Holonomy reductions of Cartan geometries and applications

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- This talk starts with a rough outline on the general theory of holonomy reductions of Cartan geometries developed in joint work with R. Gover and M. Hammerl.
- The crucial feature of such reductions is that they come with a decomposition of the underlying manifold into "curved orbits" of different dimension that inherit geometric structures of different types.
- This provides a path towards new applications of Cartan geometries to the study of various kinds of geometric compactifications.
- In particular, I will sketch applications to compactifications of homogeneous spaces and to the study of conformally compact manifolds and various analogs of this concept.

Contents

Let $(p : \mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P). Since the Cartan connection ω has no horizontal curves, there is no holonomy associated to it in a naive sense. But one can naturally extend ω to a principal connection on a bigger bundle.

Consider the principal G-bundle $\tilde{\mathcal{G}} := \mathcal{G} \times_P G$. This comes with a canonical inclusion $i : \mathcal{G} \to \tilde{\mathcal{G}}$ and there is a unique principal connection $\tilde{\omega}$ on $\tilde{\mathcal{G}}$ such that $i^*\tilde{\omega} = \omega$.

The basic idea is to define the holonomy of ω as the holonomy of $\tilde{\omega}$. This came up in the early 2000's for the canonical Cartan connections associated to conformal and projective structures under the name "conformal (projective) holonomy". There were some early results, including classification results on possible holonomy groups (S. Armstrong), but some basic geometric features remained unnoticed for a longer time.

An immediate consequence of the approach is that bundles that are associated to $\tilde{\mathcal{G}}$ and thus to actions of G ("tractor bundles") play an important role for holonomy. Any action (representation) of G can be restricted to P, and then $\mathcal{G} \times_P S \cong \tilde{\mathcal{G}} \times_G S$. By construction any bundle associated to $\tilde{\mathcal{G}}$ inherits a canonical connection and parallel sections define holonomy reductions.

Example (conformal standard tractors)

Put $G := SO_0(p+1, q+1)$ and P the stabilizer of a null-line in the standard representation $\mathbb{V}$ of G. Normal Cartan geometries of type (G, P) are equivalent to conformal structures of signature (p, q) and the bundle induced by $\mathbb{V}$ is the *conformal standard tractor bundle* $\mathcal{T}$. This inherits a bundle metric h of signature $(p+1, q+1)$ as well as a line subbundle $\mathcal{T}^1 \subset \mathcal{T}$ which is null for h. It turns out that $\mathcal{T}^1 \cong \mathcal{E}[-1]$, $\mathcal{T}/(\mathcal{T}^1)^\perp \cong \mathcal{E}[1]$ and $(\mathcal{T}^1)^\perp/\mathcal{T}^1 \cong T^*M \otimes \mathcal{E}[1]$.

Thus any section $s \in \Gamma(\mathcal{T})$ projects onto a section $\sigma := \Pi(s) \in \Gamma(\mathcal{E}[1])$, and $\nabla s = 0$ can be characterized by σ solving a linear PDE (BGG machinery). In particular, this implies that $U := \{x \in M : \sigma(x) \neq 0\}$ is a dense open subset of M. On U, one uses $1/\sigma^2$ to define a rescaling to an Einstein metric whose scalar curvature is determined by $h(s,s)$. A holonomy reduction thus gives an Einstein metric on a dense open subset of M.

Most of the early results on conformal holonomy had a similar flavor. But it turns out that the zero locus of σ is very interesting:

Theorem (Gover, '09)

Under appropriate assumptions on (p,q) and the sign of $h(s,s)$, $\Sigma := \{x : \sigma(x) = 0\}$ is a smooth embedded submanifold of M. Locally around Σ one obtains a Poincaré-Einstein metric, so Σ inherits a conformal structure. All P-E metrics arise like that.

The case of the homogeneous model

Here $G \times_P G \cong G/P \times G$ via $(g, \tilde{g}) \mapsto (gP, g\tilde{g})$ and $\tilde{\omega}$ is the flat connection determined by this trivialization. Hence a holonomy reduction here just is the choice of a subgroup $H \subset G$. But this raises a crucial issue of "*relative position*" between H and P (which usually both are considered up to conjugacy only).

For conformal standard tractors, s is defined by an element $v_0 \in \mathbb{V}$ and $h(s, s) = \langle v_0, v_0 \rangle$ (assumed to be $\neq 0$), while P is the stabilizer of a null line $\ell \subset \mathbb{V}$. The two possible relative positions are $v_0 \in \ell^\perp$ and $v_0 \notin \ell^\perp$. Correspondingly, the stabilizer H of v_0 in G acts with two orbits on the space G/P of isotropic lines in $\mathbb{V}$.

This relative position issue is a basic ingredient for the general theory of holonomy reductions in [C.–Gover–Hammerl]

Let $\mathcal{O}$ be a homogeneous space of G. Then one defines a *holonomy reduction of type* $\mathcal{O}$ of a geometry $(p : \mathcal{G} \to M, \omega)$ as a parallel section of $\mathcal{G} \times_P \mathcal{O} \cong \tilde{\mathcal{G}} \times_G \mathcal{O}$. Such a section corresponds to a P-equivariant smooth function $f : \mathcal{G} \to \mathcal{O}$.

Let $\mathcal{O} = \sqcup \mathcal{O}_i$ be the decomposition of $\mathcal{O}$ into P-orbits. Then for each $x \in M$, $f(\mathcal{G}_x) \subset \mathcal{O}$ is one of these orbits and if this is $\mathcal{O}_i$, we say that $i \in \mathcal{O}/P$ is the *P-type of x*. Correspondingly, we get a decomposition $M = \sqcup M_i$ according to P-types.

For G/P, one verifies that there is a unique holonomy reduction of type $\mathcal{O}$ for each $\alpha \in \mathcal{O}$, corresponding to the P-equivariant function $G \to \mathcal{O}$ given by $f(g) = g^{-1} \cdot \alpha$. Putting $H := G_\alpha$ (so we can identify $\mathcal{O}$ with G/H) we conclude that the decomposition of G/P according to P-types coincides with the decomposition into H-orbits. Thus P-types are indexed by $H \backslash G/P$ in general.

It is a general result that H-orbits in G/P are initial submanifolds, and of course they are homogeneous spaces of H. If G/P is compact, then one can close one H-orbit to get a compactification of a homogeneous space of H which adds homogeneous spaces as a "boundary". This is particularly interesting for open H-orbits, since these inherit a flat geometry of type (G, P) which can be used to describe the compactification. There are many examples:

Theorem (J. Wolf, 1976)

Let G be a real simple Lie group and θ an involutive automorphism of G with fixed point group $H \subset G$. Then for each parabolic subgroup $P \subset G$, H acts on G/P with finitely many orbits.

This leads to many examples of compactifications of symmetric spaces. For some of these, tractors and BGG sequences have been used to describe orbit closures as zero sets of smooth functions, to prove slice theorems, and so on [C-G-H].

The fundamental result for the curved case is proved via "normal coordinates" for Cartan geometries, which define local diffeomorphisms to the homogeneous model. For $i \in H\backslash G/P$, let us realize the corresponding H-orbit in G/P as H/L_i, so $L_i = H \cap (gPg^{-1})$ for appropriate $g \in G$. Then one proves:

Let $x \in M$ be a point of P-type i and fix a holonomy reduction of G/P for which eP has P-type i. Then

1. There are open neighborhoods U of x and V of eP and a diffeomorphism $\varphi : U \to V$ that is compatible with the decomposition into P-types. In particular, $M_i \subset M$ is an initial submanifold.
2. The initial submanifold M_i inherits a Cartan geometry of type (H, L_i) from the holonomy reduction.

This motivates the terminology "curved orbit decomposition". The curved orbits cannot look worse than the true H-orbits in G/P.

Return to $G = SO(p+1, q+1)$, P the stabilizer of a null line in $\mathbb{V}$ and holonomy reductions determined by $\{v_0 \in \mathbb{V} : \langle v_0, v_0 \rangle = 1\}$, so $H \cong SO(p, q+1)$. Here G/P decomposes into two H-orbits, one open (two copies of hyperbolic space) and one closed (the common boundary sphere). The corresponding subgroups $L_i \subset H$ are isomorphic to $SO(p, q)$ and the stabilizer of a null line, respectively.

Thus curved orbit decompositions have the form $M = M_1 \sqcup M_0$ with M_1 open in M and M_0 (if non-empty) an embedded hypersurface. The induced Cartan geometries define a Riemannian metric on M_1 and a conformal structure on M_0. Normality of ω implies that on M_1, one obtains the torsion-free Cartan geometry corresponding to an Einstein metric, while on M_0 one obtains the canonical Cartan geometry associated to the conformal structure.

All these ideas easily generalize to the case of a manifold with boundary, which we write as $\overline{M} = M \sqcup \partial M$. One proves that Poincaré-Einstein manifolds are equivalent to conformal manifolds $\overline{M}$ endowed with a holonomy reduction of this type for which the curved obit decomposition is $\overline{M} = M \sqcup \partial M$. The setup implies that the projection $\sigma = \Pi(s)$ is a defining density for ∂M.

There is an invariant differential operator $S : \Gamma(\mathcal{E}[1]) \to \Gamma(\mathcal{T})$ such that $\Pi \circ S = \mathrm{id}$ ("BGG splitting operator"). This has the property that $S(\sigma)$ is parallel if and only if σ satisfies the PDE that characterizes rescalings to Einstein metrics.

Dropping the Einstein requirement, one obtains an equivalent description of conformally compact metrics and their conformal infinities via a conformal structure on $\overline{M}$ and a defining density σ for ∂M, and $S(\sigma)$ can be used for a tractor description of both structures. This allows for natural subclasses.

There are other holonomy reductions with a similar geometric flavor. Fixing a hyperplane in $\mathbb{R}^{n+1}$ defines a reduction on $S^n = SL(n+1,\mathbb{R})/P$, for which the orbits are two open hemispheres and their common boundary S^{n-1}. They inherit a flat connection respectively a projective structure. The reduction of S^n induced by an indefinite inner product on $\mathbb{R}^{n+1}$ also leads to open orbits that inherit space form metrics of different signature and hypersurface orbits that inherit a conformal structure.

Going through curved analogs and weakening as described above in the case $\overline{M} = M \sqcup \partial M$, these two reductions turn out to be special cases for $\alpha = 1$ and $\alpha = 2$ of a concept of *projective compactness of order* α which is defined for $\alpha \in (0, 2]$. These are defined for connections on M, but via the Levi-Civita connection, they automatically apply to pseudo-Riemannian metrics.

Projective modifications of connections are parametrized by one-forms, and the definition of projective compactness is that specific modifications obtained from defining functions via $dr/\alpha r$ admit a smooth extension to the boundary. The order α turns out to be equivalent to a specific rate of volume growth towards the boundary.

Results include equivalent characterizations of projective compactness in terms of asymptotic forms as well as several surprising rigidity results. For example assume that g is an Einstein metric on M whose projective structure extends smoothly to the boundary. Then g is projectively compact of order 1 if $Ric(g) = 0$ and of order 2 otherwise.

The case $\alpha = 2$ admits an almost complex analog ("c-projective compactness" which involves (quasi-)Kähler metrics in the interior and (almost) CR structures on the boundary.

Thank you for your attention!