitive concept in submanifold geometry, which is related to the covariant derivative, probably is the notion of a geodesic. The simplest non–trivial curves in E^n are the affine lines $t \mapsto x + tv$ with $x \in E^n$, $v \in \mathbb{R}^n$ and $t \in \mathbb{R}$. For a general curve $t \mapsto c(t)$, one can view the derivative c' as a map to $\mathbb{R}^n$, so it is no problem to form the second derivative c'', which again is an $\mathbb{R}^n$–valued function. The affine lines in E^n are exactly the curves for which c' is constant or equivalently $c'' = 0$. Now if $M \subset E^n$ is a smooth submanifold, then in general M will not contain any pieces of affine lines. However, there is a nice class of curves in M, which can be thought of as the paths of particles which move freely in M. Namely, for a smooth curve $c : I \to M$, one requires that for each $t \in I$, the second derivative $c''(t)$ is perpendicular to the tangent space $T_{c(t)}M \subset \mathbb{R}^n$. Intuitively, this means that acceleration is only there to keep the curve on the submanifold. These curves are the geodesics of M, and one shows given $x \in M$ and $\xi \in T_xM$, there locally is a unique geodesic $c : I \to M$ with $c(0) = x$ and $c'(0) = \xi$.

As a slight variation, one can consider the concept of parallel transport. In E^n one can transport a tangent vector $X \in T_xE^n = \mathbb{R}^n$ parallely to all of E^n by looking at the vector field corresponding to the constant function X. To be usable for submanifolds, one has to modify this concept by only looking at it along a curve. Namely, for a curve $c : I \to E^n$, a vector field along c is a smooth function $X : I \to \mathbb{R}^n$, which we view as associating to t a tangent vector in the point $c(t)$. Then one can simply say that X is parallel along c if the function X is constant. Now this concept can be adapted to a smooth submanifold $M \subset E^n$. Given a smooth curve $c : I \to M$, one defines a *vector field along* c as a smooth map $X : I \to \mathbb{R}^n$ such that $X(t) \in T_{c(t)}M$ for all $t \in \mathbb{R}$. Then one says that X *is parallel along* c if for each $t \in I$ the derivative $X'(t)$ is perpendicular to $T_{c(t)}M$. In this sense, any tangent vector can be locally transported parallely along a curve, i.e. it can be locally extended uniquely to a vector field which is parallel along the curve.

Observe that a curve c is a geodesic if and only if $c'(t)$ (which evidently defines a vector field along c) is parallel along c. In this sense, parallel transport is easier to deal with than geodesics are. Simple examples of surfaces in E^3 show that the concept of parallel transport only makes sense along curves. Take the unit sphere S^2 and a tangent vector ξ at the north pole. Then take the great circle in S^2 obtained by intersecting the sphere with the plane orthogonal to ξ. Then along this great circle the constant vector field on E^3 corresponding to ξ is tangent to S^2, so it must be parallel along the curve. So transporting ξ parallely to the south pole along this curve, one obtains ξ. In contrast to this, if one takes the great circle emanating from the north pole in direction ξ and transports ξ parallely along this to the south pole, one obtains $-\xi$! This is another way to see that the sphere is (intrinsically) curved.

The last step is to absorb these ideas into the definition of the covariant derivative, an analog of a directional derivative for vector fields. Suppose that $M \subset E^n$ is a submanifold and $\eta \in \mathfrak{X}(M)$ is a vector field, which we can view as a smooth function $\eta : M \to \mathbb{R}^n$ such that $\eta(x) \in T_xM \subset \mathbb{R}^n$ for all $x \in M$. Now given a point $x \in M$ and a tangent vector $\xi \in T_xM$, one forms $\xi \cdot \eta \in \mathbb{R}^n$ (the directional derivative of the function η in direction ξ) and projects the result orthogonally into T_xM to obtain an element $\nabla_\xi\eta(x) \in T_xM$. This depends smoothly on the point in the sense that for $\xi, \eta \in \mathfrak{X}(M)$, one obtains a smooth vector field $\nabla_\xi\eta$ in this way. There are two crucial properties of this operation. On the one hand, taking $\eta, \zeta \in \mathfrak{X}(M)$ and their point–wise inner product, one gets

$$\xi \cdot \langle \eta, \zeta \rangle = \langle \xi \cdot \eta, \zeta \rangle + \langle \eta, \xi \cdot \zeta \rangle.$$

Since ζ and η lie in the tangent spaces to M, the inner products in the right hand side remain unchanged if one replaces $\xi \cdot \eta$ by $\nabla_\xi \eta$ and $\xi \cdot \zeta$ by $\nabla_\xi \zeta$. Hence we see that ∇ satisfies a Leibniz rule with respect to the first fundamental form.

On the other hand, consider the skew–symmetrization $\nabla_\xi \eta - \nabla_\eta \xi$ of the operation. This can be computed as the orthogonal projection of $\xi \cdot \eta - \eta \cdot \xi$ to the tangent spaces of M. However, it is well known that $\xi \cdot \eta - \eta \cdot \xi = [\xi, \eta]$, the Lie bracket, which is contained in the tangent space anyway. Hence $\nabla_\xi \eta - \nabla_\eta \xi = [\xi, \eta]$, which is referred to as *torsion–freeness* of the covariant derivative. Having the covariant derivative at hand, the fact that a vector field ξ is parallel along c can be written as $0 = \nabla_{c'(t)} \xi$ for all t. (One has to check that this also makes sense for vector fields along c.) So one can again recover the more intuitive earlier concepts.

1.9. Existence and uniqueness of the Levi–Civita connection. It turns out that it is easiest to generalize the covariant derivative to Riemannian manifolds and then derive the other concepts as consequences.

Definition 1.9. Let M be a smooth manifold.

(1) A *linear connection* on TM is an operator $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, which is bilinear over $\mathbb{R}$ and satisfies

$$\nabla_{f\xi} \eta = f \nabla_\xi \eta \qquad \nabla_\xi (f\eta) = (\xi \cdot f)\eta + f \nabla_\xi \eta$$

for all $\xi, \eta \in \mathfrak{X}(M)$ and all $f \in C^\infty(M, \mathbb{R})$.

(2) If ∇ is a linear connection on TM, then the *torsion* of ∇ is the bilinear map $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$T(\xi, \eta) := \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta].$$

The connection ∇ is called *torsion–free* if and only if its torsion vanishes identically.

(3) A linear connection ∇ on TM is said to be *metric* with respect to a Riemannian metric g on M if and only if

$$\xi \cdot g(\eta, \zeta) = g(\nabla_\xi \eta, \zeta) + g(\eta, \nabla_\xi \zeta)$$

for all $\xi, \eta, \zeta \in \mathfrak{X}(M)$.

While this is not really needed for our purposes, observe that the torsion of any linear connection actually defines a $\binom{1}{2}$–tensor field on M. To see this, we just have to prove that $T(\xi, \eta)$ is bilinear over smooth functions. Now if we replace η by $f\eta$ for $f \in C^\infty(M, \mathbb{R})$, then $\nabla_\xi(f\eta) = (\xi \cdot f)\eta + f\nabla_\xi \eta$ and $\nabla_{f\eta} \xi = f \nabla_\eta \xi$ by definition of a linear connection. On the other hand, it is well known that $[\xi, f\eta] = (\xi \cdot f)\eta + f[\xi, \eta]$, which shows that $T(\xi, f\eta) = fT(\xi, \eta)$. Since $T(\eta, \xi) = -T(\xi, \eta)$ is evident, we see that T indeed is a tensor field. This is why the torsion is an important concept.

One of the most fundamental results of Riemannian geometry is the following

Theorem 1.9. *Let (M, g) be a Riemannian manifold. Then there is a unique torsion–free linear connection on TM, which is metric for g.*

We discuss two proofs for this result, we are of quite different nature. While the first proof is entirely global, it is slightly mysterious why it works. The second proof requires some local input, but is makes the algebraic background clear. In both proofs we leave some straightforward verifications to the reader.

First proof. The global proof is based on the fact that, assuming the existence of a torsion free metric linear connection, one can derive a formula for it via a nice trick. Take three vector fields ξ, η, and ζ on M. Write out the definition of being metric three times with the vector fields cyclically permuted and taking the negative in one case, we get

$$\begin{aligned} 0 &= \xi \cdot g(\eta, \zeta) - g(\nabla_\xi \eta, \zeta) - g(\eta, \nabla_\xi \zeta) \\ 0 &= \eta \cdot g(\zeta, \xi) - g(\nabla_\eta \zeta, \xi) - g(\zeta, \nabla_\eta \xi) \\ 0 &= -\zeta \cdot g(\xi, \eta) + g(\nabla_\zeta \xi, \eta) + g(\xi, \nabla_\zeta \eta). \end{aligned}$$

Adding up these three lines, we of course get zero. We can always exchange arguments in g and then use bilinearity. Via torsion freeness, we can replace $-\nabla_\xi\zeta+\nabla_\zeta\xi$ by $-[\xi,\zeta]$, $-\nabla_\eta\zeta+\nabla_\zeta\eta$ by $-[\eta,\zeta]$, and $-\nabla_\xi\eta-\nabla_\eta\xi$ by $-2\nabla_\xi\eta+[\xi,\eta]$. Dividing by 2 and bringing the term involving $\nabla_\xi\eta$ to the other side, we arrive at the so–called *Koszul–formula*, which expresses $2g(\nabla_\xi\eta,\zeta)$ as

$$(*) \qquad \xi \cdot g(\eta,\zeta) + \eta \cdot g(\zeta,\xi) - \zeta \cdot g(\xi,\eta) + g([\xi,\eta],\zeta) - g([\xi,\zeta],\eta) - g([\eta,\zeta],\xi).$$

Observe that in the right hand side, only the Lie bracket and the action of vector fields on smooth functions is used. If we have a torsion free metric connection ∇, then this formula allows us to compute, for each $x \in M$, the value $g_x(\nabla_\xi\eta(x),\zeta(x))$. For fixed ξ and η, we can of course realize any element of T_xM as $\zeta(x)$ for an appropriate vector field ζ. Hence $\nabla_\xi\eta(x)$ is uniquely determined by these values, and since this can be done in each point, $\nabla_\xi\eta$ is uniquely determined. Since this works for arbitrary vector fields, the uniqueness part of the theorem follows.

To prove existence, we show that the formula $(*)$ can be used to define a linear connection ∇. Let us first fix two vector fields ξ and η, and view $(*)$ for arbitrary ζ as defining a map from vector fields to smooth function. This map is linear and one verifies directly that it is even linear over smooth functions, so we have actually defined an element of $\Omega^1(M)$. From 1.5 we know that this can be expressed as $g(\varphi,\zeta)$ for a uniquely determined vector field $\varphi \in \mathfrak{X}(M)$, and we define $\nabla_\xi\eta := \frac{1}{2}\varphi$.

Doing this for all vector fields ξ and η, we obtain an operator $\nabla : \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$, which is bilinear since $(*)$ is evidently linear in ξ and η. Next, one verifies that $(*)$ is linear over smooth functions in ξ. This means that

$$g(\nabla_{f\xi}\eta,\zeta) = fg(\nabla_\xi\eta,\zeta) = g(f\nabla_\xi\eta,\zeta).$$

As above, this shows that ∇ is linear over smooth functions in the first argument. On the other hand, replacing η by $f\eta$ in $(*)$, one obtains the product of $(*)$ by f plus $2(\xi\cdot f)g(\eta,\zeta)$. Bringing the function into the metric, we conclude that $\nabla_\xi(f\eta) = (\xi\cdot f)\eta + f\nabla_\xi\eta$. Hence ∇ defines a linear connection on TM.

To prove torsion–freeness, we observe that the first two summands, the last two summands and the third summand in $(*)$ are symmetric in ξ and η. Hence we obtain

$$2g(\nabla_\xi\eta - \nabla_\eta\xi,\zeta) = g([\xi,\eta],\zeta) - g([\eta,\xi],\zeta),$$

and torsion freeness follows. On the other hand, the second and third summand, the fourth and fifth summand, and the last summand in $(*)$ are skew symmetric in η and ζ. Thus we obtain

$$2(g(\nabla_\xi\eta,\zeta) + g(\eta,\nabla_\xi\zeta)) = 2\xi\cdot g(\eta,\zeta),$$

so ∇ is metric. □

Second Proof. The second proof starts by showing that for any smooth manifold M, there exist linear connections on TM and the space of all such connections can be nicely described. First, in the domain of a chart (U, u), one can represent vector fields as $\xi = \sum_i \xi^i \partial_i$ and $\eta = \sum_j \eta^j \partial_j$ and then define

$$\nabla_\xi \eta := \sum_j (\sum_i \xi^i \tfrac{\partial \eta^j}{\partial u^i}) \partial_j.$$

One immediately verifies that this defines a linear connection on TU. Now take an atlas $\{(U_\alpha, u_\alpha) : \alpha \in I\}$ for M and a subordinate partition φ_α of unity. Then for each index α consider a linear connection ∇^α on TU_α as constructed above. Now taking $\xi, \eta \in \mathfrak{X}(M)$, $\nabla^\alpha_{\varphi_\alpha \xi}\eta$ is defined on U_α and vanishes identically outside of the support of φ_α, so it can be extended by zero to a smooth vector field on M. Thus given ξ and η, $\nabla_\xi \eta := \sum_\alpha \nabla^\alpha_{\varphi_\alpha \xi}\eta$ defines a smooth vector field on M. One immediately verifies that this defines a linear connection on TM.

Given two linear connections ∇ and $\hat{\nabla}$ on TM, one considers their difference, i.e. the map $\Phi : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by $\Phi(\xi, \eta) = \hat{\nabla}_\xi \eta - \nabla_\xi \eta$. This expression is bilinear and clearly linear over smooth functions in ξ. But since both connections satisfy the same compatibility condition with respect to multiplication of η by smooth functions, their difference is linear over smooth functions in η, too. Thus, Φ is a smooth $\binom{1}{2}$–tensor field.

Conversely, if ∇ is a linear connection on TM and Φ is a $\binom{1}{2}$–tensor field, then Φ defines a map $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ which is bilinear over smooth functions, and one immediately verifies that $\hat{\nabla}_\xi \eta := \nabla_\xi \eta + \Phi(\xi, \eta)$ defines a linear connection on TM. (Technically speaking, we have shown that the space of linear connections on TM is an affine space modeled on the vector space of smooth $\binom{1}{2}$–tensor fields on M.) It is also clear, how such a modification affects the torsion. Denoting by T and $\hat{T}$ the torsions of ∇ and $\hat{\nabla}$, we of course get $\hat{T}(\xi, \eta) = T(\xi, \eta) + (\Phi(\xi, \eta) - \Phi(\eta, \xi))$. In abstract index notation, this reads as $\hat{T}^i_{jk} = T^i_{jk} + 2\Phi^i_{[jk]}$.

Next, suppose that g is a Riemannian metric on M, and that ∇ is some linear connection on TM. Then we can look at the extent to which ∇ fails to be metric for g, i.e. consider the map $\mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M, \mathbb{R})$ defined by

$$(\xi, \eta, \zeta) \mapsto A(\xi, \eta, \zeta) := \xi \cdot g(\eta, \zeta) - g(\nabla_\xi \eta, \zeta) - g(\eta, \nabla_\xi \zeta).$$

This mapping evidently is trilinear over $\mathbb{R}$, linear over smooth functions in ξ, and symmetric in η and ζ. But one also verifies readily that it is linear over smooth functions in η and ζ, too, and thus it is given by a $\binom{0}{3}$–tensor field. If we change the connection to $\hat{\nabla}$ using a $\binom{1}{2}$–tensor field Φ, then the resulting tensor field $\hat{A}$ evidently satisfies

$$\hat{A}(\xi, \eta, \zeta) = A(\xi, \eta, \zeta) - g(\Phi(\xi, \eta), \zeta) - g(\eta, \Phi(\xi, \zeta)),$$

or $\hat{A}_{ijk} = A_{ijk} - g_{\ell k}\Phi^\ell_{ij} - g_{\ell j}\Phi^\ell_{ik}$. Now if we put $\Phi^i_{jk} := \frac{1}{2} g^{ir} A_{jkr}$ then the resulting change becomes

$$-\tfrac{1}{2}\delta^r_k A_{ijr} - \tfrac{1}{2}\delta^r_j A_{ikr} = -\tfrac{1}{2}A_{ijk} - \tfrac{1}{2}A_{ikj} = -A_{ijk},$$

where in the last step we used that A_{ijk} is symmetric in the last two indices. Hence this change leads to $\hat{A} = 0$, and thus to a linear connection on TM, which is metric for g.

So finally, we can start with a linear connection ∇ on TM which is metric for g. Changing from ∇ to $\hat{\nabla}$ using Φ, we see from above that $\hat{\nabla}$ is also metric for g if and only if $0 = g_{\ell k}\Phi^\ell_{ij} + g_{\ell j}\Phi^\ell_{ik}$. Otherwise put, the $\binom{0}{3}$–tensor field $\Psi_{ijk} := \Phi^\ell_{ij} g_{k\ell}$ has to be skew symmetric in j and k. On the other hand, the change of torsion caused by this change of connection is trivial if and only if Φ^ℓ_{ij} is symmetric in i and j, i.e. if and only if Ψ_{ijk} is symmetric in i and j. But we already know from the proof of Proposition 1.1 that these symmetries force Ψ to vanish identically. Hence we see that the map sending a linear connection which is metric for g to its torsion is injective. But both the change of connection and the torsion are point–wise objects. In a point $x \in M$ the changes of metric connections are described by trilinear maps $T_xM \times T_xM \times T_xM \to \mathbb{R}$ which are skew symmetric in the last two entries. Hence this space has dimension $n\frac{n(n-1)}{2}$. On the other hand, the torsion in each point is a skew symmetric bilinear map $T_xM \times T_xM \to T_xM$, so again the space of maps has dimension $n\frac{n(n-1)}{2}$. Hence we conclude that the map between metric connections and torsions is a linear isomorphism, so there is a unique torsion free one. □

1.10. The covariant derivative in local coordinates. We first observe that a linear connection is a local operator and thus can be described in local coordinates.

Lemma 1.10. *Let M be a smooth manifold and let ∇ be a linear connection on TM. Then the operator ∇ is local in both arguments, i.e. if $U \subset M$ is open and for $\xi, \eta \in \mathfrak{X}(M)$ we either have $\xi|_U = 0$ or $\eta|_U = 0$, then $\nabla_\xi \eta$ vanishes on U.*

Moreover, ∇ is tensorial in the first argument, i.e. if ξ vanishes in some point $x \in M$, then $\nabla_\xi \eta(x) = 0$ for any $\eta \in \mathfrak{X}(M)$.

Proof. For a point $x \in U$, there is a bump function $\varphi \in C^\infty(M, \mathbb{R})$ such that $\varphi(x) = 1$ and $\operatorname{supp}(\varphi) \subset U$. If $\xi|_U = 0$, then the vector field $\varphi\xi$ vanishes identically, so $0 = \nabla_{\varphi\xi}\eta = \varphi\nabla_\xi\eta$ for any $\eta \in \mathfrak{X}(M)$. Evaluating in x, we get $0 = \varphi(x)\nabla_\xi\eta(x)$ and since $\varphi(x) = 1$, this implies that $\nabla_\xi\eta(x) = 0$.

If $\eta|_U = 0$, then $\varphi\eta = 0$, and we get $0 = \nabla_\xi(\varphi\eta) = (\xi \cdot \varphi)\eta + \varphi\nabla_\xi\eta$. Evaluating in x and using that $\eta(x) = 0$, we again get $\nabla_\xi\eta(x) = 0$.

Now assume that $\xi(x) = 0$ for some point $x \in M$ and choose a chart (U, u) with $x \in U$. Expanding $\xi|_U = \sum_i \xi^i\partial_i$, we conclude from the first part that in computing $\nabla_\xi\eta|_U$ we may replace ξ by this sum. Using the defining properties of ∇, we conclude that $\nabla_\xi\eta|_U = \sum_i \xi^i\nabla_{\partial_i}\eta$. But if $\xi(x) = 0$ then $\xi^i(x) = 0$ for all i and hence $\nabla_\xi\eta(x) = 0$. □

As usual, the lemma implies that $\nabla_\xi\eta|_U$ depends only on $\xi|_U$ and $\eta|_U$ and that $\nabla_\xi\eta(x)$ depends only on $\xi(x)$. The second fact indicates that a linear connection on TM can indeed be thought of as an analog of a directional derivative for vector fields.

This implies that we can compute the action of any linear connection on TM in local coordinates. Consider a local chart (U, u) for M with coordinate vector fields ∂_i and two vector fields $\xi, \eta \in \mathfrak{X}(M)$. Then we can expand the fields as $\xi|_U = \sum_i \xi^i\partial_i$ and $\eta = \sum_j \eta^j\partial_j$ for smooth functions $\xi^i, \eta^j : U \to \mathbb{R}$ and using the lemma, we get

$$\nabla_\xi\eta|_U = \textstyle\sum_{i,j} \nabla_{\xi^i\partial_i}(\eta^j\partial_j) = \sum_{i,j} \xi^i(\partial_i \cdot \eta^j)\partial_j + \sum_{i,j} \xi^i\eta^j\nabla_{\partial_i}\partial_j.$$

Since any vector field on U can be expanded in terms of the coordinate vector fields, there are uniquely determined smooth functions $\Gamma^k_{ij} : U \to \mathbb{R}$ for $i, j, k = 1, \dots, n = \dim(M)$ such that

$$\nabla_{\partial_i}\partial_j = \textstyle\sum_k \Gamma^k_{ij}\partial_k.$$

Knowing this functions, one has a complete description of ∇ in local coordinates via

$$\nabla_\xi\eta|_U = \textstyle\sum_{i,j} \xi^i(\partial_i \cdot \eta^j)\partial_j + \sum_{i,j,k} \xi^i\eta^j\Gamma^k_{ij}\partial_k. \tag{1}$$

Definition 1.10. The quantities Γ^k_{ij} are called the *connection coefficients* or, in particular in the case of the Levi–Civita connection of a Riemannian metric, the *Christoffel symbols* of the linear connection ∇ with respect to the chart (U, u).

Proposition 1.10. *Let M be a smooth manifold and let ∇ be a linear connection on TM.*

(1) The connection ∇ is torsion–free if and only if its connection coefficients with respect to any chart are symmetric in the lower indices, i.e. $\Gamma^k_{ij} = \Gamma^k_{ji}$ for all i, j, k.

(2) If ∇ is the Levi–Civita connection of a Riemannian metric g on M, then the Christoffel symbols are given explicitly by

$$\Gamma^k_{ij} = \tfrac{1}{2}\textstyle\sum_\ell g^{k\ell}\left(\partial_i \cdot g_{j\ell} + \partial_j \cdot g_{i\ell} - \partial_\ell \cdot g_{ij}\right),$$

where g_{ij} and g^{ij} are the components of the metric and its inverse in local coordinates.

Proof. (1) If ∇ is torsion–free, then $[\partial_i, \partial_j] = 0$ implies $\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i$ and hence symmetry of the connection coefficients. Conversely, the formula (1) for ∇ in local coordinates together with the formula for the Lie bracket in local coordinates shows that

$$(\nabla_\xi \eta - \nabla_\eta \xi)|_U = [\xi, \eta]|_U + \textstyle\sum_{i,j,k} \xi^i \eta^j (\Gamma^k_{ij} - \Gamma^k_{ji})\partial_k.$$

Thus symmetry of the connection coefficients implies torsion–freeness of ∇.

(2) We apply the Koszul formula from the first proof of Theorem 1.9 for $\xi = \partial_i$, $\eta = \partial_j$ and $\zeta = \partial_\ell$. The left hand side of this reads as

$$2g(\nabla_{\partial_i}\partial_j, \partial_\ell) = 2\textstyle\sum_m g_{m\ell}\Gamma^m_{ij},$$

so we can recover Γ^k_{ij} from this by multiplying with $g^{k\ell}$ and summing over ℓ. But using that the Lie bracket of two coordinate vector fields always vanishes, the right hand side of the formula in this case reads as

$$\partial_i \cdot g_{j\ell} + \partial_j \cdot g_{i\ell} - \partial_\ell \cdot g_{ij},$$

which immediately implies the claim. □

It is easy to compute directly how connection coefficients transform under a change of local coordinates and in particular to see that they do *not* define a tensor field. Nonetheless, it is sometimes useful to interpret them as a tensor field defined on the domain on a coordinate chart. Given a chart (U, u) one thus defines $\Gamma^U : \mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U)$ as $\Gamma^U(\xi, \eta) = \sum_{i,j,k} \xi^i \eta^j \Gamma^k_{ij}\partial_k$. Using this, one can rewrite equation (1) for a linear connection in local coordinates as

$$\nabla_\xi \eta|_U = \textstyle\sum_j (\xi \cdot \eta^j)\partial_j + \Gamma^U(\xi, \eta).$$

1.11. Parallel transport. The last formula for the covariant derivative has an important advantage. It shows that in order to compute the value of $\nabla_\xi \eta$ in a point $x \in M$ one only has to know $\eta(x)$ and the derivative of the component functions of η with respect to some chart in direction $\xi(x)$. Given a smooth curve $c : I \to M$ with $c(0) = x$ and $c'(0) = \xi$, these derivatives can be computed as $\xi \cdot \eta^j = \frac{d}{dt}|_{t=0}\eta^j(c(t))$.

Now suppose that we start with a curve $c : I \to M$ and take a vector field $\eta \in \mathfrak{X}(M)$. Then for each $t \in I$, we can look at $\nabla_{c'(t)}\eta(c(t)) \in T_{c(t)}M$. From above we see that this depends only on the restriction of η to the image of c. This allows us to generalize the next part of what was discussed in the motivation in 1.9 from embedded submanifolds to general Riemannian manifolds.

Consider an interval $I \subset \mathbb{R}$ and a smooth curve $c : I \to M$ in a manifold M. Then one defines a *vector field along* c as a smooth function $\xi : I \to TM$ such that $\xi(t) \in T_{c(t)}M$ for all $t \in I$. Observe that in the domain of a chart (U, u), we can expand the tangent vectors $\xi(t)$ in terms of the coordinate vector fields ∂_i determined by the chart. Thus we obtain smooth functions ξ^i such that $\xi(t) = \sum_i \xi^i(t)\partial_i(c(t))$. Finally observe that given a vector field ξ along c and a smooth function $f : I \to \mathbb{R}$, one can form $f\xi$ in an obvious way.

Given a linear connection ∇ on TM, the above considerations show that there is a well defined tangent vector $\nabla_{c'(t)}\xi(c(t)) \in T_{c(t)}M$ for each $t \in I$. From the coordinate formula above it follows readily that these fit together to form a smooth vector field $\nabla_{c'}\xi$ along c. The basic properties of this operation are as follows.

Proposition 1.11. *Let (M, g) be a Riemannian manifold and ∇ its Levi–Civita connection.*

(1) The covariant derivative for vector fields along a smooth curve $c : I \to M$ is a linear operator which satisfies the product rule $\nabla_{c'}(f\xi) = f'\xi + f\nabla_{c'}\xi$. In local coordinates we get

$$\nabla_{c'}\xi(t) = \textstyle\sum_i (\xi^i)'(t)\partial_i(c(t)) + \Gamma^U(c'(t), \xi(t))(c(t)).$$

(2) For two vector fields ξ and η along c, one has

$$\tfrac{d}{dt}g(\xi, \eta) = g(\nabla_{c'}\xi, \eta) + g(\xi, \nabla_{c'}\eta),$$

where $g(\xi, \eta)(t) = g(c(t))(\xi(t), \eta(t))$.

(3) Given a point $a \in I$ and a tangent vector $\xi_0 \in T_xM$, where $x = c(a)$, there is a unique vector field $\xi : I \to M$ along c such that $\xi(a) = \xi_0$ and $\nabla_{c'}\xi = 0$.

(4) In the setting of (3) suppose that $[a, b] \subset I$. Then mapping ξ_0 to $\xi(b)$ defines an orthogonal linear isomorphism $T_{c(a)}M \to T_{c(b)}M$.

Proof. (1) Linearity follows immediately from bilinearity of the covariant derivative of vector fields. The formula in local coordinates then follows directly from the considerations in the end of 1.10. Since for the components with respect to local coordinates, one clearly has $(f\xi)^i = f\xi^i$, the product rule follows from this coordinate formula.

(2) this follows immediately from the fact that ∇ is metric for g.

(3) From the coordinate formula in (1) is is clear that in local coordinates $\nabla_{c'}\xi = 0$ is a linear system of first order ordinary differential equations on the coordinate functions ξ^i. Hence this admits a unique global solution for any initial value.

(4) Linearity clearly implies that one obtains a linear map $T_{c(a)}M \to T_{c(b)}M$. From (2) it follows that if $\nabla_{c'}\xi = \nabla_{c'}\eta = 0$, then $g(\xi, \eta)$ is constant, which implies orthogonality of the map. □

Definition 1.11. (1) A vector field ξ along c is called parallel (along c) if and only if $\nabla_{c'}\xi = 0$.

(2) For $c : [a, b] \to M$, the map $T_{c(a)}M \to T_{c(b)}M$ from part (4) of the proposition is called the *parallel transport along c*.

Parallel transport is closely related to a concept called holonomy. Given a point x in a Riemannian manifold M, one considers piece–wise smooth closed curves starting and ending in x. It is easy to see that parallel transport extends to piece–wise smooth curves without problem, so each such curve gives rise to an orthogonal linear map $T_xM \to T_xM$. It is also easy to see that the resulting linear maps form a subgroup of the orthogonal group $O(T_xM)$ (compositions comes from going through two curves successively, while inversion comes from going in the opposite direction). This is called the *holonomy group* of the metric g in the point x. One further proves that for connected M, the holonomy groups in different points are isomorphic, so one can speak about the holonomy group of M. One of the reasons for the importance of the concept of holonomy is that by a classical result of M. Berger, one can completely classify (in a certain sense) the possible holonomy groups of Riemannian manifolds.

1.12. Geodesics and the exponential map. For a smooth curve $c : I \to M$, the derivative c' of course is a vector field along c. Hence it makes sense to call a curve c a *geodesic* of g, if $\nabla_{c'}c' = 0$ i.e. if c' is parallel along c. We can quickly prove some fundamental results on geodesics:

Proposition 1.12. *Let (M, g) be a Riemannian manifold.*

(1) Given $x \in M$ and $\xi \in T_xM$, there is a unique maximal interval $I \subset \mathbb{R}$ with $0 \in I$ and a unique maximal geodesic $c : I \to M$ with $c(0) = x$ and $c'(0) = \xi$.

(2) Given $x \in M$, there is an open neighborhood U of zero in T_xM such that for each $\xi \in U$, the interval I from (1) contains $[0, 1]$ and mapping ξ to $c(1)$ defines a smooth map $\exp_x : U \to M$.

(3) The map $\exp_x$ from (2) satisfies $\exp_x(0) = x$ and $T_0 \exp = \mathrm{id}_{T_xM}$ so choosing U small enough, $\exp_x$ is a diffeomorphism from U onto an open neighborhood of x in M.

(4) Let $\pi : TM \to M$ be the natural projection. There is an open neighborhood V of the zero–section in TM such that for each $\xi \in V$, $\exp_{\pi(\xi)}(\xi) \in M$ is defined. Calling the latter element $\exp(\xi)$, one obtains a smooth map $\exp : V \to M$. Choosing V small enough, $(\pi, \exp) : V \to M \times M$ is a diffeomorphism onto an open neighborhood of the diagonal in $M \times M$.

Proof. From part (1) of Proposition 1.11, we see that in local coordinates, the equation $0 = \nabla_{c'}c'$ reads as $(c^i)''(t) = -\Gamma^i_{jk}(c(t))(c^j)'(t)(c^k)'(t)$, so this is a (non–linear) system of second order ODEs, which admits unique local solutions for fixed initial values for c and c'. From this, (1) follows by piecing together unique local solutions to maximal solutions.

(2) The fact that solutions of ODEs depend smoothly on the initial data implies that there is an $\epsilon > 0$ such that for each unit vector $\xi \in T_xM$, the maximal interval on which the solution from (1) is defined contains $(-\epsilon, \epsilon)$. Now suppose that $I \subset \mathbb{R}$ is an interval containing zero and $c : I \to M$ is a geodesic. Fix a real number s and consider $\tilde{c}(t) := c(st)$. Then $\tilde{c}'(t) = sc'(st)$, and one easily concludes that $\tilde{c}$ is a geodesic with $\tilde{c}(0) = c(0)$ and $\tilde{c}'(0) = sc'(0)$. Together with the above, this shows that $\exp_x$ is defined and smooth on the ball of radius ϵ and thus on an open neighborhood of 0.

(3) Since the constant curve $c(t) = x$ is a geodesic with $c(0) = x$ and $c'(0) = 0$, we see that $\exp_x(0) = x$. Moreover, the considerations in the proof of part (2) show that the geodesic $c : I \to M$ with $c(0) = x$ and $c'(0) = \xi$ can be written as $t \mapsto \exp_x(t\xi)$ for t close enough to zero. But this shows that

$$T_0 \exp_x \cdot\xi = \tfrac{d}{dt}_{t=0} \exp_x(t\xi) = \xi,$$

so $T_0 \exp_x = \mathrm{id}(T_xM)$. Hence $\exp_x$ is a local diffeomorphism around 0.

(4) The fact that exp is well defined on an open neighborhood of the zero section in TM follows from smooth dependence of solutions of ODEs on the initial conditions as before. Hence we can consider $(\pi, \exp) : V \to M \times M$. This maps $0_x \in T_xM$ to (x, x), so the diagonal is in the image. Next we claim that $T_{0_x}(\pi, \exp) : T_{0_x}TM \to T_xM \times T_xM$ is injective and thus a linear isomorphism for dimensional reasons. The first component of this map is $T_x\pi$, so this is surjective and thus has a kernel of dimension $n = \dim(M)$. On the other hand, one can view T_xM naturally as a subspace of $T_{0_x}TM$ (via the derivatives of the curves $t \mapsto t\xi$). This is contained in $\ker(T_x\pi)$ and hence has to coincide with this subspace for dimensional reasons. But by construction, the second component of $T_{0_x} \exp$ coincides on this subspace with $T_0 \exp_x$, so the claim follows. Hence we know that $(\pi, \exp)$ is a local diffeomorphism around 0_x for each $x \in M$. A moment of thought shows that this map is injective and hence a diffeomorphism on sufficiently small open neighborhoods of the zero section. □

An important consequences is that for each $x \in M$ on can use the inverse of $\exp_x$ as a local chart around x. Choosing an orthonormal basis of T_xM, one can identify the range of this chart with $\mathbb{R}^n$ (endowed with the standard inner product), and thus get local coordinates around x. These are called *normal coordinates centered at x*.

1.13. Curvature. The last topic we discuss in this chapter is the curvature tensor of a Riemannian metric.

Proposition 1.13. *Let (M,g) be a Riemannian manifold and consider the trilinear map $\mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ defined by*

$$R(\xi,\eta)(\zeta) := \nabla_\xi\nabla_\eta\zeta - \nabla_\eta\nabla_\xi\zeta - \nabla_{[\xi,\eta]}\zeta.$$

(1) This is given by the action of a $\binom{1}{3}$–tensor field, which in abstract index notation is denoted by $R_{ij}{}^k{}_\ell$ via $R(\xi,\eta)(\zeta)^k = R_{ij}{}^k{}_\ell\xi^i\eta^j\zeta^\ell$.

(2) The tensor field R can be viewed as a two–form with values in skew–symmetric endomorphisms of the tangent bundle, i.e. $g(R(\xi,\eta)(\zeta_1),\zeta_2)$ is skew symmetric both in ξ and η and in ζ_1 and ζ_2, respectively $R_{ij}{}^k{}_\ell = R_{[ij]}{}^k{}_\ell$ and $R_{ij}{}^a{}_\ell g_{ka} = R_{ij}{}^a{}_{[\ell}g_{k]a}$.

(3) In view of the last symmetry, R_x can be viewed as a bilinear form on Λ^2T_xM, and as such a form it is symmetric, i.e. $g(R(\xi,\eta)(\zeta_1),\zeta_2) = g(R(\zeta_1,\zeta_2)(\xi),\eta)$, respectively $R_{ij}{}^a{}_k g_{\ell a} = R_{k\ell}{}^a{}_i g_{ja}$.

(4) Finally, R satisfies the first Bianchi–identity

$$0 = R(\xi,\eta)(\zeta) + R(\zeta,\xi)(\eta) + R(\eta,\zeta)(\xi),$$

respectively $R_{[ij}{}^k{}_{\ell]} = 0$.

Proof. (1) We have to show that the map we have defined is linear over smooth functions in all three entries, but since it is obviously skew–symmetric in ξ and η, it suffices to verify this linearity in η and ζ. Now the second term in the defining formula for R evidently is linear over smooth functions in η, while for the first term, we compute for $f\in C^\infty(M,\mathbb{R})$:

$$\nabla_\xi\nabla_{f\eta}\zeta = \nabla_\xi(f\nabla_\eta\zeta) = f\nabla_\xi\nabla_\eta\zeta + (\xi\cdot f)\nabla_\eta\zeta.$$

But on the other hand, $[\xi,f\eta] = f[\xi,\eta]+(\xi\cdot f)\eta$, which after inserting into the covariant derivative cancels the other contribution.

To verify linearity over smooth functions in ζ, we take $f\in C^\infty(M,\mathbb{R})$ and compute

$$\begin{aligned}\nabla_\xi\nabla_\eta f\zeta &= \nabla_\xi(f\nabla_\eta\zeta + (\eta\cdot f)\zeta)\\ &= f\nabla_\xi\nabla_\eta f\zeta + ((\xi\cdot f)\nabla\eta\zeta + (\eta\cdot f)\nabla_\xi\zeta) + (\xi\cdot\eta\cdot f)\zeta\end{aligned}$$

The middle sum is symmetric in ξ and η and thus cancels with the corresponding term coming from $-\nabla_\eta\nabla_\xi f\zeta$. On the other hand,

$$\nabla_{[\xi,\eta]}f\zeta = f\nabla_{[\xi,\eta]}\zeta + ([\xi,\eta]\cdot f)\zeta.$$

By definition of the Lie bracket $[\xi,\eta]\cdot f = \xi\cdot\eta\cdot f - \eta\cdot\xi\cdot f$, so linearity over smooth functions in ζ follows.

(2) We have already observed that $R(\xi,\eta)(\zeta)$ is skew symmetric in ξ and η, so $R_{ij}{}^k{}_\ell = R_{[ij]}{}^k{}_\ell$. On the other hand, $R_{ij}{}^a{}_\ell g_{ka}$ is just the $\binom{0}{4}$–tensor field defined by $(\xi,\eta,\zeta_1,\zeta_2)\mapsto g(R(\xi,\eta)(\zeta_1),\zeta_2)$, and we have to prove that this is skew symmetric in ζ_1 and ζ_2. Now we can compute directly as follows

$$\begin{aligned}\xi\cdot\eta\cdot g(\zeta_1,\zeta_2) &= \xi\cdot(g(\nabla_\eta\zeta_1,\zeta_2) + g(\zeta_1,\nabla_\eta\zeta_2))\\ &= g(\nabla_\xi\nabla_\eta\zeta_1,\zeta_2) + g(\nabla_\eta\zeta_1,\nabla_\xi\zeta_2) + g(\nabla_\xi\zeta_1,\nabla_\eta\zeta_2) + g(\zeta_1,\nabla_\xi\nabla_\eta\zeta_2).\end{aligned}$$

Observe that the middle two terms in the last expression are symmetric in ξ and η, hence they will vanish if we subtract the same term with ξ and η exchanged. But then if we further subtract

$$[\xi,\eta]\cdot g(\zeta_1,\zeta_2)=g(\nabla_{[\xi,\eta]}\zeta_1,\zeta_2)+g(\zeta_1,\nabla_{[\xi,\eta]}\zeta_2),$$

then the left hand side will vanish by definition of the Lie bracket, while on the right hand side we get

$$g(R(\xi,\eta)(\zeta_1),\zeta_2)+g(\zeta_1,R(\xi,\eta)(\zeta_2)).$$

(4) Expanding $R(\xi,\eta)(\zeta)+R(\zeta,\xi)(\eta)+R(\eta,\zeta)(\xi)$ according to the definition of R, the first term $\nabla_\xi\nabla_\eta\zeta$ from the first summand adds up with the second term $-\nabla_\xi\nabla_\zeta\eta$ from the second summand to $\nabla_\xi[\eta,\zeta]$ by torsion freeness. Again by torsion freeness, this adds up with the last term $-\nabla_{[\eta,\zeta]}\xi$ from the last summand to $[\xi,[\eta,\zeta]]$. This can be similarly done for the other terms to see that

$$R(\xi,\eta)(\zeta)+R(\zeta,\xi)(\eta)+R(\eta,\zeta)(\xi)=[\xi,[\eta,\zeta]]+[\zeta,[\xi,\eta]]+[\eta,[\zeta,\xi]],$$

which vanishes by the Jacobi identity for the Lie bracket of vector fields. Since $R(\xi,\eta)(\zeta)$ is skew symmetric in ξ and η, its complete alternation coincides with $1/3$ times the sum over all cyclic permutations of the arguments.

(3) This identity is a formal consequence of the other ones. Writing $S_{ijk\ell}:=R_{ij}{}^a{}_k g_{a\ell}$, we know skew symmetry in (i,j) and in (k,ℓ) from (2) and $0=S_{ijk\ell}+S_{kij\ell}+S_{jki\ell}$ from (4). Using this, we compute

$$\begin{aligned}S_{ijk\ell}&=-S_{kij\ell}-S_{jki\ell}=S_{ki\ell j}+S_{jk\ell i}=-S_{\ell kij}-S_{i\ell kj}-S_{\ell jki}-S_{k\ell ji}\\&=2S_{k\ell ij}+S_{i\ell jk}+S_{\ell jik}=2S_{k\ell ij}-S_{ji\ell k}.\end{aligned}$$

Since the last term equals $-S_{ijk\ell}$ the claimed symmetry $S_{ijk\ell}=S_{k\ell ij}$ follows. □

The Riemann curvature tensor is the fundamental invariant of a Riemannian metric. As the discussion of the symmetries shows, it is a rather complicated object, and extracting parts of the curvature, which are more easily handled is an important problem in Riemannian geometry. We will discuss some aspects of this in the next chapter.

1.14. Remarks on isometries. Let us apply the concepts discussed so far to obtain some basic facts on isometries, which are the appropriate concept of morphisms in the category of Riemannian manifolds. This discussion also shows that the concepts we have developed so far actually are naturally associated to Riemannian manifolds.

Definition 1.14. Let (M,g) and (N,h) be Riemannian manifolds of dimension n. An *isometry* between M and N is a smooth map $\Phi:M\to N$ such that for each $x\in M$ the tangent map $T_x\Phi:T_xM\to T_{\Phi(x)}N$ is orthogonal with respect to the inner products g_x and $h_{\Phi(x)}$.

Observe that by definition $T_x\Phi$ always has to be a linear isomorphism, so Φ is a local diffeomorphism. In particular, one may always pull back arbitrary tensor fields along isometries. Moreover, since Riemannian metrics can be restricted to open subsets, there is an obvious concept of a *local isometry*. For simplicity, one often restricts to the case of isometries which are diffeomorphisms.

For an isometric diffeomorphisms $\Phi:(M,g)\to(N,h)$ we have induced linear isomorphisms $\Phi^*:C^\infty(N,\mathbb{R})\to C^\infty(M,\mathbb{R})$ and like wise for all kinds of geometric objects. From the constructions in Proposition 1.4 it is clear that Φ^* maps the inverse metric to h to the inverse metric to g and is also compatible with the volume forms. In particular, we see that the maps Φ^* are always orthogonal for the L^2–inner products we have constructed in 1.5 and hence extend to isomorphisms of the Hilbertspace completions. Likewise, the pullback along Φ is compatible with the Hodge–$*$ operation, since the maps induced by Φ are compatible with the induced inner products on the spaces $\Lambda^k T_x^* M$ and $\Lambda^k T_{\Phi(x)}^* N$. Hence Φ^* is also compatible with the codifferential and the Laplace–Beltrami operator on forms.

Next, for a smooth curve $c : [a, b] \to M$, $\Phi \circ c$ is a smooth curve in N, and since $(\Phi \circ c)'(t) = T_{c(t)}\Phi \cdot c'(t)$, orthogonality of the tangent maps of Φ implies that c and $\Phi \circ c$ has the same arclength. Denoting by d_g and d_h the metrics on M and N as defined in 1.7, we conclude that $d_h(\Phi(x), \Phi(y)) = d_g(x, y)$ for all $x, y \in M$. This means that Φ is an isometry between the metric spaces (M, d_g) and (N, d_h).

Proposition 1.14. *Let (M, g) and (N, h) be Riemannian manifolds of the same dimension n with Levi–Civita connections ∇^M and ∇^N and Riemann curvature tensors R^M and R^N, and let $\Phi : M \to N$ be an isometry.*

(1) For $\xi, \eta \in \mathfrak{X}(M)$ we have $\Phi^*(\nabla^N_\xi \eta) = \nabla^M_{\Phi^*\xi} \Phi^*\eta$.

(2) Φ is compatible with the curvature tensors, i.e. $\Phi^* R^N = R^M$.

(3) Φ is compatible with the covariant derivative of smooth vector fields along smooth curves. Thus it is compatible with the parallel transport along smooth curves and maps geodesics in M to geodesics in N.

Proof. (1) This is a local question, so we may replace M and N by U and $\Phi(U)$ where Φ restricts to a diffeomorphism on U. Then consider the operation $\mathfrak{X}(\Phi(U)) \times \mathfrak{X}(\Phi(U)) \to \mathfrak{X}(\Phi(U))$ defined by $(\xi, \eta) \mapsto (\Phi^{-1})^*(\nabla^M_{\Phi^*\xi} \Phi^*\eta)$. This is evidently bilinear and since pullbacks are linear over smooth functions it follows readily that it is linear over smooth functions in the first variable. On the other hand, one uses $\Phi^*(f\eta) = (f \circ \Phi)\Phi^*\eta$ and $(\Phi^*\xi) \cdot (f \circ \Phi) = (\xi \cdot f) \circ \Phi$ to conclude that this satisfies a Leibniz rule in second variable, so we have constructed a linear connection on TU. Next, alternating this operation, we just have to use $[\Phi^*\xi, \Phi^*\eta] = \Phi^*([\xi, \eta])$ to conclude that this linear connection is torsion free.

Finally, since the tangent maps of Φ are all orthogonal, we conclude that

$$h_{\Phi(x)}(\xi(\Phi(x)), \eta(\Phi(x))) = g_x(\Phi^*\xi(x), \Phi^*\eta(x)),$$

so $h(\xi, \eta) \circ \Phi = g(\Phi^*\xi, \Phi^*\eta)$. Thus for another vector field $\zeta \in \mathfrak{X}(U)$, we can write $(\zeta \cdot h(\xi, \eta)) \circ \Phi$ as $(\Phi^*\zeta) \cdot g(\Phi^*\xi, \Phi^*\eta)$. Now apply compatibility of ∇^M with the metric and rewrite $g(\nabla^M_{\Phi^*\zeta} \Phi^*\xi, \Phi^*\eta)$ as $g((\Phi^{-1})^*(\nabla^M_{\Phi^*\zeta} \Phi^*\xi), \eta) \circ \Phi$ and likewise for the other summand. Since Φ is a diffeomorphism, we can forget about the composition with Φ and conclude that the connection we have defined is compatible with h and hence has to coincide with ∇^N by Theorem 1.9. From this, the result follows by applying Φ^*.

(2) Since Φ is a local diffeomorphism, we can realize all tangent vectors in a point x as the values of vector fields of the form $\Phi^*\xi$ for $\xi \in \mathfrak{X}(N)$. But the formula in (1) together with the definition of curvature shows that $R^M(\Phi^*\xi, \Phi^*\eta)(\Phi^*\zeta) = \Phi^*(R^N(\xi, \eta)(\zeta))$, which implies the claim.

(3) For a smooth curve c in M, $\Phi \circ c$ is a smooth curve in N and for a vector field ξ along c, $\Phi_*\xi(t) = T_{c(t)}\Phi \cdot \xi(t)$ is a vector field along $\Phi \circ c$. Now the result in (1) easily implies that $\Phi_* \nabla_{c'} \xi = \nabla_{(\Phi \circ c)'} \Phi_* \xi$. This immediately implies compatibility with the parallel transport and since by definition $(\Phi \circ c)' = \Phi_* c'$, the claim on geodesics follows, too. $\square$

This implies that isometries are rather rare in various senses. For example, consider Euclidean space E^n. By definition, the coordinate vector fields ∂_i on E^n satisfy $\nabla_\xi \partial_i = 0$ for any vector field ξ on E^n. But this easily shows that $R(\xi, \eta)(\partial_i) = 0$, so on E^n the

Riemann curvature vanishes identically. From part (2) of the proposition, we thus conclude that if (M, g) is a Riemannian manifold and $x \in M$ is a point, there may be an isometry from an open neighborhood U of x in M to E^n only if the Riemann curvature R^M vanishes identically on U.

If we endow $\mathbb{R}^n$ with an arbitrary Riemannian metric g, then the Riemann curvature tensor of g can be considered as a smooth function to (a subspace of) $\otimes^3\mathbb{R}^{n*} \otimes \mathbb{R}^n$. Now even taking into account all the symmetries of the curvature tensor, the target space is high dimensional. Suppose it is possible to choose the metric g in such a way that the curvature tensor defines an injective function. Then part (2) of the proposition implies that the only isometry between open subsets of $\mathbb{R}^n$ endowed with the restriction of this metric, is the identity map, so there are no non–trivial local isometries. Indeed, one can make this precise and show that the space of Riemannian metrics on a smooth manifold M in dimension $n \geq 3$ contains an open dense subset (in a suitable topology) consisting of metric which do not admit any non–trivial local isometries.

Finally, Proposition 1.1 shows that any isometry of Euclidean space is a Euclidean motion. The proof of (ii)$\Rightarrow$(iii) we have given actually applies more generally to show that for connected open subsets U and V of E^n any isometry $f : U \to V$ is the restriction of a Euclidean motion. So even for this simplest example of a Riemannian manifold, isometries are rather rare (they form a finite dimensional manifold). Now we can prove an analog of this result for arbitrary Riemannian manifolds.

Corollary 1.14. *Let (M, g) and (N, h) be Riemannian manifolds of dimension n such that M is connected, and let $x \in M$ be a point. Then an isometry $\Phi : M \to N$ is uniquely determined by $\Phi(x)$ and $T_x\Phi$.*

Proof. From part (3) of the proposition, we know that an isometry Φ maps geodesics to geodesics. Hence if $c : I \to M$ is a geodesic with $c(0) = x$ and $c'(0) = \xi$, then $\Phi \circ c$ is a geodesic through $\Phi(x)$ with initial direction $T_x\Phi \cdot \xi$. In terms of the exponential mapping this means that $\Phi \circ \exp_x = \exp_{\Phi(x)} \circ T_x\Phi$ holds on the domain of definition of $\exp_x$. By Proposition 1.12, $\exp_x$ restricts to a diffeomorphism from some open neighborhood of zero onto a neighborhood of x in M. But then the restriction of Φ to this neighborhood is uniquely determined by $\Phi(x)$ and $T_x\Phi$.

Given two isometries $\Phi, \Psi : M \to N$, this shows that the set

$$\{x \in M : \Phi(x) = \Psi(x) \text{ and } T_x\Phi = T_x\Psi\}.$$

is open in M. On the other hand, its complement is evidently open, hence if non–empty, this set coincides with M, since M is connected. $\square$

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