CHAPTER 2

Some more advanced topics

Having the core notions of Riemannian geometry at hand, we briefly discuss "how things go on from here" in different directions. There is a certain dependence between the different topics, but this is not too strong, so to a large extent the individual sections of this chapter can be studied independently of each other.

Moving frames – Examples

We start by discussing the fundamentals of E. Cartan's moving frame method. This gives a systematic way for computing the Levi–Civita connection and the Riemann curvature tensor of a Riemannian manifold in terms of local orthonormal frames and coframes. This is built on the calculus of differential forms.

2.1. Local orthonormal frames and coframes. One of the basic difficulties in Riemannian geometry is that it is impossible to choose local coordinates which are well adapted to a Riemannian metric. This is basically due to the fact that the Riemann curvature tensor constructed in 1.13 is a *local invariant* of a Riemannian metric, which tells us that Riemannian metrics in general do not locally look the same. For example, suppose that one has a local chart (U, u) on a Riemannian manifold such that the corresponding coordinate vector fields ∂_i form an orthonormal basis of T_xM for each $x \in U$. Then (compare with Proposition 2.7 below) u is an isometry to the subset $u(U) \subset \mathbb{R}^n$ with the restriction of the usual metric on $\mathbb{R}^n$. As observed in 1.13, such an isometry can only exist if the Riemann curvature vanishes identically on U.

A possible replacement for adapted coordinates are local orthonormal frames, which we have met in 1.4. Given a Riemannian manifold (M, g) of dimension n and an open subset $U \subset M$, a local orthonormal frame for U is a family $\{s_1, \dots, s_n\}$ of vector fields defined on U such that $g(s_i, s_j) = \delta_{ij}$ on U. This means that for each $x \in U$, the tangent vectors $s_1(x), \dots, s_n(x) \in T_xM$ form an orthonormal basis for T_xM (with respect to g_x). In Proposition 1.4 we have proved that local orthonormal frames always exist. Since there is a better calculus for differential forms available than for vector fields, it is better to use the dual concept defined as follows.

Definition 2.1. Let (M, g) be a Riemannian manifold of dimension n and let $U \subset M$ be an open subset. A *local orthonormal coframe* on U is a family $\{\sigma^1, \dots, \sigma^n\}$ of one–forms defined on U such that $g|_U = \sum_{i=1}^n \sigma^i \otimes \sigma^i$.

Lemma 2.1. *Let (M, g) be a Riemannian manifold of dimension n and let $U \subset M$ be an open subset. A family $\{\sigma^1, \dots \sigma^n\}$ of elements of $\Omega^1(U)$ is a local orthonormal coframe if and only if for each $x \in U$ the elements $\sigma^1(x), \dots, \sigma^n(x)$ form a basis for T_x^*M, for which the dual basis of T_xM is orthonormal. In particular, local orthonormal coframes always exist.*

Proof. This is just a linear algebra statement. Starting with a local orthonormal coframe, we get $g_x = \sum_i \sigma^i(x) \otimes \sigma^i(x)$, so non–degeneracy of g_x implies that for each $\xi \in T_xM$, there is at least one i such that $\sigma^i(x)(\xi) \neq 0$. This implies that the $\sigma^i(x)$ are linearly independent an thus form a basis of T_x^*M. Denoting the dual basis by s_i we conclude that $g_x(s_i,s_j)=\sum_k \sigma^k(x)(s_i)\sigma^k(x)(s_j)=\delta_{ij}$ so the dual basis is orthonormal.

Conversely, suppose that $\sigma^1,\dots,\sigma^n$ is a family of one–forms satisfying the condition on the values in x. Then g_x and $\sum_i \sigma^i(x)\otimes\sigma^i(x)$ agree whenever one inserts two elements of the basis dual to $\{\sigma^1(x),\dots,\sigma^n(x)\}$ and hence on all pairs of vectors.

In particular, we see that we can obtain a local orthonormal coframe by forming the dual basis to a local orthonormal frame in each point, so existence follows from Proposition 1.4. □

From now on, we will usually work in a local orthonormal coframe $\{\sigma^1,\dots,\sigma^n\}$ with dual orthonormal frame $\{s_1,\dots,s_n\}$, so $\sigma^i(s_j)=\delta^i_j$. This simply means that any vector field ξ in the domain of the frames can be written as $\xi=\sum_i \sigma^i(\xi)s_i$. Likewise, a one–form can, in the domain of the frames, be written as $\varphi=\sum_j \varphi(s_j)\sigma^j$, and similarly for more complicated tensor fields.

It is actually possible to develop the fundamentals of Riemannian geometry in the language of local orthonormal coframes. One defines objects in terms of such a coframe and then proves that different coframes lead to the same object. In particular, texts taking this approach contain lots of computations on how various quantities behave under a change of frame. In the approach we take, such computations are not needed, since we only compute quantities which we already know to be well defined in terms of a local coframe.

2.2. Connection and curvature in a moving frame. Consider a local orthonormal coframe $\{\sigma^1,\dots,\sigma^n\}$ for a Riemannian manifold (M,g) defined on $U\subset M$ with dual frame $\{s_1,\dots,s_n\}$. To describe the Levi–Civita connection in the frame, we observe that for each $\xi\in\mathfrak{X}(U)$ and each $i=1,\dots,n$, $\nabla_\xi s_i$ is a smooth vector field on U, so we can write it as $\sum_j \omega_i^j(\xi)s_j$ for smooth functions $\omega_i^j(\xi)$, $i=1,\dots,n$, which depend on ξ. But by definition for a smooth function $f\in C^\infty(U,\mathbb{R})$, we have $\nabla_{f\xi}s_i$, and hence $\omega_i^j(f\xi)=f\omega_i^j(\xi)$ for all i,j. Thus each ω_i^j actually is a smooth one–form on U, and it is natural to view (ω_i^j) as a matrix of one–forms on U, which is called the *matrix of connection forms* associated to the coframe $\{\sigma^i\}$.

It is even easier to describe the Riemann curvature tensor in a local frame. Namely, given vector fields $\xi,\eta\in\mathfrak{X}(U)$, we expand $R(\xi,\eta)(s_i)=\sum_j \Omega_i^j(\xi,\eta)s_j$. The fact that R is a tensor immediately implies that Ω_i^j actually is a two–form on U for each i and j. Hence we also view (Ω_i^j) as a matrix of two–forms, called the *matrix of curvature forms* associated to the coframe $\{\sigma^i\}$.

PROPOSITION 2.2. *(1) The matrix (ω_j^i) of connection forms associated to a local orthonormal coframe $\{\sigma^i\}$ is skew symmetric, i.e. $\omega_j^i=-\omega_i^j$ and for each $i=1,\dots,n$ it satisfies the equation*

$$0=d\sigma^i+\textstyle\sum_j \omega_j^i\wedge\sigma^j.$$

These two properties uniquely determine (ω_i^j).

(2) The corresponding matrix (Ω_j^i) of curvature forms is also skew symmetric and it is given by

$$\Omega_j^i=d\omega_j^i+\textstyle\sum_k \omega_k^i\wedge\omega_j^k.$$

PROOF. (1) By definition, we have

$$\omega_i^j(\xi)=\sigma^j(\nabla_\xi s_i)=g(\nabla_\xi s_i,s_j).$$

But since $g(s_i,s_j)$ is always constant, compatibility of ∇ with g implies that $0 = g(\nabla_\xi s_i, s_j) + g(s_i, \nabla_\xi s_j)$ and thus $\omega_i^j(\xi) = -\omega_j^i(\xi)$, so skew symmetry follows.

For a vector field $\eta \in \mathfrak{X}(U)$, we have noted in 2.1 that $\eta = \sum_j \sigma^j(\eta)s_j$. Hence we compute

$$\nabla_\xi \eta = \textstyle\sum_j \nabla_\xi(\sigma^j(\eta)s_j) = \sum_j (\xi \cdot \sigma^j(\eta))s_j + \sum_{j,k} \sigma^j(\eta)\omega_j^k(\xi)s_k.$$

Otherwise put, we get

$$\sigma^i(\nabla_\xi \eta) = \xi \cdot \sigma^i(\eta) + \textstyle\sum_j \sigma^j(\eta)\omega_j^i(\xi).$$

Now subtract the analogous term with ξ and η exchanged and further subtract $\sigma^i([\xi,\eta])$ from both sides. Then in the left hand side, we get zero by torsion freeness of ∇. In the right hand side, we can use the definition of the exterior derivative to conclude that

$$0 = d\sigma^i(\xi,\eta) + \textstyle\sum_j \left(\omega_j^i(\xi)\sigma^j(\eta) - \omega_j^i(\eta)\sigma^j(\xi)\right),$$

and the last term just represents $\sum_j (\omega_j^i \wedge \sigma^j)(\xi,\eta)$.

To prove the statement on uniqueness, we consider the difference of two skew symmetric matrices of one–forms, which both satisfy the equations. Then this is a matrix (τ_i^j) of one–forms such that $\tau_j^i = -\tau_i^j$ and such that $\sum_j \tau_j^i \wedge \sigma^j = 0$ for each $i = 1,\dots,n$. Now evaluate the last expression on (s_k, s_ℓ) to get $0 = \tau_\ell^i(s_k) - \tau_k^i(s_\ell)$. Hence if we put $\Phi_{ijk} := \tau_j^i(s_k)$, we get $\Phi_{ijk} = -\Phi_{ijk}$ and $\Phi_{ijk} = \Phi_{ikj}$ and we know from the proof of Proposition 1.1 that this implies $\Phi_{ijk} = 0$ and hence $\tau_j^i = 0$ for all i and j.

(2) By definition,

$$\Omega_i^j(\xi,\eta) = \sigma^j(R(\xi,\eta)(s_i)) = g(R(\xi,\eta)(s_i), s_j),$$

so skew symmetry follows from part (2) of Proposition 1.13. From the defining equation $\nabla_\eta s_i = \sum_k \omega_i^k(\eta)s_k$, we conclude that

$$\nabla_\xi \nabla_\eta s_i = \textstyle\sum_k (\xi \cdot \omega_i^k(\eta))s_k + \sum_{k,\ell} \omega_i^k(\eta)\omega_k^\ell(\xi)s_\ell,$$

and hence

$$\sigma^j(\nabla_\xi \nabla_\eta s_i) = \xi \cdot \omega_i^j(\eta) + \textstyle\sum_k \omega_k^j(\xi)\omega_i^k(\eta).$$

To obtain $\Omega_j^i(\xi,\eta)$ we have to subtract the corresponding term with ξ and η exchanged and further subtract $\sigma^j(\nabla_{[\xi,\eta]} s_i) = \omega_i^j([\xi,\eta])$. Now the result follows immediately from the definition of the exterior derivative and of the wedge product. □

2.3. Examples. (1) **Flat space**: In Euclidean space E^n, we take one of the global charts from 1.1 to identify E^n with $\mathbb{R}^n$. Then the corresponding coordinate vector fields ∂_i form a global orthonormal frame. The dual coframe is simply given by $\sigma^i = dx^i$ for $i = 1,\dots,n$. Since $d\sigma^i = 0$ for all i, we conclude that both the matrix (ω_j^i) of connection forms and the matrix (Ω_j^i) of curvature forms vanish identically in this frame.

(2) **The sphere**: Let us consider the unit sphere $S^n := \{x \in \mathbb{R}^{n+1} : \langle x, x\rangle = 1\}$ with the Riemannian metric induced from $\mathbb{R}^{n+1}$. To get simple formulae, we use a particularly nice chart, the *stereographic projection*. Let $N = e_{n+1} \in S^n$ be the north pole, put $U := S^n \setminus \{N\}$ and define $u : U \to \mathbb{R}^n$ by

$$u(x) = u(x^1,\dots,x^{n+1}) = \tfrac{1}{1-x^{n+1}}(x^1,\dots,x^n)$$

(To interpret this geometrically, one views $\mathbb{R}^n$ as the affine hyperplane through $-N$ which is orthogonal to N and one maps each point $x \in S^n$ to the intersection of the ray from N through x with that affine hyperplane.) One immediately verifies that the map

$$(u^1,\dots,u^n) \mapsto \tfrac{1}{\langle u,u\rangle+1}(2u^1,\dots,2u^n,\langle u,u\rangle - 1)$$

is inverse to u. The ith partial derivative of this mapping is given by

$$\tfrac{-2u^i}{(1+\langle u,u\rangle)^2}(2u,\langle u,u\rangle-1)+\tfrac{1}{1+\langle u,u\rangle}(2e_i,2u^i),$$

which shows that we can write $\frac{\partial}{\partial u^i}$ as

$$\tfrac{-2u^i}{(1+\langle u,u\rangle)^2}\left(\textstyle\sum_{j=1}^n 2u^j\tfrac{\partial}{\partial x^j}+(\langle u,u\rangle-1)\tfrac{\partial}{\partial x^{n+1}}\right)+\tfrac{2}{1+\langle y,y\rangle}\left(\tfrac{\partial}{\partial x^i}+u^i\tfrac{\partial}{\partial x^{n+1}}\right).$$

Now we can compute the inner products of these vector fields using that the fields $\frac{\partial}{\partial x^j}$ are orthonormal. The bracket in the first summand is independent of i and inserting it twice into the metric, one gets $4\langle u,u\rangle+(\langle u,u\rangle-1)^2=(1+\langle u,u\rangle)^2$. So the contributions to $g(\frac{\partial}{\partial u^i},\frac{\partial}{\partial u^k})$ is given by $\frac{4u^iu^k}{(1+\langle u,u\rangle)^2}$. Likewise from the second terms, one obtains a contribution of $\frac{4}{(1+\langle u,u\rangle)^2}(\delta_{ik}+u^iu^k)$. Finally, the terms mixing the two summands give a contribution of $\frac{-8u^iu^k}{(1+\langle u,u\rangle)^2}$. Altogether, we see that

$$g(\tfrac{\partial}{\partial u^i},\tfrac{\partial}{\partial u^k})=\tfrac{4}{(1+\langle u,u\rangle)^2}\delta_{ik}.$$

Putting $f(u)=\frac{1}{2}(1+\langle u,u\rangle)$, we see that $\{f(u)\frac{\partial}{\partial u^i}\}$ is a local orthonormal frame and hence the one–forms $\sigma^i:=\frac{1}{f(u)}du^i$ form a local orthonormal coframe.

Consequently, $d\sigma^i=-\frac{1}{f^2}df\wedge du^i$ and since $df=\sum_j u^jdu^j$ this can be written as $\sum_j\frac{u^j}{f^2}du^i\wedge du^j=\sum_j u^j\sigma^i\wedge\sigma^j$. This can be written as $-\sum_j\omega^i_j\wedge\sigma^j$ for $\omega^i_j=u^i\sigma^j-u^j\sigma^i=\frac{u^i}{f}du^j-\frac{u^j}{f}du^i$, which evidently satisfies $\omega^j_i=-\omega^i_j$ and thus gives the matrix of connection forms associated to our coframe.

This immediately gives

$$d\omega^i_j=-\tfrac{u^i}{f^2}df\wedge du^j+\tfrac{u^j}{f^2}df\wedge du^i+\tfrac{2}{f}du^i\wedge du^j.$$

On the other hand, using $df=\sum_k u^kdu^k$, we compute

$$\textstyle\sum_k(\frac{u^i}{f}du^k-\frac{u^k}{f}du^i)\wedge(\frac{u^k}{f}du^j-\frac{u^j}{f}du^k)=\frac{u^i}{f^2}df\wedge du^j-\frac{u^j}{f^2}df\wedge du^i-\frac{\sum(u^k)^2}{f^2}du^i\wedge du^j.$$

Hence we directly get $\Omega^i_j=\frac{1}{f^2}du^i\wedge du^j=\sigma^i\wedge\sigma^j$. To understand the form of the curvature more explicitly, we look at elements s_a of the orthonormal frame. By definition of the matrix of curvature forms, we have $R(\xi,\eta)(s_j)=\sum_i\Omega^i_j(\xi,\eta)s_i$ and hence $g(R(\xi,\eta)(s_j),s_i)=\Omega^i_j(\xi,\eta)$. Thus we can compute $g(R(s_a,s_b)(s_c),s_d)$ as

$$\Omega^d_c(s_a,s_b)=\sigma^d(s_a)\sigma^c(s_b)-\sigma^c(s_a)\sigma^d(s_a)=g(s_a,s_d)g(s_b,s_c)-g(s_a,s_c)g(s_b,s_d).$$

Since this is a tensorial expression, it holds for arbitrary vector fields instead of the elements of the frame, which shows that in abstract index notation, we have $R_{ij}{}^a{}_\ell g_{ka}=g_{ik}g_{j\ell}-g_{i\ell}g_{jk}$ respectively $R_{ij}{}^k{}_\ell=\delta^k_ig_{j\ell}-\delta^k_jg_{i\ell}$. This is the simplest way to construct a tensor with curvature symmetries out of the metric. We will later say that the sphere has constant (positive) sectional curvature.

(3) **Hyperbolic space**: Although this example is quite different from the sphere, the computations will quickly become very similar. We consider the open unit ball $\{x\in\mathbb{R}^n:\langle x,x\rangle<1\}$ and define a metric there as $g:=\frac{4}{(1-\langle x,x\rangle)^2}g_0$, where g_0 is the restriction of the flat metric. (As we define it here, this may seem rather artificial, but it arises from several other pictures in a natural way.) Putting $f(x):=\frac{1}{2}(1-\langle x,x\rangle)$ we see that the vector fields $f\partial_i$ form an orthonormal frame, and the corresponding orthonormal coframe is obtained by putting $\sigma^i:=\frac{1}{f}dx^i$. The only difference compared to the case of the sphere now is that $df=-\sum x^idx^i$, so there is a sign change compared to the case of the sphere. This sign change carries over to $d\sigma^i$ and hence to ω^i_j, so this time

we get $\omega^i_j = -x^i\sigma^j + x^j\sigma^i = -\frac{x^i}{f}dx^j + \frac{x^j}{f}dx^i$. As in the case of the sphere, one directly verifies that this leads to $\Omega^i_j = -\sigma^i \wedge \sigma^j$, so again there is a sign change compared to the sphere. As in the case of the sphere, one then verifies that $R_{ij}{}^a{}_\ell g_{ka} = -g_{ik}g_{j\ell} + g_{i\ell}g_{jk}$ respectively $R_{ij}{}^k{}_\ell = -\delta^k_i g_{j\ell} + \delta^k_j g_{i\ell}$. We will say that hyperbolic space has constant negative sectional curvature.

Geodesics, distance and completeness

One of the fundamental facts in Euclidean geometry is the fact that a line segment provides the shortest path connecting two points. Since the analogs of straight lines in general Riemannian manifolds are the geodesics, it is a natural question whether any two points can be connected by a geodesic and whether this is a (or even the) shortest curve connecting the two points.

The geodesics of a Riemannian metric also lead to a natural notion of completeness for Riemannian manifolds. It turns out that completeness is closely related to the interpretation of geodesics as shortest curves. Using this relation, this concept of completeness turns out to be equivalent to completeness in the sense of metric spaces. This result is called the Hopf–Rinow theorem, and it is one of the cornerstones of Riemannian geometry.

2.4. The first variational formula. We start with an elementary characterization of geodesics which is a first step towards identifying them as "shortest curves". As we have note in 1.7, the arclength of a curve is invariant under reparametrizations, which make it less suitable for the purpose of characterizing curves, so we use the energy instead. We study the behavior of the energy under a variation of curves. Given a smooth curve $c : [a, b] \to M$, such a variation is a smooth mapping $\gamma : [a, b] \times (-\epsilon, \epsilon) \to M$ such that $\gamma(t, 0) = c(t)$. Evidently, we can view such a variation as a smooth family $\{c_s : [a, b] \to M : |s| < \epsilon\}$ of curves by putting $c_s(t) := \gamma(t, s)$. The "direction" of such a variation can be described by $r(t) := \frac{\partial}{\partial s}|_{s=0}\gamma(t, s)$. This evidently is a vector field along c called the *variational vector field* determined by γ. A particularly interesting case is provided by variations fixing the endpoints, where one in addition requires that $\gamma(a, s) = c(a)$ and $\gamma(b, s) = c(b)$ for all s. The infinitesimal version of this condition of course is $r(a) = r(b) = 0$.

Given a variation γ of c, we can consider the resulting variation of energy, i.e. look at $E(s) := \frac{1}{2}\int_a^b g(\gamma(t, s))(\gamma'(t, s), \gamma'(t, s))dt$, where we write $\gamma'(t, s)$ for $\frac{\partial}{\partial t}\gamma(t, s)$. Evidently, this is a smooth function $(-\epsilon, \epsilon) \to \mathbb{R}$, so we can try to compute the infinitesimal variation $\frac{d}{ds}|_{s=0}E(s)$ of energy. The result is very appealing:

Proposition 2.4 (First variational formula). *Let γ be a smooth variation of $c : [a, b] \to M$ with variation vector field r. Then the infinitesimal variation of energy is given by*

$$\frac{d}{ds}|_{s=0}E(s) = -\int_a^b g(c(t))(\nabla_{c'}c'(t), r(t))dt + g(c(b))(c'(b), r(b)) - g(c(a))(c'(a), r(a)).$$

In particular, a smooth curve c is a critical point for the energy under all variations with fixed endpoints if and only if c is a geodesic.

Proof. The formula on $[a, b]$ clearly follows from the analogous formula on small sub–intervals of $[a, b]$. Thus, we may restrict to the case that γ has values in the domain U of a chart (U, u) for M. Passing to the image of that chart, we may restrict to the case that $M = \mathbb{R}^n$ but endowed with an arbitrary Riemannian metric g. Using the standard trivialization of the tangent bundle, we may view vector fields as $\mathbb{R}^n$–valued functions and g as a function with values in the space of symmetric bilinear forms which has values in the open subset of positive definite forms. Now forming

$$\tfrac{d}{ds}E(s) = \tfrac{1}{2}\tfrac{d}{ds}\int_a^b g(\gamma(t,s))(\gamma'(t,s),\gamma'(t,s)),$$

we may first exchange the derivative with the integral. But since the integrand comes from a trilinear map, we ca write $\frac{d}{ds}(g(\gamma(t,s))(\gamma'(t,s),\gamma'(t,s)))$ as

$$Dg(\gamma(t,s))(\tfrac{\partial}{\partial s}\gamma(t,s))(\gamma'(t,s),\gamma'(t,s)) + 2g(\gamma(t,s))(\tfrac{\partial}{\partial s}\gamma'(t,s),\gamma'(t,s)).$$

At $s=0$, $\frac{\partial}{\partial s}\gamma(t,s) = r(t)$ and since partial derivatives commute, we get $\frac{\partial}{\partial s}\gamma'(t,s) = r'(t)$ there. To compute the contribution of the second summand to the integral for $s=0$, we have to determine $\int_a^b g(c(t))(r'(t),c'(t))dt$. Integrating this by parts, we obtain

$$-\int_a^b \Big(Dg(c(t))(c'(t))(r(t),c'(t)) + g(c(t))(r(t),c''(t))\Big)dt + \Big[g(c(t))(r(t),c'(t))\Big]_a^b.$$

On the other hand, Proposition 1.11 shows that $\nabla_{c'}c'(t) = c''(t) + \Gamma(c(t))(c'(t),c'(t))$, where Γ is obtained from the Christoffel symbols. Finally, the formula for the Christoffel symbols in part (2) of Proposition 1.10 reads as

$$g(x)(\Gamma(\xi,\xi),\eta) = 2(Dg(x)(\xi))(\xi,\eta) - (Dg(x)(\eta))(\xi,\xi),$$

which implies the claim. □

The computation in the proof actually allows an elementary approach to the construction of the Levi–Civita connection. Motivated by the computation, one shows that, in the domain of a chart, one can write $2(Dg(x)(\xi))(\xi,\eta) - (Dg(x)(\eta))(\xi,\xi)$ as $g(x)(Q_x(\xi),\eta)$ for a quadratic form Q_x. This then determines a symmetric bilinear form Γ_x such that $Q_x(\xi) = \Gamma_x(\xi,\xi)$. Then one can use these forms to define a covariant derivative in charts and verify directly that the definitions in different charts coincide, so one obtains a globally defined covariant derivative.

2.5. Minimizing curves. Given a point x in a Riemannian manifold (M,g) we have seen in Proposition 1.12 that there is an open neighborhood of zero in T_xM on which the exponential map $\exp_x$ restricts to a diffeomorphism onto an open neighborhood of x in M. In particular, there is a number $\epsilon > 0$ such that $\exp_x$ restricts to a diffeomorphism from the ball of radius ϵ (with respect to g_x) in T_xM onto a neighborhood U of x in M. Now any point $y \in U$ can be written as $\exp(X)$ for some X in that ball, and hence $t \mapsto \exp_x(tX)$ defines a geodesic $c : [0,1] \to M$ such that $c(0) = x$ and $c(1) = y$. So any point in U can be joined to x by a geodesic.

On the other hand, for $0 < \delta < \epsilon$, we can consider the sphere of radius δ in T_xM. Its image under $\exp_x$ is called the *geodesic sphere* $S_\delta(x)$ of radius δ around x.

Lemma 2.5 (Gauß). *Let x be a point in a Riemannian manifold (M,g) and let $\epsilon > 0$ be chosen in such a way that $\exp_x$ restricts to a diffeomorphism from the ϵ–ball around 0 in T_xM onto an open neighborhood U of x in M. Then for each $0 < \delta < \epsilon$, the geodesic sphere $S_\delta(x)$ is a smooth submanifold in M and the geodesics through x intersect this submanifold orthogonally.*

Proof. Since any sphere in T_xM is a submanifold in any ball containing it, and $S_\delta(x)$ is the image of one of these spheres under a diffeomorphism, it is a submanifold, too. Now take any smooth curve $v(s)$ in the sphere of radius δ in T_xM and for $t \in [0,1]$ define $\gamma(t,s) := \exp_x(tv(s))$. This is a smooth variation of the curve $c(t) = \exp_x(tv(0))$

which is a geodesic. But it is also true that for each fixed s, the curve $c_s(t) = \exp_x(tv(s))$ is a geodesic. Thus $g(c_s(t))(c_s'(t), c_s'(t))$ is constant and its value at $t = 0$ of course is $g_x(v(s), v(s)) = \delta^2$. In particular, the energy of this variation is constant in s, so $0 = \frac{d}{ds}|_{s=0}E(s)$.

But we can also compute this infinitesimal variation using the first variational formula, and since c is a geodesic, only the boundary terms survive in this formula. Moreover, $\frac{\partial}{\partial s}\gamma(t, s) := (T_{tv(s)}) \exp_x(t\frac{d}{ds}v(s))$, so the variation vector field r satisfies $r(0) = 0$ and $r(1) = T_{v(0)} \exp_x \cdot v'(0)$. Thus the first variational formula simply tells us that $0 = g(\exp_x(v(0)))(c'(1), \xi)$ for any tangent vector ξ which can be written as $T_{v(0)} \exp_x \cdot v'(0)$. By construction, any vector tangent to $S_\delta(x)$ can be written in this form, so the whole tangent space of the geodesic sphere is orthogonal to the tangent vector $c'(1)$ of the geodesic c. □

Now by a *minimizing curve*, we mean a piece–wise smooth curve $c : [a, b] \to M$ which is a shortest connection between its endpoints, i.e. satisfies $d(c(a), c(b)) = L(c)$. We can next prove that for nearby points, minimizing curves exist and are geodesics (up to parametrization).

PROPOSITION 2.5. *Let (M, g) a Riemannian manifold, $x \in M$ a point and $\epsilon > 0$ a number such that $\exp_x$ restricts to a diffeomorphism from $B_\epsilon(0) := \{\xi \in T_xM : g_x(\xi, \xi) < \epsilon^2\}$ onto an open neighborhood U of x in M.*

(1) Let $u : [a, b] \to (0, \epsilon)$ and $v : [a, b] \to T_xM$ be smooth functions such that $g_x(v(t), v(t)) = 1$ for all t and put $c(t) := \exp_x(u(t)v(t))$. Then the arc length of c satisfies $L(c) \geq |u(b) - u(a)|$ and equality holds if and only if u is monotonous and v is constant.

(2) For $y = \exp_x(\xi) \in U$, the geodesic $t \mapsto \exp_x(t\xi)$ is a minimizing curve joining x to y, and up to reparametrizations it is the unique such curve.

PROOF. (1) By construction, we get $c'(t) = T \exp_x \cdot(u'(t)v(t)+u(t)v'(t))$. Along the line spanned by $v(t)$, the vector $T \exp_x \cdot v(t)$ is the speed vector of a geodesic, whence we conclude that $g(T \exp_x \cdot(u'(t)v(t)), T \exp_x \cdot(u'(t)v(t))) = |u'(t)|^2$. On the other hand, $T \exp_x \cdot(u(t)v'(t))$ is tangent to $S_{u(t)}(x)$ and hence orthogonal to $T \exp_x \cdot(u'(t)v(t))$ by Lemma 2.5. Hence we get $g(c'(t), c'(t)) \geq |u'(t)|^2$ with equality only for if $v'(t) = 0$.

Hence we obtain $L(c) \geq \int_a^b |u'(t)|dt \geq |\int_a^b u'(t)dt| = |u(b) - u(a)|$ as claimed. The first inequality becomes an equality if and only if $v'(t) = 0$ for all t i.e. iff v is constant. The second inequality becomes an equality if and only if $u'(t)$ has constant sign and hence u is monotonous.

(2) By assumption, $y \in S_\rho(x)$ for some $\rho < \epsilon$. Of course have $d(x, y) \leq \rho$, since the geodesic joining x to y has length ρ. From (1) we conclude that a curve joining x to y which stays in $S_\rho \cup \exp_x(B_\rho(0))$ has length at least ρ, since outside of x, any such curve can be written in the form used in (1). But any curve leaving this set has to have larger length, since the part up to the first intersection with $S_\rho(x)$ already has length ρ. This shows that $d(x, y) = \rho$, so the geodesic is a minimizing curve.

Conversely, a minimizing curve connecting x to y must stay in $S_\rho \cup \exp_x(B_\rho(0))$. Now it follows immediately from the definition that the restriction of a minimizing curve to a smaller interval is still minimizing. Applying the equality part of (1) outside of x shows that a minimizing curve there must be of the form $\exp_x(u(t)v)$ for a monotonous function u, and hence a reparametrization of the geodesic $\exp_x(tv)$. □

We can further use this to conclude that short pieces of minimizing curves always are geodesics.

Corollary 2.5. *Let $c : [a, b] \to M$ be a piece–wise smooth minimizing curve. Then for each $t \in (a, b)$, there are $a' < t < b'$ such that $c|_{(a',b')}$ is a reparametrization of a geodesic. In particular, c can be parametrized smoothly.*

Proof. Given t, we claim that we can find $a' < t$ and $\epsilon > 0$ such that $\exp_{c(a')}$ restricts to a diffeomorphism on $B_\epsilon(0) \subset T_{c(a')}M$ and such that $c(t)$ is contained in the image of this ball. Having shown that, openness implies that there is a $b' > t$ such that $c([a', b'])$ is contained in this image. As we have noticed above already, $c|_{[a',b']}$ is also minimizing, so the result follows from the last part of Proposition 2.5.

To prove the claim, recall the by part (4) of Proposition 1.12, there is an open neighborhood V of the zero section in TM on which $(\pi, \exp)$ restricts to a diffeomorphism on V. This implies that we can find an open neighborhood W of $c(t)$ in M and a number $\epsilon > 0$ such that $U := \{\xi : \pi(\xi) \in W, |\xi| < \epsilon\} \subset V$, where the norm of ξ is taken with respect to $g_{\pi(\xi)}$. Continuity of c then implies that we can choose $a' < t$ such that $c(a') \in W$ and $(c(a'), c(t)) \in (\pi, \exp)(U)$, which shows that a' and ϵ have the required properties. □

2.6. Completeness and the Hopf–Rinow theorem. In our discussion of geodesics in 1.12, we have proved existence of local solutions to the geodesic equation. The natural completeness condition coming from geodesics is that all these solutions are defined for all times.

Definition 2.6. A Riemannian metric g on a smooth manifold M is called (geodesically) *complete* if for any $x \in M$ and $\xi \in T_xM$, there exists a geodesic $c : \mathbb{R} \to M$ such that $c(0) = x$ and $c'(0) = \xi$. In this case, (M, g) is called a (geodesically) complete Riemannian manifold.

The Hopf–Rinow theorem shows that the notion of geodesic completeness is equivalent to completeness of the metric space (M, d_g) and at the same time proves an important property of complete Riemannian manifolds.

Theorem 2.6 (Hopf-Rinow). *Let (M, g) be a connected smooth Riemannian manifold and let d_g be the distance function associated to g as in Proposition 1.7. Then the following conditions are equivalent*

(i) The metric g is geodesically complete.

(ii) (M, d_g) is a complete metric space, i.e. any Cauchy–sequence converges.

(iii) (M, d_g) has the Heine–Borel property, i.e. bounded closed subsets are compact.

(iv) There exists a point $x \in M$ such that $\exp_x$ is defined on all of T_xM.

Moreover, these equivalent conditions imply

(v) For any two points $x, y \in M$, there is a minimizing geodesic connecting x to y.

Proof. It is clear that (i) implies (iv), and the fact that (iii) implies (ii) is a general result for metric spaces. (A Cauchy sequence is a bounded set, so (iii) implies that its closure is compact. Hence there is a convergent subsequence, which already implies that the initial Cauchy sequence converges.)

(ii)⇒(i): Assume that (ii) holds and that c is a geodesic in M, whose maximal interval (a, b) of definition is finite. Without loss of generality, we may assume that $g(c'(t), c'(t))$ (which is constant since c is a geodesic) is equal to one. This implies that for all $s, t \in (a, b)$ we have $d_g(c(s), c(t)) \leq |t - s|$. It suffices to show that the domain of definition of c can be extended on one side. Thus assume that $b < \infty$, choose a sequence t_i converging to b and consider the sequence $(c(t_i))$ in (M, d_g). By construction, this is a Cauchy sequence, so there is a point $x \in M$ such that $c(t_i)$

converges to x. As in the proof of Corollary 2.5 we can find an index i and a number ϵ such that $\exp_{c(t_i)}$ is defined on $B_\epsilon(0) \subset T_{c(t_i)}M$ and such that x lies in the image of this ball. Then $\gamma(s) := \exp_{c(t_i)}(sc'(t_i))$ is a well defined geodesic for $|s| < \epsilon$. But $\gamma(0) = c(t_i)$ and $\gamma'(0) = c'(t_i)$ so $\gamma(s) = c(t_i + s)$ as long as $t_i + s \in (a, b)$. But by assumption $b < t_i + \epsilon$, so we obtain an extension of the domain of definition to $(a, t_i + \epsilon)$, which is a contradiction.

We next claim that if for a point $x \in M$, $\exp_x$ is defined on all of T_xM, then for any point $y \in M$, there is a minimizing geodesic connecting x to y. Put $r = d_g(x, y)$, choose $\epsilon > 0$ such that $\exp_x$ restricts to a diffeomorphism on $B_\epsilon(0) \subset T_xM$ and fix $\delta < \epsilon$. Then the geodesic sphere $S_\delta(x) \subset M$ is the image of a compact submanifold of $B_\epsilon(0)$ under a diffeomorphism and hence compact. Thus there is a point $z \in S_\delta(x)$ at which the continuous function $d_g(\ , y)$ attains its minimum. From Proposition 2.5 we know that any point in $S_\delta(x)$ has distance δ from x. Together with the fact that any piece–wise smooth curve from x to y has to intersect $S_\delta(x)$, this easily implies that $d_g(z, y) = r - \delta$.

Now there is a unique unit vector $\xi \in T_xM$ such that $z = \exp_x(\delta\xi)$ and we consider the geodesic $c(t) := \exp_x(t\xi)$ emanating from x in direction ξ. By construction, this satisfies $g_{c(t)}(c'(t), c'(t)) = 1$, so it is parametrized by arclength. Now we define $A := \{t \in [\delta, r] : d_g(c(t), y) = r - t\}$, and we want to show that $r \in A$, which implies that $c(r) = y$, and hence the claim. As observed above, $d_g(z, y) = r - \delta$, so $\delta \in A$ and A is non–empty. Moreover, $A \subset [\delta, r]$ is the subset on which two continuous functions agree, so it is closed.

Therefore, putting $s_0 := \sup(A)$, we get $s_0 \in A$. If $s_0 < r$, then we can find a $\delta' < r$ satisfying the conditions of Lemma 2.5 for the point $c(s_0)$. As above, the geodesic sphere $S_{\delta'}(c(s_0))$ contains a point z' which has minimal distance to y, and $d_g(z', y) = r - s_0 - \delta'$. But this implies that $d_g(z', x) \geq s_0 + \delta'$. As above, we can write $z' = \exp_{c(s_0)}(\delta'\xi')$ for a unit vector $\xi' \in T_{c(s_0)}M$, and we denote by $\tilde{c}$ the corresponding unit speed geodesic emanating from $c(s_0)$. This shows that first going from x to $c(s_0)$ via c and then going to z' via $\tilde{c}$ is a minimizing curve connecting x to z'. By Corollary 2.5 this has to coincide with a geodesic on a neighborhood of s_0, which is only possible if $\xi' = c'(s_0)$. But this implies that $s_0 + \delta' \in A$, which is a contradiction. Thus the proof of the claim is complete.

Using this claim, we can now prove that (iv) implies (iii), which completes the proof of the equivalences. Indeed, if $K \subset M$ is bounded then there is a constant C such that $d_g(x, y) \leq C$ for all $y \in K$, where x is the point occurring in (iv). But by the claim, this implies that K is contained in the image of the closed ball of radius C in T_xM under $\exp_x$, which is compact by continuity of $\exp_x$. Hence if K is closed, it is compact, too.

Having the equivalence at hand, we see that if (iv) is satisfied for one point $x \in M$, it implies (i), which in turn says that (iv) is satisfied for any point of M. Hence (v) follows from the claim. □

Corollary 2.6. *(1) Any compact Riemannian manifold is complete.*

(2) If M is a closed submanifold of $\mathbb{R}^n$ for some n, and one endows M with the Riemannian metric g induced from the inner product of $\mathbb{R}^n$, then (M, g) is complete.

(3) If (M, g) is a complete Riemannian manifold, then for each $x \in M$, the exponential map defines a surjection $\exp_x : T_xM \to M$.

Proof. (1) Follows from the well known fact that compact metric spaces are automatically complete.

For (2), observe that for a smooth curve in M connecting two points x and y, the arclength is always at least the Euclidean distance between x and y. But this shows that any subset in M which is bounded with respect to d_g is also bounded with respect to Euclidean distance, so closed subsets with this property are automatically compact.

(3) immediately follows from condition (v) in the Hopf–Rinow theorem. □

It turns out that hyperbolic space as discussed in part (3) of 2.3 is a complete Riemannian manifold. This example nicely illustrates two general phenomena. Starting from the unit ball in $\mathbb{R}^n$ with the restriction g_0 of the flat metric (which evidently is not complete), we have obtained the hyperbolic metric as a so–called *conformal rescaling*, i.e. $g = fg_0$ for a positive smooth function f. Rescaling a metric conformally does change the notion of length, but it does not change the notions of angles, so in particular, one obtains the same concept of orthogonality. Now the general phenomenon mentioned above is that given an arbitrary Riemannian manifold (M, g_0), one can always find a positive smooth function $f : M \to \mathbb{R}$ such that $g := fg_0$ defines a complete Riemannian metric on M. Intuitively, one can think about this as "moving the missing points to infinity".

The second phenomenon is a kind of converse of this. By the Hopf–Rinow theorem, for a non–compact, complete Riemannian manifold (M, g), M must be unbounded with respect to the distance function d_g. In the case of hyperbolic space, we can also start with the hyperbolic metric g and view g_0 as a conformal rescaling of g, in which the manifold becomes bounded. Again this works in general, so any Riemannian metric can be rescaled to one leading to a bounded distance on M (which then has to be incomplete unless M is compact).

Covariant derivative of tensor fields

The covariant derivative and parallel transport can be extended to tensor fields, basically by requiring certain naturality properties. This for example allows us to form the covariant derivative of the curvature. Moreover, we can iterate covariant derivatives and thus construct higher order differential operators.

2.7. Basic notions. The extension of the covariant derivative is determined by requiring certain naturality properties. On the one hand, for smooth functions, one already has an appropriate operation given by the usual action of vector fields on smooth functions. Let us denote by $\mathcal{T}^\ell_k(M)$ the space of smooth $\binom{\ell}{k}$–tensor fields on a smooth manifold M. Then we want to use the Levi–Civita connection to define operators $\nabla : \mathfrak{X}(M) \times \mathcal{T}^\ell_k(M) \to \mathcal{T}^\ell_k(M)$ with properties analogous to the covariant derivative. In particular, ∇ should be linear over smooth functions in the $\mathfrak{X}(M)$ component.

It turns out that the only thing to require in addition is a compatibility with tensor products and with contractions. This then pins down the whole operation completely.

Proposition 2.7. *Suppose that ∇ is a linear connection on the tangent bundle of a smooth manifold M. Then this extends uniquely to a family of operators ∇ : $\mathfrak{X}(M) \times \mathcal{T}^k_\ell(M) \to \mathcal{T}^k_\ell(M)$ which are linear over smooth functions in the first variable, commute with contractions, and satisfy $\nabla_\xi(s \otimes t) = (\nabla_\xi s) \otimes t + s \otimes \nabla_\xi t$ as well as $\nabla_\xi f = \xi \cdot f$ for $f \in \mathcal{T}^0_0(M) = C^\infty(M, \mathbb{R})$.*

Proof. Let us first look at the case of $\mathcal{T}^0_1(M) = \Omega^1(M)$. Given $\xi, \eta \in \mathfrak{X}(M)$ and $\varphi \in \Omega^1(M)$ we can write the smooth function $\varphi(\eta)$ as the result of the only possible contraction applied to $\varphi \otimes \eta \in \mathcal{T}^1_1(M)$. If an extension with the required properties exists, then the contraction of $(\nabla_\xi\varphi) \otimes \eta + \varphi \otimes (\nabla_\xi \eta)$ has to coincide with $\xi \cdot \varphi(\eta)$. Thus we try defining $\nabla_\xi\varphi$ as a map $\mathfrak{X}(M) \to C^\infty(M, \mathbb{R})$ by

$$\nabla_\xi\varphi(\eta) := \xi \cdot \varphi(\eta) - \varphi(\nabla_\xi \eta).$$

This map is immediately seen to be linear over smooth functions in η, so we have defined $\nabla_\xi\varphi \in \Omega^1(M)$. Moreover, the definition readily implies that $\nabla_{f\xi}\varphi = f\nabla_\xi\varphi$ and that

$$(\nabla_\xi(f\varphi))(\eta) = f\nabla_\xi\varphi(\eta) + (\xi \cdot f)\varphi(\eta)$$

and hence $\nabla_\xi f\varphi = f\nabla_\xi\varphi + (\xi \cdot f)\varphi$.

Having this at hand, the general definition of the covariant derivative is motivated in the same way. Given $t \in \mathcal{T}^\ell_k$ and $\xi \in \mathfrak{X}(M)$, we define $\nabla_\xi t$ as a $(k+\ell)$–linear map $\mathfrak{X}(M)^k \times \Omega^1(M)^\ell \to C^\infty(M,\mathbb{R})$ by

$$\begin{aligned}(\nabla_\xi t)(\eta_1,\dots,\eta_k,\varphi^1,\dots,\varphi^\ell) &:= \xi \cdot t(\eta_1,\dots,\eta_k,\varphi^1,\dots,\varphi^\ell)\\ &- \textstyle\sum_{i=1}^k t(\eta_1,\dots,\nabla_\xi\eta_i,\dots,\eta_k,\varphi^1,\dots,\varphi^\ell)\\ &- \textstyle\sum_{j=1}^\ell t(\eta_1,\dots,\eta_k,\varphi^1,\dots,\nabla_\xi\varphi^j,\dots,\varphi^\ell).\end{aligned} \tag{2}$$

Similarly as above, one verifies directly that this map is linear over smooth functions in each η_i and each φ_j, so we have defined $\nabla_\xi t \in \mathcal{T}^\ell_k(M)$. We also see directly from the formula that $\nabla_{f\xi}t = f\nabla_\xi t$. As in the case of one–forms, this formula is forced from the properties we want to achieve, since one can view $t(\eta_1,\dots,\varphi_\ell)$ as an appropriate contraction of $t \otimes \eta_1 \otimes \cdots \otimes \varphi_\ell$. This shows the the required properties pin down the covariant derivative completely.

So it remains to prove the compatibility with tensor products and with contractions in general. Concerning tensor products, we take $t \in \mathcal{T}^\ell_k(M)$ and $s \in \mathcal{T}^{\ell'}_{k'}(M)$ and $\xi \in \mathfrak{X}(M)$ and expand the defining equation for $\nabla_\xi(t\otimes s)(\eta_1,\dots,\eta_{k+k'},\varphi^1,\dots,\varphi^{\ell+\ell'})$ as in (2). By definition $(t\otimes s)(\eta_1,\dots,\eta_{k+k'},\varphi^1,\dots,\varphi^{\ell+\ell'})$ is given by

$$t(\eta_1,\dots,\eta_k,\varphi^1,\dots,\varphi^\ell)s(\eta_{k+1},\dots,\eta_{k+k'},\varphi^{\ell+1},\dots,\varphi^{\ell+\ell'}).$$

Applying ξ to this product of smooth functions, we apply the Leibniz rule. The first term in the result adds up with those terms in which the covariant derivatives hits one of the first k η's or one of the first ℓ φ's to

$$(\nabla_\xi t)(\eta_1,\dots,\eta_k,\varphi^1,\dots,\varphi^\ell)s(\eta_{k+1},\dots,\eta_{k+k'},\varphi^{\ell+1},\dots,\varphi^{\ell+\ell'}).$$

This is exactly the action of $(\nabla_\xi t)\otimes s$ on the given vector fields an one–forms. In the same way, the remaining terms add up to the action of $t\otimes\nabla_\xi s$, so the compatibility with tensor products is proved.

Let us next look at the basic contraction, which can be viewed as a tensorial operator $C : \mathcal{T}^1_1(M) \to C^\infty(M,\mathbb{R})$. Given $\eta \in \mathfrak{X}(M)$ and $\varphi \in \Omega^1(M)$, we get $\eta\otimes\varphi \in \mathcal{T}^1_1(M)$ and $C(\eta\otimes\varphi) = \varphi(\eta)$. The definition of ∇ on $\Omega^1(M)$ together with compatibility with the tensor product shows that

$$C(\nabla_\xi(\eta\otimes\varphi)) = \xi\cdot\varphi(\eta) = \nabla_\xi(C(\eta\otimes\varphi)).$$

The definition in (2) also implies that the covariant derivative on tensor fields is a local operator. But locally any element of $\mathcal{T}^1_1(M)$ can be written as a finite sum of such tensor products, so compatibility of ∇ with C follows.

Now let us consider a general contraction $\mathcal{T}^\ell_k(M) \to \mathcal{T}^{\ell-1}_{k-1}(M)$, say the one contracting the ith upper index into the jth lower one. On a tensor field of the form $t\otimes\psi\otimes s$ with $t \in \mathcal{T}^{i-1}_{j-1}(M)$, $\psi \in \mathcal{T}^1_1(M)$ and $s \in \mathcal{T}^{\ell-i}_{k-j}(M)$, this contraction is given by $C(\psi)t\otimes s$. For $\xi \in \mathfrak{X}(M)$ we then conclude that the contraction of $\nabla_\xi(t\otimes\psi\otimes s)$ is given by

$$C(\psi)(\nabla_\xi t)\otimes s + C(\nabla_\xi\psi)t\otimes s + C(\psi)t\otimes\nabla_\xi s.$$

Since we have verified $C(\nabla_\xi\psi) = \xi\cdot C(\psi)$ already, we see that this coincides with $\nabla_\xi(C(\psi)t\otimes s)$. Locally, any element of $\mathcal{T}^\ell_k(M)$ can be written as a finite sum of such tensor products, so compatibility of the contraction with the covariant derivative holds in general. Since general contractions can be obtained by iterating contractions of a single pair of indices, the proof is complete. □

Remark 2.7. (1) For a smooth function f and a tensor field t, $f \otimes t$ is just the product ft, so $\nabla_\xi(ft) = (\xi \cdot f)t + f\nabla_\xi t$ holds in general as a consequence of the compatibility with tensor products.

(2) Given a tensor field $g \in T^0_2(M)$, the formula for the covariant derivative from the proof reads as

$$(\nabla_\xi g)(\eta, \zeta) = \xi \cdot g(\eta, \zeta) - g(\nabla_\xi \eta, \zeta) - g(\eta, \nabla_\xi \zeta).$$

Hence the condition that a linear connection ∇ on TM is metric with respect to a Riemannian metric g on M reads as $\nabla_\xi g = 0$ for the induced connection and any vector field ξ.

2.8. Parallel tensor fields. From the formula (2) for the covariant derivative in the proof of Proposition 2.7, we can easily derive a description in local coordinates. In the domain of a chart (U, u), a tensor field $t \in \mathcal{T}^\ell_k(M)$ is determined by the functions $t^{i_1\dots i_\ell}_{j_1\dots j_k}$ which can be obtained as

$$t^{i_1,\dots,i_\ell}_{j_1\dots j_k} = t(\partial_{j_1}, \dots, \partial_{j_k}, du^{i_1}, \dots, du^{i_\ell}).$$

Writing $\xi \in \mathfrak{X}(M)$ as $\sum_i \xi^i \partial_i$ in the domain of the chart, we by definition get $\nabla_\xi \partial_j = \sum_{i,a} \xi^i \Gamma^a_{ij} \partial_a$. Likewise, we can expand $\nabla_\xi du^i = \sum_j (\nabla_\xi du^i)(\partial_j) du^j$, which easily leads to $\nabla_\xi du^i = \sum_{j,a} \xi^j \Gamma^i_{ja} du^a$. Together, these observations immediately imply that

$$\begin{aligned}(\nabla_\xi t)^{i_1\dots i_\ell}_{j_1\dots j_k} = \xi \cdot t^{i_1\dots i_\ell}_{j_1\dots j_k} &- \textstyle\sum_{i,a} \xi^i \Gamma^a_{ij_1} t^{i_1\dots i_\ell}_{aj_2\dots j_k} - \dots - \sum_{i,a} \xi^i \Gamma^a_{ij_k} t^{i_1\dots i_\ell}_{j_1\dots j_{k-1}a} \\ &- \textstyle\sum_{j,a} \xi^j \Gamma^{i_1}_{ja} t^{ai_2\dots i_\ell}_{j_1\dots j_k} - \dots - \sum_{j,a} \xi^j \Gamma^{i_\ell}_{ja} t^{i_1\dots i_{\ell-1}a}_{j_1\dots j_k}.\end{aligned}$$

As in the case of vector fields, this implies that to compute $\nabla_\xi t(x)$, it suffices to know t along the flow line of ξ through x. Consequently, we can mimic the developments in 1.11 in the case of tensor fields. Given a smooth curve $c : I \to M$, we define $\binom{\ell}{k}$–tensor fields along c and then obtain a well defined linear operator $t \mapsto \nabla_{c'} t$ on the space of such tensor fields. In particular, there is the concept of a tensor field being parallel along a curve. Since in local coordinates being parallel is again a system of first order ODEs, for $a \in I$ and $x = c(a) \in M$, we can uniquely extend any element $t_0 \in \otimes^\ell T_x M \otimes \otimes^k T^*_x M$ to a $\binom{\ell}{k}$–tensor field along c which is parallel along c. For $[a, b] \subset I$, this gives rise to a well defined parallel transport of tensors along c. From the construction, one easily verifies that this is exactly the map which gets functorially induced by the parallel transport of vector fields.

For the Levi–Civita connection of a Riemannian manifold (M, g) we have noted above that the induced connection on $\mathcal{T}^0_2(M)$ has the property that $\nabla_\xi g = 0$ for any ξ. A tensor field with this property is called *parallel* since it is parallel along any smooth curve. Surprisingly, parallel tensor fields of any type on a Riemannian manifold can be described provided that on knows the holonomy of the metric as introduced in 1.11. Given a point $x \in M$, we have introduced there the holonomy group $\mathrm{Hol}_x(M)$ of M at x, which is a subgroup of the orthogonal group $O(T_x M)$. Observe that any linear automorphism of $T_x M$ induces a linear automorphism of each of the tensor powers $\otimes^\ell T_x M \otimes \otimes^k T^*_x M$. Hence any element of the holonomy group acts on the values of tensor fields of any type at x.

PROPOSITION 2.8. *Let (M, g) be a connected Riemannian manifold and let $x \in M$ be a point.*

(1) A parallel tensor field $t \in \mathcal{T}^\ell_k(M)$ is uniquely determined by its value $t(x) \in \otimes^\ell T_xM \otimes \otimes^k T^*_xM$.

(2) Given an element $t_0 \in \otimes^\ell T_xM \otimes \otimes^k T^*_xM$, there is a parallel tensor field $t \in \mathcal{T}^\ell_k(M)$ such that $t(x) = t_0$ if and only if t_0 is mapped to itself by any element of the holonomy group $\mathrm{Hol}_x(M)$ of M at x.

PROOF. (1) If $t \in \mathcal{T}^\ell_k(M)$ is parallel, it is parallel along each smooth curve. Given a point y in M, connectedness of M implies that there is a smooth curve $c : [a, b] \to M$ such that $c(a) = x$ and $c(b) = y$. But then we must have $t(y) = \mathrm{Pt}_c(t(x))$.

(2) The necessity of the condition follows readily since t is parallel along each smooth curve. To prove sufficiency, one observes that the fact that t_0 is preserved by any element of $\mathrm{Hol}_x(M)$ is equivalent to the fact that for two curves c and $\tilde{c}$ connecting x to some point $y \in M$, we get $\mathrm{Pt}_c(t_0) = \mathrm{Pt}_{\tilde{c}}(t_0)$. This is because transporting t_0 to y parallely along c and transporting the result back to x parallely along $\tilde{c}$ is the parallel transport along the pice–wise smooth closed curve obtained by first running through c and then backwards through $\tilde{c}$. Hence this is given by the action of an element of the holonomy group.

Knowing this, we can extend t_0 to a tensor field t by defining $t(y)$ as $\mathrm{Pt}_c(t_0)$ for any pice–wise smooth curve c connecting x to y. It is easy to see that the result is smooth and it is parallel along any smooth curve by construction. □

Note that the statement that g is parallel fits nicely into the picture, since any element of $\mathrm{Hol}_x(M)$ is orthogonal with respect to g_x and this exactly means that the induced map on $\otimes^2 T^*_xM$ preserves g_x.

2.9. Natural differential operators. We can interpret the covariant derivative as a linear differential operator (even in the case of vector fields). In this picture the covariant derivative can be iterated, thus providing the possibility to construct operators of higher order.

The first observation we need is that for a tensor field $t \in \mathcal{T}^\ell_k(M)$ we can consider the $(k + \ell + 1)$–linear map $\nabla t : \mathfrak{X}(M)^{k+1} \times \Omega^1(M)^\ell \to C^\infty(M, \mathbb{R})$ defined by

$$(\nabla t)(\eta_0, \dots, \eta_k, \varphi_1, \dots, \varphi_\ell) := (\nabla_{\eta_0} t)(\eta_1, \dots, \eta_k, \varphi_1, \dots, \varphi_\ell).$$

From Proposition 2.7 we know that $\nabla_{\eta_0} t$ is a tensor field, so this is linear over smooth functions in all entries but η_0. But in Proposition 2.7 we have also seen that $\nabla_{f\eta_0} t = f\nabla_{\eta_0} t$, so $\nabla t \in \mathcal{T}^\ell_{k+1}(M)$. But then it is clear that we can form $\nabla^2 t = \nabla(\nabla t) \in \mathcal{T}^\ell_{k+2}$, and more generally, $\nabla^r t$ for any integer r.

In these terms, there is a natural interpretation of the curvature. Namely, for $\zeta \in \mathfrak{X}(M)$, we can consider $\nabla^2\zeta \in \mathcal{T}^1_2(M)$. To compute this, we have to observe that $\nabla\zeta \in \mathcal{T}^1_1(M)$ is, as a bilinear map $\mathfrak{X}(M) \times \Omega^1(M) \to C^\infty(M, \mathbb{R})$ given by $(\nabla\zeta)(\eta, \varphi) = \varphi(\nabla_\eta\zeta)$. Consequently, we get

$$(\nabla^2\zeta)(\xi, \eta, \varphi) = (\nabla_\xi(\nabla\zeta))(\eta, \varphi) = \xi \cdot (\varphi(\nabla_\eta\zeta)) - \varphi(\nabla_{\nabla_\xi\eta}\zeta) - (\nabla_\xi\varphi)(\nabla_\eta\zeta).$$

The first and last term add up to $\varphi(\nabla_\xi\nabla_\eta\zeta)$, which implies that, as a bilinear map $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, we obtain

$$(\nabla^2\zeta)(\xi, \eta) = \nabla_\xi\nabla_\eta\zeta - \nabla_{\nabla_\xi\eta}\zeta.$$

In view of torsion–freeness, this implies that

$$R(\xi, \eta)(\zeta) = (\nabla^2\zeta)(\xi, \eta) - (\nabla^2\zeta)(\eta, \xi),$$

which interprets the curvature as the alternation of the square of a covariant derivative.

In the context of abstract index notation, one can use an expression like $\nabla_j t^{i_1\dots i_\ell}_{j_1\dots j_k}$ to denote ∇t for a tensor field $t = t^{i_1\dots i_\ell}_{j_1\dots j_k} \in T^\ell_k(M)$. This has to be handled with care, since one has to decide which terms are really differentiated. The usual convention is that if there are no brackets, then a covariant derivative acts on all terms to its right. Thus $\nabla_j\xi^i\varphi_k$ represents $\nabla(\xi\otimes\varphi)$ and the compatibility of the covariant derivative with the tensor product can be written as $\nabla_j\xi^i\varphi_k = (\nabla_j\xi^i)\varphi_k + \xi^i\nabla_j\varphi_k$. Alternatively, the first of these summands can be written as $\varphi_k\nabla_j\xi^i$.

In these terms, one can now easily describe some operators. For example, the fact that $\nabla_\xi f = \xi\cdot f$ for a smooth function f immediately implies that $\nabla f = df$. Likewise, for a one–form $\varphi = \varphi_i$, we have by definition

$$(\nabla\varphi)(\xi,\eta) = (\nabla_\xi\varphi)(\eta) = \xi\cdot\varphi(\eta) - \varphi(\nabla_\xi\eta).$$

Torsion–freeness of ∇ together with the global formula for the exterior derivative implies that

$$d\varphi(\xi,\eta) = (\nabla\varphi)(\xi,\eta) - (\nabla\varphi)(\eta,\xi),$$

so in abstract index notation the exterior derivative can be written as $\varphi_i \mapsto 2\nabla_{[i}\varphi_{j]}$. One can also verify that for a one–form $\varphi = \varphi_i$ the codifferential is given by $\delta\varphi = g^{ij}\nabla_i\varphi_j$. Together with the observation on the exterior derivative of functions from above, this shows that for a smooth function f, the Laplacian is given by $\Delta f = g^{ij}\nabla_i\nabla_j f$.

As an example of a natural differential operator, let us study the so–called *Killing operator* on one–forms. This is the operator mapping $\Omega^1(M)$ to the space of symmetric $\binom{0}{2}$–tensor fields, defined by $\varphi_i \mapsto \nabla_{(i}\varphi_{j)}$. One calls one–forms which lie in the kernel of this operator *Killing one–forms* and the vector fields dual to these (i.e. given by $\xi^i = g^{ij}\varphi_j$) are called *Killing vector fields.*

PROPOSITION 2.9. *Let (M,g) be a Riemannian manifold. Then we have*
(1) For $\varphi = \varphi_i \in \Omega^1(M)$ the following conditions are equivalent
(i) *φ is a Killing one–form*
(ii) $\nabla\varphi = \frac{1}{2}d\varphi$
(iii) *Each local flow of the dual vector field $\xi^i = g^{ij}\varphi_j$ is a local isometry for g.*

(2) If M is connected, then φ is uniquely determined by the values $\varphi(x)$ and $\nabla\varphi(x)$ for any point $x \in M$. In particular, the space of Killing one–forms has dimension at most $\frac{n(n+1)}{2}$, where $n = \dim(M)$.

PROOF. (1) By definition φ is a is a Killing one–form if and only if $\nabla_i\varphi_j$ has trivial symmetrization and thus is skew symmetric. Since we have observed already that $d\varphi = 2\nabla_{[i}\varphi_{j]}$ we see that (i) and (ii) are equivalent. On the other hand, it follows from general properties of the Lie derivative that the local flows of ξ are isometries (i.e. satisfy $(\mathrm{Fl}^\xi_t)^*g = g$ whenever the flow is defined), if and only if $\mathcal{L}_\xi g = 0$. Now the Lie derivative on tensor fields satisfies similar naturality properties as the covariant derivative, in particular,

$$(\mathcal{L}_\xi g)(\eta,\zeta) = \xi\cdot g(\eta,\zeta) - g([\xi,\eta],\zeta) - g(\eta,[\xi,\zeta]).$$

By torsion–freeness of the Levi–Civita connection, we can write $[\xi,\eta] = \nabla_\xi\eta - \nabla_\eta\xi$, and likewise for the other bracket. But then the fact that ∇ is metric shows that we end up with

$$(\mathcal{L}_\xi g)(\eta,\zeta) = g(\nabla_\eta\xi,\zeta) + g(\eta,\nabla_\zeta\xi),$$

and the right hand side is the symmetrization of $g_{ai}\nabla_j\xi^a = \nabla_j\varphi_i$. Hence we see that (i) is equivalent to (iii).

(2) Suppose that φ_i is a Killing one–form, and put $\mu_{ij} = \frac{1}{2}d\varphi = \nabla_i\varphi_j$. Now a nice trick allows us to compute $\nabla_i\mu_{jk}$ (as a consequence of the equation satisfied by φ). Namely, by construction, we have $d\mu = 0$. Similarly to the case of one–forms discussed above, one verifies that $d\mu$ can be computed as the complete alternation of $\nabla_i\mu_{jk}$. Hence we conclude that

$$\nabla_i\mu_{jk} = -\nabla_k\mu_{ij} - \nabla_j\mu_{ki} = \nabla_k\nabla_j\varphi_i - \nabla_j\nabla_k\varphi_i.$$

Similarly to the case of vector fields, one now verifies that the commutator of covariant derivatives can be expressed via the curvature. More precisely, one verifies that

$$\nabla_k\nabla_j\varphi_i - \nabla_j\nabla_k\varphi_i = R_{jk}{}^{\ell}{}_i\varphi_\ell.$$

Thus we conclude that for the pair $\binom{\varphi_i}{\mu_{jk}}$ we can compute the component–wise covariant derivative in terms of the values of the components and the (known) curvature of g. Along a smooth curve, this gives a first order ODE on the pair $\binom{\varphi_i}{\mu_{jk}}$, so the values along the curve are determined from the value of the pair in one point. Since M is connected, this implies the claim. □

Decomposing and interpreting curvature

From the discussion of curvature symmetries in 1.13, it is already visible that the Riemann curvature tensor is a rather complicated object. Therefore, constructing simpler objects out of the Riemann tensor is important for many applications.

2.10. Flat manifolds. Before we start decomposing the Riemann curvature tensor, we discuss the geometric meaning of vanishing of the curvature. Observe that by Lemma 1.10, the covariant derivative is a local operator. The definition of curvature in 1.13 then implies that the curvature is a local invariant of a Riemannian manifold, i.e. the restriction of the curvature to an open subset U depends only on the restriction of the metric to U. Hence vanishing of the curvature is a local condition, so we can only hope for local characterizations of this property. We first need a lemma, which is not related to Riemannian geometry, but rather nice in its own right.

LEMMA 2.10. *Let M be a smooth manifold of dimension n and let $x \in M$ be a point. Suppose that $\xi_1, \dots, \xi_n$ are vector fields defined on an open neighborhood of x such that $[\xi_i, \xi_j] = 0$ for all i and j and such that $\{\xi_1(x), \dots, \xi_n(x)\}$ is a basis for T_xM. Then there is a local chart (U, u) for M with $x \in U$ such that for each i, $\xi_i|_U$ coincides with the coordinate vector field ∂_i.*

PROOF. It is well known that vanishing of the Lie brackets implies that the flows of the vector fields ξ_i mutually commute, see Section 2.15 of [**DG1**]. Now consider the map

$$\varphi(t^1, \dots, t^n) := \mathrm{Fl}^{\xi_1}_{t^1} \circ \dots \circ \mathrm{Fl}^{\xi_n}_{t^n}(x),$$

which is defined on some open neighborhood of 0 in $\mathbb{R}^n$. The ith partial derivative of φ can be computed as $\frac{d}{ds}|_{s=0}\varphi(t^1, \dots, t^i + s, \dots, t^n)$. Inserting into the definition and using that $\mathrm{Fl}^{\xi_i}_{t^i+s} = \mathrm{Fl}^{\xi_i}_s \circ \mathrm{Fl}^{\xi_i}_{t^i}$ as well as the fact that the flows commute, this can be written as

$$\tfrac{d}{ds}|_{s=0}\, \mathrm{Fl}^{\xi_i}_s(\varphi(t^1, \dots, t^n)) = \xi_i(\varphi(t^1, \dots, t^n)).$$

Now the assumption that the $\xi_i(x)$ form a basis of T_xM shows that $T_0\varphi$ is invertible, so φ is a diffeomorphism on some neighborhood of 0. The inverse of this diffeomorphism then defines a chart with the required properties. □

Using this, we can now characterize vanishing of the Riemann curvature tensor.

PROPOSITION 2.10. *Let (M, g) be a Riemannian manifold of dimension n. Then for a point $x \in M$, the following conditions are equivalent*

(1) The Riemann curvature tensor vanishes on an open neighborhood of x.

(2) There is a chart (U, u) for M with $x \in U$ such that the coordinate vector fields ∂_i form an orthonormal basis of each tangent space.

(3) There are vector fields $\{\xi_1, \dots, \xi_n\}$ defined on a neighborhood of x which are all parallel, i.e. such that $\nabla_\eta \xi_i = 0$ for any $\eta \in \mathfrak{X}(M)$ and any $i = 1, \dots, n$ and such that $\{\xi_1(x), \dots, \xi_n(x)\}$ is a basis for T_xM.

(4) There is an isometric diffeomorphism from an open neighborhood of x in M onto an open subset of Euclidean space.

PROOF. (3)⇒(2): Let us first orthonormalize the basis $\{\xi_1(x), \dots, \xi_n(x)\}$ and write the corresponding orthonormal basis as $\eta_j(x) = \sum_i a^i_j \xi_i(x)$ with $a^i_j \in \mathbb{R}$. Putting $\eta_j = \sum_i a^i_j \xi_i$ we of course get $\nabla_\eta \eta_j = 0$ for each j and any $\eta \in \mathfrak{X}(M)$ since the a^i_j are constant. Since ∇ is metric, this implies that for all i, j, the functions $g(\eta_i, \eta_j)$ are constant, so the η_i are orthonormal wherever they are defined.

On the other hand, we in particular get $\nabla_{\eta_i} \eta_j = 0$ for all i and j, which by torsion freeness implies $[\eta_i, \eta_j] = 0$ for all i and j. By the lemma, there is a chart (U, u) around x, such that $\partial_i = \eta_i$ for all i, so (2) holds.

(2)⇒(4): By definition, the chart map u is a diffeomorphism from the open neighborhood U of x onto an open subset of $\mathbb{R}^n$. Moreover for $y \in U$, the tangent map $T_y u$ maps the orthonormal basis $\{\partial_i(y)\}$ to the standard basis of $\mathbb{R}^n$. Hence $T_y u$ is orthogonal, so u is an isometry.

(4)⇒(1): This is clear since by Proposition 1.14, any isometry is compatible with the Riemann curvature tensors, and the curvature vanishes on $\mathbb{R}^n$.

(1)⇒(3): Since this is a local question, it suffices to do this locally around $0 \in \mathbb{R}^n$ for an arbitrary Riemannian metric g on $\mathbb{R}^n$ with vanishing curvature. We denote by $x^1, \dots, x^n$ the standard coordinates and by ∂_i the corresponding coordinate vector fields. Choose an orthonormal basis $\xi_1(0), \dots, \xi_n(0)$ of $T_0\mathbb{R}^n$ and extend each of these tangent vectors to a vector field ξ_i on $\mathbb{R}^n$ as follows. To get $\xi_i(x^1, \dots, x^n)$, first translate $\xi_i(0)$ parallely along the line $t \mapsto (t, 0, \dots, 0)$ to the point $(x^1, 0, \dots, 0)$ then translate parallely along $t \mapsto (x^1, t, 0, \dots, 0)$ to $(x^1, x^2, 0, \dots, 0)$ and so on. So we have to prove that the resulting vector fields ξ_i are all parallel.

Now by construction ξ_i is parallel along each of the lines $t \mapsto (y^1, \dots, y^{n-1}, t)$, so $\nabla_{\partial_n} \xi_i = 0$. The same argument shows that $\nabla_{\partial_{n-1}} \xi_i$ vanishes on the subspace of all points with last coordinate equal to 0. But vanishing of the curvature together with $[\partial_{n-1}, \partial_n] = 0$ implies that $\nabla_{\partial_n} \nabla_{\partial_{n-1}} \xi_i = \nabla_{\partial_{n-1}} \nabla_{\partial_n} \xi = 0$. Hence $\nabla_{\partial_{n-1}} \xi_i$ is parallel along each of the lines $t \mapsto (y^1, \dots, y^{n-1}, t)$ and vanishes for $t = 0$, so it vanishes identically. Next $\nabla_{\partial_{n-2}} \xi_i$ vanishes in all points for which the last two coordinates are zero, and using vanishing of the curvature one first shows that this extends to all points with vanishing last coordinate and then to all of $\mathbb{R}^n$. Iteratively, we get $\nabla_{\partial_j} \xi_i = 0$ for all i and j, so the ξ_i are indeed parallel. □

2.11. Sectional curvature and space forms. The concept of sectional curvature is on the one hand motivated by the relation between Gauß curvature and the Riemann curvature tensor for surfaces in $\mathbb{R}^3$, see Proposition 3.7 in [**DG1**]. On the other hand, as we have noted in Proposition 1.13, the Riemann tensor can be interpreted as a bilinear form on $\Lambda^2 TM$, so one can look at the values of this form on the wedge product of two tangent vectors.

In view of the symmetries of the Riemann tensor, it does not make sense to insert four copies of a tangent vector into R. However, given two tangent vectors ξ and η in T_xM, there is an essentially unique way to insert them into the curvature. A slight variation of this idea with nicer properties is the following.

DEFINITION 2.11. (1) Let (M, g) be a Riemannian manifold with curvature tensor R. Then for a point $x \in M$ and two linearly independent tangent vectors $\xi, \eta \in T_xM$, one defines the *sectional curvature* by

$$K(x)(\xi, \eta) := \frac{g_x(R_x(\xi, \eta)(\eta), \xi)}{g_x(\xi, \xi)g_x(\eta, \eta) - g_x(\xi, \eta)^2} \in \mathbb{R}.$$

(2) One says that g has *constant sectional curvature* a if and only if $K(x)(\xi, \eta) = a$ for all x, ξ and η.

The motivation for the denominator in this definition is that $K(x)(\xi, \eta)$ depends only on the plane in T_xM spanned by the two vectors ξ and η. Indeed, replacing ξ and η by $a\xi + b\eta$, $c\xi + d\eta$, the skew symmetry properties of R show that the numerator of the expression for $K(x)$ gets multiplied by $(ad - bc)^2$. On the other hand, the square of the area of the parallelogram spanned by ξ and η can be computed as $|\xi|^2|\eta|^2 \sin^2(\alpha) = |\xi|^2|\eta|^2(1 - \cos^2(\alpha))$, where α is the angle between the two vectors, and this is exactly the denominator in the expression for $K(x)$. We also conclude that this denominator is non–zero provided that ξ and η are linearly independent.

This also implies that it is sufficient to consider $K(x)(\xi, \eta)$ for orthonormal tangent vectors, and $K(x)(\xi, \eta) = g_x(R_x(\xi, \eta)(\eta), \xi)$ in this case. Now the explicit formula for the curvatures of the sphere and of hyperbolic space from 2.3 show that the sphere has constant sectional curvature $+1$, while hyperbolic space has constant sectional curvature -1. Of course, $\mathbb{R}^n$ has constant sectional curvature 0. These three basic examples are called the *space forms* and in some sense they are the simplest Riemannian manifolds.

Other constant values of sectional curvature are not terribly interesting, since one may always rescale the metric by a positive constant. Since the Levi–Civita connection of g is also metric for any constant positive multiple of g, it follows that such a constant rescaling does not change the Levi–Civita connection and hence also the Riemann tensor remains unchanged. However, passing from the Riemann tensor to sectional curvature involves the metric which implies that passing from g to ag means passing from K to $\frac{1}{a}K$.

It can be shown in general that any manifold of constant sectional curvature 1 (respectively -1) is locally isometric to S^n (respectively $\mathcal{H}^n$), while manifolds of constant sectional curvature 0 are flat and thus locally isometric to $\mathbb{R}^n$ by Proposition 2.10. Finally, it turns out that if in each point, the sectional curvature has the same value for all planes in the tangent space, then the metric automatically has constant sectional curvature.

2.12. The covariant derivative of the curvature. Aparat from constant sectional curvature as discussed in 2.11, there is a second idea to define a concept of "constant curvature" for a Riemannian manifold. Namely, we can consider the Riemann curvature tensor R as a $\binom{1}{3}$–tensor field and form its covariant derivative ∇R, which then is a tensor field of type $\binom{1}{4}$. Before we study vanishing of this tensor field, we prove the so–called *second Bianchi identity* (or differential Bianchi identity) which is the last main symmetry property of the curvature tensor.

PROPOSITION 2.12 (Second Bianchi identity). *Let (M, g) be a Riemannian manifold with Riemann curvature tensor R. Then the covariant derivative of the Riemann tensor* *satisfies*

$$0 = (\nabla_\xi R)(\eta, \zeta) + (\nabla_\zeta R)(\xi, \eta) + (\nabla_\eta R)(\zeta, \xi)$$

for all $\xi, \eta, \zeta \in \mathfrak{X}(M)$. *In abstract index notation, this reads as* $0 = \nabla_{[i} R_{jk]}{}^\ell{}_m$.

Proof. This is most easliy verified in terms of the expression of the curvature in a local orthonormal frame, but also in this setting quite a bit of computation is needed. Apply the exterior derivative to the defining equation

$$\Omega^i_j = d\omega^i_j + \textstyle\sum_k \omega^i_k \wedge \omega^k_j$$

for the curvature two–forms and reinsert for the $d\omega$. This gives

$$d\Omega^i_j = \textstyle\sum_k \Omega^i_k \wedge \omega^k_j - \sum_k \omega^i_k \wedge \Omega^k_j - \sum_{k,\ell} \omega^i_\ell \wedge \omega^\ell_k \wedge \omega^k_j + \sum_{k,\ell} \omega^i_k \wedge \omega^k_\ell \wedge \omega^\ell_j,$$

and clearly the last two sums cancel. This is already the second Bianchi identity in the moving frame from, and we have to interpret it in terms of covariant derivatives. Recall from 2.2 that $\Omega^i_j(\eta, \zeta) = g(R(\eta, \zeta)(s_j), s_i)$, where the s_i are the elements of the given orthonormal frame. Differentiating this smooth function with $\xi \in \mathfrak{X}(M)$, we obtain

$$\xi \cdot \Omega^i_j(\eta, \zeta) = g(\nabla_\xi R(\eta, \zeta)(s_j), s_i) + g(R(\eta, \zeta)(s_j), \nabla_\xi s_i).$$

The naturality properties of the covariant derivative from Propostion 2.7 imply that

$$\nabla_\xi R(\eta, \zeta)(s_j) = ((\nabla_\xi R)(\eta, \zeta))(s_j) + R(\nabla_\xi \eta, \zeta)(s_j) + R(\eta, \nabla_\xi \zeta)(s_j) + R(\eta, \zeta)(\nabla_\xi s_j).$$

Now by definition $\nabla_\xi s_j = \sum_k \omega^k_j(\xi) s_k$ and likewise we can insert for $\nabla_\xi s_i$ above. Inserting all that above, we obtain

$$\begin{aligned} \xi \cdot \Omega^i_j(\eta, \zeta) = {} & g(((\nabla_\xi R)(\eta, \zeta))(s_j), s_i) + \Omega^i_j(\nabla_\xi \eta, \zeta) - \Omega^i_j(\nabla_\xi \zeta, \eta) \\ & + \textstyle\sum_k \omega^k_j(\xi) \Omega^i_k(\eta, \zeta) + \sum_k \omega^k_i(\xi) \Omega^k_j(\eta, \zeta). \end{aligned}$$

Summing this up over all cyclic permutations of ξ, η and ζ the second an third terms in the right hand side add up to

$$\Omega^i_j([\xi, \eta], \zeta) + \Omega^i_j([\eta, \zeta], \xi) + \Omega^i_j([\zeta, \xi], \eta),$$

and bringing this to the other side, we obtain $d\Omega^i_j(\xi, \eta, \zeta)$ on the left hand side. On the other hand, summing the last two terms in the right hand side and using $\omega^k_i = -\omega^i_k$, one gets

$$(\textstyle\sum_k \omega^k_j \wedge \Omega^i_k - \sum \omega^i_k \wedge \Omega^k_j)(\xi, \eta, \zeta).$$

Hence we conclude that the sum over all cyclic permutations of ξ, η and ζ of

$$g(((\nabla_\xi R)(\eta, \zeta))(s_j), s_i)$$

vanishes for all i and j, which implies the claim. □

Now let us study the condition of parallel curvature for Riemannian manifolds. It turns out that this is related to so called symmetries. Here by a symmetry in a point x of a smooth manifold M one means a local diffeomorphism $\sigma = \sigma_x$ defined on a neighborhood of x such that $\sigma(x) = x$ and $T_x\sigma = -\operatorname{id}_{T_xM}$. Note that in case that (M, g) is a Riemannian manifold and σ is an isometry for g, these conditions determine σ locally around x. Indeed, in this case, we must have $\sigma(\exp_x(\xi)) = \exp_x(-\xi)$ for all $\xi \in T_xM$ such that the left hand side is defined, compare with 1.14. Conversely, we can clearly define a local symmetry at x by $\exp_x \circ - (\exp_x)^{-1}$, which is called the *geodetic reflection* in x.

DEFINITION 2.12. Let (M, g) be a connected Riemannian manifold.

(1) (M, g) is called a *locally symmetric space* if and only if for each point $x \in M$, the geodetic reflection defines an isometry on some open neighborhood of x.

(2) (M, g) is called a *symmetric space* if and only if for each point $x \in M$, the geodetic reflection in x extends to a globally defined isometry of M.

Now it turns out that (M, g) is a locally symmetric space if and only if the Riemann curvature tensor R of g is parallel, i.e. iff $\nabla R = 0$. The necessity of this condition is easy to see. If the geodestic reflection defines an isometry σ_x on an open neighborhood U of x, then $(\sigma_x)^*(\nabla R) = \nabla R$, see Proposition 1.14. But the action of $(\sigma_x)^*(\nabla R)(x)$ on three tangent vectors in T_xM is given by hitting the tangent vectors with $T_x\sigma_x = -\operatorname{id}$, so $(\sigma_x)^*(\nabla R)(x) = -\nabla R(x)$. The sufficiency is more complicated to prove.

Second, it turns out that the difference between locally symmetric spaces and symmetric spaces comes from topology. Indeed, one can prove that a simply connected locally symmetric space automatically is a symmetric space. In particular, given a locally symmetric space (M, g) one can form the universal covering space $\widetilde{M}$. This is a simply connected space endowed with a covering map $p : \widetilde{M} \to M$. This covering map is a local homeomorphism, so one can use charts of M to make $\widetilde{M}$ into a smooth manifold in such a way that p becomes a local diffeomorphism. Further, one can pull back the tensor field g on M to $\widetilde{M}$ to obtain a Riemannian metric $\tilde{g}$ on $\widetilde{M}$ and then p becomes a local isometry. By construction, $(\widetilde{M}, \tilde{g})$ is a locally symmetric space and thus a symmetric space by simple connectedness.

To analyze symmetric spaces, one first proves that they are *homogenous*, i.e. for two points x and y in a symmetric space (M, g), there always is an isometry $f : M \to M$ such that $f(x) = y$. This follows easily from the same fact in the case that x and y can be connected by a geodesic, which is obvious since x is mapped to y by reflecting in the middle point between x and y on that geodesic. Now one can prove that the group of isometries of a Riemannian manifold is always a Lie group, so a homogeneous Riemannian manifold is realized as a homogeneous space of its isometry group. One can then study the condition of being symmetric in terms of Lie theory, which leads to a complete classification of symmetric spaces. Locally symmetric spaces are then obtained by further quotenting by discrete subgroups of the isometry group, and a lot is known about such subgroups.

Apart from the fact that they provide many interesting examples of Riemannian manifolds (indculding spheres and hyperbolic spaces, and Grassmann manifolds) they also play an important role in holonomy theory. In fact, in the classification of holonomy groups mentioned in 1.11 and 2.8, one always has to distinguish between the case of locally symmetric spaces and manifold for which the curvature tensor is not parallel. For the locally symmetric case, the classification of symmetric spaces in terms of Lie theory also gives a classification of holonomy groups, in the other case, the possible holonomy groups are classified by a classical theorem of M. Berger.

2.13. Decomposing the curvature tensor. An idea to obtain simpler objects from the Riemann curvature tensor is to try taking traces. Due to the symmetries of the curvature tensor, there is initially only one trace (up to sign) which has the potential to be non–zero. Writing the curvature tensor as $R_{ij}{}^k{}_\ell$ a contraction is defined by either choosing one of the lower indices and contracting k into it or by choosing two of the lower indices and contrating them with the inverse metric. Now the skew symmetry results from part (2) of Proposition 1.13 on the one hand imply that $g^{ij}R_{ij}{}^k{}_\ell = 0$ and

${R_{ij}}^k{}_k = 0$ as well as the fact that the remaining contractions (k into i or j, or ℓ with i or j with the inverse metric) all agree up to sign.

DEFINITION 2.13. Let (M, g) be a smooth Riemannian manifold of dimension $n \geq 3$ with Riemann curvature tensor ${R_{ij}}^k{}_\ell$.

(1) The *Ricci curvature* of g is the $\binom{0}{2}$–tensor field Ric defined by $\mathrm{Ric}_{ij} := {R_{ki}}^k{}_j$.

(2) The *scalar curvature* of g is the smooth function R on M defined by $R := g^{ij}\,\mathrm{Ric}_{ij}$.

(3) The *Schouten tensor* of g is defined by $\mathsf{P}_{ij} := \frac{1}{n-2}(\mathrm{Ric}_{ij} - \frac{1}{2(n-1)} R g_{ij})$.

(4) The *Weyl curvature* of g is the $\binom{1}{3}$–tensorfeld W defined by

$$ {W_{ij}}^k{}_\ell := {R_{ij}}^k{}_\ell - \left(2\delta^k_{[i}\mathsf{P}_{j]\ell} - 2g_{\ell[i}\mathsf{P}_{j]a}g^{ak}\right). $$

(5) The metric g is called *Ricci flat* if $\mathrm{Ric}_{ij} = 0$.

(6) The metric g is called an *Einstein metric* if its Ricci curvature (or equivalently its Schouten tensor) is proportional to the metric, i.e. if $\mathrm{Ric}_{ij} = \frac{1}{n} R g_{ij}$.

Let us next verify the basic properites of these quantitites.

PROPOSITION 2.13. *For any Riemannian manifold (M, g) the following hold.*

(1) The Ricci curvature and the Schouten tensor are both symmetric and they satisfy $\mathrm{Ric}_{ij} = (n-2)\mathsf{P}_{ij} + \mathsf{P} g_{ij}$, where $\mathsf{P} = g^{ij}\mathsf{P}_{ij} = \frac{1}{2(n-1)}R$ is the trace of the Schouten tensor.

(2) The Weyl curvature has all symmetries of the Riemann curvature tensor as in parts (2) – (4) of Proposition 1.13 and in addition is totally tracefree, i.e. we have

$$ {W_{ij}}^k{}_\ell = {W_{[ij]}}^k{}_\ell \quad {W_{ij}}^a{}_\ell g_{ka} = {W_{ij}}^a{}_{[\ell} g_{k]a} \quad {W_{ij}}^a{}_\ell g_{ka} = {W_{\ell k}}^a{}_i g_{ja} \quad {W_{[ij}}^k{}_{\ell]} = 0 \quad {W_{ki}}^k{}_j = 0 $$

PROOF. (1) By defintion, the Ricci curvature can be written as $\mathrm{Ric}_{ij} = g^{k\ell} R_{ki}{}^a_j g_{\ell a}$. From Proposition 1.13, we know that $R_{ki}{}^a_j g_{\ell a} = R_{j\ell}{}^a_k g_{ia} = R_{\ell j}{}^a_i g_{ka}$ and applying $g^{k\ell}$ to the last expression, we by definition get Ric_{ji}. Symmtry of the Schouten tensor then follows by definition.

From the definition of the Schouten tensor, it follows readily that $\mathrm{Ric}_{ij} = (n-2)\mathsf{P}_{ij} + \frac{1}{2(n-1)}Rg_{ij}$. Contracting this equations with g^{ij}, we see that $R = (n-2)\mathsf{P} + \frac{n}{2(n-1)}R$, and hence $\frac{n-2}{2(n-1)}R = (n-2)\mathsf{P}$, which implies the claim.

(2) Lowering the index k in the definition of the Weyl curvature, we see that ${W_{ij}}^a{}_\ell g_{ka}$ is obtained from ${R_{ij}}^a{}_\ell g_{ka}$ by subtracting

$$ 2g_{k[i}\mathsf{P}_{j]\ell} - 2g_{\ell[i}\mathsf{P}_{j]k}. $$

From this form it is evident that this term is skew symmetric in i and j as well as in k and ℓ. Morover, if we expand the alternations, symmetry of g and P implies that each of the resulting terms is symmetric in two of the three indices i, j and ℓ. Therefore, the complete alternation of this expression over these three indices vanishes, so ${R_{[ij}}^k{}_{\ell]} = 0$ implies ${W_{[ij}}^k{}_{\ell]} = 0$. In the proof of Proposition 1.13, we have seen that the symmetries derived so far imply that ${W_{ij}}^a{}_\ell g_{ka} = {W_{\ell k}}^a{}_i g_{ja}$, so it remains to prove that ${W_{kj}}^k{}_\ell = 0$. To do this, we expand the alternations in the definition of the Weyl curvature to obtain

$$ {W_{ij}}^k{}_\ell = {R_{ij}}^k{}_\ell - \delta^k_i \mathsf{P}_{j\ell} + \delta^k_j \mathsf{P}_{i\ell} + g_{\ell i}\mathsf{P}_{ja}g^{ak} - g_{\ell j}\mathsf{P}_{ia}g^{ak}. $$

Contracting the indices i and k, we get

$$ {W_{kj}}^k{}_\ell = \mathrm{Ric}_{j\ell} - n\mathsf{P}_{j\ell} + \mathsf{P}_{j\ell} + \mathsf{P}_{j\ell} - g_{j\ell}\mathsf{P}, $$

which equals $\mathrm{Ric}_{j\ell} - ((n-2)\mathsf{P}_{j\ell} + \mathsf{P}g_\ell) = 0$. $\square$

In view of this result we can reinterpret the definiton of the Weyl curvature as a the decomposition

$$R_{ij}{}^k{}_\ell = W_{ij}{}^k{}_\ell + \left(2\delta^k_{[i}\mathsf{P}_{j]\ell} - 2g_{\ell[i}\mathsf{P}_{j]a}g^{ak}\right)$$

of the Riemann curvature into a tracefree part and a trace–part. This trace part can be equivalently be described by Ric_{ij} or by P_{ij}, and it again splits into a tracefree part and trace part as $\mathrm{Ric}_{ij} = (\mathrm{Ric}_{ij} - \frac{1}{n}Rg_{ij}) + \frac{1}{n}Rg_{ij}$ and similarly for P_{ij}. By definition, the metric is Einstein if and only if the tracefree part of the Ricci curvature vanishes identically.

Forming a constraction of the second Bianchi identity from Proposition 2.12, one sees that $\nabla_i R = \frac{1}{2}g^{jk}\nabla_j \mathrm{Ric}_{ik}$. In the case of an Einstein metric, the right hand side becomes $\frac{1}{2n}\nabla_i R$, so we conclude that for an Einstein metric, the scalar curvature is constant. This constant value is referred to as the Einstein–constant of the metric, it is mainly of interest whether this is postive, negative or zero ("Ricci–flat metrics").

Example 2.13. Consider the metric on the sphere S^n from example (2) of 2.3, so $R_{ij}{}^k{}_\ell = \delta^k_i g_{j\ell} - \delta^k_j g_{i\ell}$. This gives $\mathrm{Ric}_{j\ell} = (n-1)g_{j\ell}$ and $R = n(n-1)$, which implies that the metric on the sphere is Einstein with positive scalar curvature.

Inserting this into the definitions, we obtain $\mathsf{P}_{ij} = \frac{1}{n-2}((n-1)g_{ij} + \frac{n}{2}g_{ij}) = \frac{1}{2}g_{ij}$. Inserting into the definition shows that the Weyl curvature of the sphere vanishes. So also from our current point of view, these are the simplest possible curvature tensors.

Likewise, hyperbolic space has vanishing Weyl curvature, and is Einstein with negative scalar curvature $R = -n(n-1)$.

The part of the curvature tensor which is most easily to interpret is the Weyl curvature. This is related to the concept of conformal rescaling that we have met in 2.6. There we said that two metrics g and $\hat{g}$ on a manifold M are conformal to each other if and only if there is a positive smooth function f on M such that $\hat{g} = fg$. This defines an equivalence relation on the set of Riemannian metrics on M. It turns out that conformal metrics have the same Weyl curvature, so one says that the Weyl curvature is a *conformal invariant*. It further turns out that the Weyl curvature vanishes identically if and only if the metric is (locally) conformally flat, i.e. if each point in M admits an open neighborhood on which the metric is conformal to a flat metric as characterized in Proposition 2.10.

This gives a simple explanation why the Weyl curvatures of the sphere and of hyperbolic space vanish. For hyperbolic space, we have defined the hyperbolic metric as a conformal rescaling of the flat metric on the ball. Likewise, the computation in 2.3 shows that in the chart defined by stereographic projection, the metric on a sphere is a conformal rescaling of the flat metric on $\mathbb{R}^n$, so again this is evidently conformally flat.

2.14. Curvature and normal coordinates. We complete this part by a short discussion of the relation between normal coordinates and the curvature tensor. This is useful for understanding "how well" normal coordinates are adapted to the Riemannian manifold in a point. On the other hand, it provides explanations for the meanings of the values in a point of several curvature quantities.

Recall from 1.12 that normal coordinates centered at x are obtained by using the inverse of $\exp_x$ as a chart and an orthonormal basis of T_xM to identify this space with $\mathbb{R}^n$. In the resulting local coordinates, the point x corresponds to $0 \in \mathbb{R}^n$ and we consider the local coordinate expression g_{ij} of the metric in these coordinates. Now first of all, since $T_0 \exp_x = \mathrm{id}_{T_xM}$, we see that $g_{ij}(0) = \delta_{ij}$. Second, we know that the radial lines in normal coordinates correspond to geodesics. This means that if

X is a linear combination of the coordinate vector fields ∂_i with constant coefficients, then $\nabla_X X(0) = 0$. This implies that $\Gamma^U(X,X)$ vanishes in the point 0, and since Γ^U is symmetric, polarization implies that Γ^U vanishes in 0. Hence all the Christoffel symbols Γ^k_{ij} vanish at the origin. From the definition in 1.11, we conclude that this implies that $\partial_i \cdot g_{j\ell} + \partial_j \cdot g_{i\ell} - \partial_\ell \cdot g_{ij}$ vanishes in 0 for all indices i, j and ℓ. Adding the same term with i and ℓ exchanged, we see that $2\partial_j \cdot g_{i\ell}$ vanishes in 0, so all partial derivatives of the components g_{ij} vanish at the origin.

This says that the flat metric in normal coordinates approximates g in x to first order, but this is already as good as things can get. We can see this by deriving the coordinate expression for the curvature tensor, which shows that the values of its components in 0 can be computed from the Christoffel symbols and their partial derivatives in 0. Hence they depend only on the partial derivatives of the g_{ij} up to second order, so we cannot have vanishing second order partials in 0 unless the curvature vanishes in x.

Lemma 2.14. *In arbitray local coordinates, the Riemann curvature tensor is in terms of the Christoffel symbols given by*

$$R(\partial_i,\partial_j)(\partial_\ell) = \textstyle\sum_k \left(\partial_i \cdot \Gamma^k_{j\ell} - \partial_j \cdot \Gamma^k_{i\ell} + \sum_a \left(\Gamma^a_{j\ell}\Gamma^k_{ia} - \Gamma^a_{i\ell}\Gamma^k_{ja}\right)\right).$$

Proof. By definition of the Christoffel symbols, $\nabla_{\partial_j}\partial_\ell = \sum_k \Gamma^k_{j\ell}\partial_k$, and hence

$$\nabla_{\partial_i}\nabla_{\partial_j}\partial_\ell = \textstyle\sum_k \left((\partial_i \cdot \Gamma^k_{j\ell})\partial_k - \Gamma^k_{j\ell}\nabla_{\partial_i}\partial_k\right).$$

Expanding the covariant derivative in terms of Christoffel symbols, and using that $[\partial_i,\partial_j] = 0$, the claimed formula then follows from the definition of curvature. □

In the special case of normal coordinates, we see that the components of the curvature tensor are given by $R_{ij}{}^k{}_\ell(0) = \partial_i \cdot \Gamma^k_{j\ell}(0) - \partial_j \cdot \Gamma^k_{i\ell}(0)$. In fact it turns out that the relation between the curvature and the second derivatives of the functions g_{ij} is much simpler than one would expect. To formulate this, we consider the functions $R_{ijk\ell} := g(R(\partial_i,\partial_j)(\partial_k),\partial_\ell)$. The values of these functions in a point are exactly the first non–trivial Taylor coefficients of the functions g_{ij}:

Theorem 2.14. *The Taylor expansion of the components g_{ij} of the metric in normal coordinates $(u^1,\dots,u^n)$ centered in x in the point $u = 0$ is given by*

$$g_{ij}(u) = \delta_{ij} + \tfrac{1}{3}R_{ik\ell j}(x)u^iu^j + O(|u|^3).$$

The proof of this and the following consequences is beyond the scope of this course, we refer to ??. Having derived this Taylor development, one can construct various expansions which lead to the values of various curvature quantities at x as Taylor coefficients. We list these expansions without detailed proofs.

Let us start with sectional curvature as discussed in 2.11. Here we have to specify a two–dimensional subspace $E \subset T_xM$, and the sectional curvature associated to this plane is given by inserting an orthonormal basis $\{\xi,\eta\}$ of E into the formula from Definition 2.11. We denote the resulting value by $K(x)(E)$. To interpret this, we take a small radius $r > 0$ and let $C_r \subset M$ be the image under $\exp_x$ of the circle of Radius r in $E \subset T_xM$. Let $L(r)$ denote the arclength of this smooth closed curve in M. Then it turns out that

$$L(r) = 2\pi r - \tfrac{\pi}{3}K(x)(E)r^3 + O(r^4).$$

In particular, for positive sectional curvature, the circles are shorter than their Euclidean counterparts while for negative sectional curvature they are longer.

Next, the Ricci curvature in x measures the infinitesimal growth of the volume density $\sqrt{\det(g_{ij}(u))}$. More precisely, one has

$$\sqrt{\det(g_{ij}(u))} = 1 - \tfrac{1}{6}\operatorname{Ric}_{ij}(x)u^i u^j + O(|u|^3).$$

So positive definite Ricci curvature (as in the case of the sphere) means that the volume element gets smaller when leaving the origin.

Finally, scalar curvature $R(x)$ can be interpreted in terms of the growth of volumes of geodesic balls and spheres. Let us denote by ω_n the volume of the unit ball in $\mathbb{R}^n$. Then for sufficiently small r, we let $B_r(x)$ denote the image under $\exp_x$ of the ball of radius r in T_xM, while by $S_r(x)$ we denote the geodesic sphere of radius r. Then it turns out that the volume of $B_r(x)$ and the area of $S_r(x)$ grow as

$$\begin{aligned} \operatorname{Vol}(B_r(x)) &= \omega_n r^n\left(1 - \tfrac{1}{6(n+2)}R(x)r^2 + O(r^3)\right) \\ \operatorname{Vol}(S_r(x)) &= n\omega_n r^{n-1} - \tfrac{1}{6}R(x)\omega_n r^{n+1} + O(r^{n+2}). \end{aligned}$$