Spinors and Dirac Operators

lecture notes

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CHAPTER 1

Motivation – low dimensions

There are two basic motivations for the theory of spinors. The original motivation, coming from physics, is the question of finding a "square root" of the Laplacian. Since this was first successfully done by P.A.M. Dirac, operators of this type are now called Dirac operators. The second basic motivation comes from representation theory of the special orthogonal groups and of their Lie algebras. The two motivations are connected by the concept of a Clifford algebra and of its representations. In dimensions three and four, both motivations can be nicely treated in an elementary way using quaternions. We will also briefly discuss dimensions five and six, at least on the level of special orthogonal groups.

Dimension three

We start with a quick discussion of the quaternions as introduced by Hamilton in the 1850s and their relation to linear algebra and geometry in dimension three.

1.1. The quaternions. Recall that the field of complex numbers can be realized within the algebra $M_2(\mathbb{R})$ of real 2×2-matrices as the space of matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Aiming for a complex analog of this, we consider $M_2(\mathbb{C})$ and in there the set $\mathbb{H}$ of all matrices of the form $\begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}$. Observe that the determinant of such a matrix equals $|z|^2 + |w|^2$, so non-zero matrices in $\mathbb{H}$ are always invertible. By definition, these matrices form a real subspace of $M_2(\mathbb{C})$ of real dimension 4, which is not a complex subspace, however. A simple direct computation shows that for matrices $A, B \in \mathbb{H}$ also the matrix product AB lies in $\mathbb{H}$, so we conclude that $\mathbb{H}$ is an associative real algebra of real dimension 4.

Next, it is easily verified that for $A \in \mathbb{H}$, also the adjoint (conjugate transpose) A^* lies in $\mathbb{H}$ and that $AA^* = A^*A = \det(A)\mathbb{I}$. Since we have observed that any non-zero matrix in A has non-zero determinant, we conclude that for $A \neq 0$, we have $A^{-1} = \frac{1}{\det(A)}A^* \in \mathbb{H}$. Since the unit matrix $\mathbb{I}$ lies in $\mathbb{H}$, we see that $\mathbb{H}$ has all the properties of a field, except for commutativity of the multiplication, so it is a *skew field*. To get closer to the classical picture of the quaternions, we complete the unit element $1 := \mathbb{I}$ of $\mathbb{H}$ to a basis by defining

$$i := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad j := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad k := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

One immediately verifies that these matrices satisfy the relations $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $ik = -ki = -j$, and $jk = -kj = i$. Then an element of $\mathbb{H}$ can be uniquely written as $a + bi + cj + dk$ for real numbers a, b, c, d and the product of two such expressions can be computed from the relations among the basis elements and bilinearity.

Now we will change the point of view and correspondingly our notation to emphasize the analogy to complex numbers. We will view $\mathbb{H}$ as an abstract algebra, view $\mathbb{R}$ as the subalgebra formed by real multiples of the unit element, and denote elements by lower case letters. We write the operation corresponding to $A \mapsto A^*$ as $q \mapsto \overline{q}$ and call it *(quaternionic) conjugation.* This evidently implies that $\overline{pq} = \bar{q}\bar{p}$ for all $p, q \in \mathbb{H}$. Likewise, we denote $A \mapsto \sqrt{\det(A)}$ as $q \mapsto |q| \in \mathbb{R}$, which implies $|pq| = |p||q|$ for all p, q. In terms of this operation, we get $q\overline{q} = \overline{q}q = |q|^2$ and thus $q^{-1} = \frac{1}{|q|^2}\overline{q}$ for $q \neq 0$.

Next, one calls $q \in \mathbb{H}$ *real* if $\overline{q} = q$ and *purely imaginary* if $\overline{q} = -q$. By definition, 1 is real, while i, j, and k are purely imaginary. Thus the real quaternions are exactly the elements of $\mathbb{R} \subset \mathbb{H}$, while the purely imaginary ones are exactly the real linear combinations of i, j, and k. We denote the three-dimensional space of purely imaginary quaternions by $\mathrm{im}(\mathbb{H})$. For $q \in \mathbb{H}$, we have $\mathrm{Re}(q) = \frac{1}{2}(q+\overline{q}) \in \mathbb{R}$ and $\mathrm{im}(q)) = \frac{1}{2}(q-\overline{q}) \in \mathrm{im}(\mathbb{H})$ and $q = \mathrm{Re}(q) + \mathrm{im}(q)$. This is the decomposition of q into its *real part* and its *imaginary part.*

By construction, $q \mapsto |q|^2 = q\overline{q}$ defines a positive definite real quadratic form on $\mathbb{H}$, and linear algebra tells us that this can be polarized to an inner product on $\mathbb{H}$. Explicitly, one obtains $\langle p, q\rangle = \frac{1}{2}(p\overline{q} + q\overline{p}) = \mathrm{Re}(p\overline{q})$, which again corresponds to a familiar fact for complex numbers. From this definition, one readily sees that $\mathrm{im}(\mathbb{H})$ is the orthocomplement of $\mathbb{R}$ in $\mathbb{H}$ and that the basis $\{1, i, j, k\}$ is orthonormal. Now we can restrict the inner product to $\mathrm{im}(\mathbb{H})$, thus obtaining a three-dimensional Euclidean vector space, and by linear algebra there is only one such space up to isomorphism. Using this we can deduce a basic relation that will be crucial in what follows and clarify the relation to the operations on $\mathbb{R}^3$ which are familiar from linear algebra.

PROPOSITION 1.1. *Take $a, b \in \mathbb{R}$ and $X, Y \in \mathbb{R}^3 = \mathrm{im}(\mathbb{H})$.*

(1) The product $(a + X)(b + Y)$ in $\mathbb{H}$ has real part $ab - \langle X, Y\rangle$ and imaginary part $bX + aY + X \times Y$ for the usual cross product in $\mathbb{R}^3$.

(2) For $p, q \in \mathrm{im}(\mathbb{H})$ we get $pq + qp = -2\langle p, q\rangle$, and in particular $q^2 = -|q|^2$. Hence the elements of any orthonormal basis of $\mathrm{im}(\mathbb{H})$ satisfy analogous commutation relations to i, j, k, and we get a natural orientation on $\mathrm{im}(\mathbb{H})$.

PROOF. (1) By definition of the unit element, we get $a(b + Y) = ab + aY$, and similarly for $(a + X)b$, so by bilinearity, it suffices to compute the product XY. Taking the standard basis $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$, we get $\langle e_\ell, e_m\rangle = \delta_{\ell m}$, while the cross product is skew symmetric and satisfies $e_\ell \times e_m = e_n$ whenever (ℓ, m, n) is a cyclic permutation of $(1, 2, 3)$. These are exactly the same relations as for the real and imaginary parts of the products of two of the elements i, j and k. Thus the claimed equation holds whenever we insert two elements of our chosen bases and (1) follows by bilinearity of all involved products.

(2) By definition, we have $\overline{p} = -p$ and $\overline{q} = -q$. Thus we can compute $pq + qp$ as $-p\overline{q} - q\overline{p} = -2\,\mathrm{Re}(p\overline{q}) = -2\langle p, q\rangle$. Hence for an orthonormal basis $\{p_1, p_2, p_3\}$ of $\mathrm{im}(\mathbb{H})$ we get $(p_\ell)^2 = -|p_\ell| = -1$ and $p_\ell p_m = -p_m p_\ell$ for $\ell \neq m$. From part (1) we know that p_1p_2 is perpendicular to both p_1 and p_2 and $|p_1p_2| = 1$, so we must have $p_1p_2 = \pm p_3$. Now positive orientation corresponds to $p_1p_2 = p_3$ and in this case we get the same commutation relations as for the ordered basis formed by i, j and k. □

1.2. The Dirac operator on $\mathbb{R}^3$. We can now discuss the original motivation for considering spinors, at least in the case of $\mathbb{R}^3$. We won't discuss the physics background involved, but just take a mathematical formulation of the problem, which is easy to understand. Consider the Laplace operator on smooth functions on $\mathbb{R}^3$, defined by

$\Delta(f) := -\sum_{\ell=1}^{3} \frac{\partial^2 f}{\partial {x_\ell}^2}$. (The sign is chosen in order to get a positive operator.) The original question raised by P.A.M. Dirac was whether there is a "square root" of the Laplacian, i.e. a first order differential operator D such that $D \circ D = \Delta$. As we shall see soon, such an operator does not exist on real valued functions, but it is no problem to extend Δ to a functions with values in $\mathbb{R}^m$ or $\mathbb{C}^m$ for any m and ask the same question there. In that form, we can produce solutions rather easily. To simplify notation, we will write ∂_ℓ for $\frac{\partial}{\partial x_\ell}$ in what follows.

PROPOSITION 1.2. *Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and assume that for some $m \geq 2$ we find matrices $A, B \in M_m(\mathbb{K})$, which satisfy $A^2 = B^2 = -\mathbb{I}$ and $AB = -BA$.*

Then the differential operator D on $C^\infty(\mathbb{R}^3, \mathbb{K}^m)$ defined on by

$$Df := A\partial_1 f + B\partial_2 f + AB\partial_3 f$$

has the property that $D \circ D = \Delta$.

PROOF. This is a simple direct computation. For a matrix C, the partial derivatives of Cf clearly are given by $\partial_\ell Cf = C\partial_\ell f$. Thus we can simply compute $D(Df)$ in terms of the second partial derivatives of f. Here the coefficients of $\partial_\ell\partial_\ell f$ simply are given by A^2, B^2 and $ABAB = -A^2B^2$ for $\ell = 1, 2, 3$, so these three summands just produce $\Delta(f)$. Since the iterated partial derivatives commute, there are just three further summands corresponding to $\partial_1\partial_2 f$, $\partial_1\partial_3 f$ and $\partial_2\partial_3 f$. The coefficients of these are $AB + BA = 0$, $AAB + ABA = AAB - AAB = 0$ and $BAB + ABB = -ABB + ABB = 0$, so the result follows. □

We see from the computation in the proof, that anti-commuting objects are needed to obtain a square root of the Laplacian. In particular, we see from the proof that things cannot work out for $m = 1$. Spinors and Dirac operators are one of the basic origins of "super structures", for example Lie superalgebras, in mathematics.

There is an immediate interpretation of Proposition 1.2 in terms of quaternions. If we have found matrices $A, B \in M_m(\mathbb{K})$ as in the Proposition, then we can consider the unique linear map $\mathbb{H} \to M_m(\mathbb{K})$, which sends 1 to the unit matrix $\mathbb{I}$, i to A, j to B and k to AB. Then one immediately checks that this map is a homomorphisms of algebras, thus defining a *representation* of the associative algebra $\mathbb{H}$ on $\mathbb{K}^m$. Indeed, all the multiplicative relations between i, j and k follow from $i^2 = j^2 = -1$ and $ij = -ji = k$. For example $ik = iij = -j$ and $ki = iji = -iij = j$, and so on. Hence any representation of $\mathbb{H}$ on $\mathbb{K}^m$ defines a square root of the Laplacian on smooth functions with values in $\mathbb{K}^m$. In fact, there is a kind of converse to this result that we shall discuss in detail later on.

The most basic example of such an operator comes from the representation of $\mathbb{H}$ on $\mathbb{C}^2$ that we have used to define $\mathbb{H}$. The resulting operator D on $C^\infty(\mathbb{R}^3, \mathbb{C}^2)$ is called the *Dirac operator* of $\mathbb{R}^3$. Writing $f : \mathbb{R}^3 \to \mathbb{C}^2$ as $f = \binom{f_1}{f_2}$ for complex valued functions f_1 and f_2 the operator is explicitly given by

$$\begin{aligned} D\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} \partial_1 f_1 \\ \partial_1 f_2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \partial_2 f_1 \\ \partial_2 f_2 \end{pmatrix} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} \partial_3 f_1 \\ \partial_3 f_2 \end{pmatrix} \\ &= \begin{pmatrix} i\partial_1 f_1 + \partial_2 f_2 + i\partial_3 f_2 \\ -i\partial_1 f_2 - \partial_2 f_1 + i\partial_3 f_1 \end{pmatrix}. \end{aligned}$$

Observe that, while the Laplacian simply acts component-wise by definition, the Dirac operator mixes components in an intricate way. Also the appearance of $\mathbb{C}^2$ is very surprising in our context, and it is not at all clear initially how this is related to the basic geometry of $\mathbb{R}^3$ (i.e. the flat Riemannian metric defined by the standard inner product) which determines the Laplace operator. Understanding these issues will be a main goal of this course.

1.3. The Clifford algebra of $\mathbb{R}^3$. It is rather evident that the result of Proposition 1.2 is only a special case. Suppose that one has three matrices $A, B, C \in M_m(\mathbb{K})$ such that $A^2 = B^2 = C^2 = -\mathbb{I}$ and which anti-commute pairwise, i.e. satisfy $AB = -BA$, $AC = -CA$ and $BC = -CB$. Then the proof of Proposition 1.2 shows the the operator D on $C^\infty(\mathbb{R}^3, \mathbb{K}^m)$ defined by $Df := A\partial_1 f + B\partial_2 f + C\partial_3 f$ again satisfies $D \circ D = \Delta$. Now the necessary property can be neatly formulated in terms of the Euclidean vector space $(\mathbb{R}^3, \langle\ ,\ \rangle)$: Mapping the elements of the standard basis to A, B, and C, we obtain a linear map $\varphi : \mathbb{R}^3 \to M_m(\mathbb{K})$. The required commutation relations simply mean that for $v, w \in \{e_1, e_2, e_3\}$, we get

$$\varphi(v)\varphi(w) + \varphi(w)\varphi(v) = -2\langle v, w\rangle\mathbb{I}. \tag{1.1}$$

Since φ is a linear map and matrix multiplication and the inner product are bilinear, we readily see that both sides of this equation are bilinear in v and w. This immediately implies that this equations holds for all elements of a basis of $\mathbb{R}^3$ if and only if it holds for all $v, w \in \mathbb{R}^3$. Equation (1.1) is usually phrased as the fact that "φ satisfies the *Clifford relations*".

So to define an analog of the Dirac operator on $\mathbb{K}^m$-valued functions, we need a linear map $\varphi : \mathbb{R}^3 \to M_m(\mathbb{K})$ which satisfies the Clifford relations. Now observe that the Clifford relations make sense in any associative algebra $\mathcal{A}$ which has a unit element. Hence we can ask for linear maps to such algebras which satisfy the Clifford relations. Surprisingly, on that level, this problem has a universal solution, which in the case of $\mathbb{R}^3$ can be easily described explicitly. Namely, we consider the space $\mathbb{H} \oplus \mathbb{H}$, which is an associative algebra under the component-wise operations and has $(1, 1)$ as its unit element. (This is not a skew-field, however, since only elements for which both components are non-zero are invertible.) Now we define $\varphi : \mathbb{R}^3 \to \mathbb{H} \oplus \mathbb{H}$ by $\varphi(e_1) = (i, -i)$, $\varphi(e_2) = (j, -j)$ and $\varphi(e_3) = (k, -k)$.

Proposition 1.3. *(1) The map φ satisfies the Clifford relations.*

(2) If $\mathcal{A}$ is a unital associative algebra and $\psi : \mathbb{R}^3 \to \mathcal{A}$ satisfies the Clifford relations, then there is a unique homomorphism $\tilde{\psi} : \mathbb{H} \oplus \mathbb{H} \to \mathcal{A}$ of unital associative algebras such that $\psi = \tilde{\psi} \circ \varphi$.

Proof. (1) As we have noted above, it suffices to verify that the images of the elements of the standard basis square to -1 and the images of different elements anti-commute. This can be verified by simple direct computations. For example $(i, -i) \cdot (i, -i) = (-1, -1)$, while $(i, -i) \cdot (j, -j) = (k, k)$ and $(j, -j) \cdot (i, -i) = (-k, -k)$ by the standard quaternion relations.

(2) Completing the computations from (1), we see that

$$\varphi(e_1)\varphi(e_2)\varphi(e_3) = (k, k) \cdot (k, -k) = (-1, 1).$$

Hence we conclude that the elements 1, $\varphi(e_\ell)$ for $\ell = 1, 2, 3$, $\varphi(e_\ell)\varphi(e_m)$ for $1 \leq \ell < m \leq 3$ and $\varphi(e_1)\varphi(e_2)\varphi(e_3)$ form a basis for the 8-dimensional space $\mathbb{H} \oplus \mathbb{H}$.

Now suppose that we have given a unital associative algebra $\mathcal{A}$ and a linear map $\psi : \mathbb{R}^3 \to \mathcal{A}$, which satisfies the Clifford relations. Then we define $\tilde{\psi} : \mathbb{H} \times \mathbb{H} \to \mathcal{A}$ on the above basis in an obvious way, which is forced if we want it to be a homomorphism: We send 1 to the unit of $\mathcal{A}$, $\varphi(e_\ell)$ to $\psi(e_\ell)$, $\varphi(e_\ell) \cdot \varphi(e_m)$ to $\psi(e_\ell) \cdot \psi(e_m)$ (product in $\mathcal{A}$) and likewise for the triple product. Of course, this defines a linear map $\tilde{\psi}$ as required and by construction $\psi = \tilde{\psi} \circ \varphi$. Thus it remains to verify that $\tilde{\psi}$ is an algebra homomorphism and by bilinearity it is sufficient to check that it is compatible with the products of basis elements.

This can be sorted out directly by observing that all such products are completely determined by the Clifford relations. For example, we have $\tilde{\psi}(\varphi(e_\ell)) = \psi(e_\ell)$ by definition and likewise for e_m. Then compatibility of $\tilde{\psi}$ with the product $\varphi(e_\ell) \cdot \varphi(e_m)$ holds by definition if $\ell < m$, by the fact that the corresponding elements square to -1 in both algebras for $\ell = m$ and by the fact that the anti-commute in both algebras and by definition for $\ell > m$. Similarly, one can deal with products of the form $\varphi(e_\ell) \cdot (\varphi(e_m) \cdot \varphi(e_n))$ for $m < n$. If $\ell = m$, then the Clifford relations shows that this equals $-\varphi(e_n)$ and if $\ell = n$, we obtain $\varphi(e_m)$. Finally if (ℓ, m, n) is a permutation of $(1, 2, 3)$ then the Clifford relations show that the product equals the sign of that permutation times $\varphi(e_1) \cdot \varphi(e_2) \cdot \varphi(e_3)$. Since the same relations hold for the corresponding elements in $\mathcal{A}$, we again get compatibility with the products, and so on. $\square$

Similarly to the discussion in Section 1.2, this shows that to get a square root of the Laplacian on $\mathbb{K}^m$-valued functions on $\mathbb{R}^3$ as discussed above, one needs a homomorphism $\mathbb{H} \oplus \mathbb{H} \to M_m(\mathbb{K})$ of unital associative algebras. The algebra $\mathbb{H} \oplus \mathbb{H}$ is called the *Clifford Algebra* of $(\mathbb{R}^3, \langle\ ,\ \rangle)$, and a homomorphism as above defines a representation of this algebra on $\mathbb{K}^m$.

1.4. Quaternions and $SO(3)$. We have already noted in Section 1.1 that on the three dimensional space $\mathrm{im}(\mathbb{H})$ we have a natural inner product and an orientation. This shows that the quaternions are related to inner product geometry in dimension 3. We next show that this relation also has a nice interpretation in terms of group theory.

Consider the set $Sp(1) := \{q \in \mathbb{H} : |q| = 1\}$ of *unit quaternions*, which evidently closed under quaternionic multiplication and contains the unit element 1. For a unit quaternion q, we get $q^{-1} = \overline{q} \in Sp(1)$, so we conclude that $Sp(1)$ is a group under quaternion multiplication. By definition, $Sp(1)$ is the unit sphere in $\mathbb{H} \cong \mathbb{R}^4$, so we can naturally view it as the submanifold $S^3 \subset \mathbb{R}^4$. Since the multiplication of quaternions is bilinear over $\mathbb{R}$, its restriction defines a smooth map $S^3 \times S^3 \to S^3$, and hence $Sp(1)$ is a Lie group. Alternatively, in the presentation as matrices we started from, it is readily verified that $Sp(1) \cong SU(2)$, the group of unitary 2×2-matrices of determinant 1. We can also readily read off the Lie algebra $\mathfrak{sp}(1)$ of this group. Since the tangent space to S^3 in a point is just the hyperplane orthogonal to that point, we see that $\mathfrak{sp}(1) = \mathrm{im}(\mathbb{H})$.

Since quaternionic multiplication is $\mathbb{R}$-bilinear, we can determine the adjoint action of $Sp(1)$ on its Lie algebra as in the case of matrix groups, so this is simply given by $Ad(q)(p) = qpq^{-1} = qp\bar{q}$. Observe that $\overline{p} = -p$ immediately implies that $\overline{qp\overline{q}} = -qp\bar{q}$, so we do not leave the space of purely imaginary quaternions. Differentiating this (using bilinearity once more), we conclude that the Lie bracket on $\mathfrak{sp}(1)$ is given by $[p, q] = pq - qp$. As before, one checks immediately that $p, q \in \mathrm{im}(\mathbb{H})$ implies $pq - qp \in \mathrm{im}(\mathbb{H})$. Using this, we can now describe the relation of $Sp(1)$ to the special orthogonal group of $\mathrm{im}(\mathbb{H})$, which is isomorphic to $SO(3)$.

Theorem 1.4. *For any unit quaternion $q \in Sp(1)$ the endomorphism $\mathrm{Ad}(q)$ of $\mathrm{im}(\mathbb{H})$ defined by $p \mapsto qp\overline{q}$ is orthogonal. Mapping $q \to \mathrm{Ad}(q)$ defines a surjective smooth homomorphism $Sp(1) \to SO(\mathrm{im}(\mathbb{H}))$ of Lie groups, whose kernel equals $\{\pm 1\}$.*

Hence $SO(3)$ is isomorphic to $Sp(1)/\mathbb{Z}_2 = SU(2)/\mathbb{Z}_2$ as a Lie group and to the projective space $\mathbb{R}P^3$ as a manifold.

Proof. Since $|q| = 1$ we also get $|\overline{q}| = 1$ and hence $|qp\overline{q}| = |p|$ for each $p \in \mathrm{im}(\mathbb{H})$. This already implies that the map $\mathrm{Ad}(q)$ is orthogonal for each $q \in Sp(1)$. For $q_1, q_2 \in$

$Sp(1)$, we get $\overline{q_1q_2} = \bar{q}_2\bar{q}_1$, which immediately shows that $\mathrm{Ad}(q_1q_2) = \mathrm{Ad}(q_1) \circ \mathrm{Ad}(q_2)$ so we obtain a homomorphism from $Sp(1)$ to the orthogonal group $O(\mathrm{im}(\mathbb{H}))$. As a manifold, $Sp(1) \cong S^3$ and thus is connected, which implies that the image has to be contained in the connected component $SO(\mathrm{im}(\mathbb{H}))$ of the identity.

The derivative of $\mathrm{Ad} : Sp(1) \to GL(\mathfrak{sp}(1))$ is $\mathrm{ad} : \mathfrak{sp}(1) \to \mathfrak{gl}(\mathfrak{sp}(1))$, so this maps $p_1 \in \mathrm{im}(\mathbb{H})$ to $p_2 \mapsto [p_1, p_2] = p_1p_2 - p_2p_1$. Putting $p = ai + bj + ck$ one immediately computes directly that the matrix representation of $\mathrm{ad}(p)$ with respect to the orthonormal basis $\{i, j, k\}$ of $\mathrm{im}(\mathbb{H})$ is given by $\begin{pmatrix} 0 & 2c & 2b \\ -2c & 0 & -2a \\ -2b & 2a & 0 \end{pmatrix}$. This shows that ad defines a linear isomorphism from $\mathfrak{sp}(1)$ onto the space $\mathfrak{o}(\mathrm{im}(\mathbb{H}))$ of skew symmetric endomorphisms. It is a basic result of Lie theory, that this implies that Ad is surjective, a local diffeomorphism, and induces an isomorphism between $Sp(1)/\ker(\mathrm{Ad})$ and $SO(3)$.

The kernel of Ad consists of those $q \in Sp(1)$, such that $qp = pq$ for all $p \in \mathrm{im}(\mathbb{H})$. Expanding q as a linear combination of 1, i, j and k one sees immediately that q commutes with i if and only if has trivial j-component and k-component. Commuting with i and j thus implies that q has to be real, so we see that $\ker(\mathrm{Ad}) = Sp(1) \cap \mathbb{R} = \{\pm 1\}$. As a manifold $Sp(1)/\{\pm 1\}$ is obtained by taking S^3 and identifying each point $x \in S^3$ with $-x \in S^3$, which implies the last statement of the theorem. □

Given a homomorphism $SO(3) \to H$ for some Lie group H, we can always compose with the quotient homomorphism $Sp(1) \to SO(3)$ to obtain a homomorphism $Sp(1) \to H$. This composition of course maps $-\mathbb{I}$ to the neutral element e. Conversely, given a homomorphism $Sp(1) \to H$ with this property, it factorizes to $Sp(1)/\{\pm\mathbb{I}\} \cong SO(3)$. In particular, this applies to representations, so representations of $SO(3)$ can be identified with those representations of $Sp(1)$, in which $-\mathbb{I}$ acts trivially, so this is a proper subclass. Indeed, using that $Sp(1) \cong SU(2)$ is a compact real form of $SL(2, \mathbb{C})$, it is easy to see that the irreducible complex finite dimensional representations of $SU(2)$ are exactly the symmetry powers $S^\ell\mathbb{C}^2$ of the standard representation $\mathbb{C}^2$ for $\ell \geq 0$. Evidently, $-\mathbb{I}$ acts trivially in $S^\ell\mathbb{C}^2$ if and only if ℓ is even, so loosely speaking, there are only half as many complex representations of $SO(3)$ as of $Sp(1)$. As we shall see in detail later on, representations of these groups give rise to geometric objects on Riemannian manifolds respectively on Riemannian spin manifolds in dimension 3. The simplest example of an "additional" representation of $Sp(1)$ is $\mathbb{C}^2$ itself, and this gives rise to the geometric objects on which the basic Dirac operator acts.

Finally recall that the sphere S^3 (and indeed each sphere S^n for $n \geq 2$) is a simply connected topological space. Hence $Sp(1)$ is a simply connected Lie group, so Lie theory tells us that there is a perfect correspondence between homomorphisms from $\mathfrak{sp}(1)$ to the Lie algebra $\mathfrak{g}$ of any Lie group G and Lie group homomorphisms $Sp(1) \to G$. The above discussion shows that this fails to be true for $SO(3)$.

Dimensions 4 to 6

All the phenomena we have discussed in dimension 3 above, have analogs in all higher dimensions. In particular, there always is a quotient homomorphism from a simply connected Lie group $Spin(n)$ onto $SO(n)$ whose kernel has two elements. This is the *spin group*, which is the *universal covering group* of $SO(n)$. This group then has representations which do not correspond to representations of $SO(n)$. A general construction of these groups requires a general version of Clifford algebras, which are again related to questions of defining square roots of Laplace operators and so on. However, up to dimension 6, there are direct descriptions of the spin groups as other classical Lie groups, and we briefly discuss those next.

1.5. Dimension 4. The universal covering group $Spin(4)$ of $SO(4)$ can also be described in terms of quaternions. It turns out that in dimension 4 (and only in that dimension) this universal covering group decomposes as a product of two subgroups. This gives special features to Riemannian geometry in dimension 4 which also play an important role in theoretical physics.

PROPOSITION 1.5. *For $q_1, q_2 \in Sp(1)$, the linear map $\varphi_{q_1,q_2} : \mathbb{H} \to \mathbb{H}$ defined by $\varphi_{q_1,q_2}(p) := q_1 p \overline{q_2}$ is orthogonal. Mapping (q_1, q_2) to φ_{q_1,q_2} defines a surjective homomorphism $Sp(1) \times Sp(1) \to SO(4)$ of Lie groups whose kernel consists of the two elements $(1, 1)$ and $(-1, -1)$. Hence $SO(4)$ is isomorphic to $(SU(2) \times SU(2))/\mathbb{Z}_2$.*

PROOF. Since $|q_1| = |q_2| = 1$, we see that $|q_1 p \overline{q_2}| = |p|$, which already shows that φ_{q_1,q_2} is orthogonal. The definition readily implies that $(q_1, q_2) \mapsto \varphi_{q_1,q_2}$ is a group homomorphism and smoothness clearly follows from bilinearity of the multiplication of quaternions. Since $Sp(1) \times Sp(1)$ is connected, we see that the image of this homomorphism must be contained in $SO(4)$. The Lie algebra of $Sp(1) \times Sp(1)$ is $\mathfrak{sp}(1) \times \mathfrak{sp}(1)$ and the derivative of φ sends a pair (p_1, p_2) of purely imaginary quaternions to the map φ'_{p_1,p_2} defined by $q \mapsto p_1 q - q p_2$. If $\varphi'_{p_1,p_2} = 0$, then evaluating on $q = 1$ shows that $p_1 = p_2$. But restricting $\varphi'_{p,p}$ to $\operatorname{im}(\mathbb{H})$, we get the map $\operatorname{ad}(p)$ from Theorem 1.4, which vanishes for $p = 0$ only.

So we see that the derivative φ' is injective, and since $\dim(\mathfrak{so}(4)) = 6$ it has to be bijective. Lie theory then implies that φ maps onto $SO(4)$ and induces an isomorphism $(Sp(1) \times Sp(1))/\ker(\varphi) \to SO(4)$. The kernel of φ clearly contains $(1, 1)$ and $(-1, -1)$. Conversely, if $q_1 p \overline{q_2} = p$ for all $p \in \mathbb{H}$, we can first put $p = 1$ to conclude $q_1 = q_2$. But then $\varphi_{q,q}|_{\operatorname{im}(\mathbb{H})} = \operatorname{Ad}(q)$ and this is the identity if an only if $q = \pm 1$. □

Similarly as for dimension three, this result implies that homomorphisms from $SO(4)$ to any Lie group G are in bijective correspondence with homomorphisms $Sp(1) \times Sp(1) \to G$ for which $(-1, -1)$ lies in the kernel. In particular, we can apply this to representations, showing that again, roughly speaking, $SO(4)$ has half as many representations as $Sp(1) \times Sp(1)$. In particular, there are two basic "new" representations, namely that standard representations of the two factors $Sp(1) \cong SU(2)$ on $\mathbb{C}^2$. These are commonly denoted by S_+ and S_- and called the *half-spin representations*. Observe that these two representations carry Hermitian inner products which are invariant under the action of $Sp(1) \cong SU(2)$. Now the natural action of (g_1, g_2) on $L_{\mathbb{C}}(S_-, S_+)$ is given by $((g_1, g_2) \cdot f)(v) = g_1 \cdot f(g_2^{-1} \cdot v)$. Hence we see that the basic representation $\mathbb{R}^4 = \mathbb{H}$ of $SO(4)$ can be viewed as a subspace of $L_{\mathbb{C}}(S_-, S_+)$ of real dimension 4 which is invariant under the natural action of $Sp(1) \times Sp(1)$.

The fact that the universal covering group of $SO(4)$ decomposes into a products of subgroups comes as a big surprise, nothing like that is available in any other dimension. Representation theory tells us that representations of $Sp(1) \times Sp(1)$ can be built up from tensor products of representations of the individual factors. So a typical representation has the form $V \otimes W$ with $(g_1, g_2) \cdot (v \otimes w) = (g_1 \cdot v) \otimes (g_2 \cdot w)$. Restricting to complex representations we are left with the cases $V = S^\ell \mathbb{C}^2$ and $W = S^m \mathbb{C}^2$ and we see that $(-1, -1)$ acts trivially in $V \otimes W$ if and only if $\ell + m$ is even, thus confirming the above claim that half of the representations descend to $SO(4)$.

We can also describe the 4-dimensional analogs of the other things we did in dimension 3 directly. To define a square root of the Laplacian on $C^\infty(\mathbb{R}^4, \mathbb{K}^m)$, we now need 4 matrices $A_\ell \in M_m(\mathbb{K})$ which all square to $-\mathbb{I}$ and pairwise anti-commute. Given these we can define $Df := \sum_{\ell=1}^4 A_\ell \partial_\ell f$ as before and compute directly that

$D^2f = -\sum_{\ell=1}^4(\partial_\ell)^2 f = \Delta(f)$. The simplest example this time is the basic Dirac operator defined on $C^\infty(\mathbb{R}^4,\mathbb{C}^4)$, we $\mathbb{C}^4$ should be viewed as $S_+ \oplus S_-$. Thus we write $f : \mathbb{R}^4 \to \mathbb{C}^4$ as $f = \binom{f_1}{f_2}$, where now the components are $\mathbb{C}^2$-valued and define $D := \begin{pmatrix} 0 & D_+ \\ D_- & 0 \end{pmatrix}$, where $D_\pm$ is the operator on $C^\infty(\mathbb{R}^4,\mathbb{C}^2)$ defined by

$$D_\pm(f) := \pm\partial_0 f + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\partial_1 f + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\partial_2 f + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\partial_3 f.$$

Using the commutation relations between the basic matrices one easily verifies directly that $D_+ \circ D_- = D_- \circ D_+ = \Delta$ on $C^\infty(\mathbb{R}^4,\mathbb{C}^2)$, which easily implies that $D^2 = \Delta$ as claimed. Of course, we could collect all that together to give an explicit formula for D, but this becomes a bit tedious.

We can recast these computations in a slightly different way, which leads directly to the description of the Clifford algebra of $\mathbb{R}^4$. In order to do this, we first observe that there is no problem in considering the space $M_n(\mathbb{H})$ of $n\times n$-matrices, whose entries are quaternions, and multiply such matrices in the usual way (taking care that quaternions do not commute). Indeed, viewing $\mathbb{H}^n$ as a *right* module over $\mathbb{H}$, the usual arguments from linear algebra show that $M_n(\mathbb{H})$ can be identified with the space of maps $\mathbb{H}^n \to \mathbb{H}^n$ which are linear over the quaternions. Matrix multiplication then again corresponds to composition of linear maps, thus making $M_n(\mathbb{H})$ into an associative algebra with unit element $\mathbb{I}$.

In particular, we can consider the algebra $M_2(\mathbb{H})$, which has real dimension 16. For any $q \in \mathbb{H}$, we can consider the matrix $A_q := \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$, and verify directly that $(A_q)^2 = -|q|^2\mathbb{I}$. Simple direct computations then show that the matrices A_1, A_i, A_j and A_k are pair-wise anti-commutative. This shows that the map $\mathbb{H} \to M_2(\mathbb{H})$ defined by $q \mapsto A_q$ satisfies the 4-dimensional version of the Clifford relations, namely

$$A_{q_1}A_{q_2} + A_{q_2}A_{q_1} = -2\langle q_1, q_2\rangle\mathbb{I}. \tag{1.2}$$

It is also easy to verify directly that the unit matrix $\mathbb{I}$ together with the ordered products of different elements from $\{A_1, A_i, A_j, A_k\}$ define a basis of the vector space $M_2(\mathbb{H})$. Using this, one proves the 4-dimensional analog of Proposition 1.3: Given any unital associative algebra $\mathcal{A}$ and a linear map $\varphi : \mathbb{H} \to \mathcal{A}$ which satisfies the Clifford relations (1.2), there is a unique homomorphism $\tilde{\varphi} : M_2(\mathbb{H}) \to \mathcal{A}$ such that $\varphi(q) = \tilde{\varphi}(A_q)$ for all $q \in \mathbb{H}$.

1.6. Dimensions 5 and 6. For completeness, we briefly explain how two-fold covering groups for $SO(5)$ and $SO(6)$ can be constructed from classical groups. One can also describe basic Dirac operators and Clifford algebras (which happen to be isomorphic to $M_4(\mathbb{C})$ in dimension 5 and to $M_8(\mathbb{R})$ in dimension 6) directly, but we do not go into that.

Here the basic idea applies directly to dimension 6 and with some additional input to dimension 5. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, consider the vector space $\mathbb{K}^4$ and its second exterior power $\Lambda^2\mathbb{K}^4$ which has dimension $\binom{4}{2} = 6$. The wedge product defines a bilinear map $\Lambda^2\mathbb{K}^4 \times \Lambda^2\mathbb{K}^4 \to \Lambda^4\mathbb{K}^4$, which is symmetric since the wedge product of 2-forms is commutative. Now the natural representation of $SL(4,\mathbb{K})$ on $\mathbb{K}^4$ gives rise to representations on $\Lambda^2\mathbb{K}^4$ and $\Lambda^4\mathbb{K}^4$. The latter space has dimension 1, so the second representation is trivial. Fixing a non-zero element of $\Lambda^4\mathbb{K}^4$ (which essentially means fixing a volume form on $\mathbb{K}^4$) we can view the wedge product as a symmetric bilinear form on $\Lambda^2\mathbb{K}^4 \cong \mathbb{K}^6$. By definition of the action, we get $g\cdot(v_1\wedge v_2) = (g\cdot v_1)\wedge(g\cdot v_2)$, which easily implies that the action of $SL(4,\mathbb{K})$ is orthogonal for this bilinear form. Hence, we get a homomorphism from $SL(4,\mathbb{K})$ to the orthogonal group of a $\mathbb{K}$-bilinear form in dimension 6.

To get to dimension 5, one fixes a symplectic inner product on $\mathbb{K}^4$, i.e. a skew symmetric bilinear map $\omega : \mathbb{K}^4 \times \mathbb{K}^4 \to \mathbb{K}$, which is non-degenerate. By linear algebra, there is a unique such form on each vector space of even dimension up to isomorphism. Next, one defines $Sp(4,\mathbb{K})$ to be the subgroup of $SL(4,\mathbb{K})$ consisting of all g such that $\omega(g\cdot v, g\cdot w) = \omega(v,w)$ for all $v,w \in \mathbb{K}^4$. This clearly is a closed subgroup and thus a Lie subgroup of $SL(4,\mathbb{K})$ which turns out to be connected. The corresponding Lie algebra $\mathfrak{sp}(4,\mathbb{K})$ consists of those $X \in \mathfrak{sl}(4,\mathbb{K})$, for which $\omega(Xv,w)+\omega(v,Xw) = 0$. This easily implies that the dimension of $Sp(4,\mathbb{K})$ is 10. Now we can view ω as defining a linear map $\Lambda^2\mathbb{K}^4 \to \mathbb{K}$ via $v_1 \wedge v_2 \mapsto \omega(v_1, v_2)$, whose kernel is a subspace $\Lambda^2_0\mathbb{K}^4 \subset \Lambda^2\mathbb{K}^4$ of dimension 5. By construction, this subspace is invariant under the action of $Sp(4,\mathbb{K})$, so one obtains a homomorphism from $Sp(4,\mathbb{K})$ to the orthogonal group of a symmetric bilinear form on a 5-dimensional space.

One can carry out this construction directly for $SL(4,\mathbb{R})$ and $Sp(4,\mathbb{R})$, but the result is not quite what we need. It is easy to verify directly that the wedge product is non-degenerate both on $\Lambda^2\mathbb{R}^4$ and on $\Lambda^2_0\mathbb{R}^4$, but the signatures are $(3,3)$ and $(2,3)$, respectively. So we obtain homomorphisms $SL(4,\mathbb{R}) \to O(3,3)$ and $Sp(4,\mathbb{R}) \to O(2,3)$ whose images have to be contained in the connected component of the identity. In both cases, one verifies directly that the derivatives of the homomorphism is injective and since both groups have dimensions 15, respectively 10, it has to be a linear isomorphism. This implies that the homomorphisms are surjective onto the connected components of the identity and in both cases, one verifies that the kernel consists of $\pm\,\mathrm{id}$.

To get the analogous results for definite signature, one has to work with real subgroups of $SL(4,\mathbb{C})$ and $Sp(4,\mathbb{C})$, which leave appropriate real subspaces in $\Lambda^2\mathbb{C}^4 \cong \mathbb{C}^6$ invariant. For dimension 6, we consider $SU(4) \subset SL(4,\mathbb{C})$. With a bit of work, one shows that $\Lambda^2\mathbb{C}^4$ decomposes into the direct sum of two real subspaces of real dimension 6, which are invariant under the action of $SU(4)$ and on which the wedge product is positive definite respectively negative definite. Restricting to the first subspace gives rise to a homomorphism $SU(4) \to O(6)$ and similarly as described above, one proves that this is a surjection onto $SO(6)$ with kernel $\{\pm\,\mathrm{id}\}$ (observing that both groups have real dimension 15).

The story in dimension 5 is again related to quaternions. The inclusion $\mathbb{H} \hookrightarrow M_2(\mathbb{C})$ we started from in Section 1.1 defines an inclusion $M_2(\mathbb{H}) \hookrightarrow M_4(\mathbb{C})$ (just think of a 4×4-matrix written as a collection of 4 blocks of size 2×2). Now for $A \in M_2(\mathbb{H})$, we define A^* as the conjugate (in the quaternionic sense) transpose of A. It is easy to see the this operation still satisfies $(AB)^* = B^*A^*$ and thus $Sp(2) := \{A \in M_2(\mathbb{H}) : A^*A = \mathbb{I}\}$ can be viewed as a closed subgroup of $GL(4,\mathbb{C})$ and thus is a Lie group. One then proves that $Sp(2)$ has real dimension 10 and is contained in the subgroup $Sp(4,\mathbb{C})$, so it has a natural action on $\Lambda^2_0\mathbb{C}^4 \cong \mathbb{C}^5$. Then one shows that $\Lambda^2_0\mathbb{C}^4$ splits as a direct sum of two real subspaces of dimension 5, which are invariant under the action of $Sp(2)$ and on which the wedge product is positive definite and negative definite, respectively. Restricting to the first subspace defines a homomorphism $Sp(2) \to O(5)$ and similarly as before, one verifies that this maps onto $SO(5)$ and has kernel $\{\pm\mathbb{I}\}$.

CHAPTER 2

The geometric perspective

In this chapter, we explain why Riemannian manifolds are the appropriate general setting for analog of the Laplace operators on $\mathbb{R}^3$ and $\mathbb{R}^4$ that we considered in Section 1.2 and 1.5. We will phrase this in terms of vector bundles associated to the orthonormal frame bundle. This approach has the advantage that it directly indicates how a two–fold covering of the special orthogonal group leads to additional geometric objects and to analogs of the Dirac operators from Sections 1.2 and 1.5.

2.1. Riemannian manifolds. Let us return to the Laplacian on $\mathbb{R}^n$, defined by $\Delta f := -\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$. As indicated in Section 1.2, this not only makes sense on real valued functions but also on functions with values in $\mathbb{R}^m$ and we will deal with smooth functions only. Trying to generalize this to smooth manifolds, let us first check which ingredients are actually needed on $\mathbb{R}^n$. The most natural description there is that $\Delta(f)$ is minus the trace of the second derivative $D^2 f$. Note however, that for each $x \in \mathbb{R}^n$, $D^2 f(x)$ is a symmetric bilinear form on $\mathbb{R}^n$ and not a linear map $\mathbb{R}^n \to \mathbb{R}^n$. Forming a trace of a bilinear form b on $\mathbb{R}^n$ is only possible using the inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$.

There are several equivalent definitions of $\operatorname{tr}(b) \in \mathbb{R}$. Either one observes that there is a unique linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $b(v, w) = \langle v, A(w)\rangle$ and defines $\operatorname{tr}(b) := \operatorname{tr}(A)$. Equivalently, one may observe that for an orthonormal basis $\{v_1, \dots, v_n\}$ of $\mathbb{R}^n$, the value $\sum_{i=1}^n b(v_i, v_i)$ is independent of the basis and coincides with $\operatorname{tr}(b)$. The latter formula also shows that $\operatorname{tr}(b)$ depends only on the symmetric part of b and that $\Delta(f)(x) = -\operatorname{tr}(D^2 f(x))$. In either formulation, we can proceed in the same way if f is only defined on an open subset $U \subset \mathbb{R}^n$, since then still $D^2 f(x)$ is a symmetric bilinear form on $\mathbb{R}^n$ for each $x \in U$.

Trying to generalize this to manifolds, the last observation shows that the natural replacement for the vector space $\mathbb{R}^n$ is provided by the tangent spaces of M. Hence instead of the inner product on $\mathbb{R}^n$, we should use a family of inner products on the tangent spaces on M. Requiring that these inner products depend smoothly on the base point in an obvious sense, one arrives at the notion of a Riemannian metric:

Definition 2.1. Let M be a smooth manifold.

(1) A *pseudo-Riemannian metric* on M is a function g, which assigns to each point $x \in M$ a non-degenerate, symmetric bilinear form $g_x : T_xM \times T_xM \to \mathbb{R}$ such that for any vector fields $\xi, \eta \in \mathfrak{X}(M)$, the map $g(\xi, \eta) : M \to \mathbb{R}$ defined by $x \mapsto g_x(\xi(x), \eta(x))$ is smooth.

(2) We say that g is a *Riemannian metric* if for each $x \in M$, the bilinear form g_x is positive definite.

(3) A (pseudo-)Riemannian manifold (M, g) is a smooth manifold M together with a (pseudo-)Riemannian metric g on M.

In the usual language, this just means that g is a smooth $\binom{0}{2}$-tensor field on M, for which all values are symmetric and non-degenerate or positive definite, respectively. Given a pseudo-Riemannian metric g on M, then for any smooth $\binom{0}{2}$-tensor field b on

M, one defines $\operatorname{tr}(b)$ in each point as above and then concludes that $\operatorname{tr}(b) : M \to \mathbb{R}$ is a smooth function.

This is not quite enough to define a Laplacian on functions on a pseudo-Riemannian manifold (M, g). It is no problem to form the derivative of a smooth function $f : M \to \mathbb{R}$ on a smooth manifold M which to each $x \in M$ assigns a linear map $T_xM \to \mathbb{R}$. This can either be viewed as the tangent map $Tf : TM \to T\mathbb{R} = \mathbb{R} \times \mathbb{R}$ for which the first component is f, or as the exterior derivative $df \in \Omega^1(M)$, which equivalently encodes the second component of Tf. To form a second derivative, one would have to pass to $TTf : TTM \to TT\mathbb{R}$, which then encodes f and its first two derivatives. The double tangent bundles occurring in this formulation are not easy to handle, and in particular it is unclear how to form something like a trace in this situation. On functions (and also on differential forms) one can bypass this problem and use constructions from linear algebra together with the exterior derivative to define a Laplacian, compare with Section 1.6 of [**Riem**].

However, Riemannian geometry offers a nice solution to this problem, which can be used for arbitrary tensor fields. The right concept here is the one of a natural vector bundle $E \to M$, which we will discuss in more detail below. For each point $x \in M$, such a bundle has a fiber E_x which is a finite dimensional vector space and a section s is a map which associates to each $x \in M$ an element $s(x) \in E_x$ which depends smoothly on x. Now there is the concept of the *covariant derivative* which can be applied to any such section. For a section s of E, the covariant derivative ∇s is a section of the bundle $T^*M \otimes E$, whose fiber at a point x is $(T_xM)^* \otimes E_x = L(T_xM, E_x)$. This turns out to be a natural vector bundle again, so one can apply the covariant derivative again and form $\nabla^2 s = \nabla(\nabla s)$, which is a section of the bundle $T^*M \otimes T^*M \otimes E$. The fiber of the latter bundle at $x \in M$ is the space of bilinear maps $T_xM \times T_xM \to E_x$. The above discussion can be easily modified to associate to such a bilinear map a trace (with respect to g_x) which is an element of E_x. The upshot of this is that one can define a section Δs of E via $\Delta s(x) = -\operatorname{tr}(\nabla^2 s(x)) \in E_x$ and thus a Laplace operator on sections of E.

Fiber bundles and vector bundles

2.2. Bundles. To start making these considerations precise, we first introduce the basic concepts related to bundles. Basically, a fiber bundle over M is a manifold that locally looks like a product of M with a fixed manifold S (the standard fiber of the bundle). An important class of such bundles are vector bundles, for which the standard fiber is a vector space, and the isomorphism to a product can be chosen in a way compatible with the vector space structure. The standard example of a vector bundle over M is the tangent bundle TM.

DEFINITION 2.2. Let M, E and S be smooth manifolds and let $p : E \to M$ be a smooth map.

(1) A *fiber bundle chart* (U, ψ) for $p : E \to M$ with standard fiber S is an open subset $U \subset M$ together with a diffeomorphism $\psi : p^{-1}(U) \to U \times S$ such that $\mathrm{pr}_1 \circ \psi = p$.

(2) For two fiber bundle charts (U_1, ψ_1) and (U_2, ψ_2) such that $U_{12} := U_1 \cap U_2 \neq \emptyset$, the *transition function* $\psi_{12} : U_{12} \times S \to S$ is the smooth map characterized by the fact that for $x \in U_{12}$ and $y \in S$, one has $\psi_1(\psi_2^{-1}(x, y)) = (x, \psi_{12}(x, y))$.

(3) A *fiber bundle atlas* for $p : E \to M$ is a collection $\{(U_\alpha, \psi_\alpha) : \alpha \in I\}$ of fiber bundle charts such that the sets U_α form an open covering of M. If such an atlas exists, then $p : E \to M$ is called a *fiber bundle* with *total space* E, *base* M, *standard fiber* S and *bundle projection* p. For $x \in M$, the *fiber* of E over x is $E_x := p^{-1}(x) \subset E$.

(4) A *smooth section* of a fiber bundle $p : E \to M$ is a smooth map $s : M \to E$ such that $p \circ s = \mathrm{id}_M$. Otherwise put, s associates to each point $x \in M$ an element $s(x) \in E_x$. A *local smooth section* of $p : E \to M$ is a smooth map $s : U \to E$ defined on an open subset $U \subset M$ such that $p \circ s = \mathrm{id}_U$.

Let us now specialize to the case that as a standard fiber we take a finite dimensional vector space V over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then in the context of the following notions, we use the name *vector bundle chart* rather than fiber bundle chart.

(5) Two vector bundle charts (U_1, ψ_1) and (U_2, ψ_2) are called *compatible* if either $U_{12} = U_1 \cap U_2$ is empty or the transition function $\psi_{12} : U_{12} \times V \to V$ is linear in the second variable.

(6) A *vector bundle atlas* is a fiber bundle atlas consisting of vector bundle charts which are mutually compatible. Two vector bundle atlases are called *equivalent* if their charts are mutually compatible. A $\mathbb{K}$-vector bundle $p : E \to M$ is then a fiber bundle endowed with an equivalence class of vector bundle atlases.

From these definitions, it follows readily that the bundle projection $p : E \to M$ of any fiber bundle is a surjective submersion (since the projection onto one factor in a product has this property). This in turn implies that the fiber E_x over each point $x \in M$ is a smooth submanifold of E which is diffeomorphic to the standard fiber S. If (U, ψ) is a fiber bundle chart for $p : E \to M$ and $f : U \to S$ is any smooth map, then $x \mapsto \psi^{-1}(x, f(x))$ defines a local smooth section of E. However, in general a fiber bundle does not admit global smooth sections. It is common to call bundle charts *local trivializations* and talk about *locally trivial bundles*. A bundle is called trivial, if it is globally isomorphic to a product.

Let us also point out here that a vector bundle is not just a fiber bundle, whose standard fiber is a finite dimensional vector space V. In a fiber bundle $p : E \to M$ with standard fiber V, we see from above that each fiber E_x is diffeomorphic to V, but trying to carry over the vector space structure from V to E_x, one obtains different results for different fiber bundle charts. In the case of a vector bundle, the situation is different. Taking $x \in M$ and two elements $y, z \in E_x$ and $\lambda \in \mathbb{K}$, we can choose a vector bundle chart (U, ψ) with $x \in U$. Then $\psi(y) = (x, v)$ and $\psi(z) = (x, w)$ for some elements $v, w \in V$ and we define $y + \lambda z := \psi^{-1}(x, v + \lambda w)$. A short computation shows that the definition is chosen in such a way that any vector bundle chart compatible to (U, ψ) will lead to the same element, so $y + \lambda z$ is unambiguously defined in a vector bundle. Thus any fiber E_x is canonically a $\mathbb{K}$-vector space, which is linearly isomorphic to V.

This also has important consequences for sections. If $p : E \to M$ is a $\mathbb{K}$-vector bundle, then for sections $s_1, s_2 : M \to E$, one puts $(s_1 + s_2)(x) := s_1(x) + s_2(x)$ and immediately observes that this defines a smooth section of E. Similarly, one can multiply sections by elements of $\mathbb{K}$, but these may even depend on the point. Explicitly, for a section s of E and a smooth function $f : M \to \mathbb{K}$, $fs(x) := f(x)s(x)$ defines a smooth section of E. This shows that the space $\Gamma(E)$ of all smooth sections of E is a $\mathbb{K}$-vector space and a module over the commutative ring $C^\infty(M, \mathbb{K})$, and implies that any vector bundle admits many global smooth sections. Indeed, given a vector bundle chart (U, ψ), we have seen above that local sections of E defined on U are in bijective correspondence with smooth functions $U \to V$. Given $x \in U$, we can find an open neighborhood W of x in M such that $\overline{W} \subset U$ and a bump function $f : M \to [0, 1]$ whose support is contained in U and which is identically one on W. Given a local section s of E defined on U, we can form fs, which again is a smooth section defined on U. But since the support of f is contained in U, we can extend it by zero outside of U, to define a global section fs of E, which agrees with s on W. This shows that the space $\Gamma(E)$ is always infinite dimensional. In fact, the usual technique of gluing local smooth objects to global smooth objects using partitions of unity applies without problems to sections of any vector bundle.

Example 2.2. As mentioned above, the standard example of a vector bundle is the tangent bundle $p : TM \to M$. The standard constructions of the tangent bundle also illustrate that one can use a vector bundle atlas to make the total space into a smooth manifold rather than starting from a manifold structure on the total space. This is a line of argument that we will use frequently. To review how TM is constructed, recall that one first defines, for each point $x \in M$, the tangent space T_xM at x. Then one defines TM to be the (disjoint) union of these tangent spaces and endows it with the obvious projection $p : TM \to M$, which sends T_xM to x. Now consider a chart (U, u) for M, so $U \subset M$ is open and u is a diffeomorphism from U onto an open subset $u(U) \subset \mathbb{R}^n$, where $n = \dim(M)$. Then the tangent map Tu is a bijection between $TU = p^{-1}(U) \subset TM$ and $u(U) \times \mathbb{R}^n$.

Now starting from a countable atlas $\{(U_i, u_i) : i \in \mathbb{N}\}$ for M, one defines a topology on TM by declaring a subset $W \subset TM$ to be open, if and only if $Tu_i(W \cap p^{-1}(U_i))$ is open in $u_i(U_i) \times \mathbb{R}^n$ for each $i \in \mathbb{N}$. It is easy to verify that this defines a topology on TM which is Hausdorff and second countable, that each of the sets $p^{-1}(U_i)$ is open in TM and that each Tu_i is a homeomorphism. Now by definition, $U_{ij} = U_i \cap U_j$ is open, and if it is non-empty, the chart changes $u_{ij} : u_i(U_{ij}) \to u_j(U_{ij})$, which are characterized by $u_i(y) = u_{ij}(u_j(y))$ for all $y \in U_{ij}$, are all smooth. Moreover, if for $x \in U_{ij}$ and $\xi \in T_xM$, we denote the second component of $Tu_j(\xi)$ by $v \in \mathbb{R}^n$, $Tu_i(\xi) = (u_{ij}(u_j(x)), Du_{ij}(u_j(x))(v))$. Since both u_{ij} and Du_{ij} are smooth maps, we can use the maps Tu_i as charts, thus making TM into a smooth manifold such that $p : TM \to M$ is smooth. It is easy to see that this manifold structure is independent of the atlas we have chosen initially.

But now we can just use the second components of these charts to define diffeomorphisms $p^{-1}(U_i) \to U_i \times \mathbb{R}^n$, which are given by $\xi \mapsto (p(\xi), T_{p(\xi)}u_i \cdot \xi)$. The transition functions between two such charts are clearly given by $(x, v) \mapsto (x, Du_{ij}(u_j(x))(v))$, so they are linear in the second variable. Thus we have found a family of compatible vector bundle charts on TM and it is easy to see that any other atlas for M gives rise to an equivalent vector bundle atlas for TM. So we see that we have made TM into a vector bundle in a canonical way.

2.3. Local frames. Recall that a chart (U, u) on M gives rise to a family of local vector fields $\partial_i := \frac{\partial}{\partial u^i}$ defined on U. These are characterized by the fact that $T_xu \cdot \partial_i(x) = (x, e_i)$ for each $x \in U$, where e_i denotes the ith vector in the standard basis of $\mathbb{R}^n$. This implies that the fields $\partial_i(x)$ form a basis for T_xM for any $x \in U$, which depends smoothly on x. Consequently, given a vector field $\xi \in \mathfrak{X}(M)$, there are smooth functions $\xi^i : U \to \mathbb{R}$ for $i = 1, \dots, n$ such that $\xi|_U = \sum_{i=1}^n \xi^i \partial_i$. This is a special case of a local frame for a vector bundle:

Definition 2.3. (1) Let $p : E \to M$ be a smooth $\mathbb{K}$-vector bundle with standard fiber V and put $m := \dim(V)$. Then a *local frame* for E defined on an open subset $U \subset M$ is a family $\{\sigma_1, \dots, \sigma_m\}$ of local sections of E defined on U such that for each $x \in U$ the values $\sigma_1(x), \dots, \sigma_m(x)$ form a $\mathbb{K}$-basis for the vector space E_x. A *global frame* for E is a local frame defined on all of M.

(2) In the case of the tangent bundle TM, local frames obtained from charts for M as discussed above are called *holonomic*, other local frames are called *non-holonomic*.

To formulate the next result, recall that the group $GL(m,\mathbb{K})$ of invertible $m\times m$-matrices forms an open subset of the space $M_m(\mathbb{K})$ of all $m\times m$-matrices which can be identified with $\mathbb{K}^{m^2}$. Hence $GL(m,\mathbb{K})$ is a smooth manifold and a map A from a smooth manifold M to $GL(m,\mathbb{R})$ is smooth if and only if for $A(x)=(a^i_j(x))$ (upper index numbers rows, lower index columns) each of the a^i_j is a smooth map $M\to\mathbb{K}$. This readily implies that matrix multiplication is smooth as a map $GL(m,\mathbb{K})\times GL(m,\mathbb{K})\to GL(m,\mathbb{K})$, while matrix inversion is smooth as a map $GL(m,\mathbb{K})\to GL(m,\mathbb{K})$. (Indeed, matrix multiplication is polynomial in the entries of the matrix by definition, while matrix inversion is a rational function by Cramer's rule.) This says that $GL(m,\mathbb{K})$ is a Lie group and a fundamental result of Lie theory says that any closed subgroup of $GL(m,\mathbb{K})$ automatically is a smooth submanifold and thus itself a Lie group, see Theorem 1.11 of [**LieG**].

THEOREM 2.3. *Let $p:E\to M$ be a $\mathbb{K}$-vector bundle with standard fiber V and $\dim(V)=m$, fix a $\mathbb{K}$-basis $\{v_1,\dots,v_m\}$ for V, let $U\subset M$ be an open subset.*

(1) If $\varphi:p^{-1}(U)\to U\times V$ is a vector bundle chart for E defined on U, then for $i=1,\dots,m$ putting $\sigma_i(x)=\varphi^{-1}(x,v_i)$ one obtains a smooth local frame $\{\sigma_i\}$ for E defined on U.

(2) Conversely, given a smooth local frame $\{\sigma_i\}$ for E defined on U, there a unique vector bundle chart φ defined on U, for which (1) leads to the given local frame $\{\sigma_i\}$. Moreover, for any local smooth section s of E defined on U, there are smooth functions $s^1,\dots,s^m:U\to\mathbb{K}$ such that $s=\sum_i s^i\sigma_i$.

(3) Given a local frame $\{\sigma_i\}$ defined on U and a smooth function $A:U\to GL(m,\mathbb{K})$, $A(x)=(a^i_j(x))$, then putting $\tau_j:=\sum_k a^k_j\sigma_k$ defines a local frame $\{\tau_i\}$ on U. Moreover, any local frame for E defined on U is of this form.

PROOF. (1) Since $x\mapsto(x,v_i)$ is a smooth map $U\to U\times V$ and φ is a diffeomorphism, $\sigma_i:U\to E$ is a smooth map. By definition, we have $p(\varphi^{-1}(x,v))=x$ for each $v\in V$ and thus $p\circ\sigma_i=\mathrm{id}_U$, so each σ_i is a smooth section of E defined on U. The definition of the vector space structure on E_x exactly says that mapping $y\in E_x$ to the second component of $\varphi(y)$ is a linear isomorphism. Since this linear isomorphism maps the set $\{\sigma_i(x)\}$ to the basis $\{v_i\}$ of V, it is a basis. Thus the σ_i form a local frame, and the proof of (1) is complete.

Let us next assume that $W\subset M$ is any open subset and that $\{\tau_1,\dots,\tau_m\}$ is a local frame for E defined on W. Then for a local section $s:W\to E$ of E and each $x\in W$, we have $s(x)\in E_x$, so there are unique elements $s^i(x)\in\mathbb{K}$ for $i=1,\dots,n$ such that $s(x)=\sum_i s^i(x)\tau_i(x)$. This defines functions $s^i:W\to\mathbb{K}$ and we claim that these are smooth. This is a local question, so for $x_0\in W$, we can take a vector bundle chart (U,φ) for E with $x_0\in U\subset W$. Let $\{\sigma_i\}$ be the local frame for E obtained from (U,φ) as in (1). Then the second component of $\varphi\circ s|_U$ is a smooth map $U\to V$. Writing this as $\sum_i f^iv_i$ for smooth functions $f:U\to\mathbb{K}$ we get $s|_U=\sum_i f^i\sigma_i$.

Similarly, for each $j=1,\dots,m$, we obtain smooth functions $a^i_j:U\to\mathbb{K}$ such that $\tau_j|_U=\sum_i a^i_j\sigma_i$. Evaluating this in a point $x\in U$ the fact that both $\{\sigma_i(x)\}$ and $\{\tau_j(x)\}$ are bases of E_x implies that the matrix $(a^i_j(x))$ is invertible, so $A(x)=(a^i_j(x))$ defines a smooth function $U\to GL(m,\mathbb{K})$. From above we know that matrix inversion is smooth, so writing $A(x)^{-1}=(b^i_j(x))$ each of the functions $b^i_j:U\to\mathbb{K}$ is smooth, too. But by construction, we get $\sigma_i=\sum_j b^j_i\tau_j$ and hence $s=\sum_{i,j}f^ib^j_i\tau_j$. But this shows that $s^j|_U=\sum_i(f^ib^j_i)$, so this is smooth. This completes the proof of the second statement in (2).

The first statement in (3) is obvious, since each $\tau_j : U \to E$ is a smooth section of E by construction and invertibility of $A(x)$ implies that $\{\tau_j(x)\}$ is a basis of E_x for each $x \in U$. For the second statement, we assume that we have given a second fram τ_j and apply the second part of (2) to obtain a smooth function $A = (a^i_j) : U \to M_n(\mathbb{R})$ such that $\tau_j = \sum_i a^i_j \sigma_i$. As above, we conclude that the matrix $A(x)$ is invertible for each x, so we get a smooth function to $GL(m, \mathbb{K})$.

So it remains to prove the first statement in (2). Assuming that $\{\tau_j\}$ is a smooth local frame defined on an arbitrary open subset $W \subset M$. Then we can define a smooth map $W \times V \to p^{-1}(W)$ by mapping $(x, \sum_i \lambda_i v_i)$ to $\sum_i \lambda_i \tau_i(x) \in E_x$. Since this is evidently bijective, we can take its inverse $\varphi : p^{-1}(W) \to W \times V$, which by construction satisfies $\mathrm{pr}_1 \circ \varphi = p$. To complete the proof, we have to show that φ is smooth and compatible with the vector bundle charts for E. So given $x_0 \in W$, we take a vector bundle chart (U, ψ) with $x_0 \in U \subset W$ and the local frame $\{\sigma_i\}$ determined by the chart (U, ψ). By part (3), we know that there are smooth functions $a^i_j : U \to \mathbb{K}$ such that $\sigma_j = \sum_i a^i_j \tau_i|_U$. But this exactly says that $\varphi(\psi^{-1}(x, v_j)) = (x, \sum_i a^i_j(x) v_i)$ for each $x \in U$, and the second component in the right hand side equals $A(x)^t v_j$. This implies both smoothness of φ on $p^{-1}(U)$ and compatibility with the chart (U, ψ). □

This already indicates that general vector bundle charts for TM are more flexible than those obtained from local charts for M. For example, on the unit circle S^1, there exists a nowhere vanishing vector field, which forms a global frame for TS^1. Consequently, there is a global vector bundle chart $TS^1 \to S^1 \times \mathbb{R}$, but of course the smooth manifold S^1 does not admit a global chart. Similar arguments apply to the tangent bundle of the torus and of any Lie group.

Principal bundles and associated bundles

2.4. Frame bundles and principal bundles. Given a $\mathbb{K}$-vector bundle $p : E \to M$, we next construct a bundle whose local sections are exactly the local frames of E. As we shall soon see, we can not only recover E from that bundle but also get a whole class of related bundles.

For a point $x \in M$, we define $\mathcal{P}_x E$ to be the set of all $\mathbb{K}$-linear isomorphisms $u_x : V \to E_x$, where V is the standard fiber of E. Observe that for a linear automorphism $A \in GL(V)$ and $u_x \in \mathcal{P}_x E$, we also have $u_x \circ A \in \mathcal{P}_x E$. Moreover, if v_x is another element of $\mathcal{P}_x E$, then $A := (u_x)^{-1} \circ v_x \in GL(V)$ and $v_x = u_x \circ A$, so choosing a point in $\mathcal{P}_x E$ gives rise to a bijection $\mathcal{P}_x E \cong GL(V)$. Now define $\mathcal{P}E$ to be the (disjoint) union of the spaces $\mathcal{P}_x E$ and endow this with the obvious projection $\pi : \mathcal{P}E \to M$, which sends $\mathcal{P}_x E$ to x.

Next suppose that (U, φ) is a vector bundle chart for E, and consider the inverse $\varphi^{-1} : U \times V \to p^{-1}(U)$. Restricted to each of the spaces $\{x\} \times V$ for $x \in U$, this defines a linear isomorphism $u_x : V \to E_x$. Consequently, we obtain a bijection $\psi^{-1} : U \times GL(V) \to \pi^{-1}(U)$ by sending (x, A) to $u_x \circ A \in \mathcal{P}_x E$ and this satisfies $\pi \circ \psi^{-1} = \mathrm{pr}_1$. Now we proceed similarly as in the case of the tangent bundle described in Section 2.2. We take a countable atlas (U_i, φ_i) for E (which exists since M is a Lindelöff space) and use the induced bijections $\psi_i : \pi^{-1}(U_i) \to U_i \times GL(V)$ to defined a topology on $\mathcal{P}E$, which is easily seen to be second countable.

Next it is easy to compute the transition functions between the local trivializations of $\mathcal{P}E$ obtained from compatible vector bundle charts for E. Let U be the intersection of the domains of definition of two such charts φ_1 and φ_2 with transition function $\varphi_{12} : U \times V \to V$. By definition, this is linear in the second variable, so we can view it as a smooth map $U \to L(V,V)$ and clearly the values actually lie in $GL(V)$. Writing this function as $f : U \to GL(V)$, we get $\varphi_1(\varphi_2^{-1}(x,v)) = (x, f(x)(v))$, which in the language introduced above means that $(u_x^1)^{-1} \circ u_x^2 = f(x)$ or $u_x^2 = u_x^1 \circ f(x)$. Otherwise put, $\psi_1(\psi_2^{-1}(x,A)) = \psi_1(u_x^1 \circ f(x) \circ A) = (x, f(x) \circ A)$, and the transition functions are given by left translations in the group $GL(V)$, so in particular, they are smooth. Loosely speaking, E and $\mathcal{P}E$ have the "same transition functions".

Thus, we can use the countable vector bundle atlas from above to make $\mathcal{P}E$ into a smooth manifold, and then clearly $\pi : \mathcal{P}E \to M$ is a smooth fiber bundle with standard fiber $GL(V)$. As above, any vector bundle chart for E gives rise to a local trivialization of $\mathcal{P}E$, and for any two local trivializations obtained in this way, the transition functions are given by a left translation by a smooth function with values in $GL(V)$. The bundle $\mathcal{P}E$ is called the *frame bundle of* E. The reason for this is that, as we have seen above, the inverse of any vector bundle chart defines a local smooth section of $\mathcal{P}E$ and conversely, local smooth sections of $\mathcal{P}E$ give rise to vector bundle charts for E. But (after a choice of basis for the standard fiber) such charts are equivalent by Theorem 2.3 to local frames of E. Explicitly, given a local section σ of $\mathcal{P}E$ defined on $U \subset M$, each $\sigma(x)$ is a linear isomorphism $V \to E_x$ and the frame on U corresponding to σ consists of the sections s_i defined by $s_i(x) = \sigma_i(x)(v_i)$, where $\{v_1, \dots, v_n\}$ is the fixed basis of V.

In particular, we can apply this construction to the tangent bundle TM of an n-dimensional manifold M. The resulting bundle is denoted by $\mathcal{P}M$, it has structure group $GL(m, \mathbb{R})$ and is called the (linear) *frame bundle of* M. All these frame bundles are special instances of the following general concept

Definition 2.4. Let G be a Lie group and let $p : P \to M$ be a smooth fiber bundle with standard fiber G.

(1) Two local trivializations (U_1, ψ_1) and (U_2, ψ_2) for P are said to be *compatible principal bundle charts* if either $U_{12} = U_1 \cap U_2$ is empty or the corresponding transition function $\psi_{12} : U_{12} \times G \to G$ has the form $\psi_{12}(x,g) = f(x) \cdot g$ for a smooth function $f : U_{12} \to G$.

(2) A *principal bundle atlas* is a fiber bundle atlas consisting of compatible principal bundle charts. Two such atlases are said to be equivalent if their charts are mutually compatible. A *principal fiber bundle with structure group* G (or, for short a *principal* G*-bundle*) is a fiber bundle $p : P \to M$ with fiber G together with an equivalence class of principal bundles atlases.

In contrast to the case of vector bundles, the fibers of a principal G-bundle $p : P \to M$ do *not* inherit a Lie group structure, although they are diffeomorphic to G. The reason for this simply is that the transition functions are left translations in G and these are not group homomorphisms. The best way to think about the fibers is as the Lie group analogs of affine spaces, which are identified with G after selecting a base point. (The usual technical term is that the fibers are principal homogeneous spaces of G.) The example one should keep in mind is the set of all bases for a vector space V, which can be identified with $GL(V)$ after selecting one basis. Still there are interesting structures on a principal bundle $p : P \to M$ derived from the group structure on G, namely the *principal right action* and the *fundamental vector fields.*

Proposition 2.4. *Let $p : P \to M$ be a principal fiber bundle with structure group G, and let $\mathfrak{g}$ be the Lie algebra of G.*

(1) There is a natural smooth right action $r : P \times G \to G$ written as $r(u,g) := u \cdot g$, which is characterized by the fact that for each principal bundle chart (U, ψ) and each

$x \in U$, we have $\psi^{-1}(x,h)\cdot g = \psi^{-1}(x,hg)$. The orbits of this action are exactly the fibers of the bundle.

(2) For any $X \in \mathfrak{g}$, there is a smooth vector field $\zeta_X \in \mathfrak{X}(P)$ such that for each $u \in P$, we get $\zeta_X(u) = \frac{d}{dt}|_{t=0} u\cdot \exp(tX)$, where $\exp : \mathfrak{g} \to G$ is the exponential map of G. For each $u \in P$, the map $X \mapsto \zeta_X(u)$ defines a linear isomorphism from $\mathfrak{g}$ onto the subspace $\ker(T_u p) \subset T_u P$.

Proof. (1) Consider a point $u \in P$, an element $g \in G$ and two compatible principal bundle charts (U_i, ψ_i) with $i = 1, 2$ and put $x := p(u) \in U_{12}$. Then for an appropriate function $\psi_{12} : U_{12} \to G$, we get $\psi_2^{-1}(x,h) = \psi_1^{-1}(x, \psi_{12}(x)\cdot h)$. This immediately shows that both charts lead to the same element $u \cdot g$. Hence we get a well defined map $P \times G \to P$, such that $v\cdot e = v$ and $v\cdot(gh) = (v\cdot g)\cdot h$. Since the claim on the orbits is evidently true, it remains to show smoothness to complete the proof of (1). This is a local question and since for a principal bundle chart $\psi : p^{-1}(U) \to U \times G$ we get $\psi \circ \rho = (\mathrm{id}_U \times \mu) \circ (\psi \times \mathrm{id}_G)$, where $\mu : G \times G \to G$ is the multiplication map, smoothness follows.

(2) From (1), we conclude that for $u \in P$, $g \mapsto u\cdot g$ defines a smooth map $r_u : G \to P$ such that $p\circ r_u$ is the constant map to $x = p(u)$. This shows that $\zeta_X(u)$ is a well defined tangent vector in u, which is given by $T_e r_u \cdot X$ for $X \in \mathfrak{g} = T_e G$. Thus $X \mapsto \zeta_X(u)$ is a linear map $\mathfrak{g} \to T_u P$ and since $p \circ r_u$ is constant, this has values in $\ker(T_u p)$. For a principal bundle chart (U, ψ) with $x \in U$ suppose that $\psi(u) = (x, h)$. Then the second component of $\psi \circ r_u$ maps g to hg, so this is left translation by h and thus a diffeomorphism, which shows that $T_e r_u$ is injective. Since p is a submersion, we get $\dim(\ker(T_u p)) = \dim(P) - \dim(M) = \dim(\mathfrak{g})$, so $X \mapsto \zeta_X(u)$ is a linear isomorphism from $\mathfrak{g}$ onto $\ker(T_u p)$.

For fixed $X \in \mathfrak{g}$, ζ_X defines a map $P \to TP$ such that $\zeta_X(u)$ lies in $T_u P$, so we only have to verify that this is smooth to complete the proof. But from the description above, it easily follows that we can write $\zeta_X : P \to TP$ as the composition of the smooth map $T\rho : TP \times TG \to TP$ with the map $P \to TP \times TG$ which sends each $u \in P$ to $(0_u, X) \in T_u P \times T_e G$. Since the latter map is clearly smooth, we get the result. □

Using the principal right action, we can easily show that for a principal bundle local smooth sections are equivalent to local principal bundle charts. In particular, this shows that principal bundles usually do not admit global smooth sections.

Corollary 2.4. *Let G be a Lie group with neutral element e and let $p : P \to M$ be a principal G-bundle.*

Then for a principal chart (U, ψ), $\sigma(x) := \psi^{-1}(x, e)$ is a local smooth section of P defined on U. Conversely, a smooth local section of P defined on U gives rise to a local trivialization $\psi : p^{-1}(U) \to U \times G$.

Proof. The first part immediately follows from smoothness of ψ. Conversely, given a local section σ of P defined on U, we define $\alpha : U \times G \to p^{-1}(U)$ by $\alpha(x,g) = \sigma(x)\cdot g$. Then α clearly is a smooth bijection, so it suffices to prove that α^{-1} is smooth, too. This is a local question, so given $x \in U$, we can choose a principal bundle chart $(\tilde{U}, \tilde{\psi})$ with $x \in \tilde{U} \subset U$ and prove that $\alpha^{-1} \circ \tilde{\psi}^{-1} : \tilde{U} \times G \to \tilde{U} \times G$ is smooth. But smoothness of σ shows that $\tilde{\psi} \circ \sigma(x) = (x, f(x))$ for a smooth map $f : \tilde{U} \to G$. Thus $\tilde{\psi} \circ \alpha(x,g) = (x, f(x)g)$ and hence $\alpha^{-1} \circ \tilde{\psi}^{-1}(x,g) = (x, f(x)^{-1}g)$ and this is smooth since the inversion in G is smooth. □

2.5. Principal subbundles. An advantage of the approach via frame bundles is that one can easily encode additional structures on a manifold (or, more generally on a vector bundle) via subbundles of the frame bundle. Let us first give the general definition.

DEFINITION 2.5. Let G be a Lie group, $H \subset G$ be a closed Lie subgroup and let $p : P \to M$ be a G-principal bundle.

A *principal subbundle* with structure group H is a subset $Q \subset P$ with the following properties:

(i) For each $u \in Q$ with $p(u) = x$, the intersection $p^{-1}(x) \cap Q$ coincides with $\{u \cdot h : h \in H\}$.
(ii) For each $x \in M$, there exists an open neighborhood U of x in M and a local smooth section σ of P such that $\sigma(y) \in Q$ for all $y \in U$.

Alternatively, Q is called a *reduction* of P to the structure group $H \subset G$.

Applying Corollary 2.4 to local sections σ as in part (ii) of the definition, one obtains local charts $\psi : p^{-1}(U) \to U \times G$ which restrict to bijections $p^{-1}(U)\cap Q \to U \times H$. This shows that Q is a smooth submanifold of P and that the restriction of p to Q makes Q into a principal H-bundle over M. Alternatively, one can define a reduction of structure group of $p : P \to M$ to be a principal H-bundle $q : Q \to M$ together with a smooth map $F : Q \to P$ such that $p \circ F = q$ and such that $F(u \cdot h) = F(u) \cdot h$ for each $u \in Q$ and each $h \in H$. From these properties, one readily verifies that F is injective and that $F(Q) \subset P$ satisfies all conditions of Definition 2.5.

We discuss the relation between structures and reductions only in the case of the tangent bundle and just in a few simple examples. We consider the subgroups $GL^+(n, \mathbb{R}) := \{A : \det(A) > 0\}$ and $O(n) = \{A : A^tA = \mathbb{I}\}$ of $GL(n, \mathbb{R})$ and their intersection, which equals $SO(n)$ since for $A \in O(n)$ one has $\det(A) = \pm 1$. The key issue to observe that these are the subgroups of those linear automorphisms of $\mathbb{R}^n$, which preserve the standard orientation, the standard inner product, and both these data, respectively.

PROPOSITION 2.5. *Let M be a smooth manifold of dimension n and let $\mathcal{P}M \to M$ be its frame bundle, which has structure group $G := GL(n, \mathbb{R})$.*

(1) M is orientable if and only if there is a principal subbundle $\mathcal{P}^+M \subset PM$ corresponding to $GL^+(n, \mathbb{R})$, and an orientation on M is equivalent to the choice of such a subbundle.

(2) A Riemannian metric on M is equivalent to a principal subbundle $\mathcal{O}M \subset \mathcal{P}M$ corresponding to the orthogonal group $O(n) \subset G$.

(3) On an orientable manifold M, the choice of an orientation and a Riemannian metric is equivalent to the choice of a principal subbundle $\mathcal{SO}M \subset \mathcal{P}M$ corresponding to $SO(n) \subset G$.

PROOF. (1) It suffices to prove the second part, since orientability just means existence of an orientation. We use the definition of an orientation as a family of orientations of all tangent spaces which are compatible in the sense that locally around each point $x \in M$, there is a local frame whose values in each point is positively oriented. Given such an orientation, we define $\mathcal{P}^+M \subset \mathcal{P}M$ by saying that a linear isomorphism $u : \mathbb{R}^n \to T_xM$ lies in $\mathcal{P}^+M$ if and only if the image of the standard basis of $\mathbb{R}^n$ under u is a positively oriented basis of T_xM. It is well known that then there exist local charts which provide positively oriented local frames. Via Corollary 2.4, any principal bundle chart for $\mathcal{P}M$ constructed from such a local chart provides a local smooth section of $\mathcal{P}M$ which has values in $\mathcal{P}^+M$. On the other hand, fixing $u \in \mathcal{P}^+M$ with $p(u) = x$, a linear isomorphism $\hat{u} : \mathbb{R}^n \to T_xM$ lies in $\mathcal{P}^+M$ if and only if $A = u^{-1} \circ \hat{u} : \mathbb{R}^n \to \mathbb{R}^n$ lies in $GL^+(n, \mathbb{R})$. Since $\hat{u} = u \circ A$, this verifies property (i) of Definition 2.5.

Conversely, given a principal subbundle $\mathcal{P}^+M \subset \mathcal{P}M$ we define a basis of T_xM to be positively oriented if and only if the unique linear isomorphism $\mathbb{R}^n \to T_xM$ that maps the standard basis to the given one lies in $\mathcal{P}^+M$. One easily verifies that this defines an orientation on T_xM and that these orientations are compatible.

(2) If g is a Riemannian metric on M, then for each $x \in M$, g_x is an inner product on T_xM. Linear algebra implies that there exists a linear isomorphism $u_x : \mathbb{R}^n \to T_xM$ which is orthogonal with respect to the standard inner product on $\mathbb{R}^n$ and g_x on T_xM. Equivalently, one chooses a basis for T_xM which is orthogonal for g_x and defines u_x to be the unique linear isomorphism mapping the standard basis of $\mathbb{R}^n$ to that basis. Given g, we define $\mathcal{O}M \subset \mathcal{P}M$ to be the subset of all orthogonal isomorphisms. It then easily follows as above that for $u_x \in \mathcal{O}M$ a linear isomorphism $\hat{u}_x : \mathbb{R}^n \to T_xM$ lies in $\mathcal{O}M$ if and only if $A := (u_x)^{-1} \circ \hat{u}_x \in O(n)$.

Here the existence of local smooth sections is slightly more difficult, since these cannot come form local coordinates in general. By Theorem 2.3, what we have to construct are local frames $\{\xi_1, \dots, \xi_n\}$ for TM, whose values in each point are orthonormal. But this can be easily done starting from any local frame using the Gram–Schmidt process. So let us start with a local coordinate frame $\{\partial_1, \dots, \partial_n\}$ defined on an open subset $U \subset M$. Then $f := g(\partial_1, \partial_1) : U \to \mathbb{R}$ is a positive smooth function so also $\frac{1}{\sqrt{f}}$ is smooth. Thus $\xi_1 := \frac{1}{\sqrt{f}}\partial_1$, satisfies $g(\xi_1, \xi_1) = 1$. Next, define $\tilde{\xi}_2 := \partial_2 - g(\xi_1, \partial_2)\xi_1$, which clearly is smooth and and by construction satisfies $g(\xi_1, \tilde{\xi}_2) = 0$. Moreover, $\tilde{\xi}_2$ is nowhere vanishing, since ∂_1 and ∂_2 are linearly independent in each point, so we can norm it as above to obtain ξ_2. Proceeding similarly, we obtain an orthonormal frame, which then defines a smooth local section of $\mathcal{P}M$ which has values in $\mathcal{O}M$.

Conversely, if we have given $\mathcal{O}M \subset \mathcal{P}M$ and a point $x \in M$, we choose a linear isomorphism $u_x : \mathbb{R}^n \to T_xM$ lying in $\mathcal{O}M$ and define g_x on T_xM by $g_x(\xi, \eta) := \langle (u_x)^{-1}(\xi), (u_x)^{-1}(\eta)\rangle$. This evidently is a positive definite inner product on T_xM and since any other such isomorphism is of the form $u_x \circ A$ with $A \in O(n)$, g_x is independent of the choice of u_x. So it remains to show that $x \mapsto g_x$ is smooth, i.e. that for smooth vector fields $\xi, \eta \in \mathfrak{X}(M)$, the function $x \mapsto g_x(\xi(x), \eta(x))$ is smooth. This is a local question, so we may restrict to an open subset $U \subset M$ on which there is a local smooth section σ of $\mathcal{P}M$ which has values in $\mathcal{O}M$. This exactly means that σ defines a local frame $\{\xi_1, \dots, \xi_n\}$ for TM defined on U, which is orthonormal for g, i.e. such that the functions $g(\xi_i, \xi_j)$ are identically zero for $i \neq j$ and identically 1 for $i = j$. But then we can write $\xi|_U$ as $\sum f_i\xi_i$ and $\eta|_U$ as $\sum_i g_i\xi_i$ for smooth functions $f_1, \dots, f_n, g_1, \dots, g_n : U \to \mathbb{R}$, and $g(\xi, \eta) = \sum f_i g_i$ and this is smooth.

(3) This is just a combination of (1) and (2). □

In view of (2), any Riemannian manifold (M, g) comes with a canonical principal bundle $\mathcal{O}M \to M$ with structure group $O(n)$, which is called the *orthonormal frame bundle* of (M, g). For oriented Riemannian manifolds, we obtain the *oriented orthonormal frame bundle* $\mathcal{SO}M$, which has structure group $SO(n)$.

2.6. Associated bundles. Given a vector bundle $p : E \to M$ with standard fiber V and the frame bundle $\pi : \mathcal{P}E \to M$ of E as constructed above, there is an easy way to reconstruct E from $\mathcal{P}E$. For a point $x \in M$, an element $u \in \mathcal{P}E$ with $\pi(u) = x$ by definition is a linear isomorphism $V \to E_x$. Hence there is an obvious map $\mathcal{P}E \times V \to E$ defined by $(u, v) \mapsto u(v)$ for $u \in \mathcal{P}E$ and $v \in V$. This map is evidently surjective and since $u(v) \in E_{p(u)}$, we see that $u_1(v_1) = u_2(v_2)$ is only possible if $p(u_1) = p(u_2)$. If this is the case, then we know that $u_2 = u_1 \circ A$ for a unique element $A \in GL(V)$, and then

$u_2(v_2) = u_1(Av_2)$, so this equals $u_1(v_1)$ if and only if $v_2 = A^{-1}v_1$. This means that we can identify E with the set of equivalence classes of the equivalence relation on $\mathcal{P}E \times V$ defined by $(u_1, v_1) \sim (u_2, v_2)$ iff there is an element $A \in GL(V)$ such that $u_2 = u_1 \circ A$ and $v_2 = A^{-1}v_1$. Otherwise put, this is the space of orbits of the right action of $GL(V)$ on $\mathcal{P}E \times V$ defined by $(u, v) \cdot A = (u \circ A, A^{-1}v)$.

Now it turns out that it is easy to also deduce the vector bundle structure on E from this description. Moreover, the whole result vastly generalizes in more than one way. We can not only replace the frame bundle of a vector bundle by an arbitrary principal G-bundle, but also replace the natural action of $GL(V)$ on V by an arbitrary representation of G on a vector space or even by a smooth left action of G on a manifold. In this way, a single principal fiber bundle (or a single vector bundle via its frame bundle) gives rise to a large family of bundles over the same base manifold.

Recall that a smooth *left action* of G on a manifold S is a smooth map $\ell : G \times S \to S$, which we also write as $(g, y) \mapsto g \cdot y$ such that $e \cdot y = y$ and $(gh) \cdot y = g \cdot (h \cdot y)$ holds for all $g, h \in G$ and $y \in S$. A *representation* of G on a finite dimensional K-vector space V is a smooth left action of G on V by linear maps. Equivalently, such a representation can be described as a smooth homomorphism $G \to GL(V)$. Using these concepts, we can now describe the general version of the construction.

Consider a principal G-bundle $p : P \to M$ and a smooth left action $\ell : G \times S \to S$. Then we define a smooth map $(P \times S) \times G \to P \times S$ by $(u, y) \cdot g := (u \cdot g, g^{-1} \cdot y)$, where we use the principal right action in the first component and the given action ℓ in the second component. One immediately verifies that this is a smooth right action of G, and we define $P \times_G S$ to be the set of orbits of this action, i.e. the quotient of $P \times S$ by the equivalence relation $(u_1, y_1) \sim (u_2, y_2) \Leftrightarrow \exists g \in G : (u_2, y_2) = (u_1, y_1) \cdot g$. Writing $[(u, y)]$ for the equivalence class of (u, y), there are natural maps $q : P \times F \to P \times_G F$ and $\pi : P \times_G F \to M$ defined by $q(u, y) = [(u, y)]$ and $\pi([(u, y)]) = p(u)$.

Theorem 2.6. *Let $p : P \to M$ be a principal fiber bundle with structure group G, and let $\ell : G \times S \to S$ be a smooth left action of G on a manifold S. Then we have:*

(1) The space $P \times_G S$ naturally is a smooth manifold and $\pi : P \times_G S \to M$ is a smooth fiber bundle with standard fiber S.

(2) If we deal with a representation $G \times V \to V$, then $\pi : P \times_G V \to M$ canonically is a vector bundle over M.

Proof. The basic point here is that any principal bundle chart $\psi : p^{-1}(U) \to U \times G$ for P induces a fiber bundle chart $\varphi : \pi^{-1}(U) \to U \times S$, which is a vector bundle chart in the case of a representation. Consider the composition $\tilde{\varphi}$ defined as

$$p^{-1}(U) \times S \overset{\psi \times \mathrm{id}_S}{\longrightarrow} U \times G \times S \overset{\mathrm{id}_U \times \ell}{\longrightarrow} U \times S.$$

This is clearly smooth and since $(\psi^{-1}(x, e), y)$ is mapped to (x, y), it is surjective. Moreover, the first component of $\tilde{\varphi}(u, y)$ is $p(u)$. Thus, (u_1, y_1) and (u_2, y_2) can have the same image only if $p(u_2) = p(u_1)$ and hence $u_2 = u_1 \cdot g$ for some $g \in G$. But then denoting $\psi(u_1)$ by (x, h), we get $\psi(u_2) = (x, hg)$, so $\tilde{\varphi}(u_1, y_1) = \tilde{\varphi}(u_1, y_2)$ if and only if $hg \cdot v_2 = h \cdot v_1$ or, equivalently, $v_2 = g^{-1} \cdot v_1$. This shows that $\tilde{\varphi}$ descends to a bijection $\varphi : \pi^{-1}(U) \to U \times S$.

For two compatible principal bundle charts related as $\psi_1(\psi_2^{-1}(x, h)) = (x, \psi_{12}(x)h)$ with a smooth G-valued map ψ_{12}, the corresponding charts are related as $\varphi_1(\varphi_2^{-1}(x, y)) = (x, \psi_{12}(x) \cdot y)$, where the action in the right hand side is ℓ. This immediately shows that the chart changes are smooth and they are are linear in the second variable if we start from a representation. At this stage, we can proceed as before, defining a topology on

$P \times_G S$ either as a quotient of $P \times S$ or via the charts and then use the charts to induce a manifold structures. For this structure π is visibly smooth and defines a fiber bundle respectively a vector bundle. □

DEFINITION 2.6. For a principal G-bundle $p : P \to M$ and a left action $\ell : G \times S \to S$ of G, the bundle $\pi : P \times_G S \to M$ constructed in Theorem 2.6 is called the *associated bundle* to P with respect to the action ℓ. In the case of a representation, one uses the terminology "*associated vector bundle*" or "induced vector bundle".

A crucial fact for our purposes now is that most of the geometric objects one meets in differential geometry can be interpreted as sections of associated bundles to the linear frame bundle of a manifold. Let us describe this in a few examples.

EXAMPLE 2.6. Let M be a smooth manifold of dimension n and let $p : \mathcal{P}M \to M$ be the linear frame bundle of M, which has structure group $G := GL(n, \mathbb{R})$. In the beginning of this section, we have seen that for the obvious action of G on $\mathbb{R}^n$ we by definition get $\mathcal{P}M \times_G \mathbb{R}^n \cong TM$ with the isomorphism induced by $(u, v) \mapsto u(v) \in T_{p(u)}M$.

Another simple example of a representation of G is provided by the dual space $\mathbb{R}^{n*} = L(\mathbb{R}^n, \mathbb{R})$. In order to obtain a left action of G, one has to define $(A \cdot \lambda)(v) := \lambda(A^{-1}v)$ for $A \in G$, $\lambda \in \mathbb{R}^{n*}$ and $v \in \mathbb{R}^n$. Otherwise put, we have $A \cdot \lambda = \lambda \circ A^{-1}$. But given a point $u \in \mathcal{P}M$ with $p(u) =: x$ and a linear functional $\lambda \in \mathbb{R}^{n*}$, we can form $\lambda \circ u^{-1}$, which is a linear map $T_xM \to \mathbb{R}$. Since $u \cdot A = u \circ A$ and $A^{-1} \cdot \lambda = \lambda \circ A$, we readily see that the pair $(u \cdot A, A^{-1} \cdot \lambda)$ leads to the same linear functional as (u, A). This easily implies that the fiber of $\mathcal{P}M \times_G \mathbb{R}^{n*}$ over $x \in M$ is the dual space $(T_xM)^*$ and hence a section of this bundle associates to each $x \in M$ a linear functional $T_xM \to \mathbb{R}$. It is easy to see the smoothness of such a section exactly means that we get a one-form on M, so $\mathcal{P}M \times_G \mathbb{R}^{n*}$ is the cotangent bundle T^*M.

This easily generalizes in several ways. For $1 < k \leq n$, we can consider the vector space $\Lambda^k\mathbb{R}^{n*}$ of k-linear, alternating functions $(\mathbb{R}^n)^k \to \mathbb{R}$. This is a representation of G via $(A \cdot \alpha)(v_1, \dots, v_k) := \alpha(A^{-1}v_1, \dots, A^{-1}v_k)$. As above, a point $u \in \mathcal{P}M$ with $p(u) = x$ and an element $\varphi \in \Lambda^k\mathbb{R}^{n*}$ define a k-linear, alternating map $\varphi \circ (u^{-1})^k : (T_xM)^k \to \mathbb{R}$. Further, sections of the associated bundle $\mathcal{P}M \times_G \Lambda^k\mathbb{R}^{n*}$ are exactly k-forms on M, so this is the bundle Λ^kT^*M of k-forms.

Taking general multilinear maps on $\mathbb{R}^n$ one identifies general tensor fields of type $\binom{\ell}{k}$ with sections of the associated bundle $\mathcal{P}M \times_G (\otimes^\ell\mathbb{R}^n \otimes \otimes^k\mathbb{R}^{n*})$. Any G-invariant subspace in $\otimes^\ell\mathbb{R}^n \otimes \otimes^k\mathbb{R}^{n*}$, for example maps having appropriate (anti-)symmetry properties, gives rise to a smooth subbundle and geometric objects as sections.

2.7. Functorial properties. One may expect that the construction of associated bundles is functorial in two ways, namely with respect to the principal bundle and with respect to the space used to induce the bundle. We only discuss this for the case of associated vector bundles, the case of associated fiber bundles is similar. We have not yet discussed morphisms of fiber bundles, but the concept is rather obvious: If $p : E \to M$ and $\tilde{p} : \tilde{E} \to \tilde{M}$ are arbitrary fiber bundles, the one defines a *bundle map* as a smooth map $F : E \to \tilde{E}$, which maps fibers to fibers. Explicitly, this can be formulated as the fact that there is a set map $f : M \to \tilde{M}$, called the *base map* of F such that $\tilde{p} \circ F = f \circ p$. Assuming this, the facts that $\tilde{p} \circ F$ is smooth and that p is a surjective submersion imply that f is smooth, too.

If E and $\tilde{E}$ are principal bundles with the same structure group G, then a bundle map $F : E \to \tilde{E}$ is called a *principal bundle map* if it is equivariant for the principal right actions of G, i.e. if $F(u \cdot g) = F(u) \cdot g$ for each $u \in E$ and $g \in G$. For bundles with different structure group, one can similarly require equivariancy with respect to a fixed homomorphism. For example, the alternative definition of a reduction of structure group to a subgroup $H \subset G$ from Section 2.5 just is as a principal bundle map with respect to the inclusion homomorphism $H \to G$ and with base map id_M.

Next, consider $\mathbb{K}$-vector bundles E and $\tilde{E}$ and a bundle map $F : E \to \tilde{E}$ with base map f. Then by definition, for each $x \in M$, F maps the fiber $E_x = p^{-1}(x)$ to the fiber $\tilde{E}_{f(x)}$. One calls F a *vector bundle map* or a *vector bundle homomorphism* if each of the induced maps between the fibers is linear over $\mathbb{K}$.

A basic example of a vector bundle homomorphism is given by the tangent map $Tf : TM \to TN$ of a smooth map $f : M \to N$, which then is the base map of Tf. We can also use this to obtain a fundamental example of a principal bundle map: Consider two smooth manifolds M and N, which both have dimension n and a local diffeomorphism $f : M \to N$, i.e. a smooth map such that for each $x \in M$, the tangent map $T_x f$ is a linear isomorphism. Then we define a map $\mathcal{P}f : \mathcal{P}M \to \mathcal{P}M$ between the frame bundles of the two manifolds with base map f as follows. A point of $\mathcal{P}M$ in the fiber over $x \in M$ by definition is a linear isomorphism $u : \mathbb{R}^n \to T_x M$. But then $\mathcal{P}f(u) := T_x f \circ u$ is a linear isomorphism $\mathbb{R}^n \to T_{f(x)}N$ and thus an element of $\mathcal{P}N$ which lies in the fiber over $f(x)$. One easily checks that $\mathcal{P}f$ is smooth and thus defines a fiber bundle map with base map f. But the principal action of $G = GL(n, \mathbb{R})$ on the frame bundles is given by composition, so $u \cdot A = u \circ A$, so $\mathcal{P}f(u \cdot A) = T_x f \circ u \circ A = \mathcal{P}f(u) \cdot A$, so $\mathcal{P}f$ is a principal bundle map.

Proposition 2.7. *Let $p : E \to M$ and $\tilde{p} : \tilde{E} \to \tilde{M}$ be principal G-bundles and let V and W be representations of the Lie group G.*

(1) A principal bundle map $F : E \to \tilde{E}$ with base map f naturally induces a vector bundle homomorphism $F_V : E \times_G V \to \tilde{E} \times_G V$ with base map f. The restriction of F_V to each fiber of $E \times_G V$ is a linear isomorphism.

(2) A G-equivariant linear map $\varphi : V \to W$ naturally induces a vector bundle homomorphism $\varphi_M : E \times_G V \to E \times_G W$ with base map id_M.

Proof. (1) We can form $F \times \mathrm{id}_V : E \times V \to \tilde{E} \times V$ and observe that this maps $(u \cdot g, g^{-1} \cdot v)$ to $(F(u) \cdot g, g^{-1} \cdot v)$ by equivariancy of F. Thus there is a well defined map F_V between the orbit spaces. The definitions imply that the natural projection $E \times V \to E \times_G V$ is a surjective submersion, which implies that F_V is smooth, and by construction it has base map f. The identification of V with a fiber of $E \times_G V$ over $x \in M$ is given by $v \mapsto [(u_0, v)]$ for some element $u_0 \in E_x$. Likewise, $\tilde{u}_0 \in \tilde{E}_{f(x)}$ leads to the identification of the fiber of $\tilde{E} \times_G V$ with V defined by $[(\tilde{u}_0, w)] \mapsto w$. But then $F(u_0) = \tilde{u}_0 \cdot g$ for some $g \in G$ and $F_V([(u_0, v)]) = [(\tilde{u}_0 \cdot g, v)] = [(\tilde{u}_0, g \cdot v)]$. Hence the map between the fibers of the associated bundles corresponds to the map $V \to V$ defined by $v \mapsto g \cdot v$, so this is a $\mathbb{K}$-linear isomorphism.

(2) Equivariancy of φ shows that the map $\mathrm{id}_E \times \varphi : E \times V \to E \times W$ sends $(u \cdot g, g^{-1} \cdot v)$ to $(u \cdot g, g^{-1} \cdot \varphi(v))$, so again there is an induced map φ_M between the orbit spaces. As above, we conclude that φ_M is smooth and clearly it is a bundle map with base map id_M. But then for identifications of the fibers as in the proof of part (1), it is clear that the maps on the fibers induced by φ_M correspond to φ and thus are linear. □

This result provides us with a nice interpretation of the action of local diffeomorphisms on tensor fields. From Example 2.6, we know that any tensor bundle over an n-dimensional manifold M can be realized as $EM := \mathcal{P}M \times_G V$ for appropriate representations V of the group $G := GL(n, \mathbb{R})$. Given another n-manifold N and a local diffeomorphism $f : M \to N$, we can form the principal bundle map $\mathcal{P}f : \mathcal{P}M \to \mathcal{P}N$, and then by part (1) of the Proposition get an induced map $\mathcal{P}f_V =: Ef : EM \to EN$, which restricts to a linear isomorphism on each fiber of EM. This allows us to define a pullback operation sending sections of EN to sections of EM. Given a section $\sigma \in \Gamma(EN)$ and a point $x \in M$, we can form $\sigma(f(x)) \in E_{f(x)}N$ and since Ef restricts to a linear isomorphism $E_xM \to E_{f(x)}N$, there is a unique element $f^*\sigma(x) \in E_xM$ which is mapped to $\sigma(f(x))$ by Ef. This defines a map $f^*\sigma : M \to EM$ mapping each point into the fiber over that point and this is characterized by $Ef \circ f^*\sigma = \sigma \circ f$. It is easy to verify that $f^*\sigma$ is indeed a smooth section of EM.

In the case tensor bundles, this leads to the usual action of local diffeomorphisms. Let us consider the case $V = \mathbb{R}^n$, so $EM = TM$. The isomorphism $\mathcal{P}M \times_G \mathbb{R}^n \cong TM$ comes from $[(u, v)] \mapsto u(v)$, and $\mathcal{P}F_V([(u, v)]) = [(\mathcal{P}f(u), v)] = [(T_xf \circ u, v)]$, where x is the base point of u, so this is mapped to $T_xf(u(v))$. This shows that $\mathcal{P}f_V = Tf$ so we get the right induced bundle map here. Similar considerations apply to more general tensor bundles.

Second, these considerations lead to an interesting perspective on Riemannian geometry. For a smooth manifold M with linear frame bundle $\mathcal{P}M \to M$, we know from Proposition 2.5 that the choice of a Riemannian metric g on M is equivalent to the choice of a principal subbundle $\mathcal{O}M \subset \mathcal{P}M$ with structure group $O(n) \subset GL(n, \mathbb{R})$. Now any representation V of $GL(n, \mathbb{R})$ can simply be restricted to a representation of the subgroup $O(n)$ and from the definitions we readily see that the associated bundles $\mathcal{P}M \times_{GL(n,\mathbb{R})} V$ and $\mathcal{O}M \times_{O(n)} V$ can be naturally identified. Thus, on a Riemannian manifold, we can view all tensor bundles as associated bundles to the orthonormal frame bundle.

But it may happen that two non-isomorphic representations V and W of $GL(n, \mathbb{R})$ are isomorphic as representations of the subgroup $O(n)$. The simplest example is provided by the representations $\mathbb{R}^n$ and $\mathbb{R}^{n*}$, which are isomorphic as representations of $O(n)$ via $v \mapsto \langle v, \ \rangle$. Indeed for $A \in O(n)$, we get $\langle Av, w\rangle = \langle v, A^{-1}w\rangle$, which exactly says that $\langle Av, \ \rangle = \langle v, \ \rangle \circ A^{-1}$ and thus our map is $O(n)$-equivariant. Visibly, this map is not $GL(n, \mathbb{R})$-equivariant, and indeed $\mathbb{R}^n$ and $\mathbb{R}^{n*}$ are non-isomorphic as representations of $GL(n, \mathbb{R})$. Now part (2) of Proposition 2.7 readily implies that an isomorphism between two representations induces an isomorphism between the corresponding associated bundles. Thus we see that for a Riemannian manifold the bundles $TM \cong \mathcal{O}M \times_{O(n)} \mathbb{R}^n$ and $T^*M \cong \mathcal{O}M \times_{O(n)} \mathbb{R}^{n*}$ are naturally isomorphic, but the isomorphism depends on the Riemannian metric. This of course generalizes to other tensor bundles, compare with Section 1.5 of [**Riem**].

Similarly, there may be other linear maps between representations of $GL(n, \mathbb{R})$ which are $O(n)$-equivariant but not $GL(n, \mathbb{R})$-equivariant. Consider the representation $S^2\mathbb{R}^{n*}$ of symmetric bilinear forms on $\mathbb{R}^n$, compare with Example 2.6. As described in Section 2.1, the inner product on $\mathbb{R}^n$ gives rise to a trace map $\mathrm{tr} : S^2\mathbb{R}^{n*} \to \mathbb{R}$, and it is easy to verify that this map is $O(n)$-equivariant (but not $GL(n, \mathbb{R})$-equivariant). Thus it defines a bundle map between the corresponding associated bundles, which exactly leads to the definition of the trace of a symmetric $\binom{0}{2}$-tensor field on a Riemannian manifold as described in Section 2.1. This leads to a finer decomposition of the bundle S^2T^*M on a Riemannian manifold corresponding to the $O(n)$-invariant decomposition $S^2\mathbb{R}^{n^*} = \ker(\mathrm{tr}) \oplus \mathbb{R} \cdot \langle\ ,\ \rangle$.

Applications to Riemannian geometry

We close the chapter by a discussion of the Levi-Civita connection in the description of Riemannian metrics via the orthonormal frame bundle from Proposition 2.5. This will directly lead to a Laplace operator on sections of any natural vector bundle over a Riemannian manifold as well as to the concept of generalized Laplacians. We will also sketch applications of these operators in geometry and topology, thus providing perspective for the use of (generalized) Dirac operators.

2.8. The Levi-Civita connection. The last crucial bit of information that we need is that the Levi-Civita connection of a Riemannian manifold can be nicely expressed in terms of the orthonormal frame bundle. The first step towards this is that one can describe (local) sections of an associated bundle in terms certain smooth functions on the total space of the principal bundle.

PROPOSITION 2.8. *Let $p : P \to M$ be a principal G-bundle, $\ell : G \times S \to S$ a left action of G and $\pi : P \times_G S \to M$ the corresponding associated bundle. Then for any open subset $U \subset M$, local sections of $P \times_G S \to M$ defined on U are in bijective correspondence with smooth functions $f : p^{-1}(U) \to S$, which are G-equivariant in the sense that $f(u \cdot g) = g^{-1} \cdot f(u)$ for all $u \in p^{-1}(U)$.*

PROOF. The main step here is to understand that this works point wise, then one just has to verify that smoothness has the same meaning in the two pictures. So suppose that we have given $x \in U$ and points $u \in P$ and $z \in P \times_G S$ such that $p(u) = \pi(z) = x$. We can write $z = [(\tilde{u}, y)]$ for some $\tilde{u} \in P$ with $p(\tilde{u}) = x$. But then there is a unique element $g \in G$ such that $u = \tilde{u} \cdot g$ and $z = [(u, g^{-1} \cdot y)]$. On the other hand, since $u = u \cdot g$ implies $g = e$, we see that $[(u, y)] = [(u, \tilde{y})]$ implies $y = \tilde{y}$.

Now we can establish the claimed bijection. Given a local section s of $P \times_G S$ defined on U and a point $u \in p^{-1}(U)$, we define $f(u) \in S$ to be the unique element such that $s(p(u)) = [(u, f(u))]$. Since $p(u) = p(u \cdot g)$, this implies $[(u \cdot g, f(u \cdot g))] = [(u, f(u))]$ and thus $f(u \cdot g) = g^{-1} \cdot f(u)$. Smoothness of f can be verified locally, so we can work in a principal bundle chart ψ and the corresponding chart φ for $P \times_G S$. On the domain of this chart, we get $(\varphi \circ s)(x) = (x, \alpha(x))$ for a smooth S-valued map α. But $\alpha(x)$ by definition equals $[(\psi^{-1}(x, e), \alpha(x))]$, so we see that $f(\psi^{-1}(x, g)) = g^{-1} \cdot \alpha(x)$, so this is smooth, too.

Conversely, given an equivariant smooth function $f : p^{-1}(U) \to S$, then for each $x \in U$ all elements $u \in P_x$ lead to the same class $[(u, f(u))]$ and we define this class to be $s(x)$. Clearly, this defines a map $s : U \to \pi^{-1}(U)$ such that $\pi \circ s = \mathrm{id}_U$. Smoothness of s can be verified locally. But in terms of a local smooth section σ of P and the natural map $q : P \times S \to P \times_G S$, we can write $s = q \circ (\sigma, f \circ \sigma)$, so this is smooth, too. □

Now functions on any manifold with values in some vector space can be differentiated using vector fields on this manifold. (Just use a basis to obtain real valued functions, differentiate those and observe that the result is independent of the choice of basis.) The derivative of an equivariant function $f : P \to V$ along a general vector field $\xi \in \mathfrak{X}(P)$ is not equivariant. However, if one in addition requires that ξ is G-invariant in the sense that $(r^g)^*\xi = \xi$ for each $g \in G$, then $\xi \cdot f$ is also equivariant: Invariance means that $\xi(u \cdot g) = T_{r^g} \cdot \xi(u)$ and applying this tangent vector to f, we get $\xi(u) \cdot (f \circ r^g) = \xi(u) \cdot (\lambda(g^{-1}) \circ f)$, where we denote by $\lambda(g^{-1}) : V \to V$ the action of g^{-1} on V. But this is a linear map, so we end up with $\xi(u \cdot g) \cdot f = \lambda(g^{-1})(\xi(u) \cdot f)$, which is exactly the required equivariancy condition.

Suppose that $\tilde{\xi} \in \mathfrak{X}(P)$ is a G-invariant vector field on a principal G-bundle $p : P \to M$. Then for two points $u, \hat{u} \in P$ in the fiber over $x \in M$, we get an element $g \in G$ such that $\hat{u} = u \cdot g$, and hence $\tilde{\xi}(\hat{u}) = T_{r^g} \cdot \tilde{\xi}(u)$. Since $p \circ r^g = p$, we conclude that $T_{\hat{u}}p \cdot \tilde{\xi}(\hat{u}) = T_u p \cdot \tilde{\xi}(u) \in T_x M$. Denoting this element by $\xi(x)$, we get a map $\xi : M \to TM$ such that $\xi(x) \in T_x M$ for all $x \in M$ and such that $\xi \circ p = Tp \circ \tilde{\xi} : P \to TM$. In local trivializations, one easily verifies that ξ is smooth and thus defines a vector field $\xi \in \mathfrak{X}(M)$. This says that $\tilde{\xi} \in \mathfrak{X}(P)$ is a *projectable* vector field with *projection* $\xi \in \mathfrak{X}(M)$.

There is a general gadget, called a *principal connection*, which can be chosen on any principal G-bundle $p : P \to M$ and that allows one to revert this process, i.e. to associate to each vector field $\xi \in \mathfrak{X}(M)$ a G-invariant vector field $\xi^{\text{hor}} \in \mathfrak{X}(P)$ that projects onto ξ. Such a principal connection can be equivalently described in two ways. On the one hand, one may choose in each tangent space $T_u P$ a linear subspace $\mathcal{H}_u$, which is complementary to $\ker(T_u p) \subset T_u P$. This choice should be G-equivariant in the sense that for each $u \in P$ and $g \in G$, we get $\mathcal{H}_{u \cdot g} = T_u r^g(\mathcal{H}_u)$. Moreover, the subspaces $\mathcal{H}_u$ should define a *smooth distribution* on P, in the sense that each point $u \in P$ should have an open neighborhood $U \subset P$ for which there are smooth local vector fields $\xi_1, \dots, \xi_n$ whose values in each point of U span the distinguished subspace. Since on often refers to $\ker(T_u p)$ as the *vertical subspace* of the tangent space, the family $\mathcal{H}_u$ of subspaces is called the *horizontal distribution* of the principal connection.

The second description of a principal connection is as a differential form $\gamma \in \Omega^1(P, \mathfrak{g})$ with values in the Lie algebra $\mathfrak{g}$ of G. This means that γ assigns to each $u \in P$ a linear map $T_u P \to \mathfrak{g}$ and this assignment depends smoothly on u in the usual sense. In addition, γ is required to reproduce the generators of fundamental vector fields, i.e. for each $X \in \mathfrak{g}$ with fundamental vector field ζ_X (see Proposition 2.4), one has $\gamma(\zeta_X(u)) = X$ for all $u \in P$. Finally, γ should be G-equivariant in the sense that $(r^g)^*\gamma = \mathrm{Ad}(g^{-1}) \circ \gamma$ for all $g \in G$.

The relation between the two pictures is more simple than it may look at first sight. Having give $\mathcal{H}_u \subset T_u P$, one can uniquely decompose each $\xi \in T_u P$ as the sum of an element of $\mathcal{H}_u$ and an element of $\ker(T_u p)$, and the latter component can be uniquely written as $\zeta_X(u)$ for an element $X \in \mathfrak{g}$. Putting $\gamma(u)(\xi) = X$ defines a form $\gamma \in \Omega^1(P, \mathfrak{g})$, which evidently reproduces the generators of fundamental vector fields. Conversely, having give $\gamma \in \Omega^1(P, \mathfrak{g})$ such that $\gamma(u) : T_u P \to \mathfrak{g}$ is surjective for each u, one may define $\mathcal{H}_u := \ker(\gamma(u))$. One then verifies directly that the equivariancy conditions in the two pictures correspond to each other.

By construction, for each $u \in P$, the tangent map $T_u p$ restricts to a linear isomorphism $\mathcal{H}_u \to T_{p(u)} M$. Thus given a vector field $\xi \in M$, there is a unique element $\xi^{\text{hor}}(u) \in \mathcal{H}_u$ such that $T_u p \cdot \xi^{\text{hor}}(u) = \xi(p(u))$. Equivariancy of $\mathcal{H}$ shows that $T_u r^g \cdot \xi^{\text{hor}}(u) \in \mathcal{H}_{u \cdot g} \subset T_{u \cdot g} P$ and since this projects on $\xi(p(u))$ is has to coincide with $\xi^{\text{hor}}(u \cdot g)$, so ξ^{hor} is G-invariant. Finally, one easily verifies that ξ^{hor} indeed defines a smooth vector field on P, called the *horizontal lift* of ξ.

Now the fundamental theorem on the Levi-Civita connection from Riemannian geometry can be formulated as follows:

THEOREM 2.8. *For any Riemannian manifold (M, g) of dimension n, there is a canonical principal connection on the orthonormal frame bundle $\mathcal{O}M$. This is characterized by the fact that for vector fields $\xi, \eta \in \mathfrak{X}(M)$ the covariant derivative $\nabla_\xi \eta$*

corresponds to the $O(n)$-equivariant function $\xi^{hor}\cdot f$, where $\xi^{hor}\in\mathfrak{X}(\mathcal{O}M)$ is the horizontal lift of ξ and $f:\mathcal{O}M\to\mathbb{R}^n$ is the $O(n)$-equivariant function corresponding to η.

Sketch of proof. This sketch is intended for people familiar with the basic facts about the Levi-Civita covariant derivative in the usual language of Riemannian geometry. An independent proof of existence an uniqueness of the connection building on the orthonormal frame bundle is given in Appendix A.

The covariant derivative ∇ defines an operator $\nabla:\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$, written as $(\xi,\eta)\mapsto\nabla_\xi\eta$, which is linear over smooth functions in the first variable and satisfies a Leibniz rule in the second variable. In particular, for a point $x\in M$, $\nabla_\xi\eta(x)\in T_xM$ depends only on $\xi(x)\in T_xM$. Now take a point $u\in\mathcal{O}M$ over x and choose any tangent vector $\tilde{\xi}\in T_u\mathcal{O}M$ such that $T_up\cdot\tilde{\xi}=\xi(x)$. Now for a vector field $\eta\in\mathfrak{X}(M)$, let us temporarily denote the corresponding equivariant function $\mathcal{O}M\to\mathbb{R}^n$ as f^η. Then consider the map $\Psi:\mathfrak{X}(M)\to\mathbb{R}^n$ defined by $\Psi(\eta):=\tilde{\xi}\cdot f^\eta-u^{-1}(\nabla_\xi\eta(x))$. One easily verifies that this is linear over $\mathbb{R}$.

Next, for a smooth function $\varphi:M\to\mathbb{R}$ consider $\varphi\eta\in\mathfrak{X}(M)$. From the definitions, one easily verifies that $f^{\varphi\eta}=(\varphi\circ p)f^\eta$, and thus

$$\tilde{\xi}\cdot f^{\varphi\eta}=((T_up\cdot\tilde{\xi})\cdot\varphi)f^\eta+(\varphi\circ p)\tilde{\xi}\cdot f^\eta=(\xi\cdot\varphi)(x)f^\eta+\varphi(x)\tilde{\xi}\cdot f^\eta.$$

On the other hand, the Leibniz rule says that $\nabla_\xi\varphi\eta(x)=\varphi(x)\nabla_\xi\eta(x)+(\xi\cdot\varphi)(x)\eta(x)$. Applying u^{-1} to the second summand, we simply get $(\xi\cdot\varphi)(x)f^\eta(u)$, which together with the above shows that $\Psi(\varphi\eta)=\varphi(x)\Psi(\eta)$.

Now similarly to the characterization of tensor fields, one shows that this implies that $\Psi(\eta)$ only depends on $\eta(x)$, so there is a linear map $X:\mathbb{R}^n\to\mathbb{R}^n$ such that $\Psi(\eta)=X(u^{-1}(\eta(x)))$. Next, we claim that X is skew symmetric with respect to the standard inner product on $\mathbb{R}^n$. The idea here is that for $\eta_1,\eta_2\in\mathfrak{X}(M)$, the function $\langle f^{\eta_1},f^{\eta_2}\rangle:\mathcal{O}M\to\mathbb{R}$ equals $g(\eta_1,\eta_2)\circ p$. Differentiating this equation by $\tilde{\xi}$, we get $\tilde{\xi}\cdot\langle f^{\eta_1},f^{\eta_2}\rangle=(\xi\cdot g(\eta_1,\eta_2))(x)$. Bilinearity of the inner product shows that the left hand side equals $\langle\tilde{\xi}\cdot f^{\eta_1},f^{\eta_2}(u)\rangle+\langle f^{\eta_1}(u),\tilde{\xi}\cdot f^{\eta_2}\rangle$. Compatibility of the Levi-Civita connection with the metric says that the right hand side can be written as

$$g_x(\nabla_\xi\eta_1(x),\eta_2(x))+g_x(\eta_1(x),\nabla_\xi\eta_2(x)).$$

The first summand can be written as $\langle u^{-1}(\nabla_\xi\eta_1(x)),f^{\eta_2}(u)\rangle$ and similarly for the second one. Rearranging terms then immediately implies that

$$\langle X(f^{\eta_1}(u)),f^{\eta_2}(u)\rangle=-\langle f^{\eta_1}(u),X(f^{\eta_2}(u))\rangle$$

and thus the claimed skew symmetry.

But this means that we can form $\xi^{\mathrm{hor}}(u):=\tilde{\xi}+\zeta_X(u)\in T_u\mathcal{O}M$, and $T_up\cdot\xi^{\mathrm{hor}}(u)=\xi(x)$. Equivariancy of f^η easily implies that $\zeta_X(u)\cdot f^\eta=-X(f^\eta(u))$ and thus $\xi^{\mathrm{hor}}(u)\cdot f^\eta=u^{-1}(\nabla_\xi\eta(x))$. Sending $\xi(x)$ to $\xi^{\mathrm{hor}}(u)$ defines a map $T_xM\to T_u\mathcal{O}M$, which is right inverse to T_up and easily seen to be linear. Now one defines $\mathcal{H}_u$ to be the image of this map and then verifies that these spaces satisfy all the properties of a principal connection. □

2.9. (Generalized) Laplacians and some applications. Now we have all ingredients at hand to define a broad class of analogs of the Laplacian on vector valued functions on $\mathbb{R}^n$. Namely, let (M,g) be an n-dimensional Riemannian manifold with orthonormal frame bundle $\mathcal{O}M$ and let V be any representation of $O(n)$. Then we can consider the associated natural vector bundle $EM:=\mathcal{O}M\times_{O(n)}V$ and the principal connection on $\mathcal{O}M$ induced by the Levi-Civita connection. This provides the horizontal lift ξ^{hor} for any vector field $\xi \in \mathfrak{X}(M)$. For a smooth section $\sigma \in \Gamma(EM)$ corresponding to $f^\sigma : \mathcal{O}M \to V$, also the function $\xi^{\mathrm{hor}} \cdot f^\sigma : \mathcal{O}M \to V$ is equivariant and thus corresponds to a section $\nabla_\xi \sigma \in \Gamma(EM)$. The definitions readily imply that for a smooth function $\varphi : M \to \mathbb{R}$, the horizontal lift of $\varphi\xi$ is given by $(\varphi \circ p)\xi^{\mathrm{hor}}$, which readily implies that $\nabla_{\varphi\xi}\sigma = \varphi\nabla_\xi\sigma$. On the other hand $f^{\varphi\sigma} = (\varphi \circ p)f^\sigma$ which implies that $\nabla_\xi\varphi\sigma = (\xi \cdot \varphi)\sigma + \varphi\nabla_\xi\sigma$. This means that ∇ defines a *covariant derivative* on EM.

Linearity over smooth functions in ξ again shows that $\nabla_\xi\sigma(x) \in E_xM$ depends only on $\xi(x) \in T_xM$. Thus we can view $\nabla\sigma(x)$ as defining a linear map $T_xM \to E_x$, or otherwise put, view $\nabla\sigma$ as a section of the bundle $\mathcal{O}M \times_{O(n)} L(\mathbb{R}^n, V)$. In this form, the covariant derivative can be iterated, i.e. we can form $\nabla(\nabla\sigma)$ which is a section of the bundle induced by the representation $L(\mathbb{R}^n, L(\mathbb{R}^n, V))$ which is isomorphic to the space of bilinear maps $\mathbb{R}^n \times \mathbb{R}^n \to V$. Evaluating such a map on the elements of an orthonormal basis and summing up defines a linear map $\operatorname{tr} : L(\mathbb{R}^n, L(\mathbb{R}^n, V)) \to V$, which is easily seen to be $O(n)$-equivariant. Hence we can define the *Laplace operator* $\Delta : \Gamma(EM) \to \Gamma(EM)$ by $\Delta(\sigma) := -\operatorname{tr}(\nabla(\nabla\sigma))$. It turns out that there are several small variations of this idea, so it is better to use a more general concept.

DEFINITION 2.9. Let (M, g) be a Riemannian manifold and let EM be an associated bundle to the orthonormal frame bundle $\mathcal{O}M$. A *generalized Laplace operator* on EM is an operator $\Delta : \Gamma(EM) \to \Gamma(EM)$, which is the sum of $-\operatorname{tr}(\nabla(\nabla\sigma))$ and some differential operator of order at most one.

For what follows, it will be important that there are two different strains of applications of such operators, which build on different points of view. The more obvious strain is formed by applications of such operators to Riemannian geometry. In fact, already the study of the (standard) Laplacian $\Delta(f) = -\operatorname{tr}(\nabla(\nabla f))$ on smooth functions on a Riemannian manifold (M, g) forms a substantial part of Riemannian geometry. It turns out that Δ can be extended to an essentially self-adjoint, non-negative, unbounded operator on the Hilbert space $L^2(M)$ of L^2-functions. If M is compact, then this extension has discrete spectrum consisting of eigenvalues with finite multiplicity. This spectrum contains an amazing amount of information on (M, g), and *spectral geometry* is devoted to the study of these relations, see e.g. [**La15**]. For non-compact manifolds, the spectrum of the Laplacian becomes much more complicated, with both eigenvalues and essential spectrum showing up. But studying this spectrum still is a very active area of research, involving topics like scattering theory.

The second line of applications is less obvious but equally important. Here the main interest is in the topology of a smooth manifold M, on which one chooses a Riemannian metric is as an auxiliary ingredient. Alternatively, the results can also go in the direction that existence of Riemannian metrics on M with certain properties imply restrictions on the topology of M. The model example for this line of application is provided by the Laplace–Beltrami operator on differential forms on an oriented Riemannian manifold M. Recall that for any smooth manifold M and any $k = 0, \dots, \dim(M) - 1$, one has the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$. These operators satisfy $d \circ d$ and the quotient $\ker(d)/\operatorname{im}(d)$ in degree k is called the kth *de Rham cohomology* $H^k(M)$ of M. These de Rham cohomology spaces of M are fundamental topological invariants.

Let us sketch the construction of the Laplace–Beltrami operator, see Section 1.6 of [**Riem**] for details: If M is oriented and endowed with a Riemannian metric g, then one defines a tensorial operation $*$, the *Hode–$*$–operator*. This maps k-forms to $(n-k)$-forms, where n is the dimension of M. The *codifferential* $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ is then defined as $\delta := (-1)^{nk+n+1} * \circ d \circ *$. Having this at hand, the Laplace–Beltrami operator $\Delta : \Omega^k(M) \to \Omega^k(M)$ is defined by $\Delta = d \circ \delta + \delta \circ d$. This is easily seen to be a generalized Laplacian. The upshot of this construction is that, on a compact manifold, δ is adjoint to d with respect to natural inner products on the spaces of forms, so Δ is symmetric with respect to these inner products and $\ker(\Delta) = \ker(d) \cap \ker(\delta)$, see Proposition 1.6 of [**Riem**].

This result also shows that the projection $\ker(d) \to H^k(M)$ restricts to an injection on $\ker(\Delta)$. On a finite dimensional space adjointness of d and δ would directly show that $\ker(\delta)$ is the orthocomplement of $\mathrm{im}(d)$ and thus $\ker(d) \cap \ker(\delta)$ is complementary to $\mathrm{im}(d)$ in $\ker(d)$. Using functional analysis, it can be proved that, for a compact manifold M, things also work out for the infinite dimensional space $\Omega^k(M)$, so the projection restricts to a linear isomorphism $\Omega^k(M) \supset \ker(\Delta) \to H^k(M)$. Since the spaces $H^k(M)$ turn out to be finite dimensional for a compact manifold M, the essential quantity is their dimension, which coincides with the dimension of the kernel of the Laplace Beltrami operator. This topological interpretation also shows that the dimension of the kernel of the Laplace–Beltrami operator is independent of the Riemannian metric in question, which is a very surprising result in its own right.

CHAPTER 3

Spin structures

In this chapter, we introduce the fundamental concepts of spin geometry as an "extension" of Riemannian geometry. We start by describing the general properties of spin groups and their spin representations abstractly. For dimensions 3 to 6 we have found these groups and representations in Chapter 1, for higher dimensions we will for now assume their existence and describe a general construction in Chapter 4 below. From the description of Riemannian geometry via the orthonormal frame bundle, one is lead to the notion of a spin structure as an extension of that bundle. Having given such a spin structure, we can prove existence of spinor bundles, the spin connection and the basic Dirac operator rather quickly. We then turn to the questions of existence and uniqueness of spin structures which are related to algebraic topology.

3.1. Properties of spin groups. Let us collect the properties of the spin groups we will use in what follows and check that we have indeed found all these data in low dimensions in Chapter 1. For the rest of this chapter, we will assume that for each dimension $n \geq 3$ there is

- A connected Lie group $Spin(n)$, called the *spin group* endowed with a surjective homomorphism $\rho : Spin(n) \to SO(n)$ with two-element kernel contained in the center of $Spin(n)$.
- A specific faithful complex unitary representation $\mathcal{S}$ of $Spin(n)$.
- A bilinear map $* : \mathbb{R}^n \times \mathcal{S} \to \mathcal{S}$ called *Clifford multiplication*, which satisfies the Clifford relations from Section 1.3 in the sense that $v*(w*z)+w*(v*z) = -2\langle v, w\rangle z$ for all $v, w \in \mathbb{R}^n$ and $z \in \mathcal{S}$. Moreover, this map is equivariant for the natural action of $Spin(n)$ coming from the given representation on $\mathcal{S}$ and the representation on $\mathbb{R}^n$ defined by ρ.

Recall that a representation is called faithful if only the neutral element of the group acts as the identity map and it is called unitary if there is a positive definite Hermitian inner product on the representation space for which the group elements act by unitary maps. Observe that since $\mathcal{S}$ is faithful, the non-trivial element of $\ker(\rho)$ acts non-trivially on $\mathcal{S}$. As discussed in Sections 1.4 and 1.5, this shows that $\mathcal{S}$ does not come from a representation of $SO(n)$.

Let us first verify that we have indeed found all that in dimensions 3 and 4 in Chapter 1. We started by defining $\mathbb{H}$ as an associative subalgebra of $M_2(\mathbb{C})$. For $n = 3$, we have found $Spin(3) := SU(2) \subset M_2(\mathbb{C})$, so this comes with a natural faithful unitary representation on $\mathcal{S} := \mathbb{C}^2$. Further, we have viewed $\mathbb{R}^3$ as the subspace $\operatorname{im}(\mathbb{H})$ so this already sits in $L(\mathcal{S}, \mathcal{S})$. Explicitly, $v \in \mathbb{R}^3$ corresponds to the matrix $A_v := \begin{pmatrix} iv_1 & v_2 + iv_3 \\ -v_2 + iv_3 & -iv_1 \end{pmatrix}$. The considerations in Section 1.3 or a simple direct computation show that $A_vA_w + A_wA_v = -2\langle v, w\rangle \mathbb{I}_2$. This exactly means that the map $\mathbb{R}^3 \times \mathbb{C}^2 \to \mathbb{C}^2$ defined by $v * z := A_vz$ satisfies the Clifford relations. Finally, the homomorphism $SU(2) \to SO(3)$ from Section 1.4 was characterized by $A_{\rho(B)v} =$

BA_vB^{-1}, which shows that $(\rho(B)v) * Bz = BA_vB^{-1}Bz = B(v * z)$ which is the natural equivariancy condition.

For $n = 4$ we have found $Spin(4) = SU(2) \times SU(2)$. Denoting the standard representations of the two factors on $\mathbb{C}^2$ by $\mathcal{S}^+$ and $\mathcal{S}^-$ we get a natural faithful unitary representation of $Spin(4)$ on $\mathcal{S} := \mathcal{S}^+ \oplus \mathcal{S}^-$. Further, we have viewed $\mathbb{R}^4$ as $\mathbb{H} \subset M_2(\mathbb{C})$. Interpreting $M_2(\mathbb{C})$ as describing linear maps from $\mathcal{S}^-$ to $\mathcal{S}^+$, we can thus associate to $v \in \mathbb{R}^4$ a map $f_v : \mathcal{S}^- \to \mathcal{S}^+$. Denoting by $f_v^* : \mathcal{S}^+ \to \mathcal{S}^-$ the adjoint map, we define $* : \mathbb{R}^4 \times \mathcal{S} \to \mathcal{S}$ by $v * \binom{z^+}{z^-} := \binom{f_v(z^-)}{-f_v^*(z^+)}$. By definition, we get

$$v * \left(w * \begin{pmatrix} z^+ \\ z^- \end{pmatrix}\right) + w * \left(v * \begin{pmatrix} z^+ \\ z^- \end{pmatrix}\right) = \begin{pmatrix} -(f_v f_w^* + f_w f_v^*)(z^+) \\ -(f_v^* f_w + f_w^* f_v)(z^-) \end{pmatrix}.$$

Written in terms of quaternions, the terms in the right hand side are $-(p\bar{q} + q\bar{p})$ and $-(\bar{p}q + \bar{q}p)$. These both give $-2\langle p, q\rangle$ and thus the Clifford relations are satisfied. The homomorphism $SU(2) \times SU(2) \to SO(4)$ from Section 1.5 is characterized by $f_{\rho(B_1,B_2)v} = B_1 f_v B_2^*$ and hence $f^*_{\rho(B_1,B_2)v} = B_2 f_v^* B_1^*$. Using this, we compute

$$(\rho(B_1, B_2)v) * \begin{pmatrix} B_1 z^+ \\ B_2 z^- \end{pmatrix} = \begin{pmatrix} B_1 f_v B_2^* B_2 z^- \\ -B_2 f_v^* B_1^* B_1 z^+ \end{pmatrix} = \begin{pmatrix} B_1(f_v(z^-)) \\ -B_2 f_v^*(z^+) \end{pmatrix},$$

and the right hand side is the action of (B_1, B_2) on $v * \binom{z^+}{z^-}$.

As indicated by this discussion, there is a difference between even and odd dimensions. For an odd dimension, let us write $n = 2m - 1$. Then it turns out that the spin representation always is a complex irreducible representation $\mathcal{S}$ of complex dimension 2^m. In particular, we have seen in Section 1.6 that $Spin(5) \cong Sp(2) \subset Sp(4, \mathbb{C})$. The spin representation $\mathcal{S}$ is just the restriction of the standard representation (on $\mathbb{C}^4$) of $Sp(4, \mathbb{C})$ to $Sp(2)$. It is also possible to describe the Clifford multiplication in this picture, but we don't go into that. In even dimensions $n = 2m$, the spinor representation $\mathcal{S}$ is a direct sum $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ of two *half-spin representations* which are irreducible. Clifford multiplication exchanges the two factor, so it maps $\mathcal{S}^+$ to $\mathcal{S}^-$ and vice versa. In dimension 6, we have seen in Section 1.6 that $Spin(6) \cong SU(4)$ and the half-spin representations are the basic representation of $SU(4)$ on $\mathbb{C}^4$ and its dual $\mathbb{C}^{4*}$.

Spin structures and the Dirac operator

3.2. Definition of spin structures. Let (M, g) be a smooth, oriented Riemannian manifold of dimension $n \geq 3$. Then from Proposition 2.5 we get the oriented orthonormal frame bundle $\mathcal{SO}M$, which is a principal bundle with structure group $SO(n)$. Now a spin structure on M is defined as an "extension" of this bundle to a principal bundle with structure group $Spin(n)$ in a natural sense.

DEFINITION 3.2. (1) A *spin structure* on an oriented Riemannian manifold (M, g) is given by

- A principal fiber bundle $Q \to M$ with structure group $Spin(n)$.
- A fiber bundle homomorphism $\Phi : Q \to \mathcal{SO}M$ to the oriented orthonormal frame bundle with base map id_M, which is equivariant over the quotient homomorphism $\rho : Spin(n) \to SO(n)$ in the sense that $\Phi(u \cdot g) = \Phi(u) \cdot \rho(g)$ for each $u \in Q$ and each $g \in Spin(n)$.

(2) Given a spin structure $Q \to M$, the *spinor bundle* $\Sigma \to M$ is defined as the associated bundle $Q \times_{Spin(n)} \mathcal{S}$ corresponding to the spin representation $\mathcal{S}$.

At this stage it is not clear whether a given oriented Riemannian manifold (M, g) does admit a spin structure, but given such a structure, we can derive several consequences: Via $\rho : Spin(n) \to SO(n)$, any representation of $SO(n)$ can also be viewed as a representation of $Spin(n)$. In particular, any natural vector bundle on (M, g) in the sense of Example 2.6 can also be viewed an associated bundle to the $Spin(n)$-principal bundle $Q \to M$. By definition, the spin representation $\mathcal{S}$ does not descend to $SO(n)$ since the non-trivial element in $\ker(\rho)$ acts non-trivially on $\mathcal{S}$. Hence the spinor bundle $\Sigma \to M$ can be viewed as providing "new" geometric objects associated to a spin structure. Note that for even dimensions, the decomposition $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ gives associated bundles $\Sigma^\pm \to M$ such that $\Sigma = \Sigma^+ \oplus \Sigma^-$.

It is almost as easy to see that the Levi-Civita connection gives rise to a principal connection on any spin structure over a Riemannian manifold.

Proposition 3.2. *Let (M, g) be a oriented Riemannian manifold and let $\Phi : Q \to \mathcal{SO}M$ be a spin structure. Then Φ is a local diffeomorphism and a two-fold covering. Hence the Levi-Civita connection on $\mathcal{SO}M$ canonically lifts to a principal connection on Q, which defines the same covariant derivative on all vector bundles bundles coming from representations of $SO(n)$.*

Proof. Let $U \subset M$ be an open subset such that Q admits a local section σ defined on U. Since Φ has base map id_M, we see that $\Phi \circ \sigma : U \to \mathcal{SO}M$ defines a local smooth section, too. By Corollary 2.4, we obtain corresponding principal bundle charts $\tilde{\varphi}$ for Q and φ for $\mathcal{SO}(M)$ such that $\varphi \circ \Phi \circ \tilde{\varphi}^{-1} : U \times Spin(n) \to U \times SO(n)$ satisfies $(x, e) \mapsto (x, e)$ and thus $(x, g) \mapsto (x, \rho(g))$ by equivariancy. Hence each tangent map of Φ has to be bijective, so it is a local diffeomorphism. Moreover, by construction Φ is surjective and for each point in $\mathcal{SO}M$, the pre-image consists of exactly two elements, so Φ is a two-fold covering.

For each $u \in Q$, we have just seen that $T_u\Phi : T_uQ \to T_{\Phi(u)}\mathcal{SO}M$ is a linear isomorphism and this preserves the vertical subspaces since Φ is a fiber bundle map. Hence we can define $\mathcal{H}_u \subset T_uQ$ as the pre-image of the horizontal subspace of the Levi-Civita connection under $T_u\Phi$. For each u, this is complementary to the vertical subspace. Moreover, for $g \in Spin(n)$ we have $\Phi \circ r^g = r^{\rho(g)} \circ \Phi$, which implies that $T_{u\cdot g}\Phi(T_ur^g(\mathcal{H}_u)) = T_{\Phi(u)}r^{\rho(g)}(T_u\Phi(\mathcal{H}_u))$. Equivariancy of the Levi-Civita connection shows that the right hand side coincides with the horizontal subspace of the Levi-Civita connection in $\Phi(u) \cdot \rho(g) = \Phi(u \cdot g)$. Thus we conclude that $T_ur^g(\mathcal{H}_u) = \mathcal{H}_{u\cdot g}$, so we have indeed defined a principal connection on Q.

Now given a representation V of $SO(n)$, we view it as a representation of $Spin(n)$ via ρ. As discussed above, we can identify the bundles $Q \times_{Spin(n)} V$ and $\mathcal{SO}M \times_{SO(n)} V$ via $[(u, v)] \mapsto [(\Phi(u), v)]$. Denoting that bundle by E, Proposition 2.8 tells us that a section $s \in \Gamma(E)$ corresponds to an $SO(n)$-equivariant smooth function $f : \mathcal{SO}(M) \to V$. Similarly, it corresponds to a $Spin(n)$-equivariant function $Q \to V$, but from the construction it is clear that this function is just $f \circ \Phi$. For $\xi \in \mathfrak{X}(M)$, we have the horizontal lift $\xi^{\mathrm{hor}} \in \mathfrak{X}(Q)$ and the covariant derivative of s with respect to the lifted connection corresponds to $u \mapsto \xi^{\mathrm{hor}}(u) \cdot (f \circ \Phi) = (T_u\Phi(\xi^{\mathrm{hor}}(u))) \cdot f$. But from the description above it is clear that $T_u\Phi(\xi^{\mathrm{hor}}(u))$ is the horizontal lift of ξ in $\Phi(u)$ with respect to the Levi-Civita connection. So we just get the value in $\Phi(u)$ of the function corresponding to the covariant derivative with respect to the Levi-Civita connection. □

The principal connection on Q is often called the *spin-connection*, but we will just view it as an extension of the Levi-Civita connection. Therefore we will also denote all covariant derivatives with respect to either of the two principal connections by ∇.

3.3. The Dirac operator. Given a spin structure $Q \to M$ over an oriented Riemannian manifold (M, g), we can form the spin bundle $\Sigma = Q \times_{Spin(n)} \mathcal{S}$ over M which is a complex vector bundle. From Proposition 3.2 we know that there is a natural covariant derivative on sections of this spin bundle, which are usually called *spinors*. As discussed in Section 2.9 for tensor bundles, given a section $\psi \in \Gamma(\Sigma)$, we can interpret the covariant derivative $\nabla\psi$ as a section of the vector bundle $L(TM, \Sigma) = Q \times_{Spin(n)} L(\mathbb{R}^n, \mathcal{S})$. Hence it is no problem to form another covariant derivative to get $\nabla\nabla\psi$, which is a section of the $Q \times_{Spin(n)} L(\mathbb{R}^n, L(\mathbb{R}^n, \mathcal{S}))$. Continuing as in Section 2.9, there is a $Spin(n)$-equivariant trace map $L(\mathbb{R}^n, L(\mathbb{R}^n, \mathcal{S})) \to \mathcal{S}$. Using this, we can define the Laplace operator on Spinors as $\Delta\psi := -\operatorname{tr}(\nabla^2\psi)$.

But in the case of spinors, there is another way to proceed. By linear algebra, $L(\mathbb{R}^n, \mathcal{S}) \cong \mathbb{R}^{n*} \otimes \mathcal{S}$ and $\mathbb{R}^{n*} \cong \mathbb{R}^n$ as a representation of $SO(n)$ and hence of $Spin(n)$. The Clifford multiplication from Section 3.1 defines a $Spin(n)$-equivariant bilinear map $* : \mathbb{R}^n \times \mathcal{S} \to \mathcal{S}$ which corresponds to a linear map $\mathbb{R}^n \otimes \mathcal{S} \to \mathcal{S}$. Hence we obtain a $Spin(n)$-equivariant map $L(\mathbb{R}^n, \mathcal{S}) \to \mathcal{S}$, which we can describe more explicitly as follows.

Lemma 3.3. *For a linear map $\alpha : \mathbb{R}^n \to \mathcal{S}$ and a positively oriented orthonormal basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$, consider the expression $C(\alpha) := \sum_{i=1}^n e_i * \alpha(e_i) \in \mathcal{S}$. Then this is independent of the choice of basis and defines a $Spin(n)$-equivariant map $C : L(\mathbb{R}^n, \mathcal{S}) \to \mathcal{S}$.*

Proof. Any positively oriented orthonormal basis is of the form $\{A(e_1), \dots, A(e_n)\}$ for a matrix $A \in SO(n)$. Writing $A = (a^i_j)$ we get $A(e_i) = \sum_j a^j_i e_j$, and thus

$$\textstyle\sum_i A(e_i) * \alpha(A(e_i)) = \sum_{i,j,k} a^j_i a^k_i e_j * \alpha(e_k),$$

where we have used linearity of α. But since A is orthogonal, it satisfies $A^t = A^{-1}$, so $\sum_i a^j_i a^k_i = \delta^{jk}$, so we see that we again get $\sum_j e_j * \alpha(e_j)$.

This is also the key towards equivariancy. Indeed, for $h \in Spin(n)$, we have

$$(h \cdot \alpha)(v) = h \cdot \alpha(\rho(h)^{-1}(v))),$$

and this is mapped to $\sum_i e_i * h \cdot \alpha(\rho(h)^{-1}(e_i))$. Replacing the basis $\{e_i\}$ by $\{\rho(h)(e_i)\}$, we see that this expression equals $\sum_i(\rho(h)(e_i) * (h \cdot \alpha(e_i))$. Equivariancy of the Clifford multiplication tells us that this equals $\sum_i h \cdot (e_i * \alpha(e_i))$. Since h acts by linear maps, this equals $h \cdot (\sum_i e_i * \alpha(e_i))$ which is exactly the claimed equivariancy. □

We will call the map $C : L(\mathbb{R}^n, \mathcal{S}) \to \mathcal{S}$ the *Clifford action*, since in essence, it is just the Clifford multiplication. This is all we need to define the Dirac operator.

Definition 3.3. Given a spin structure $Q \to M$ on an oriented Riemannian manifold (M, g) with spinor bundle $\Sigma \to M$, we define the Dirac operator $\not{D} : \Gamma(\Sigma) \to \Gamma(\Sigma)$ by $\not{D}(\psi) := \mathcal{C}(\nabla\psi)$. Here $\mathcal{C} : L(TM, \Sigma) \to \Sigma$ is the vector bundle map induced by the Clifford action.

Proposition 3.3. *Let $Q \to M$ be a spin structure on an oriented Riemannian manifold (M, g), let $\Sigma \to M$ be the associated spinor bundle and $\not{D} : \Gamma(\Sigma) \to \Gamma(\Sigma)$ be the Dirac operator.*

(1) For a local orthonormal frame $\{\xi_1, \dots, \xi_n\}$ defined on $U \subset M$ and $\psi \in \Gamma(\Sigma)$, we get $\not{D}(\psi)|_U = \sum_i \xi_i * \nabla_{\xi_i}\psi$.

(2) For even n, the Dirac operator maps $\Gamma(\Sigma^+)$ to $\Gamma(\Sigma^-)$ and vice versa.

(3) The square of the Dirac operator is a generalized Laplacian, i.e. the operators $\not{D} \circ \not{D}$ and Δ differ by a first order operator.

PROOF. By definition, $\not{D}(\psi)$ corresponds to the function $Q \to \mathcal{S}$ given by $C \circ f$, where $f : Q \to L(\mathbb{R}^n, \mathcal{S})$ corresponds to $\nabla\psi$. On the other hand, denoting by $\tilde{U} \subset Q$ the preimage of U, the vector fields ξ_i correspond to functions $f_i : \tilde{U} \to \mathbb{R}^n$, whose values in each point of $\tilde{U}$ form an orthonormal basis of $\mathbb{R}^n$. By Lemma 3.3, we thus get $C(f(u)) = \sum_i f_i(u) * f(u)(f_i(u))$ for each $u \in \tilde{U}$. But by construction, the function $\tilde{U} \to \mathcal{S}$ defined by $u \mapsto f(u)(f_i(u))$ represents $\nabla_{\xi_i}\psi|_U$, so (1) follows.

(2) A section $\psi \in \Gamma(\Sigma^+)$ corresponds to a function with values in $\mathcal{S}^+ \subset \mathcal{S}$ and of course than also the derivative of this function in direction of a vector field has values in $\mathcal{S}^+$. Thus we see that $\nabla\psi \in \Gamma(L(TM, \mathcal{S}^+))$. Computing locally using (1), we see that $\nabla_{\xi_i}\psi \in \Gamma(\Sigma^+)$ and the result follows since the Clifford multiplication exchanges the spinor bundles.

(3) The main ingredient here is differtiating bilinear maps. For $\xi, \eta \in \mathfrak{X}(M)$ and $\psi \in \Gamma(\Sigma)$ we first claim that $\nabla_\xi(\eta * \psi) = (\nabla_\xi\eta) * \psi + \eta * (\nabla_\xi\psi)$. Denoting by $f : Q \to \mathbb{R}^n$ and $g : Q \to \mathcal{S}$ the functions corresponding to η and ψ, we see that $\eta * \psi$ corresponds to $* \circ (f, g)$, where $*$ is the Clifford multiplication $\mathbb{R}^n \times \mathcal{S} \to \mathcal{S}$. To get the function corresponding to $\nabla_\xi(\eta * \psi)$ we have to differentiate this with respect to the horizontal lift ξ^{hor} of ξ. Since $*$ is bilinear, we get $\xi^{\text{hor}} \cdot (* \circ (f, g)) = * \circ (\xi^{\text{hor}} \cdot f, g) + * \circ (f, \xi^{\text{hor}} \cdot g)$, which implies our claim.

Next, we claim that $(\nabla\nabla\psi)(\xi, \eta) = \nabla_\xi\nabla_\eta\psi - \nabla_{\nabla_\xi\eta}\psi$. Here we use bilinearity evaluation map $L(\mathbb{R}^n, \mathcal{S}) \times \mathbb{R}^n \to \mathcal{S}$ as above to conclude that

$$\nabla_\xi((\nabla\psi)(\eta)) = (\nabla_\xi(\nabla\psi))(\eta) + (\nabla\psi)(\nabla_\xi\eta),$$

which exactly gives the claimed formula.

Now we compute locally using (1) to get

$$\not{D}(\not{D}\psi) = \textstyle\sum_i \xi_i * (\nabla_{\xi_i}(\sum_j \xi_j * \nabla_{\xi_j}\psi) = \sum_{i,j} \xi_i * \left((\nabla_{\xi_i}\xi_j) * \nabla_{\xi_j}\psi + \xi_j * \nabla_{\xi_i}\nabla_{\xi_j}\psi\right).$$

Expanding the sum in the last bracket according to bilinearity of $*$, the first term evidently produces a first order operator, so we can ignore it and only deal with the term $\sum_{i,j} \xi_i * (\xi_j * \nabla_{\xi_i}\nabla_{\xi_j}\psi)$. Now the function corresponding to $\nabla_{\xi_i}\nabla_{\xi_j}\psi$ is obtained by differentiating the function corresponding to ψ with two horizontal lifts of vector fields. Subtracting the same term with the two vector fields exchanged, one gets the derivative with respect to the Lie bracket, which is a first order operator. Hence up to a first order operator, we may replace $\nabla_{\xi_i}\nabla_{\xi_j}\psi$ by $1/2(\nabla_{\xi_i}\nabla_{\xi_j}\psi + \nabla_{\xi_j}\nabla_{\xi_i}\psi)$. But then

$$\begin{aligned} \tfrac{1}{2}\textstyle\sum_{i,j} \xi_i * \xi_j * (\nabla_{\xi_i}\nabla_{\xi_j}\psi + \nabla_{\xi_j}\nabla_{\xi_i}\psi) &= \tfrac{1}{2}\textstyle\sum_{i,j}(\xi_i * \xi_i * \nabla_{\xi_i}\nabla_{\xi_j}\psi + \xi_j * \xi_i * \nabla_{\xi_i}\nabla_{\xi_j}\psi) \\ &= -\textstyle\sum_{i,j} g(\xi_i, \xi_j)\nabla_{\xi_i}\nabla_{\xi_j}\psi = -\sum_i \nabla_{\xi_i}\nabla_{\xi_i}\psi, \end{aligned}$$

where we have used the Clifford relation in the last but one step. But from above, we see that the last expression differs from $-\operatorname{tr}(\nabla\nabla\psi) = \Delta\psi$ by a first order operator. □

REMARK 3.3. It turns out that the relation between $\not{D}^2$ and Δ is significantly simpler than the above computations suggest. Indeed, invoking a bit of Riemannian geometry it is visible that things can be simplified. We have used an arbitrary local orthonormal frame $\{\xi_i\}$ in the computation, but working in a point $x \in M$, one can do better. By starting with normal coordinates around x and then orthonormalizing the resulting coordinate frame, one may assume that $\nabla_{\xi_i}\xi_j(x) = 0$. This shows that many of the first order contributions occurring in our computation actually vanish in x. For the last part of the computation, it is already visible (for those who know about these things) that a curvature term will show up in the computation. Indeed things work out very nicely, which is a major reason for the importance of the Dirac operator in Riemannian geometry: The resulting curvature term turns out to be simply given by multiplication by a multiple the scalar curvature r of the underlying Riemannian metric. This is the famous *Schrödinger–Lichnerowicz formula*, $\not{D}^2 = \Delta + \frac{r}{4}$, see Section 3.6 of [**BGV92**] or Theorem 17 of [**Ma12**].

This result has immediate strong consequences for metrics of positive scalar curvature (i.e. manifolds which satisfy $r \geq 0$ and $r \neq 0$). It is easy to see that for the natural Hermitian inner product on spinors, one has $\langle \Delta\psi, \psi\rangle = \langle \nabla\psi, \nabla\psi\rangle \geq 0$. For a metric of positive sectional curvature, we can use this to see that $\int_M \not{D}^2\psi > 0$, so in this case the Dirac operator must have trivial kernel. Via the index theorem, which we will discuss in a bit more detail later, this leads to topological obstructions to the existence of metrics of positive scalar curvature.

There are lots of other applications of the Dirac operator in Riemannian geometry. On the one hand, there is a big theory of differential equations on spinors derived from the Dirac operator and on the geometric meaning of existence of solutions to these equations. On the other hand, the ideas about the spectrum of the Laplace operator have a counterpart for the Dirac operator and a lot is known in that direction. Discussions of applications of Dirac operators in Riemannian geometry can for example be found in the books [**BHMMM15**] and [**Fr00**].

Existence and uniqueness of spin structures

3.4. Open coverings and Čech cochains. The questions of existence and uniqueness of spin structures are closely related to algebraic topology. For our purposes it will be most appropriate to use an approach to cohomology theory which was introduced by E. Čech and is based on open coverings. We will not go into details on the relation to other approaches to cohomology.

We need a notion of open coverings for which the same open set may occur several times in a covering. Hence we define an *open covering* of a topological space X as a function which associated to each element i of some index set I an open subset $U_i \subset X$ in such a way that $\cup_{i\in I} U_i = X$. We will also write the cover as a tuple, $\mathcal{U} = (U_i)_{i\in I}$. Given two open coverings $\mathcal{U} = (U_i)_{i\in I}$ and $\mathcal{V} = (V_j)_{j\in J}$ of X, we write $\mathcal{V} \leq \mathcal{U}$ and say that $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if each of the sets in $\mathcal{V}$ is contained on of the sets in $\mathcal{U}$. Otherwise put, there has to be a *refinement map* $\mu : J \to I$ such that $V_j \subset U_{\mu(j)}$ for any $j \in J$.

Definition 3.4. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering of a topological space X.

(1) For $k \geq 0$, a *k-simplex* σ of $\mathcal{U}$ is a $k+1$-tuple $(i_0, \dots, i_k)$ of elements of I such that the open subset $|\sigma| := \cap_{\ell=0}^k U_{i_\ell}$ is non-empty. The set $|\sigma| \subset X$ is called the support of σ.

(2) Given an abelian group $(G, +)$, a Čech *k-cochain of $\mathcal{U}$* with coefficients in G is a function c, which associates to each k-simplex σ of $\mathcal{U}$ a locally constant function $c(\sigma) = c_\sigma : |\sigma| \to G$. The space of all such cochains, which clearly is an abelian group under pointwise addition, is denoted by $C^k(\mathcal{U}, G)$.

(3) Given a k-simplex $\sigma = (i_0, \dots, i_k)$ of $\mathcal{U}$ and an integer j with $0 \leq j \leq k$. Then the *jth side σ_j of* σ is the $(k-1)$-simplex $(i_0, \dots, i_{j-1}, i_{j+1}, \dots, i_k)$, so we simply leave out the index i_j. Observe that by definition $|\sigma| \subset |\sigma_j|$ for each j.

(4) The *Čech coboundary operator* $\partial = \partial_k : C^k(\mathcal{U}, G) \to C^{k+1}(\mathcal{U}, G)$ is defined by

$$\partial c(\sigma) = \textstyle\sum_{j=0}^{k+1} (-1)^j c(\sigma_j)|_{|\sigma|}.$$

In what follows, we will use the convention that putting a hat over an element in a tuple means that this element has to be left out. So in this notation, we can write that jth side of $\sigma = (i_0, \dots, i_k)$ as $\sigma_j = (i_0, \dots, \widehat{i_j}, \dots, i_k)$. To get things started, we have to prove that the Čech coboundary operator is a differential, i.e. applying it twice, we get zero.

LEMMA 3.4. *Each Čech coboundary operator is a homomorphism and they satisfy* $\partial_k \circ \partial_{k-1} = 0$. *Thus for each k, we have subgroups* $\operatorname{im}(\partial_{k-1}) \subset \ker(\partial_k) \subset C^k(\mathcal{U}, G)$.

PROOF. The fact that each ∂_k is a homomorphism is clear from the definition. Next, take a $k+1$-simplex $\sigma = (i_0, \dots, i_{k+1})$ of $\mathcal{U} = (U_i)_{i\in I}$ and consider its jth side σ_j, as well as the ℓth side $(\sigma_j)_\ell$ of σ_j for $0 \le j \le k+1$ and $0 \le \ell \le k$. If $\ell < j$, this simply is $(i_0, \dots, \widehat{i_\ell}, \dots, \widehat{i_j}, \dots, i_k)$ while for $\ell \ge j$, it is $(i_0, \dots, \widehat{i_j}, \dots, \widehat{i_{\ell+1}}, \dots, i_k)$. This readily shows that for $\ell < j$ we get $(\sigma_j)_\ell = (\sigma_\ell)_{j-1}$. Using this and suppressing restrictions, we compute for $c \in C_{k-1}(\mathcal{U}, G)$ as follows:

$$\begin{aligned}\partial(\partial c)(\sigma) &= \textstyle\sum_{j=0}^{k+1}(-1)^j \partial c(\sigma_j) = \sum_{j=0}^{k+1}\sum_{\ell=0}^{k}(-1)^{\ell+j} c((\sigma_j)_\ell)\\ &= \textstyle\sum_{j=0}^{k+1}\sum_{\ell<j}(-1)^{\ell+j} c((\sigma_j)_\ell) + \sum_{j=0}^{k+1}\sum_{\ell\ge j}(-1)^{\ell+j} c((\sigma_\ell)_{j-1}).\end{aligned}$$

Now we can change the order of summation in the second term and then rename $j-1$ to j, to see that this equals $\sum_{\ell=0}^{k+1}\sum_{j<\ell}(-1)^{\ell+j+1}(\sigma_\ell)_j$ and thus cancels with the first sum. The last statement then obviously follows. □

Elements of $\ker(\partial_k) \subset C^k(\mathcal{U}, G)$ are called *k-cocycles*, and elements of $\operatorname{im}(\partial_{k-1}) \subset C^k(\mathcal{U}, G)$ are called *k-coboundaries* of the covering $\mathcal{U}$ with coefficients in G. The quotient space $H^k(\mathcal{U}, G) := \ker(\partial_k)/\operatorname{im}(\partial_{k-1})$ is called the *kth Čech cohomology group* of the covering $\mathcal{U}$ with coefficients in G.

EXAMPLE 3.4. (1) It is easy to determine $H^0(\mathcal{U}, G)$, which even turns out to be independent of the covering $\mathcal{U}$. Indeed, a 0-simplex of $\mathcal{U} = (U_i)_{i\in I}$ is just one index $i \in \mathcal{U}$, so a zero cochain $c \in C^0(\mathcal{U}, G)$ simply is a collection of locally constant functions $c_i : U_i \to G$. In degree zero, there are no coboundaries, so $H^0(\mathcal{U}, G) = \ker(\partial_0)$. A 1-simplex of U is just a pair (i, j) of indices such that $U_{ij} := U_i \cap U_j \neq \emptyset$. Now by definition, $\partial c((i, j)) = c_j|_{U_{ij}} - c_i|_{U_{ij}}$. Supposing that this vanishes, we define a function $f : X \to G$ as follows: For $x \in X$, there exist an index $i \in I$ such that $x \in U_i$ and we define $f(x) = c_i(x)$. Since $\partial c = 0$, this is independent of the choice of i, and since $f|_{U_i} = c_i$, we see that f is locally constant.

Conversely, we see that for a locally constant function $f : X \to G$, the rule $c_i := f|_{U_i}$ defines a 0-cochain, which clearly is a cocycle. So we conclude that $H^0(\mathcal{U}, G)$ is isomorphic to the space of locally constant functions $X \to G$. Observe in particular, that in case that X is a smooth manifold and $G = \mathbb{R}$, this coincides with the de Rham cohomology in degree zero, since for $f \in C^\infty(M, \mathbb{R})$ the equation $df = 0$ is equivalent to f being locally constant.

(2) Another instructive example is the covering $\mathcal{U} = (U_0, U_1, U_2)$ of S^1 formed by 3 arcs of a bit more of 120 degrees. Thus U_{01}, U_{02}, and U_{12} are small arcs (and thus topologically trivial), while the intersection of all three sets is empty. (For those who know about these things, this is the covering dual to the standard triangulation of S^1 as a triangle.) Hence we see that $C^k(\mathcal{U}, G) = G^3$ for $k = 0, 1$ and zero for all other degrees. In this identification, the only non-trivial coboundary operator is ∂_0, which is explicitly given by $\partial_0(a, b, c) = (b - a, c - a, c - b)$. As we have seen above, its kernel is $\{(a, a, a) : a \in G\} \cong G$. On the other hand, the map $C^1(\mathcal{U}, G) \to G$ given by $(u, v, w) \mapsto u - v + w$ is a homomorphism and visibly vanishes on $\operatorname{im}(\partial_0)$. Conversely, an

element in the kernel of this homomorphism is of the form $(u, u+w, w) = \partial(0, u, u+w)$. So we see that $H^k(\mathcal{U}, G)$ is isomorphic to G for $k = 0, 1$ and trivial for all other k. Again, this coincides with the de Rham cohomology of S^1 for $G = \mathbb{R}$.

3.5. Refinements and Čech cohomology. We next have to relate Čech cochains corresponding to different coverings. The first step is to look at a refinement $\mathcal{V} = (V_j)_{j\in J}$ of a covering $\mathcal{U} = (U_i)_{i\in I}$ and consider a refinement map $\mu : J \to I$, i.e. a map such that $V_j \subset U_{\mu(j)}$ for each $j \in J$. Now suppose that $\sigma = (j_0, \dots, j_k)$ is a k-simplex of $\mathcal{V}$. Then for each $\ell = 0, \dots, k$, we have $\mu(j_\ell) \in I$ and $\emptyset \neq \cap_\ell V_{j_\ell} \subset \cap_\ell U_{\mu(j_\ell)}$. Thus we see that $\mu(\sigma) := (\mu(j_0), \dots, \mu(j_k))$ is a k-simplex of $\mathcal{U}$ and that $|\sigma| \subset |\mu(\sigma)|$. By construction, it is clear that for the sides we get $\mu(\sigma_\ell) = (\mu(\sigma))_\ell$ for each $\ell = 0, \dots k$.

This readily implies that for any abelian group G and any cochain $c \in C^k(\mathcal{U}, G)$, we can define $\mu^*c \in C^k(\mathcal{V}, G)$ by $(\mu^*c)(\sigma) := c(\mu(\sigma))|_{|\sigma|}$, thus obtaining a homomorphism $\mu^* : C^k(\mathcal{U}, G) \to C^k(\mathcal{V}, G)$. Moreover, the observation on sides shows that $\mu^*\circ\partial = \partial\circ\mu^*$, so in particular for a cocycle c, also μ^*c is a cocycle and for a coboundary c, also μ^*c is a coboundary. Hence there is an induced map in cohomology, and we can prove a technical lemma which implies that this map is independent of the choice of the refinement map μ.

Lemma 3.5. *Let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering of a topological space X, let $\mathcal{V} = (V_j)_{j\in J}$ be a refinement of $\mathcal{U}$ and consider two refinement maps $\mu, \nu : J \to I$. Then for any abelian group G and any $k \geq 0$, we can construct a map $h_k : C^k(\mathcal{U}, G) \to C^{k-1}(\mathcal{V}, G)$ such that for each $c \in C^k(\mathcal{U}, G)$ we get*

$$\nu^*c - \mu^*c = \partial(h_k(c)) + h_{k+1}(\partial c).$$

In particular, if c is a cocycle, then ν^*c and μ^*c represent the same class in $H^k(\mathcal{V}, G)$.

Proof. Let $\sigma = (j_0, \dots, j_{k-1})$ be a $(k-1)$-simplex of $\mathcal{V}$. Then for each $\ell = 0, \dots, k-1$, we get $V_{j_\ell} \subset U_{\mu(j_\ell)} \cap U_{\nu(j_\ell)}$, so $\widetilde{\sigma}^\ell := (\mu(j_0), \dots, \mu(j_\ell), \nu(j_\ell), \dots, \nu(j_{k-1}))$ is a k-simplex of $\mathcal{U}$ such that $|\sigma| \subset |\widetilde{\sigma}^\ell|$. Thus we can define $h_k : C^k(\mathcal{U}, G) \to C^{k-1}(\mathcal{V}, G)$ by

$$(h_k(c))(\sigma) := \textstyle\sum_{\ell=0}^{k-1}(-1)^\ell c(\widetilde{\sigma}^\ell)|_{|\sigma|}.$$

To see that this has the required property, we have to look at sides. By definition, $(\widetilde{\sigma}^\ell)_i$ is $\widetilde{(\sigma_i)}^{\ell-1}$ for $i < \ell$ and $\widetilde{(\sigma_{i-1})}^{\ell}$ for $i > \ell + 1$, while

$$(\widetilde{\sigma}^\ell)_\ell = (\mu(j_0), \dots, \mu(j_{\ell-1}), \nu(j_\ell), \dots, \nu(j_{k-1}))$$
$$(\widetilde{\sigma}^\ell)_{\ell+1} = (\mu(j_0), \dots, \mu(j_\ell), \nu(j_{\ell+1}), \dots, \nu(j_{k-1})).$$

Using this, we can now compute for a k-simplex σ of $\mathcal{V}$ (suppressing the restrictions to appropriate subsets) as follows.

$$\begin{aligned}(h_{k+1}(\partial c))(\sigma) &= \textstyle\sum_{\ell=0}^{k}(-1)^\ell \partial c(\widetilde{\sigma}^\ell) = \sum_{\ell=0}^{k}\sum_{i=0}^{k+1}(-1)^{i+\ell}c((\widetilde{\sigma}^\ell)_i)\\ &= \textstyle\sum_{\ell=0}^{k}\sum_{0\leq i<\ell}(-1)^{i+\ell}\widetilde{\sigma}_i^{\ell-1} + \sum_{\ell=0}^{k}\sum_{i>\ell+1}(-1)^{i+\ell}\widetilde{\sigma_{i-1}}^{\ell}\\ &\quad + \textstyle\sum_{\ell=0}^{k} c((\mu(j_0), \dots, \mu(j_{\ell-1}), \nu(j_\ell), \dots, \nu(j_k)))\\ &\quad - \textstyle\sum_{\ell=0}^{k} c((\mu(j_0), \dots, \mu(j_\ell), \nu(j_{\ell+1}), \dots, \nu(j_k)))\end{aligned}$$

The last two lines together form a telescopic sum, so only the very first and very last terms remain, and these exactly give $\nu^*c(\sigma) - \mu^*c(\sigma)$. In the two remaining sums in the right hand side we swap the sequence of summation. In the first of the two, we in addition replace $\ell - 1$ by ℓ to get $\sum_{i=0}^{k-1}\sum_{\ell=i}^{k}(-1)^{i+\ell+1}c(\widetilde{(\sigma_i)}^{\ell})$. In the other sum, we in addition replace $i-1$ by i and obtain $\sum_{i=0}^{k+1}\sum_{\ell=0}^{i-1}(-1)^{i+\ell+1}c(\widetilde{(\sigma_i)}^{\ell})$, so these two terms by definition add up to $-\partial(h_k(c))(\sigma)$, which proves the claimed identity.

If $\partial c=0$, then we get $\nu^*c-\mu^*c=\partial(h_k(c))$, which says that ν^*c and μ^*c represent the same cohomology class. □

The upshot of this is that for a refinement $\mathcal{V}$ of a covering $\mathcal{U}$ of X and any abelian group G, there is a *canonical* homomorphism $H^k(\mathcal{U},G)\to H^k(\mathcal{V},G)$ induced by any choice of refinement map. Using this, one could formally define the Čech cohomology of X with coefficients in G as the direct limit of the directed family $\{H^k(\mathcal{U},G)\}$ of sets, which is indexed by all open coverings of X. The following definition just expresses this in elementary terms.

Definition 3.5. Let X be a topological space and G an abelian group. Then we consider the set of all Čech k-cocycles c for some open covering $\mathcal{U}$ of X with coefficients in G. On this set, we define a relation by $c_1\sim c_2$ iff for the corresponding coverings $\mathcal{U}_1$ and $\mathcal{U}_2$, there is some joint refinement $\mathcal{V}$ with refinement maps $\mu_i:\mathcal{V}\to\mathcal{U}_i$ for $i=1,2$ such that $\mu_1^*c_1$ and $\mu_2^*c_2$ represent the same class in $H^k(\mathcal{V},G)$. Since this is independent of the refinement maps by Lemma 3.5, this is an equivalence relation. The set $H^k(X,G)$ of equivalence classes is the *kth Čech cohomology of* X with coefficients in G

From this definition we can easily conclude that the cohomologies have functorial properties. Suppose that G and H are abelian groups and that $\alpha:G\to H$ is a homomorphism. Then for a topological space X, an open covering $\mathcal{U}$ of X, and a cochain $c\in C^k(\mathcal{U},G)$ we can define a cochain $\alpha_*c\in C^k(\mathcal{U},H)$: We simply put $(\alpha_*c)(\sigma):=\alpha\circ c(\sigma):|\sigma|\to H$ and this clearly is a locally constant function. Thus we get a map $\alpha_*:C^k(\mathcal{U},G)\to C^k(\mathcal{U},H)$, and since α is a group homomorphism, the map α_* is a group homomorphism, too.

On the other hand, suppose that $f:X\to Y$ is a continuous map between topological spaces. Given an open covering $\mathcal{U}=(U_i)_{i\in I}$ of Y, we define $f^{-1}(\mathcal{U}):=(f^{-1}(U_i))_{i\in I}$ and clearly this is on open covering of X. A k-simplex σ of $f^{-1}(\mathcal{U})$ is a tupel $(i_0,\dots,i_k)$ such that $\cap_\ell f^{-1}(U_{i_\ell})\neq\emptyset$. This intersection equals $f^{-1}(\cap_\ell U_{i_\ell})$, so we can also view $(i_0,\dots,i_k)$ as a k-simplex $\hat\sigma$ of $\mathcal{U}$ and $|\sigma|=f^{-1}(|\hat\sigma|)$. Given $c\in C^k(\mathcal{U},G)$, we see that $(f^*c)(\sigma):=c(\hat\sigma)\circ f$ defines a locally constant function $|\sigma|\to G$. Thus we have constructed a map $f^*:C^k(\mathcal{U},G)\to C^k(f^{-1}(\mathcal{U}),G)$, which evidently is a group homomorphism.

Proposition 3.5. *Let X and Y be topological spaces and let G and H be abelian groups.*

(1) For a homomorphism $\alpha:G\to H$ and any covering $\mathcal{U}$ of X the homomorphisms $\alpha_*:C^k(\mathcal{U},G)\to C^k(\mathcal{U},H)$ have the property that $\partial\circ\alpha_*=\alpha_*\circ\partial$. Thus for a cocycle c, also $\alpha_*(c)$ is a cocycle and mapping the equivalence class $[c]$ of c to $[\alpha_*(c)]$ gives rise to a well defined homomorphism $\alpha_\#:H^k(X,G)\to H^k(X,H)$ for each $k\geq 0$.

(2) For a continuous map $f:X\to Y$ and any covering $\mathcal{U}$ of Y, the homomorphisms $f^*:C^k(\mathcal{U},G)\to C^k(f^{-1}(\mathcal{U}),G)$ have the property that $\partial\circ f^*=f^*\circ\partial$. Mapping $[c]$ to $[f^*(c)]$ gives rise to a well defined homomorphism $f^\#:H^k(Y,G)\to H^k(X,G)$ for each $k\geq 0$.

Proof. These are simple verifications:

(1) Since α is a homomorphism, $\alpha_*(\partial c)$ maps a simplex σ of $\mathcal{U}$ to

$$\alpha(\textstyle\sum_{\ell=0}^{k}(-1)^\ell c(\sigma_\ell))=\sum_{\ell=0}^{k}(-1)^\ell\alpha(c(\sigma_\ell)),$$

and this visibly coincides with $\partial(\alpha_*(c))(\sigma)$. In particular, $\partial c=0$ implies $\partial(\alpha_*(c))=0$, so for a cocycle $c\in C^k(\mathcal{U},G)$, $\alpha_*(c)$ defines a class in $H^k(X,H)$. Now suppose that

c_1 and c_2 represent the same class in $H^k(X,G)$. Then there is a refinement $\mathcal{V}$ of $\mathcal{U}$ with refinement map μ such that $\mu^*(c_1)=\mu^*(c_2)+\partial c$ for some $c\in C^{k-1}(\mathcal{V},G)$. But of course, we also have α_* acting on cochains of any degree for $\mathcal{V}$, and from the definitions it follows readily that $\alpha_*(\mu^*(c_i))=\mu^*(\alpha_*(c_i))$ for $i=1,2$. Thus applying α_* to the above equation, we get $\mu^*(\alpha_*(c_1))=\mu^*(\alpha_*(c_2))+\partial(\alpha_*(c))$, and thus $\alpha_*(c_1)\sim\alpha_*(c_2)$.

(2) As we have noted already, a simplex $\sigma=(i_0,\dots,i_k)$ of $f^{-1}(\mathcal{U})$ also is a simplex $\hat\sigma$ of $\mathcal{U}$ and $f(|\sigma|)=|\hat\sigma|$. Clearly, for any side σ_j the simplex $\widehat{\sigma_j}$ is exactly the jth side of $\widehat\sigma$. Thus $f^*(c)(\sigma_j)=c(\hat\sigma_j)\circ f$ for each $c\in C^{k-1}(\mathcal{U},G)$ and each j, which readily implies that $\partial(f^*(c))=f^*(\partial c)$. A refinement map μ identifying $\mathcal{V}$ as a refinement of $\mathcal{U}$ also identifies $f^{-1}(\mathcal{V})$ as a refinement of $f^{-1}(\mathcal{U})$. By construction, we then get $\mu^*(f^*(c))=f^*(\mu^*(c))$ and using this, one completes the proof as in (1). □

If we have two homomorphism $\alpha:G\to H$ and $\beta:H\to K$, then the definitions imply that $(\beta\circ\alpha)_*=\beta_*\circ\alpha_*:C^k(\mathcal{U},G)\to C^k(\mathcal{U},K)$ for each $\mathcal{U}$ and each k. Thus for the homomorphisms in cohomology, we get $(\beta\circ\alpha)_\#=\beta_\#\circ\alpha_\#$. In particular, if $\alpha:G\to H$ is an isomorphism and $\beta=\alpha^{-1}:H\to G$, then $\beta\circ\alpha=\mathrm{id}_G$ and $\alpha\circ\beta=\mathrm{id}_H$. Since the identity homomorphism induces the identity in cohomology, we see that in this case $\alpha_\#$ and $\beta_\#$ are inverse isomorphisms between the cohomologies $H^k(X,G)$ and $H^k(X,H)$.

Similarly, if we take two continuous maps $f:X\to Y$ and $g:Y\to Z$ and a covering $\mathcal{U}$ of Z, then $f^{-1}(\mathcal{U})$ is an open covering of Y and $(g\circ f)^{-1}(\mathcal{U})=g^{-1}(f^{-1}(\mathcal{U}))$ is an open covering of X. Further, we get $(g\circ f)^*=f^*\circ g^*:C^k(\mathcal{U},G)\to C^k((g\circ f)^{-1}(\mathcal{U}),G)$ for each k. As before, this implies that in cohomology we get $(g\circ f)^\#=f^\#\circ g^\#$ and that inverse homeomorphisms between two spaces induce inverse isomorphisms in cohomology. Thus the Čech cohomology groups are indeed topological invariants of spaces.

Remark 3.5. If one looks at the constructions we have done so far, it is visible, that things work in a more general setting. We have used cochains which assign to each simplex σ of a covering a locally constant function $|\sigma|\to G$. Instead, one could assign to σ an element of an abelian group which depends on $|\sigma|$ as long as there is a reasonable notion of "restriction" between the groups associated to nested open sets. This is formalized in the concept of a sheaf of abelian groups, and Čech cohomology is one of many possibilities of computing the cohomology of such sheaves. In some of the things we will do, this more general perspective will be visible. Moreover, if one restricts to cohomology in degree 1, it is even possible to extend to extend the ideas to non-commutative groups, and again this will be visible in some things we will do.

3.6. Example: Orientations. To prepare for the discussion of spin structures, we do a simpler example for the use of Čech cohomology. Suppose that we have given a real vector bundle $E\to M$ on a smooth manifold M or equivalently a principal bundle with structure group $GL(n,\mathbb{R})$ for some $n\geq 1$. We want to decide when the bundle E is orientable and how many orientations it admits.

Proposition 3.6. *A real vector bundle $p:E\to M$ canonically determines a cohomology class $w_1(E)\in H^1(M,\mathbb{Z}_2)$, which vanishes if and only if the bundle E is orientable. If $w_1(E)=0$, then the possible orientations of E are parametrized by $H^0(M,\mathbb{Z}_2)$.*

Proof. Choose a vector bundle atlas $\{(U_i,\varphi_i):i\in I\}$ for E and let $\mathcal{U}=(U_i)_{i\in I}$ be the corresponding covering of M. A 1-simplex σ of $\mathcal{U}$ is a pair (i,j), such that $|\sigma|=U_i\cap U_j\neq\emptyset$. Thus there is the transition function $\pi_{ij}:|\sigma|\to GL(n,\mathbb{R})$ and we define $c(\sigma)(x) := \frac{\det(\varphi_{ij}(x))}{|\det(\varphi_{ij}(x))|} \in \{-1, 1\}$. Since φ_{ij} is smooth, this is a continuous and thus locally constant function from U_{ij} to the multiplicative group $\mathbb{Z}_2$.

Now suppose that we have three elements of $\mathcal{U}$ such that $U_{ijk} \neq \emptyset$. Then for the corresponding charts and each $x \in U_{ijk}$, we have $\varphi_i(\varphi_j^{-1}(x, v)) = (x, \varphi_{ij}(x)v)$ and similarly for the other combination of indices. Computing $\varphi_i(\varphi_k^{-1}(x, v))$ as $\varphi_i(\varphi_j^{-1}(\varphi_j(\varphi_k^{-1}(x, v))))$ shows that $\varphi_{ij}(x)\varphi_{jk}(x) = \varphi_{ik}(x)$. Putting $\sigma = (i, j, k)$ and taking the signs of the determinants, this reads as $c(\sigma_2)c(\sigma_0) = c(\sigma_1)$ and hence $c(\sigma_0)c(\sigma_1)^{-1}c(\sigma_2) = 1$. This exactly says that $\partial c(\sigma) = 0$, so $c \in C^1(\mathcal{U}, \mathbb{Z}_2)$ is indeed a cocycle and we can form the cohomology class $[c] \in H^1(X, \mathbb{Z}_2)$.

Next, we observe that for a refinement $\mathcal{V} = (V_j)_{j\in J}$ of $\mathcal{U}$ with refinement map $\mu : J \to I$, we get an associated vector bundle atlas for E. For $j \in J$ with $\mu(j) = i$ we have $V_J \subset U_i$, and the restriction ofφ_i to $p^{-1}(V_j)$ defines a vector bundle chart. The resulting transition functions are simply restrictions of the functions φ_{ij}, so they lead to the cocycle μ^*c.

To compare the cocycles associated to two different vector bundle atlases, we can first pass to a joint refinement of the coverings defined by the atlases. Given two atlases $\{(U_i, \varphi_i)\}$ and $\{(U_i, \psi_i)\}$ corresponding to the same covering $\mathcal{U}$, we see that $\psi_i \circ \varphi_i^{-1} : U_i \times \mathbb{R}^n \to U_i \times \mathbb{R}^n$ has to be of the form $(x, v) \mapsto (x, \omega_i(x)(v))$ for a smooth function $\omega_i : U_i \to GL(n, \mathbb{R})$. Defining $\tilde{c} \in C^0(\mathcal{U}, \mathbb{Z}_2)$ by $\tilde{c}_i(x) = \frac{\det(\omega_i(x))}{|\det(\omega_i(x))|}$ we get a function $U_i \to \mathbb{Z}_2$ which is continuous and hence locally constant, so these define an element $\tilde{c} \in C^0(\mathcal{U}, \mathbb{Z}_2)$. Given U_j with $U_{ij} \neq \emptyset$, we can compute (on U_{ij}) $\psi_i \circ \varphi_j^{-1}$ either as $\psi_i \circ \psi_j^{-1} \circ \psi_j \circ \varphi_j^{-1}$ or as $\psi_i \circ \varphi_i^{-1} \circ \varphi_i \circ \varphi_j^{-1}$, which leads to $\psi_{ij}(x)\omega_j(x) = \omega_i(x)\varphi_{ij}(x)$ for all $x \in U_{ij}$. Taking signs of determinants, we can commute terms and get for $\sigma = (i, j)$ the equation $\tilde{c}(\sigma_0)|_{|\sigma|}(\tilde{c}(\sigma_1)|_{|\sigma|})^{-1} = c^\psi(\sigma)(c^\varphi(\sigma))^{-1}$. Since the left hand side is $\partial\tilde{c}(\sigma)$, this exactly says that the cocycles constructed from the two atlases represent the same class of $H^1(M, \mathbb{Z}_2)$ and we define this to be $w_1(E)$.

If E is oriented, then we can take an oriented atlas, which directly leads to the trivial cocycle $c \in C^1(\mathcal{U}, \mathbb{Z}_2)$, which sends any simplex to the constant function 1, so $w_1(E)$ is trivial. Conversely, if $w_1(E)$ is trivial, we can take any atlas $\{(U_i, \varphi_i)\}$ for E and form the corresponding cocycle c as above. Passing to a refinement and the corresponding atlas if necessary, we may assume that $c = \partial\tilde{c}$ for some $\tilde{c} \in C^0(\mathcal{U}, \mathbb{Z}_2)$. This means that for a simplex $\sigma = (i, j)$ and all $x \in U_{ij}$, we get

$$c(\sigma)(x) = \frac{\det(\varphi_{ij}(x))}{|\det(\varphi_{ij}(x))|} = \tilde{c}_j(x)\tilde{c}_i(x)^{-1}.$$

For each i, the function $\tilde{c}_i : U_i \to \{-1, 1\}$ is locally constant and thus constant on each connected component of U_i. Now we modify the chart $\varphi_i : p^{-1}(U_i) \to U_i \times \mathbb{R}^n$ by multiplying its first component by $\tilde{c}_i^{-1}$ and leave the rest unchanged, and call the result $\hat{\varphi}_i$. This is again a vector bundle atlas, and the transition functions satisfy $\det(\hat{\varphi}_{ij}(x)) = \det(\varphi_{ij}(x))\tilde{c}_j(x)\tilde{c}_i(x)^{-1}$, so these are constant equal to one and thus define an oriented atlas.

If E is orientable, then one may choose one of two possible orientations on each connected component of M, so fixing one orientation, these choices correspond to locally constant functions $M \to \mathbb{Z}_2$ and hence to $H^0(M, \mathbb{Z}_2)$. □

The cohomology class $w_1(E)$ is called the *first Stiefel–Whitney class* of E. For a manifold M, one defines the first Stiefel–Whitney class $w_1(M)$ of M as $w_1(TM)$, so this vanishes if and only if M is orientable. These are the simplest examples of *characteristic classes* of vector bundles and of manifolds. Looking at real line bundles (i.e. real vector bundles with one-dimensional fibers), it turns out that $E \mapsto w_1(E)$ defines an identification of $H^1(M, \mathbb{Z}_2)$ with the set of isomorphism classes of real line bundles over M.

3.7. Existence and uniqueness of spin structures. Let (M, g) be a Riemannian manifold. We want to construct a cohomology class $w_2(M, g) \in H^2(M, \mathbb{Z}_2)$ which vanishes if and only if there is a spin structure for (M, g). We will not carry out all the necessary identifications in detail but only sketch some of the steps. The idea here is to consider the oriented orthonormal frame bundle $\mathcal{SOM} \to M$ and some principal bundle atlas $\{(U_i, \varphi_i) : i \in I\}$ for this bundle with transition functions $\varphi_{ij} : U_{ij} \to SO(n)$. It turns out that any open covering of M admits a refinement for which each of the open sets in the covering as well as each non-empty intersection of two such sets is simply connected. Thus it suffices to consider only coverings with that property here.

We have already observed that the homomorphism $\rho : Spin(n) \to SO(n)$ is a two-fold covering map. This implies that each of the maps φ_{ij} admits a smooth lift $\tilde{\varphi}_{ij} : U_{ij} \to Spin(n)$, i.e. we have $\varphi_{ij} = \rho \circ \tilde{\varphi}_{ij}$. Now consider a 2-simplex $\sigma = (i, j, k)$ of $\mathcal{U}$, i.e. we have $U_{ijk} \neq \emptyset$. Then as in the proof of Proposition 3.6, we see that for each $x \in U_{ijk}$, we get $\varphi_{ik}(x) = \varphi_{ij}(x)\varphi_{jk}(x)$. Defining $c(\sigma) : U_{ijk} \to Spin(n)$ by $c(\sigma)(x) := \tilde{\varphi}_{ij}(x)\tilde{\varphi}_{jk}(x)\tilde{\varphi}_{ik}(x)^{-1}$ we conclude that $\rho(c(\sigma)(x)) = \mathbb{I}$ for each x, so $c(\sigma)$ actually has values in $\ker(\rho) = \mathbb{Z}_2$. By construction $c(\sigma)$ is smooth and thus locally constant, thus defining an element of $C^2(\mathcal{U}, \mathbb{Z}_2)$.

On the other hand, given that one spin structure exists, we want to understand how many of them there are. We want to understand this up to the natural concept of isomorphism, which is as follows: Two Spin structures $\Phi_1 : Q_1 \to \mathcal{SOM}$ and $\Phi_2 : Q_2 \to \mathcal{SOM}$ are isomorphic iff there is an isomorphism $\Psi : Q_1 \to Q_2$ of principal bundles with base map id_M such that $\Phi_2 \circ \Psi = \Phi_1$. Using this we can now formulate.

THEOREM 3.7. *The cochain c constructed above is a cocycle and the class $[c] \in H^2(M, \mathbb{Z}_2)$ is independent of the choice of lifts made in the construction and of the principal bundle atlas one starts from. This cohomology class vanishes if and only if (M, g) admits a spin structure. If this is the case, then the set of isomorphism classes of spin structures is parametrized by the Čech cohomology group $H^1(M, \mathbb{Z}_2)$.*

PROOF. Let $\sigma = (i, j, k, \ell)$ be a 3-simplex of $\mathcal{U}$, so $U_{ijk\ell} \neq \emptyset$. We have to prove that (using multiplicative notation) for each $x \in U_{ijk\ell}$ we get

$$c_{ij\ell}(x)c_{ik\ell}(x)^{-1}c_{ijk}(x)^{-1}c_{jk\ell}(x) = 1,$$

where we have permuted factors, which is possible by commutativity of $\mathbb{Z}_2$. When expanding the product of the two middle terms using the definition of c, we get

$$\tilde{\varphi}_{i\ell}(x)\tilde{\varphi}_{k\ell}(x)^{-1}\tilde{\varphi}_{ik}(x)^{-1}\tilde{\varphi}_{ik}(x)\tilde{\varphi}_{jk}(x)^{-1}\tilde{\varphi}_{ij}(x)^{-1},$$

and there is an evident cancellation in this product. Similarly, the expansion of the first term ends with $\tilde{\varphi}_{i\ell}(x)^{-1}$, which again causes a cancellation. So our product equals

$$\tilde{\varphi}_{ij}(x)\big(\tilde{\varphi}_{j\ell}(x)\tilde{\varphi}_{k\ell}(x)^{-1}\tilde{\varphi}_{jk}(x)^{-1}\big)\tilde{\varphi}_{ij}(x)^{-1}\tilde{\varphi}_{jk}(x)\tilde{\varphi}_{k\ell}(x)\tilde{\varphi}_{j\ell}(x)^{-1}.$$

Mapping the expression in brackets to $SO(n)$, one visibly gets the identity element, so this bracket lies in $\mathbb{Z}_2 \subset Spin(n)$ and thus commutes with any other element of $Spin(n)$. But commuting it with $\tilde{\varphi}_{ij}(x)^{-1}$ the whole product cancels, so c is indeed a cocycle.

Fixing the initial atlas, the only choice made in the construction was the choice of smooth lifts $\tilde{\varphi}_{ij} : U_{ij} \to Spin(n)$ of the transition functions φ_{ij}. Any other lift is of the form $\hat{\varphi}_{ij}(x) = \varphi_{ij}(x)\alpha_{ij}(x)$ for a smooth and thus locally constant function

$\alpha_{ij} : U_{ij} \to \mathbb{Z}_2$. Taking these together, we get $\alpha \in C^1(\mathcal{U}, \mathbb{Z}_2)$. Using these in the construction we get

$$\hat{c}_{ijk}(x) = \tilde{\varphi}_{ij}(x)\alpha_{ij}(x)\tilde{\varphi}_{jk}(x)\alpha_{jk}(x)\alpha_{ik}(x)^{-1}\tilde{\varphi}_{ik}(x)^{-1}.$$

Using the fact that $\mathbb{Z}_2$ lies in the center of $Spin(n)$, this immediately implies that $\hat{c} = c\partial\alpha$ (in multiplicative notation) so $[\hat{c}] = [c]$.

To prove independence of the principal bundle atlas we have used in the construction, we may assume the both atlases have the same underlying covering by passing to a joint refinement, which in addition satisfies our technical condition. Denoting these by (U_i, φ_i) and (U_i, ψ_i) we argue as in the proof of Proposition 3.6 to see that for each $x \in U_i$ and $g \in SO(n)$, we get $\psi_i(\varphi_i^{-1}(x, g)) = (x, \omega_i(x) \cdot g)$ for a smooth function $\omega_i : U_i \to SO(n)$. Since each of the sets U_i is assumed to be simply connected, there is a smooth lift $\tilde{\omega}_i : U_i \to Spin(n)$ for each of these maps. Again as in the proof of Proposition 3.6 the transition functions of the two principal bundle atlases are related by $\psi_{ij}(x)\omega_j(x) = \omega_i(x)\varphi_{ij}(x)$ for each $x \in U_{ij}$. Having chosen lifts $\tilde{\varphi}_{ij}$ of the transition functions, we can now define $\tilde{\psi}_{ij}(x) = \tilde{\omega}_i(x)\tilde{\varphi}_{ij}(x)\tilde{\omega}_j(x)^{-1}$ for each $x \in U_{ij}$ and see that $\rho(\tilde{\psi}_{ij}(x)) = \psi_{ij}(x)$ for all x. In view of the above, we only have to compare the cocycles c^φ and c^ψ obtained from the two families $\tilde{\varphi}_{ij}$ and $\tilde{\psi}_{ij}$.

But this is rather straightforward. For a 2-simplex $\sigma = (i, j, k)$ of $\mathcal{U}$ and $x \in U_{ijk}$ we get $c^\psi(\sigma)(x) = \tilde{\psi}_{ij}(x)\tilde{\psi}_{jk}(x)\tilde{\psi}_{ik}(x)^{-1}$. Inserting the definitions of the functions $\tilde{\psi}$, one immediately sees that this equals $\tilde{\omega}_i(x)c^\varphi(\sigma)(x)\tilde{\omega}_i(x)^{-1}$. But we know that $c^\varphi(\sigma)(x)$ lies in $\ker(\rho) \subset Spin(n)$ and thus commutes with any element of $Spin(n)$, so $c^\varphi = c^\psi$.

If (M, g) admits a Spin structure $\Phi : Q \to \mathcal{SO}M$, we can start from an atlas $\{(U_i, \tilde{\varphi}_i) : i \in I\}$ for Q such that the covering $\{U_i\}$ satisfies our technical condition. As we have seen in the proof of Proposition 3.2, this gives rise to an atlas (U_i, φ_i) for $\mathcal{SO}M$ such that $\varphi_i(\Phi(\tilde{\varphi}_i^{-1}(x, g))) = (x, \rho(g))$ for each $x \in U_i$ and $g \in Spin(n)$. But this immediately implies that the transition functions of these two atlases are related by $\varphi_{ij}(x) = \rho(\tilde{\varphi}_{ij}(x))$ for arbitrary i, j and all $x \in U_{ij}$. This shows that starting from the atlas $\{(U_i, \varphi_i) : i \in I\}$ for $\mathcal{SO}M$, we can base our constructions on the lifts $\tilde{\varphi}_{ij}$. But these are transition functions for a principal bundle, so they satisfy $\tilde{\varphi}_{ij}(x)\tilde{\varphi}_{jk}(x) = \tilde{\varphi}_{ik}(x)$ and thus lead to the cocycle c for which each $c(\sigma)$ is the constant function 1, and hence to the trivial cohomology class.

Conversely, assume that we have given (M, g) leading to a trivial cohomology class. Passing to an appropriate refinement, we may assume that we have an atlas $\{(U_i, \varphi_i) : i \in I\}$ for $\mathcal{SO}M$ with transition functions φ_{ij} and lifts $\tilde{\varphi}_{ij}$ for which the resulting cocycle c can be written as $\partial\tilde{c}$ for some $\tilde{c} \in C^1(\mathcal{U}, \mathbb{Z}_2)$, where $\mathcal{U} = \{U_i : i \in I\}$. By definition $\tilde{c}$ associates to each 1-simplex (i, j) of $\mathcal{U}$ a locally constant function $\tilde{c}_{ij} : U_{ij} \to \mathbb{Z}_2$, which we can also view as having values in $\ker(\rho) \subset Spin(n)$. Now define $\hat{\varphi}_{ij}(x) = \tilde{\varphi}_{ij}(x)\tilde{c}_{ij}(x)^{-1}$ for each $x \in U_{ij}$. Then these are again smooth functions $U_{ij} \to Spin(n)$ which satisfy $\rho \circ \hat{\varphi}_{ij} = \varphi_{ij}$ for all i, j, so they are lifts of the transition functions, too. Now by definition, for a 2-simplex $\sigma = (i, j, k)$ and $x \in U_{ijk}$ we get

$$c(\sigma)(x) = \tilde{\varphi}_{ij}(x)\tilde{\varphi}_{jk}(x)\tilde{\varphi}_{ik}(x)^{-1} = \partial\tilde{c}(\sigma)(x) = \tilde{c}_{ij}(x)\tilde{c}_{jk}(x)\tilde{c}_{ik}(x)^{-1}.$$

Since the elements of $\mathbb{Z}_2$ commute with all elements of $Spin(n)$, this can be easily rewritten as $\hat{\varphi}_{ij}(x)\hat{\varphi}_{jk}(x) = \hat{\varphi}_{ik}(x)$.

To complete the discussion of existence, we show that we can use these functions to construct a spin structure. We consider the set $\{(i, x, g) : i \in I, x \in U_i, g \in Spin(n)\}$ and define a relation by $(i, x, g) \sim (j, y, h)$ iff $x = y$ (and hence lies in U_{ij}) and $g = \hat{\varphi}_{ij}(x) \cdot h$. Using the above property of the functions $\hat{\varphi}_{ij}$, one immediately verifies that this is an equivalence relation, and we define Q to be the set of all equivalence classes. Writing $[(i,x,g)]$ for such a class, we define a map $\Phi : Q \to \mathcal{SO}M$ by $[(i,x,g)] \mapsto \varphi_i^{-1}(x,\rho(g))$. This is well defined, since for $g = \hat{\varphi}_{ij}(x)\cdot h$, we get $\rho(g) = \varphi_{ij}(x)\cdot\rho(h)$ and thus $\varphi_i^{-1}(x,\rho(g)) = \varphi_j^{-1}(x,\rho(h))$. The composition $q := p\circ\Phi : Q \to M$ is clearly given by $[(i,x,g)] \mapsto x$, so if $[(j,x,g)] \in q^{-1}(U_i)$, then $x \in U_{ij}$. Thus we can define a map $\hat{\varphi}_i : q^{-1}(U_i) \to U_i \times Spin(n)$ by $\hat{\varphi}_i([(j,x,h)]) := (x, \hat{\varphi}_{ij}(x)\cdot h)$. One easily verifies that these are bijective and the chart changes are described by the maps $\hat{\varphi}_{ij}$. Then one can use this atlas to define a topology and the structure of a smooth manifold on Q and to make $q : Q \to M$ into a principal $Spin(n)$-bundle. But then it is obvious that $\Phi : Q \to \mathcal{SO}M$ is a spin structure for (M,g).

From the above discussion, it is clear that any spin structure with an atlas compatible to a given atlas $\{(U_i,\varphi_i) : i \in I\}$ for $\mathcal{SO}(M)$ is described by a family of lifts $\tilde{\varphi}_{ij} : U_{ij} \to Spin(n)$ of the transition functions $\varphi_{ij} : U_{ij} \to SO(n)$ such that for all i,j,k and all $x \in U_{ijk}$ we get $\tilde{\varphi}_{ij}(x)\tilde{\varphi}_{jk}(x) = \tilde{\varphi}_{ik}(x)$. Given two spin structures, we may assume that the correspond to the same atlas for $\mathcal{SO}M$ by passing to a joint refinement, so it remains to discuss when two such lifts, say $\tilde{\varphi}_{ij}$ and $\hat{\varphi}_{ij}$ lead to isomorphic spin structures. Since for each i,j we have $\rho\circ\tilde{\varphi}_{ij} = \rho\circ\hat{\varphi}_{ij}$, there must be a smooth and hence locally constant function $e_{ij} : U_{ij} \to \ker(\rho) = \mathbb{Z}_2$ such that $\hat{\varphi}_{ij} = \tilde{\varphi}_{ij}\cdot e_{ij}$. This defines $e \in C^1(M,\mathbb{Z}_2)$ and since both families satisfy the above condition, we get for each i,j,k and each $x \in U_{ijk}$ that $e_{ij}(x)e_{jk}(x) = e_{ik}(x)$, which says that $\partial e = 0$. Conversely, given one lift $\tilde{\varphi}_{ij}$ and a cocycle e, we can define $\hat{\varphi}_{ij} = \tilde{\varphi}_{ij}\cdot e_{ij}$ to get a new lift and thus, as above, a new spin structure.

To complete the proof, we show that the two spin structures are isomorphic if and only if e is a coboundary. If we have an isomorphism $\Psi : Q_1 \to Q_2$, then for each i we consider the corresponding charts $(U_i,\tilde{\varphi}_i)$ and $(U_i,\hat{\varphi}_i)$ for the two bundles. Then there must be a smooth map $\omega_i : U_i \to Spin(n)$ such that $\hat{\varphi}_i(\Psi(\tilde{\varphi}_i^{-1}(x,g))) = (x,\omega_i(x)\cdot g)$. But recall that in these charts, Φ is given by $\mathrm{id}\times\rho$, so Φ_2 maps that element to $(x,\rho(\omega_i(x)\cdot g))$ whereas $\Phi_1(\tilde{\varphi}_i^{-1}(x,g)) = (x,\rho(g))$. Since $\Phi_2\circ\Psi = \Phi_1$, we see that ω_i has values in $\ker(\rho) = \mathbb{Z}_2$. As in the proof of Proposition 3.6, we then get $\hat{\varphi}_{ij}(x)\omega_j(x) = \omega_i(x)\tilde{\varphi}_{ij}(x)$ for all $x \in U_{ij}$. This implies that $e_{ij}(x) = \omega_j(x)^{-1}\omega_i(x)$ for all $x \in U_{Ij}$ and hence $e = \partial\omega$. Conversely, if $e = \partial\omega$, we can use the functions $\omega_i : U_i \to \mathbb{Z}_2 \subset Spin(n)$ to define an isomorphism $Q_1 \to Q_2$ in charts in an analogous way. □

Definition 3.7. The cohomology class $[c]$ is called the *second Stiefel–Whitney class* of (M,g).

It turns out that this cohomology class does actually not depend on the Riemannian metric but is a topological invariant of the manifold M. Basically, this is due to the fact that two Riemannian metrics on a smooth manifold can always be deformed into each other smoothly, which in turn is a consequence of the fact that the space of inner products on a vector space (or equivalently, the space of positive definite symmetric matrices) is convex. Since the Stiefel–Whitney class has values in a discrete group, it looks plausible that it does not change under small deformations of the metric, which then implies the result.

As mentioned already, the Stiefel–Whitney classes are basic examples of characteristic classes of vector bundles. There are several equivalent descriptions of such classes, which look very different from the outset. This is even more true for characteristic classes like Chern classes and Pontryagin classes, which have values in the cohomology $H^*(M,\mathbb{Z})$. These also admit nice interpretations in terms of de Rham cohomology.

3.8. Remarks. There are several more general results an principles that are visible in the background of what we have done in Sections 3.6 and 3.7 and we conclude the chapter with some remarks on these.

First of all, as mentioned in Remark 3.5, Čech cohomology makes sense in the more general setting of sheaves. In particular given a commutative Lie group (or topological group) G and an open covering $\mathcal{U}$ of a topological space X, one may look at cochains which associate to each simplex σ a continuous map $|\sigma| \to G$. The coboundary operator ∂ can be defined for such chains in exactly the same way as in Definition 3.4, and one can define the cohomology groups $H^k(X, C(\_, G))$ in the same way as in Sections 3.4 and 3.5. Similarly, one may work with smooth maps in the setting of a Lie group and a smooth manifold, to obtain $H^k(M, C^\infty(\_, G))$. Indeed, this reduces to the concepts we have discussed for discrete groups, for a map to a discrete space both smoothness and continuity are equivalent to being locally constant.

It was also mentioned in Remark 3.5 that in low degrees one can even drop the requirement that the group in question is commutative. The problem with non-commutativity in general is that defining the coboundary map ∂, one has to worry about the succession of factors, and the proof that $\partial \circ \partial = 0$ does not work in general degrees without the commutativity assumption. This is no problem in degrees 0 an 1. Given a 0-cochain c and a 1-simplex $\sigma = (i, j)$, one simply defines $\partial c(\sigma)(x) = c_i(x)c_j(x)^{-1}$ for $x \in U_{ij}$. Similarly, for a 1-cochain c and a 2-simplex $\sigma = (i, j, k)$ one puts $\partial c(\sigma)(x) = c_{ij}(x)c_{jk}(x)c_{ik}(x)^{-1}$ for all $x \in U_{ijk}$, and immediately verifies that $\partial \circ \partial = 0$. This allows one to define $H^1(M, C^\infty(\_, G))$ as a set even in the non-commutative case, but there is no natural group structure available, since $\mathrm{im}(\partial)$ is not a normal subgroup of $\ker(\partial)$ in general.

For a Lie group G, a smooth manifold M and a principal G-bundle $p : P \to M$, the transition functions φ_{ij} of any principal bundle atlas for P with underlying open covering $\mathcal{U}$ define a cocycle in $C^1(\mathcal{U}, C^\infty(\_, G))$. As in Proposition 3.6, one shows that the cocycles associated to different atlases define the same class in $H^1(M, C^\infty(\_, G))$ and in the same way, isomorphic principal bundles lead to the same cohomology class. Conversely, given a cocycle in $C^1(\mathcal{U}, C^\infty(\_, G))$ one defines an equivalence relation on $\{(i, x, g) : x \in U_i, g \in G\}$ as in the proof of Theorem 3.7 to obtain a principal G-bundle with an atlas having the given cocycle as its transition functions. This shows that one may identify the set $H^1(M, C^\infty(\_, G))$ with the set of isomorphism classes of principal G-bundles over M. (However, this is more of a formal observation and not really helpful in understanding the set of isomorphism classes of principal bundles in general.)

One may use this to get some nice results in the case of discrete groups, however. If G is discrete, then a principal G-bundle $p : P \to M$ is a covering (in the topological sense) of M. In particular, basic results of algebraic topology imply that such a bundle is always isomorphic to $M \times G$ provided that M is simply connected. Thus, one obtains

Proposition 3.8. *If M is a simply connected smooth manifold, then $H^1(M, G) = \{0\}$ for any discrete group G. In particular, simply connected manifolds are orientable and if a simply connected manifold admits a spin structure, then this structure is unique up to isomorphism.*

The second general principle that lies behind the proof of Theorem 3.7 is the so-called long exact cohomology sequence. Suppose that we have a discrete commutative group G and a subgroup $H \subset G$ and consider the quotient G/H. Then the inclusion $i : H \hookrightarrow G$ and the projection $\pi : G \to G/H$ are homomorphisms and thus induce homomorphisms

$$H^k(X,H) \xrightarrow{i_\#} H^k(X,G) \xrightarrow{\pi_\#} H^k(X,G/H)$$

for each topological space X and each k. It turns out that the image of $i_\#$ always coincides with the kernel of $\pi_\#$. Now one may define a homomorphism $\delta : H^k(X,G/H) \to H^{k+1}(X,H)$ as follows. Given $c \in C^k(\mathcal{U},G/H)$ with $\partial c = 0$, one can always choose $\tilde{c} \in C^k(\mathcal{U},G)$ such that $\pi_*\tilde{c} = c$. Then $\partial\tilde{c} \in C^{k+1}(\mathcal{U},G)$ is non-zero in general, but it certainly satisfies that $\pi_*\partial\tilde{c} = \partial\pi_*\tilde{c} = 0$. Thus $\partial\tilde{c}$ has values in H and hence defines an element of $C^{k+1}(\mathcal{U},H)$, which is immediately seen to be a cocycle. Next, one proves that the class of this cocycle in $H^{k+1}(X,H)$ depends only on the class $[c] \in H^k(X,G/H)$ so one can define this to be $\delta([c])$. This *connecting homomorphism* has the congenial property that its kernel coincides with the image of $\pi_\#$ while its image coincides with the kernel of $i_\#$.

The proof of Theorem 3.7 can be viewed as extending this in degree one to a partially non-commutative situation. Namely, putting $H := \ker(\rho) \subset Spin(n) =: G$, we get $G/H \cong SO(n)$. Then construction in the proof of Theorem 3.7 can be applied to general principal $SO(n)$-bundles thus defining a connecting homomorphism $\delta : H^1(M, C^\infty(\_, SO(n))) \to H^2(M,\mathbb{Z}_2)$. This associates to each principal $SO(n)$-bundle P a cohomology class $w_2(P) \in H^2(M,\mathbb{Z}_2)$ called the *second Stiefel–Whitney class* of P. The map $\rho_\# : H^1(M, C^\infty(\_, Spin(n))) \to H^1(M, C^\infty(\_, SO(n)))$ has a simple interpretation in terms of principal bundles: For any $Spin(n)$-principal bundle, there is an underlying $SO(n)$-principal bundle (defined either as $Q/\ker(\rho)$ or via transition functions) and $\rho_\#$ just is the induced map between isomorphism classes of principal bundles. The proof of Theorem 3.7 essentially shows that in this situation, one again has $\ker(\delta) = \mathrm{im}(\rho_\#)$, i.e. a principal $SO(n)$-bundle P comes from a principal $Spin(n)$-bundle (i.e. it admits a spin structure) if and only if $w_2(P) = 0$. In addition, it shows that for each element in $\mathrm{im}(\rho_\#) \subset H^1(M, C^\infty(\_, SO(n)))$ the pre-image in $H^1(M, C^\infty(\_, Spin(n)))$ is determined by the image of $i_\#$, which again resembles statements from the long exact cohomology sequence.

At this point, we should mention another fundamental application of these ideas towards characteristic classes of complex vector bundles. Consider the subgroup $\mathbb{Z} \subset \mathbb{C}$. The map $z \mapsto e^{2\pi i z}$ induces an isomorphism between the quotient $\mathbb{C}/\mathbb{Z}$ and the multiplicative group $\mathbb{C} \setminus \{0\}$. Now for a complex line bundle $L \to M$ (i.e. a complex vector bundle with 1-dimensional fiber), the transition functions of a vector bundle atlas define a cocycle in $C^1(\mathcal{U}, C^\infty(\_, \mathbb{C}\setminus\{0\}))$. Restricting to appropriate open coverings $\mathcal{U}$, one may locally write $\varphi_{ij}(x) = e^{2\pi i a_{ij}(x)}$ for smooth function $a_{ij} : U_{ij} \to \mathbb{C}$. These can then be used to define a cocycle c via $c(\sigma)(x) = a_{ij}(x)a_{jk}(x)a_{ik}(x)^{-1}$ for $\sigma = (i,j,k)$ and $x \in U_{ijk}$. This is easily seen to have values in $\mathbb{Z}$ and its class in $H^2(M,\mathbb{Z})$ turns out to be independent of all choices. This is called the *first Chern class* $c_1(L)$ of L and $L \mapsto c_1(L)$ defines a bijection from the set of isomorphism classes of complex line bundles on M to $H^2(M,\mathbb{Z})$. (In the proof, one shows using partitions of unity that $H^1(M, C^\infty(\_, \mathbb{C})) = 0$.) For general complex vector bundles, the first Chern class is defined analogously starting from the cocycle $x \mapsto \det(\varphi_{ij}(x))$.

CHAPTER 4

Clifford algebras and spin groups

In this chapter, we describe the general construction of the Spin groups $Spin(n)$ for all $n \geq 3$. This builds on a general construction of Clifford algebras, which is based on multilinear algebra. It turns out that this construction extends without additional difficulties to a much more general setting than we need. The discussion follows Sections III.3 and IV.4 of [**Ka78**] in several places, some parts have been taken from [**LM89**]. The full classiciation of real Clifford algebras can be found in [**Ka78**] and in [**BT88**], which also explains the terminology for spinors and Clifford algebras used in Physics.

Definition and structure of Clifford algebras

4.1. Clifford algebras. Let V be a vector space over a field $\mathbb{K}$, which we only assume to be of characteristic different from 2, and let $\beta : V \times V \to \mathbb{K}$ be a symmetric, $\mathbb{K}$-bilinear form. Consider an associative unital algebra A over $\mathbb{K}$, i.e. a vector space endowed with a $\mathbb{K}$-bilinear, associative multiplication $A \times A \to A$, which admits a unit element $1 \in A$. The we say that a linear map $\varphi : V \to A$ satisfies the Clifford relations if and only if

$$\forall v, w \in V : \quad \varphi(v) \cdot \varphi(w) + \varphi(w) \cdot \varphi(v) = -2\beta(v, w)1. \tag{4.1}$$

As we have noted in Chapter 1 already, bilinearity of the these relations implies that they are satisfied for all pairs of elements of V if they are satisfied for all pairs of elements of a fixed basis for V. Observe in particular, that the realations say that $\varphi(v)^2 = -\beta(v, v)1$ for all $v \in V$, while $\beta(v, w) = 0$ implies that $\varphi(v)$ and $\varphi(w)$ anti-commute. Since both sides of (4.1) are symmetric in v and w, we also conclude that $\varphi(v)^2 = -\beta(v, v)1$ for all $v \in V$ implies (4.1) for all $v, w \in V$. Observe further that for a homomorphsims $f : A \to B$ of unital associative algebras and a linear map $\varphi : V \to A$, which satisfies the Clifford relations also $f \circ \varphi : V \to B$ satisfies the Clifford relations. Using tools from multilinear algebra, it is rather easy to show that the problem of finding linear maps that satisfy the Clifford relations has a universal solution in this sense:

Proposition 4.1. *Let V be a vector space over $\mathbb{K}$ and let β be a symmetric, $\mathbb{K}$-bilinear form. There there is an associative, unital algebra $C\ell(V, \beta)$ together with a linear map $j : V \to C\ell(V, \beta)$, which satisfies (4.1) with the following universal property. For any unital associative $\mathbb{K}$-algebra A and any linear map $\varphi : V \to A$, which satisfies (4.1), there is a unique homomorphism $\tilde{\varphi} : C\ell(V, \beta) \to A$ of unital associative algebras such that $\varphi = \tilde{\varphi} \circ j$. This determines $(C\ell(V, \beta), j)$ uniquely up to isomorphism.*

Suppose that V is finite dimensional and $\{e_1, \dots, e_n\}$ is a basis for V, which is orthogonal for β in the sense that $\beta(e_i, e_j) = 0$ for $i \neq j$. Then the unit 1 and the products $j(e_{1_1}) \cdot j(e_{i_2}) \cdots j(e_{i_k})$ with $1 \leq i_1 < i_2 < \dots < i_k \leq n$ together span the vector space $C\ell(V, \beta)$. In particular, $\dim(C\ell(V, \beta)) \leq 2^n$ in this case.

Proof. Recall that there is the tensor algebra $T(V) = \oplus_{k \geq 0} \otimes^k V$ which is a unital associative algebra that has a universal property for the obvious inclusion $i : V = \otimes^1 V \hookrightarrow T(V)$. Namely, if $\varphi : V \to A$ is any linear map with values in a unital associative algebra, there is a unique homomorphism $\hat{\varphi} : T(V) \to A$ of such algebras such that $\varphi = \hat{\varphi} \circ i$. (One just has to define $\hat{\varphi}(v_1 \otimes \cdots \otimes v_k)$ as the product $\varphi(v_1) \cdots \varphi(v_k)$ in A.)

Now one defines $I \subset T(V)$ to be the ideal generated by all elements of the form $v \otimes w + w \otimes v + 2\beta(v, w)1$ with $v, w \in V$ and puts $C\ell(V, \beta) := T(V)/I$. This is a unital associative $\mathbb{K}$-algebra and we define $j : V \to C\ell(V, \beta)$ as $p \circ i$, where $p : T(V) \to C\ell(V, \beta)$ is the surjective quotient homomorphism. In $T(V)$ we have $i(v) \cdot i(w) = v \otimes w$, so we conclude that $j(v) \cdot j(w) + j(w) \cdot j(w) = p(v \otimes w + w \otimes v)$. By defintion of I, this coincides with $p(-2\beta(v, w)1)$, so j satisfies the Clifford relations. On the other hand, if a linear map $\varphi : V \to A$ satsifies (4.1), then we get

$$\hat{\varphi}(v \otimes w + w \otimes v + 2\beta(v, w)1) = \varphi(v) \cdot \varphi(w) + \varphi(w) \cdot \varphi(v) + 2\beta(v, w)1 = 0.$$

Thus all generators of I lie in the kernel of $\hat{\varphi}$ and since this kernel is an ideal, we see that $I \subset \ker(\hat{\varphi})$. Hence there is a unique homomorphism $\tilde{\varphi} : C\ell(V, \beta) \to A$ such that $\hat{\varphi} = \tilde{\varphi} \circ p$ and hence $\varphi = \hat{\varphi} \circ i = \tilde{\varphi} \circ j$. Thus we see that $C\ell(V, \beta)$ has the claimed universal property, and standard arguments show that this pins down the pair $(C\ell(V, \beta), j)$ up to isomorphism.

For the last part, we observe that the elements $e_{i_1} \otimes \cdots \otimes e_{i_k}$ for arbitrary indices i_j span $\otimes^k V$ for each k. This means that any element of $C\ell(V, \beta)$ can be written as a linear combination of products of the elements $j(e_i)$. But since j satisfies the Clifford relations, get $j(e_i)j(e_k) = -j(e_k)j(e_i)$ for $i \neq k$ and $j(e_i)^2 = -\beta(e_i, e_i)1$. Hence we can order each such product at the expense of a sign and leave out elements which occur twice at the expense of a multiple, which completes the argument. □

This result has immediate consequences. Let $f : (V, \beta) \to (W, \gamma)$ be a $\mathbb{K}$-linear map which is orthogonal for the bilinear forms in the sense that $\gamma(f(v_1), f(v_2)) = \beta(v_1, v_2)$ holds for arbitrary elements $v_1, v_2 \in V$. Then consider $j_W \circ f : V \to C\ell(W, \gamma)$, which satisfies

$$j_W(f(v_1)) \cdot j_W(f(v_2)) + j_W(f(v_2)) \cdot j_W(f(v_1)) = -2\gamma(f(v_1), f(v_2)) = -2\beta(v_1, v_2).$$

Thus it induces a homomorphism $C\ell(f) : C\ell(V, \beta) \to C\ell(W, \gamma)$ such that $C\ell(f) \circ j_V = j_W \circ f$. Together with uniqueness in the universal property, this immediately implies that $C\ell(g \circ f) = C\ell(g) \circ C\ell(f)$ and $C\ell(\mathrm{id}) = \mathrm{id}$. In particular, for $W = V$, we obtain a homomorphism from the orthgonal group $O(V, \beta)$ to the group $\mathrm{Aut}(C\ell(V, \beta))$ of automorphisms of the associative Algebra $C\ell(V, \beta)$. In particular, we can apply this to $-\mathrm{id}_V$, which gives rise to an involutive automorphism α of $C\ell(V, \beta)$. Since $\alpha^2 = \mathrm{id}$, we obtain a splitting $C\ell(V, \beta)$ into eigenspaces of α with eigenvalues ± 1, which we denote by $C\ell_0(V, \beta)$ and $C\ell_1(V, \beta)$, respectively. Since α is a homomorphism, we conclude that $C\ell_i \cdot C\ell_j \subset C\ell_{i+j}$ where we interpret the sum in $\mathbb{Z}_2$, so this makes $C\ell(V, \beta)$ into a $\mathbb{Z}_2$-graded algebra. In particular, $C\ell_0(V, \beta)$ is a subalgebra in $C\ell(V, \beta)$.

4.2. On the structure of Clifford algebras. For the next steps, we need some facts on tensor products of algebras. Given two unital associative $\mathbb{K}$-algebras A and B, one can form the tensor product of the underlying vector spaces and endow it with the multiplication characterized by $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$. This makes $A \otimes B$ into an associative algebra with unit element $1_A \otimes 1_B$. The algebras A and B can be identified with the subalgebras of $A \otimes B$ formed by the elements of the form $a \otimes 1_B$ and $1_A \otimes b$, respectively, and by construction any element from the first subalgebra commutes with any element of the second one. This tensor product has a nice universal property: Suppose that C is any associative unital $\mathbb{K}$-algebra and that $\varphi : A \to C$ and $\psi : B \to C$ are homomorphisms. Then $(a, b) \mapsto \varphi(a) \cdot \psi(b)$ is a bilinear map $A \times B \to C$, so it induces a linear map $A \otimes B \to C$, which maps $a \otimes 1_B$ to $\varphi(a)$ and $1_A \otimes b$ to $\psi(b)$. One immediately verifies that this induced map is an algebra homomorphism if and only if $\varphi(a) \cdot \psi(b) = \psi(b) \cdot \varphi(a)$ for all $a \in A$ and $b \in B$.

For $\mathbb{Z}_2$-graded algebras, there is a modification of this construction, which we will denote by $\widehat{\otimes}$. Take two $\mathbb{Z}_2$-graded algebras $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$. Then any element of $A \otimes B$ can be written as a finite sum of elements of the form $a \otimes b$ with a and b lying in one of the grading components. Denoting their degrees by $|a|, |b| \in \mathbb{Z}_2$ we decree that $a \otimes b$ has degree $|a| + |b|$. This makes $A \otimes B$ into a $\mathbb{Z}_2$-graded vector space with $(A \otimes B)_0 = (A_0 \otimes B_0) \oplus (A_1 \otimes B_1)$, while $(A \otimes B)_1 = (A_1 \otimes B_0) \oplus (A_0 \otimes B_1)$. Now we modify the definition of the multiplication by taking the gradings into account. For elements a_i and b_j contained in one grading component, we put $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) := (-1)^{|b_1||a_2|}(a_1 a_2) \otimes (b_1 b_2)$. This is compatible with the $\mathbb{Z}_2$-grading, associative and has $1_A \otimes 1_B$ as a unit, and we denote the resulting algebra by $A \widehat{\otimes} B$.

The universal property here works for a $\mathbb{Z}_2$-graded algebra C and homomrophisms $\varphi : A \to C$ and $\psi : B \to C$ which respect the gradings. As above, these induce a linear map $A \widehat{\otimes} B \to C$ which is compatible with the gradings. This is a homomorphism if for $a \in A$ and $b \in B$, the elements $\varphi(a)$ and $\psi(b)$ *commute in the graded sense.* This means that for a and b contained in one grading component, we must have $\varphi(a)\psi(b) = (-1)^{|a||b|}\psi(b)\varphi(a)$. Using this, we can prove the first core result on the structure of Clifford algebras.

Theorem 4.2. *Starting from (V, β), assume that $V = V_1 \oplus V_2$ for two linear subspaces $V_1, V_2 \subset V$, which are orthogonal with respect to β, i.e. such that $\beta(v_1, v_2) = 0$ for each $v_1 \in V_1$ and $v_2 \in V_2$. Denoting by β_i the restriction of β to a bilinear form on V_i for $i = 1, 2$, we get $C\ell(V, \beta) \cong C\ell(V_1, \beta_1) \widehat{\otimes} C\ell(V_2, \beta_2)$ as a $\mathbb{Z}_2$-graded algebra.*

Proof. Any element $v \in V$ can be uniquely written as $v = v_1 + v_2$ with $v_i \in V_i$ and we define $\varphi : V \to C\ell(V_1, \beta_1) \widehat{\otimes} C\ell(V_2, \beta_2)$ as $\varphi(v) := j_1(v_1) \otimes 1 + 1 \otimes j_2(v_2)$, where $j_i : V_i \to C\ell(V_i, \beta_i)$ are the canonical maps. Given another element $w \in V$ we get $\beta(v, w) = \beta_1(v_1, w_1) + \beta_2(v_2, w_2)$ and

$$\varphi(v)\varphi(w) = j_1(v_1)j_1(w_1) \otimes 1 + j_1(v_1) \otimes j_2(w_2) - j_1(w_1) \otimes j_2(v_2) + 1 \otimes j_2(v_2)j_2(w_2).$$

Since the two middle terms cancel with the conttibution form $\varphi(w)\varphi(v)$, we conclude that φ satisfies the Clifford relations. Thus there is an induced homomorphism $\tilde{\varphi} : C\ell(V, \beta) \to C\ell(V_1, \beta_1) \widehat{\otimes} C\ell(V_2, \beta_2)$.

In the other direction, the inclusions $\iota_1 : V_1 \to V$ and $\iota_2 : V_2 \to V$ are orthogonal, and thus induce homomorphisms $C\ell(\iota_1) : C\ell(V_1, \beta_1) \to C\ell(V, \beta)$ and similarly for $C\ell(\iota_2)$. Choosing orthogonal bases $\{e^i_k\}$ of the spaces V_i, the union of the two bases is an orthogonal basis for V. By Proposition 4.1, we know that any element of $C\ell(V_1, \beta_1)$ can be written as a linear combination of products of elements $j_1(e^1_k)$ and for a homogeneous element, these products either have all an even number of factors or have all an odd number of factors. By construction, $C\ell(\iota_1)$ sends the product $j_1(e^1_{k_1}) \cdots j_1(e^1_{k_\ell})$ to $j(e^1_{k_1}) \cdots j(e^1_{k_\ell})$. We can describe elements from the image of $C\ell(\iota_2)$ similarly. Now by the Clifford relations, any of the elements $j(e^1_i)$ anti-commutes with any of the elements $j(e^2_i)$. This shows that the images of $C\ell(\iota_1)$ and $C\ell(\iota_2)$ satsify the condition needed for them to induce a homomorphism $C\ell(V_1, \beta_1) \widehat{\otimes} C\ell(V_2, \beta_2) \to C\ell(V, \beta)$.

We claim that these homomorphisms are inverse to each other. Starting from $C\ell(V, \beta)$, the element $j(v)$ for $v = v_1 + v_2$ is mapped to $j_1(v_1) \otimes 1 + 1 \otimes j_2(v_2)$, which then is mapped further to $j(v_1) + j(v_2) = j(v)$. But a homomorphism from $C\ell(V, \beta)$ to itself which sends each $j(v)$ to itself has to be the identity by the uniqueness in the universal property. For the other composition, consider an elemente of the form $j_1(v_1)\otimes 1$. By definition, this is mapped to $j(v_1)$ and since $0\otimes 1=0$, this is mapped back to $j_1(v_1)\otimes 1$. Similarly, elements of the form $1\otimes j_2(v_2)$ are mapped to themselves. But by Proposition 4.1 and standard properties of the tensor product, products of finitely many elements of these two forms span $C\ell(V_1,\beta_1)\widehat{\otimes}C\ell(V_2,\beta_2)$. Thus, the other composition is the identity map, too, and the proof is complete. □

Corollary 4.2. *Suppose that $\mathbb{K}$ has characteristic different from 2 and that V is an n-dimensional vector space over $\mathbb{K}$ endowed with some symmetric bilinear form β. Then the canonical map $j:V\to C\ell(V,\beta)$ is injective and* $\dim(C\ell(V,\beta))=2^n$.

Proof. We first prove the statement on $\dim((V,\beta))$ by induction on n. For $n=1$, let $v\in V$ be a non-zero element. Then v is a basis for V and, more generally, the tensor product of k copies of v is a basis for $\otimes^k V$. This implies that mapping x to v induces an isomorphism $\mathbb{K}[x]\to T(V)$ of algebras. Hence $C\ell(V,\beta)$ is isomorphic to $\mathbb{K}[x]/I$ where I is the ideal generated by $x^2-\beta(v,v)1$. It is well known that the classes of 1 and x form a basis for this quotient and hence $\dim(C\ell(V,\beta))=2$.

For $\dim(V)=n>1$, we first claim that we either have $\beta=0$ or there is an element $v_1\in V$ such that $\beta(v_1,v_1)\neq 0$. Indeed, we get $\beta(v+w,v+w)=\beta(v,v)+\beta(w,w)+2\beta(v,w)$ for all $v,w\in V$. Since the characteristic of $\mathbb{K}$ is different from 2, this shows that $0=\beta(v,v)$ for all $v\in V$ implies $0=\beta(v,w)$ for all $v,w\in V$. If $\beta=0$, we choose any decompostion $V=V_1\oplus V_2$ with $\dim(V_1)=1$ and $\dim(V_2)=n-1$ and this is automatically orthogonal. Otherwise we choose $v_1\in V$ with $\beta(v_1,v_1)\neq 0$, define V_1 to be the subspace spanned by V_1 and $V_2:=\{v\in V:\beta(v_1,v)=0\}$. One immediately verifies that V is the orthgonal direct sum $V_1\oplus V_2$, so we always get such a decomposition. Theorem 4.1 then shows that $C\ell(V,\beta)\cong C\ell(V_1,\beta_1)\hat{\otimes}C\ell(V_2,\beta_2)$ so its dimension is the product of the dimensions of the two factors and this completes the induction.

Now let $\dim(V)=n$ and assume that j is not injective. Then for a basis $v_1,\dots,v_n$ for V, the elements $j(v_1),\dots,j(v_n)\in C\ell(V,\beta)$ would be linearly dependent. Without loss of generality, this would imply that $j(v_n)$ can be written as a linear combination of $j(v_1),\dots,j(v_{n-1})$. But as in the proof of Propostion 4.1, this would show that $\dim(C\ell(V,\beta))\leq 2^{n-1}$, a contradiction. □

Injectivity of the map $j:V\to C\ell(V,\beta)$ implies that we can view V as a linear subspace of $C\ell(V,\beta)$ and we will do this from now on without further mentioning. In particular for elements $v_1,\dots,v_k$, we have an element $v_1\cdot v_2\cdots v_k\in C\ell(V,\beta)$, which lies in $C\ell_0(V,\beta)$ if k is even and in $C\ell_1(V,\beta)$ if k is odd.

Remark 4.2. It turns out that there is a nice relation between the Clifford algebra $Cl(V,\beta)$ and the exterior algebra Λ^*V for any symmetric bilinear form β. The decomposition $T(V)=\oplus_{k\geq 0}\otimes^k V$ makes $T(V)$ into a graded algebra, i.e. the product of $\otimes^k V$ and $\otimes^\ell V$ is contained in $\otimes^{k+\ell}V$. The generators of the ideal I used to construct $Cl(V,\beta)$ in Proposition 4.1 are not homogeneous for this grading, since they mix elements of degree two and zero. Hence there is no induced grading on $Cl(V,\beta)$, although the $\mathbb{Z}_2$-grading from Section 4.1 is obtained from the grading of $T(V)$. But there is another way to use the grading of $T(V)$. Denoting by $p:T(V)\to Cl(V,\beta)$ the quotient homomorphism, we consider for each $i\geq 0$, linear subspace $\mathcal{F}^i:=p(\oplus_{k=0}^i\otimes^k V)\subset Cl(V,\beta)$. By construction, these spaces satisfy $\mathcal{F}^i\subset\mathcal{F}^{i+1}$ for each i and $\mathcal{F}^i\cdot\mathcal{F}^j\subset\mathcal{F}^{i+j}$, thus making $Cl(V,\beta)$ into a *filtered algebra*. From Proposition 4.1 it also follows that $\mathcal{F}^i=C\ell(V,\beta)$ for all $i\geq n$.

For such a filtered algebra, one can form the *associated graded* algebra. Consider the vector space $\oplus_{i\geq 0}\mathcal{F}^i/\mathcal{F}^{i-1}$, which in our case ends with $i=n$ and can be endowed with a multiplication as follows. Given elements of $x\in\mathcal{F}^i/\mathcal{F}^{i-1}$ and $y\in\mathcal{F}^j/\mathcal{F}^{j-1}$, we choose representatives $\hat{x}\in\mathcal{F}^i$ and $\hat{y}\in\mathcal{F}^j$, form $\hat{x}\cdot\hat{y}\in\mathcal{F}^{i+j}$ and project it to $\mathcal{F}^{i+j}/\mathcal{F}^{i+j-1}$. Choosing different representatives, we get $\hat{x}+x'$ and $\hat{y}+y'$ with $x'\in\mathcal{F}^{i-1}$ and $y'\in\mathcal{F}^{j-1}$ and multiplying them gives $\hat{x}\cdot\hat{y}+x'\cdot\hat{y}+\hat{x}\cdot y'+x'\cdot y'$. But the last three terms lie in $\mathcal{F}^{i+j-1}$ respectively even in $\mathcal{F}^{i+j-2}$, so the class in the quotient is independent of the choice of representatives. Clearly, this associated graded algebra is again associative and $1\in\mathcal{F}^0$ is a unit element.

Now we can view $v\in V$ as an element of $\mathcal{F}^1$, and then consider its class in $\mathcal{F}^1/\mathcal{F}^0$. This defines a linear map φ from V to the associated graded algebra. The Clifford relations for j immediately imply that $\varphi(v)\varphi(w)+\varphi(w)\varphi(v)=0$ for all $v,w\in V$, so there is a unique algebra homomorphism from Λ^*V to this associated graded algebra. Since $C\ell(V,\beta)$ is generated by the elements v it easily follows that this homomorphism has to be surjective, so since both algebras have the same dimension, it is a linear isomorphism. Thus for any β, the associated graded algebra to $C\ell(V,\beta)$ is isomorphic to Λ^*V.

Spin groups and the spin representations

4.3. Spin groups. For any associative unital algebra A one can consider the subset A^* of invertible elements, i.e. those $a\in A$ for which there exists an element $a^{-1}\in A$ such that $a^{-1}\cdot a=a\cdot a^{-1}=1$. It follows immediately that the inverse is uniquely determined. Morover, for invertible elements a and b also the product $a\cdot b$ is invertible and $(a\cdot b)^{-1}=b^{-1}\cdot a^{-1}$, so A^* naturally is a group. An invertible element $a\in A$ is not a zero-divisor, i.e. if for some $b\in A$ either $a\cdot b=0$ or $b\cdot a=0$, then $b=0$. If A is finite dimensional, then the converse holds, i.e. if a is not a zero-divisor, then a is invertible. Indeed, if a is not a zero-divisor, then left and right multiplication by a are linear maps $A\to A$ with trivial kernel, so they have to be surjective. Hence there are unique elements $b,c\in A$ such that $a\cdot b=1$ and $c\cdot a=1$ and multiplying the latter equation by b from the right, we get $c=b$.

If we assume that $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, which we will do from now on, then for finite dimensional A, the subset $A^*\subset A$ is open and thus a Lie group of dimension $\dim(A)$. Indeed, the map seding $a\in A$ to left multiplication by a is a linear map $A\to L(A,A)$ and from above we know that A^* is the pre-image of the open subset $GL(A)$ under this map. Via the implicit function theorem, this also implies that the inversion map is smooth as a map $A^*\to A^*$.

Let us further specialize to the Clifford algebra $C\ell(V,\beta)$ of a finite dimensional, real vector space V endowed with a non-degenerate, symmetric bilinear form β. As in Section 4.1 we denote by α the automorphism of $C\ell(V,\beta)$ whose ± 1-eigenspaces are $C\ell_0(V,\beta)$ and $C\ell_1(V,\beta)$. To construct the associated spin group, we need one more bit of structure on the Clifford algebra.

Lemma 4.3. *There is a unique involutive anti-automorphism of $C\ell(V,\beta)$ which we denote by $x\mapsto\bar{x}$ (so this is linear, $\overline{x\cdot y}=\bar{y}\cdot\bar{x}$ and $\bar{\bar{x}}=x$) such that $\bar{v}=v$ for all $v\in V\subset C\ell(V,\beta)$. This satisfies $\alpha(\bar{x})=\overline{\alpha(x)}$ and the function $N:C\ell(V,\beta)\to C\ell(V,\beta)$ defined by $N(x):=\alpha(\bar{x})\cdot x$ satisfies $N(x\cdot y)=\alpha(\bar{y})\cdot N(x)\cdot y$ and $N(v)=\beta(v,v)\cdot 1$ for all $v\in V$.*

Proof. Let A be the opposite algebra to $C\ell(V,\beta)$, i.e. it has the same underlying vector space but the product $x\cdot y$ in A equals the product $y\cdot x$ in $C\ell(V,\beta)$. Viewing the canonical map $j : V \to C\ell(V,\beta)$ as a linear map to A, it satisfies the Clifford relations, so there is a unique homomorphism $C\ell(V,\beta) \to A$ sending $v \in V \subset C\ell(V,\beta)$ to v viewed as an element of A. A homomorphism $C\ell(V,\beta) \to A$ simply is an anti-homomorphism $C\ell(V,\beta) \to C\ell(V,\beta)$, so we have defined the map $x \mapsto \bar{x}$. Moreover, since α is the homomorphism induced by $-\mathrm{id}$ both the homomorphisms $C\ell(V,\beta) \to A$ given by $x \mapsto \alpha(\bar{x})$ and $x \mapsto \overline{\alpha(x)}$ send $v \in V$ to $-v \in A$ so they agree by uniqueness in the universal property. Similarly, $x \mapsto \bar{\bar{x}}$ is a homomorphism from $C\ell(V,\beta)$ to itself which maps each $v \in V$ to itself, so it has to be the identity.

By definition $N(x \cdot y) = \alpha(\bar{y} \cdot \bar{x}) \cdot x \cdot y$. Using that α is a homomorphism, it follows immediately that this equals $\alpha(\bar{y}) \cdot N(x) \cdot y$. For $v \in V$, we have $\bar{v} = v$ and $\alpha(v) = -v$, so $N(v) = -v \cdot v$, which equals $\beta(v,v) \cdot 1$ by the Clifford relations. □

Definition 4.3. (1) The map N defined in Lemma 4.3 is called the *spinorial norm.*
(2) We define subsets

$$Spin(V,\beta) \subset Pin(V,\beta) \subset \tilde{\Gamma}(V,\beta) \subset C\ell(V,\beta)^*$$

in the group of invertible elements in $C\ell(V,\beta)$ as follows. An element x lies in $\tilde{\Gamma}(V,\beta)$ iff for each $v \in V$ we have $\alpha(x)vx^{-1} \in V \subset C\ell(V,\beta)$, it lies in $Pin(V,\beta)$ if in addition $N(x) = \pm 1$, and it lies in $Spin(V,\beta)$ if it in addition lies in $C\ell_0(V,\beta)$. These are called the *twisted Clifford group*, the *pin group*, and the *spin group* of (V,β), respectively. By $Spin(n)$, we denote the spin group of the standard inner product on $\mathbb{R}^n$.

Theorem 4.3. *All three subsets defined above are closed subgroups of $C\ell(V,\beta)^*$ and thus Lie groups. For $x \in \tilde{\Gamma}(V,\beta)$, defining $\rho_x : V \to V$ as $\rho_x(v) = \alpha(x)vx^{-1}$ defines a smooth representation of $\tilde{\Gamma}(V,\beta)$ on V. This representation restricts to surjective homomorphisms $Pin(V,\beta) \to O(V,\beta)$ and $Spin(V,\beta) \to SO(V,\beta)$, each of which has kernel $\{\pm 1\}$. Finally, if β is positive definite, then $Spin(V,\beta)$ is connected.*

Proof. For $x \in C\ell(V,\beta)^*$, we define a linear map $\tilde{\rho}_x : C\ell(V,\beta) \to C\ell(V,\beta)$ by $\tilde{\rho}_x(y) := \alpha(x)yx^{-1}$. Clearly this defines a smooth homomorphism $\tilde{\rho} : C\ell(V,\beta)^* \to GL(C\ell(V,\beta))$. The invertible maps that send V to itself form a closed subgroup in $GL(C\ell(V,\beta))$, whose pre-image is by definition is $\tilde{\Gamma}(V,\beta) \subset C\ell(V,\beta)^*$, so this is a closed subgroup, too. It is also clear from this description that we obtain a smooth representation $\rho : \tilde{\Gamma}(V,\beta) \to GL(V)$ via $\rho_x(v) = \alpha(x)vx^{-1}$. Now for $x \in \tilde{\Gamma}(V,\beta)$ and $v \in V$, we have $\alpha(x)vx^{-1} \in V$ so this element is mapped to its negative by α. Since α is a homomorphism, we get $\alpha(x^{-1}) = \alpha(x)^{-1}$, so we conclude that $xv\alpha(x)^{-1} \in V$, which shows that $\alpha(x) \in \tilde{\Gamma}(V,\beta)$.

By definition, for $t \in \mathbb{R} \setminus \{0\}$, we get $t1 \in \tilde{\Gamma}(V,\beta)$ and $\rho_{t1} = \mathrm{id}_V$. We claim that these are the only elements in the kernel of ρ. Suppose that $x \in \tilde{\Gamma}(V,\beta)$ satisfies $\rho_x = \mathrm{id}_V$, so we have $\alpha(x)vx^{-1} = v$ and hence $\alpha(x)v = vx$ for all $v \in V$. Now we can write $x = x_0 + x_1$ with $x_i \in C\ell_i(V,\beta)$ for $i = 0,1$. Looking at the components of the equation $\alpha(x)v = vx$ in $C\ell_i(V,\beta)$ for $i = 0,1$, we obtain $x_0 v = vx_0$ and $-x_1 v = vx_1$. Now choose an orthogonal basis $\{e_i\}$ of V such that $\beta(e_i,e_i) \neq 0$ for all i. Then we can write x_0 as a linear combination of a multiple of 1 and of products of an even number of the e_i. Fixing an index i_0 we can split this into the linear combination of those products which do not involve e_{i_0} and those which involve this vector, arranged in such a way that e_{i_0} comes first. This gives a representation $x_0 = a_{i_0} + e_{i_0} \cdot b_{i_0}$ with $a_{i_0} \in C\ell_0$ and $b_{i_0} \in C\ell_1$ both being linear combinations of products which do not involve e_{i_0}. This readily implies that e_{i_0} commutes with a_{i_0} and anti-commutes with b_{i_0}. Thus we obtain $(a_{i_0} + e_{i_0}b_{i_0})e_{i_0} = a_{i_0}e_{i_0} - e_{i_0}^2 b_{i_0}$ and $e_{i_0}(a_{i_0} + e_{i_0}b_{i_0}) = e_{i_0}a_{i_0} + e_{i_0}^2 b_{i_0}$, so $x_0 e_{i_0} = e_{i_0} x_0$

implies $b_{i_0} = 0$. But this means that x_0 does not contain products involving e_{i_0} and since this works for each index, x_0 has to be a multiple of 1.

Simlarly, we write $x_1 = a_{i_0} + e_{i_0} b_{i_0}$ where now $a_{i_0} \in C\ell_1$ and $b_{i_0} \in C\ell_0$ so e_{i_0} anti-commutes with a_{i_0} and commutes with b_{i_0}. Similarly as above, $x_1 e_{i_0} = -e_{i_0} x_1$ implies $b_{i_0} = 0$, so x_1 may not contain any product involving e_{i_0}. Since this works for each index, and x_1 contains only products with an odd number of factors, this implies $x_1 = 0$, and the claim follows.

We next claim that for $x \in \tilde{\Gamma}(V, \beta)$, the spinorial norm $N(x)$ is a non-zero multiple of 1. For any $x \in C\ell(V, \beta)^*$ we have already observed that $\alpha(x^{-1}) = \alpha(x)^{-1}$ and similarly, we get $\overline{x^{-1}} = (\bar{x})^{-1}$. Now for $v \in V$, we have $\bar{v} = v$ and $\alpha(v) = -v$, so $\bar{x} \cdot v = \bar{x} \cdot \bar{v} = \overline{v \cdot x}$ and hence

$$\bar{x} \cdot v \cdot \alpha(\bar{x})^{-1} = \overline{v \cdot x} \cdot \overline{\alpha(x^{-1})} = \overline{\alpha(x^{-1}) v x}.$$

Assuming that $x \in \tilde{\Gamma}(V, \beta)$, we also have $x^{-1} \in \tilde{\Gamma}(V, \beta)$ and hence $\alpha(x^{-1}) v x \in V$. Hence our computation shows that $\alpha(\bar{x}) \in \tilde{\Gamma}(V, \beta)$ and that $\rho_{\alpha(\bar{x})} = \rho_{x^{-1}}$. Thus also $N(x) = \alpha(\bar{x}) x \in \tilde{\Gamma}(V, \beta)$ and $\rho_{N(x)} = \rho_{\alpha(\bar{x})} \circ \rho_x = \mathrm{id}_V$ which proves the claim.

Since $N(x)$ is a multiple of 1 for $x \in \tilde{\Gamma}(V, \beta)$ it commutes with any element of $C\ell(V, \beta)$, so Lemma 4.3 shows that $N(xy) = N(x)N(y)$ for any $y \in C\ell(V, \beta)$. In particular, N defines a homomorphism from $\tilde{\Gamma}(V, \beta)$ to the multiplicative group $\mathbb{R} \setminus \{0\}$. Since $\{\pm 1\}$ is a closed subgroup in $\mathbb{R} \setminus \{0\}$, the pre-image $Pin(V, \beta)$ is a closed subgroup in $\tilde{\Gamma}(V, \beta)$. Moreover, since $N(1) = 1$, we get $N(x^{-1}) = N(x)^{-1}$ and further $N(x) = \alpha(N(x))$, which by definition equals $N(\alpha(x))$. This shows that α restricts to a group automorphism of $Pin(V, \beta)$ such that $\alpha^2 = \mathrm{id}$ and thus the set of fixed points of α forms a closed subgroup of $Pin(V, \beta)$, which by definition coincides with $Spin(V, \beta)$.

Now for $v \in V$ we get $N(\rho_x(v)) = N(\alpha(x) v x^{-1})$ and expanding this, one immediately verifies that this coincides with $N(v)$ and using Lemma 4.3, this shows that $\beta(\rho_x(v), \rho_x(v)) = \beta(v, v)$. Thus $\rho_x \in O(V, \beta)$ and ρ is a homomorphism $\tilde{\Gamma}(V, \beta) \to O(V, \beta)$. Now suppose that $v \in V$ is such that $\beta(v, v) = N(v) = \pm 1$. Then $v^2 = -\beta(v, v) 1$ so $v^{-1} = \mp v$ and hence $\alpha(v) v v^{-1} = \pm v^3 = \mp \beta(v, v) v = -v$. On the other hand, if $w \in V$ is such that $\beta(v, w) = 0$, then $\alpha(v) w v^{-1} = -v w v^{-1} = w v v^{-1} = w$. But this shows that $\tilde{\rho}_v$ preseves the subspace V, so $v \in Pin(V, \beta)$ and that ρ_v is the reflection in the hyperplane $v^\perp$ perpendicular to v.

It is a classical result that any element of $O(V, \beta)$ can be written as a product of at most $n = \dim(V)$ such reflections, so $\rho : \tilde{\Gamma}(V, \beta) \to O(V, \beta)$ and even its restriction to $Pin(V, \beta)$ is surjective. The kernel of ρ in $\tilde{\Gamma}(V, \beta)$ are the non-zero multiples of 1, so the kernel of the restriction to $Pin(V, \beta)$ coincides with $\{\pm 1\} \subset Spin(V, \beta)$. Together these results imply that any element of $Pin(V, \beta)$ can be written as $x = \pm v_1 \dots v_k$ for some elements $v_i \in V$ such that $\beta(v_i, v_i) = \pm 1$. Such an element lies in $Spin(V)$ if and only if k is even, which is equivalent to ρ_x being a composition of an even number of reflections and thus lying in $SO(V, \beta)$.

We finally claim that there is a smooth curve in $Spin(V, \beta)$ that connects 1 and -1. If β is positive definite, then $SO(V, \beta)$ is connected, so given an element $x \in Spin(V, \beta)$, there is a continuous curve that connects $\rho(x)$ to the identity. Since ρ is a covering map, this lifts to a curve that starts in x and ends in $\ker(\rho) = \{\pm 1\}$. Together with the above, this shows that $Spin(V, \beta)$ is connected. In general, it shows that $Spin(V, \beta)$ has as many connected components as $SO(v, \beta)$ (namely two) and $Pin(V, \beta)$ has as many components as $O(V, \beta)$ (namely four). To prove our claim, we take two elements $v_1, v_2 \in V$ such that $\beta(v_1, v_2) = 0$ and $\beta(v_1, v_1) = \beta(v_2, v_2) = \pm 1$ and for $t \in \mathbb{R}$ consider the element $x_t := \cos(t) \cdot 1 + \sin(t) v_1 v_2 \in C\ell_0(V, \beta)$. Now by construction

$(v_1v_2)^2 = -v_1^2v_2^2 = -1$, which immediately implies that $\cos(t)\cdot 1 - \sin(t)v_1v_2$ is inverse to x_t. If $v \in V$ is orthogonal to both v_1 and v_2, then it commutes with v_1v_2 and hence with $x_t = \alpha(x_t)$, so $\alpha(x_t)vx_t^{-1} = v$. On the other hand, one computes directly that $\alpha(x_t)v_1x_t^{-1}$ and $\alpha(x_t)v_2x_t^{-1}$ both are linear combinations of v_1 and v_2 so $x_t \in \tilde{\Gamma}(V,\beta)$. The definitions also readily imply that $\overline{x_t} = x_t^{-1}$ and hence $N(x_t) = 1$ for all t. Hence $x_t \in Spin(V)$ and of course $x_0 = 1$ and $x_\pi = -1$. $\square$

4.4. Complex and real Clifford algebras. To obtain the spin representations of the spin groups, we have to analyze the structure of complex Clifford algebras in a bit more detail. We will also indicate how things look in the real case. Recall from linear algebra that over $\mathbb{C}$, there is up to isomorphism a unique non-degenerate, symmetric bilinear form on any finite dimensional vector space. From Section 4.1, we conclude that an orthogonal isomorphism of vector spaces induces an isomorphism of the corresponding Clifford algebras. Thus it suffices to determine the Clifford algebra for the standard bilinear form $(z,w) \mapsto \sum_{k=1}^n z_kw_k$ on $\mathbb{C}^n$, which we denote by $C\ell(n,\mathbb{C})$. Over $\mathbb{R}$, non-degenerate, symmetric bilinear forms are classified by signature (p,q) by Sylvester's theorem. A representative form on $\mathbb{R}^{p+q}$ is given by $(x,y) \mapsto \sum_{k=1}^p x_ky_k - \sum_{k=p+1}^{p+q} x_ky_k$ and we denote the corrresponding Clifford algebra by $C\ell(p,q)$.

From the proof of Corollary 4.2, we see that $C\ell(1,0)$ and $C\ell(0,1)$ are isomorphic to the quotient of $\mathbb{R}[x]$ by the ideals generated by x^2+1 and x^2-1, respectively. The first of these is of course isomorphic to $\mathbb{C}$ via mapping x to i, so $C\ell(1,0) \cong \mathbb{C}$ with $C\ell_0(1,0)$ spanned by 1 and $C\ell_1(1,0)$ spanned by i. Similarly, $C\ell(0,1) \cong \mathbb{R}\oplus\mathbb{R}$ (with component-wise operations) via the map sending x to $(1,-1)$, so the latter element spans $C\ell_1(0,1)$ while the unit element $(1,1)$ spans $C\ell_0(0,1)$. For $\mathbb{C}$, we use $\mathbb{C}[x]/x^2+1$ but in the complex case, this can be identified with $\mathbb{C}\oplus\mathbb{C}$ by sending x to $(i,-i)$. So we get $C\ell(1,\mathbb{C}) = \mathbb{C}\oplus\mathbb{C}$ with $C\ell_0$ and $C\ell_1$ spanned by $(1,1)$ and $(1,-1)$, respectively. Using this, we can now inductively derive a description of all complex Clifford algebras.

THEOREM 4.4. *For each $n \geq 1$ there are isomorphisms of algebras*

$$C\ell(2n,\mathbb{C}) \cong C\ell_0(2n+1,\mathbb{C}) \cong M_{2^n}(\mathbb{C})$$
$$C\ell(2n-1,\mathbb{C}) \cong C\ell_0(2n,\mathbb{C}) \cong M_{2^{n-1}}(\mathbb{C}) \oplus M_{2^{n-1}}(\mathbb{C})$$

PROOF. From above, we know that $C\ell(1,\mathbb{C})$ admits a basis $\{1,x\}$ such that $x^2 = -1$ and from Theorem 4.2, we know that $C\ell(n+1,\mathbb{C}) \cong C\ell(n,\mathbb{C})\widehat{\otimes}C\ell(1,\mathbb{C})$. Now for $z,w \in \mathbb{C}^n$ viewed as elements of $C\ell(n,\mathbb{C})$, we get $(z\otimes x)\cdot(w\otimes x) = -zw\otimes x^2 = zw\otimes 1$. Since all these tensor products lie in the degree 0 part, we see that $z \mapsto z\otimes x$ defines a map from $\mathbb{C}^n$ to the alsgebra $C\ell_0(n+1,\mathbb{C})$, which satisfies the Clifford relations. By Proposition 4.1, this induces an algebra homomorphism $C\ell(n,\mathbb{C}) \to C\ell_0(n+1,\mathbb{C})$. By construction $C\ell_0(n+1,\mathbb{C})$ is spanned by elements of the form $a\otimes x$ and $b\otimes 1$, where a is a products of an odd number of elements of $\mathbb{C}^n$ while b is a product of an even number of such elements. This shows that our homomorphism is surjective, and since both algebras have the same dimension, it is an isomorphism. Hence it remains to describe the full Clifford algebras.

For $n=1$, we get $C\ell(2,\mathbb{C}) = C\ell(1,\mathbb{C})\widehat{\otimes}C\ell(1,\mathbb{C})$, so this has a basis consisting of $1\otimes 1$, $x\otimes 1$, $1\otimes x$, and $x\otimes x = (x\otimes 1)\cdot(1\otimes x) = -(1\otimes x)\cdot(x\otimes 1)$, where $x^2 = -1$. Now consider the matrices $A := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $B := \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$. These satisfy $A^2 = B^2 = -\mathbb{I}$ and $AB = -BA = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, so $\{\mathbb{I},A,B,AB\}$ is a basis of $M_2(\mathbb{C})$. Clearly, the map sending $1\otimes 1$ to $\mathbb{I}$, $x\otimes 1$ to A, $1\otimes x$ to B and $x\otimes x$ to AB is an isomorphism of algebras, so $C\ell(2,\mathbb{C}) \cong M_2(\mathbb{C})$.

Next, we claim that $C\ell(n+2,\mathbb{C}) \cong C\ell(n,\mathbb{C}) \otimes M_2(\mathbb{C})$ as an algebra. View $\mathbb{C}^{n+2}$ as $\mathbb{C}^n \oplus \mathbb{C}^2$ and take an orthonormal basis $\{w_1, w_2\}$ for $\mathbb{C}^2$. Using the matrices A and B from above, we define $\varphi : \mathbb{C}^{n+2} \to C\ell(n,\mathbb{C}) \otimes M_2(\mathbb{C})$ by $\varphi(v) = v \otimes iBA$ for $v \in \mathbb{C}^n$, $\varphi(w_1) := 1 \otimes A$ and $\varphi(w_2) = 1 \otimes B$. This shows that $\varphi(v_1) \cdot \varphi(v_2) = v_1 v_2 \otimes \mathbb{I}$, $\varphi(w_1)^2 = \varphi(w_2)^2 = -1 \otimes \mathbb{I}$ and $\varphi(w_1)$ and $\varphi(w_2)$ anti-commute with each other and with any $\varphi(v)$. Hence φ satsifies the Clifford relations, so there is an induced homomorphism $C\ell(n+2,\mathbb{C}) \to C\ell(n,\mathbb{C}) \otimes M_2(\mathbb{C})$ of algebras. Since $\varphi(v)\varphi(w_1)\varphi(w_2)$ is a non-zero multiple of $v \otimes 1$, we conclude similarly as above that this has to be surjective and thus a linear isomorphism. In particular, for $n = 1$, we get $C\ell(3,\mathbb{C}) \cong (\mathbb{C} \oplus \mathbb{C}) \otimes M_2(\mathbb{C})$ and clearly, this is isomorphic to $M_2(\mathbb{C}) \oplus \mathbb{M}_2(\mathbb{C})$.

To complete the proof by induction, it suffices to show that $M_k(\mathbb{C}) \otimes M_2(\mathbb{C}) \cong M_{2k}(\mathbb{C})$. This is an easy exercise, as the isomorphism can be defined explicitly. Viewing a $(2k) \times (2k)$-matrix as decomposed into 4 blocks of size $k \times k$, an isomorphism is given by

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \mapsto A_{11} \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + A_{12} \otimes \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) + A_{21} \otimes \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) + A_{22} \otimes \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

□

Remark 4.4. It is possible to describe the real Clifford algebras using similar methods, but one has to be more careful about signatures. As we have seen above, $C\ell(1,0) \cong \mathbb{C}$ and $C\ell(0,1) \cong \mathbb{R} \oplus \mathbb{R}$. Similarly to the description of $C\ell(2,\mathbb{C})$ in the proof of Theorem 4.4 one easily verifies that $C\ell(1,1) \cong M_2(\mathbb{R})$, while $C\ell(2,0) \cong \mathbb{H}$. It turns out that also $C\ell(0,2)$ is isomorphic to $M_2(\mathbb{R})$ but with a different $\mathbb{Z}_2$-grading than $C\ell(1,1)$. Similarly to the second part of the proof of Theorem 4.4, one verifies that $C\ell(n,0) \otimes C\ell(0,2) \cong C\ell(0,n+2)$, $C\ell(p,q) \otimes C\ell(1,1) \cong C\ell(p+1,q+1)$ and so on.

For the complex Clifford algebras, Theorem 4.4 shows that there is a distinction between even and odd dimensions of the underlying vector space. This can be viewed as a phenomenon of periodicity with period two. Indeed, for a $\mathbb{K}$-algebra A, one can also interpret $A \otimes M_n(\mathbb{K})$ as the algebra $M_n(A)$ of $n \times n$-matrices with entries from A (with the usual matrix multiplication). In these terms, Theorem 4.4 says that $C\ell(n+2,\mathbb{C})$ is (isomorphic to) a matrix algebra over $C\ell(n,\mathbb{C})$. There is a similar periodicity phenomenon for real Clifford algebras, but this time the period is 8. In particular $C\ell(p+8,q)$ and $C\ell(p,q+8)$ are both isomorphic to $M_{16}(C\ell(p,q))$.

The periodicity in complex and real Clifford algebras is deeply connected to Bott periodicity in algebraic topology, which is related to homotopy groups of the orthgonal and unitary groups and to real and complex K-theory, see [**Ka78**].

4.5. The spin representation(s). We next discuss how to obtain the spin representations of the group $Spin(n)$ and prove that they have the properties listed in Section 3.1 in general. The first property listed there is connectedness of $Spin(n)$, which we have not verified so far, also because it is not true in indefinite signature (where also the special orthogonal group is not connected). But for $n \geq 2$ the group $SO(n)$ is connected, so since $SO(n) = Spin(n)/\{\pm 1\}$ we see that $Spin(n)$ can have at most two connected compoents and to prove connectedness, it suffices to show that 1 and -1 can be joined by a smooth path lying in $Spin(n)$. But such a path can be easily written out explicitly by taking two orthormal vectors $v, w \in \mathbb{R}^n$ and the path in $C\ell_0(n,0)$ defined by $t \mapsto \cos(2t) + \sin(2t)vw$. This evidently equals 1 for $t = 0$ and -1 for $t = \pi/2$, so it suffices to show that the whole path is contained in $Spin(n)$. But this is true, since it can be written as $(\cos(t)v + \sin(t)w) \cdot (\sin(t)w - \cos(t)v)$ and the vectors multiplied here are unit vectors for all t, so they lie in in the pin group and thus their product lies in $Spin(n)$.

To proceed towards the Spin representations, we consider $V := \mathbb{R}^n$ endowed with the standard inner product $\langle\ ,\ \rangle$. The obvious inclusion of $\mathbb{R}^n$ into $\mathbb{C}^n$ with the standard complex bilinear form is orthogonal and thus induces a homomorphism $C\ell(n,0) \to C\ell(n,\mathbb{C})$ of real associative algebras which is compatible with the $\mathbb{Z}_2$-gradings on the two algebras. Moreover, the images of an orthonormal basis of V generate $C\ell(n,\mathbb{C})$ as a complex unital algebra. Now in Theorem 4.3 we have obtained $Spin(n)$ as a subgroup of the group of invertible elements of $C\ell_0(n,0)$, so this is mapped to a subgroup of the group of invertible elements in $C\ell_0(n,\mathbb{C})$. To proceed further, we have to distinguish between even and odd dimensions, and we start with the simpler case of odd dimensions.

If n is odd, then Theorem 4.4 tells us that $C\ell_0(n,\mathbb{C}) \cong M_N(\mathbb{C})$ (where $N = 2^m$ if $n = 2m+1$) so its group of invertible elements is $GL(N,\mathbb{C})$. Thus, we directly obtain a faithful representation of $Spin(n)$ on $\mathbb{C}^N$, on which -1 acts as $-\mathbb{I}$ and thus non-trivially. This is the spin representation $\mathcal{S}$ in odd dimensions. Since $Spin(n)$ is compact it follows from general results that there is a positive definite Hermitian inner product on $\mathbb{C}^N$ which is invariant under the action of $Spin(n)$.

Still for odd n, we know from Theorem 4.4 that the full Clifford algebra $C\ell(n,\mathbb{C})$ is isomorphic to $M_N(\mathbb{C}) \oplus M_N(\mathbb{C})$. Thus this comes with two (non-isomorphic) representations on $\mathbb{C}^N$, say $\mathcal{S}^+$ and $\mathcal{S}^-$, and $\mathbb{R}^n \subset \mathbb{C}^n \subset C\ell(n,\mathbb{C})$ naturally acts on both these representations. Now it is easy to see that each of these representations is non-trivial when restricted to $C\ell_0(n,\mathbb{C})$ and then it is an easy fact that $M_N(\mathbb{C})$ has only one non-trivial representation in dimension N. Hence as representations of $C\ell_0(n,\mathbb{C})$, both $\mathcal{S}^+$ and $\mathcal{S}^-$ are isomorphic to $\mathcal{S}$ and we can view the action of $\mathbb{R}^n$ as $* : \mathbb{R}^n \times \mathcal{S} \to \mathcal{S}$. For $x \in Spin(n) \subset C\ell_0(n,0)$, the action on $\mathbb{R}^n$ from the proof of Theorem 4.3 reduces to $\rho_x(v) = xvx^{-1}$. But this exactly says that $\rho_x(v) * (x \cdot \psi) = xvx^{-1}x \cdot \psi = x \cdot (v * \psi)$, which exactly is the equivariancy condtion from Section 3.1.

For even dimension, the discussion is similar with small changes. Here $C\ell_0(n,\mathbb{C}) \cong M_N(\mathbb{C}) \oplus M_N(\mathbb{C})$ and $C\ell(n,\mathbb{C}) \cong M_{2N}(\mathbb{C})$ with $N = 2^{m-1}$ if $n = 2m$. This shows that $Spin(n)$ comes with two basic complex N-dimensional irreducible representations $\mathcal{S}^+$ and $\mathcal{S}^-$ in even dimensions. Moreover, the representation $\mathcal{S}$ of $C\ell(n,\mathbb{C})$ on $\mathbb{C}^{2N}$ splits as $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ when restricted to the subalgebra $C\ell_0(n,\mathbb{C})$ and thus also over $Spin(n)$. Via the inclusion $\mathbb{R}^n \hookrightarrow C\ell(n,\mathbb{C})$ we get an action on this sum. Since any element of $\mathbb{R}^n$ together with $C\ell_0(n,\mathbb{C})$ generates $C\ell(n,\mathbb{C})$ one concludes that the action of each $v \in \mathbb{R}^n$ has to map $\mathcal{S}^+$ to $\mathcal{S}^-$ and vice versa. Thus we get a Clifford multiplication $* : \mathbb{R}^n \times \mathcal{S} \to \mathcal{S}$ which exchanges the two summands, and equivariancy follows exactly as in the odd dimensional case.

Remark 4.5. The construction of the spin representations as complex representations suggests a generalization of the spin groups, for which there still is a spin representation. Namely, one defines a Lie group $Spin^c(n)$ as the quotient of $Spin(n) \times U(1)$ by the two-element subgroup consisting of $(1,1)$ and $(-1,-1)$. The natural homomorphism $Spin(n) \to SO(n)$ induces a homomorphism $\rho^c : Spin^c(n) \to SO(n)$ whose kernel is isomorphic to $U(1)$. On the other hand, the complex spin representation of $Spin(n)$ can be extended to a representation of $Spin^c(n)$ by letting $U(1)$ act by complex multiplication on the representation space (and observing that $-1 \in Spin(n)$ acts as $-\mathrm{id}$ in the spin representation.

Now similar to spin structures as discussed in Chapter 3 there is the concept of a spin^c-structure on a Riemannian manifold (M,g). This is given by a $Spin^c(n)$-principal bundle $Q^c \to M$ together with a bundle map to the orthonormal frame bundle $\mathcal{O}M$

with base map id_M which is equivariant with respect to the homomorphism ρ^c. Having given such a spin^c-structure, one defines the spinor bundle as the associated bundle with respect to the Spin representation. Via the homomorphism ρ^c, one obtains a Clifford multiplication and then defines the Dirac operator exactly as in the case of spin structures.

It is easy to see that any spin structure $Q \to M$ can be extended to a spin^c-structure $Q^c \to M$ basically by forming a product with $U(1)$ and factorizing appropriately. But the key about the whole idea is that there are Riemannian manifolds that do admit a spin^c structure but not a spin structure, so one obtains a Dirac operator in a more general situation. In particular, it turns out that in dimension 4, *any* orientable Riemannian manifold does admit a spin^c-structure, which is the basis for so-called Seiberg–Witten invariants.

In general, the question of existence of spin^c-structures can be answered in terms of Čech cohomology. In Theorem 3.7 we have met the second Stifel–Whitney class $w_2(M, g) \in H^2(M, \mathbb{Z}_2)$, whose vanishing is equivalent to the existence of a spin structure. On the other hand, we have seen in Section 3.5 that the quotient homomorphism $\mathbb{Z} \to \mathbb{Z}_2$ induces a homomorphism $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}_2)$. It turns out that (M, g) admits a spin^c-structure if and only if $w_2(M, g)$ lies in the image of this homomorphism. The idea for proving this is related to the discussion in the end of Section 3.8. There we have discussed the first Chern class in the context of complex line bundles, but this can be equivalently phrased as describing principal bundles with structure group $U(1)$. Taking an element of $H^2(M, \mathbb{Z})$ which is mapped to $w_2(M, g)$, one may form a corresponding $U(1)$-principal bundle which can then be used to "correct" local lifts of the transition functions of $\mathcal{O}M$ as transitions functions with values in $Spin^c(n)$.

4.6. Generalized Dirac operators. We conclude this chapter with a short sketch of the general concept of Dirac operators and of the local index theorem, which provides a very powerful application of these ideas. As we have seen in Section 4.1 any orthogonal map between two innner product spaces induces an algebra homomorphism between the associated Clifford algebras. In particular, we obtain a natural representation of the group $O(n)$ on the vector space $C\ell(n, 0)$ for which each element of $O(n)$ acts by an algebra homomorphism. Given a Riemannian manifold (M, g), we can thus take the orthonormal frame bundle $\mathcal{O}M$ and form the associated bundle $\mathcal{O}M \times_{O(n)} C\ell(n, 0)$. By construction, the transitions functions of this bundle are algebra homomoprhisms, so we can define a fiber-wise multiplication in local trivializations. From the construction it follows easily that the fiber of this bundle over $x \in M$ can be naturally identified with the Clifford algebra $C\ell(T_xM, g_x)$. Thus we call this the *bundle of Clifford algebras* and denote it by $C\ell(TM, g)$. By construction, the Levi-Civita connection induces a covariant derivative on the bundle $C\ell(TM, g)$ which is compatible with the fiber-wise multiplication.

Now one defines a *Clifford module* on M to be a vector bundle $E \to M$ together with a smooth family of bilinear maps $* : T_xM \times E_x \to E_x$ which satisfies the Clifford relations in the sense that $\xi * \eta * v + \eta * \xi * v = g_x(\xi, \eta)v$ for all $\xi, \eta \in T_xM$ and $v \in E_x$. This defines a homomorphims $C\ell(T_xM, g_x) \to L(E_x, E_x)$ thus making E into a bundle of modules over the bundle $C\ell(TM, g)$ of algebras. If we assume that E is associated to either the orthonormal frame bundle $\mathcal{O}M$ or, in case that M is oriented, to a fixed spin structure $Q \to M$, then we also have an induced connection on E, and we denote all these connections by ∇. Then we can define a Dirac operator on $\Gamma(E)$ similarly as in Section 3.3 via $D(s) = \sum_i \xi_i * \nabla_{\xi_i} s$, where $\{\xi_i\}$ is a local orthonormal frame for TM.

As in the proof of Proposition 3.3 one verifies that $D \circ D$ is a generalized Laplacian on E.

To proceed towards the index theorem, one assumes that E comes with a $\mathbb{Z}_2$-gading $E = E_0 \oplus E_1$ such that $*$ maps $TM \times E_0$ to E_1 and vice versa. (This corresponds to the concept of a $\mathbb{Z}_2$-graded module over a $\mathbb{Z}_2$-graded algebra.) Assuming this, we see that the resulting Dirac operator is built up from $D_0 : \Gamma(E_0) \to \Gamma(E_1)$ and $D_1 : \Gamma(E_1) \to \Gamma(E_0)$ while D^2 is the sum of two operators preserving sections of each of the subbundles. Under small technical assumptions one can the use the fact that D^2 is a generalized Laplacian to prove that on a compact manifold M, both D_0 and D_1 have finite dimensional kerel, so there is a well defined integer $ind(D) := \dim(\ker(D_0)) - \dim(\ker(D_1))$, called the *index* of the Dirac operator D.

The Dirac operators $\not{D}$ on Spinors discussed in Section 3.3 plays a central role in this theory. Basically, it turns out that on a manifold admitting a spin structure, any Clifford module can be obtained from spinors as introduced in Section 3.3. For example, given any natural vector bundle $W \to M$, one can form the tensor product $\mathcal{S} \otimes W$ and define $*$ via the action on the first factor. This leads to the *twisted Dirac operator* $\not{D}_W$ and many important examples of Dirac operators are obtained in that way.

Surprisingly, it turns out that one can use tools from functional analysis, Riemannian geometry, and algebraic topology to compute the index of such a generalized Dirac operator as an integral of a certain n-form over the manifold M. This n-form is built up in a (rather involved) universal way from the Riemann curvature of g and the Clifford module E, This is the content of the *local index theorem*, which is discussed in detail in the book [**BGV92**]. Again, the twisted Dirac operators $\not{D}_W$ provide a very important special case in these considerations. The local index theorem for generalized Dirac operators then implies the general index theorem for elliptic operators, which is one of the cornerstones in the area between (global) analysis, differential geometry, and algebraic topology.

APPENDIX A

The Levi–Civita connection

In this appendix, we sketch a construction of the Levi-Civita connection associated to a Riemannian metric as a principal connection on the orthonormal frame bundle. Background from Riemannian geometry is not formally needed here.

A.1. The soldering form. Recall the description of the orthonormal frame bundle $\mathcal{O}M$ of a Riemannian n-manifold (M,g) from Proposition 2.5. For a point $x \in M$, the fiber $\mathcal{O}_xM$ is the set of all those linear isomorphisms $u : \mathbb{R}^n \to T_xM$, which are orthogonal with respect to the standard inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$ and the inner product g_x on T_xM. The topology and smooth structure on $\mathcal{O}M = \cup_{x\in M}\mathcal{O}_xM$ comes from the linear frame bundle $\mathcal{P}M$, so in particular local smooth sections of $\mathcal{O}M$ correspond to smooth local frames for TM with are orthonormal with respect to g.

Denoting by $p : \mathcal{O}M \to M$ the projection, we have, for each $u \in \mathcal{O}M$, the tangent map $T_up : T_u\mathcal{O}M \to T_xM$, where we put $x = p(u)$. Thus, for each $u \in \mathcal{O}M$, we can define a natural linear map $\theta_u : T_u\mathcal{O}M \to \mathbb{R}^n$ as $\theta_u := u^{-1} \circ T_up$, which evidently satisfies $\ker(\theta_u) = \ker(T_up)$, the vertical subspace in $T_u\mathcal{O}M$. It also follows readily from the definitions that for $\xi, \eta \in T_u\mathcal{O}M$, we have $g_x(T_up \cdot \xi, T_up \cdot \eta) = \langle \theta_u(\xi), \theta_u(\eta)\rangle$. Finally, it is clear from the construction that for a smooth vector field $\xi \in \mathfrak{X}(\mathcal{O}M)$, the function $\theta(\xi) : \mathcal{O}M \to \mathbb{R}^n$ defined by $\theta(\xi)(u) := \theta_u(\xi(u))$ is always smooth, so θ defines a $\mathbb{R}^n$-valued one-form on $\mathcal{O}M$.

Definition A.1. The form $\theta \in \Omega^1(\mathcal{O}M, \mathbb{R}^n)$ is called the *soldering form* on the orthonormal frame bundle.

The soldering form satisfies an obvious compatibility condition with the principal right action of $O(n)$ on $\mathcal{O}M$, which can be nicely phrased in terms of pullbacks. For $A \in O(n)$, consider the principal right action r^A by A as a smooth map $\mathcal{O}M \to \mathcal{O}M$. Then one can define the pullback $(r^A)^*\theta$ as for real valued forms by $((r^A)^*\theta)(u)(\xi) = \theta(r^A(u))(T_ur^A \cdot \xi)$. But since $p \circ r^A = p$, we get $T_{u\cdot A}(T_ur^A \cdot \xi) = T_up \cdot \xi$, so we conclude that $((r^A)^*\theta)(u) = (u \circ A)^{-1}(T_up \cdot \xi) = A^{-1}(\theta_u(\xi))$. This says that $(r^A)^*\theta = A^{-1} \circ \theta$, which is usually phrased as "θ is $O(n)$-equivariant".

Remark A.1. The soldering form is just an equivalent way to encode the inclusion of $\mathcal{O}M$ into $\mathcal{P}M$, which identifies $\mathcal{O}M$ as a reduction of structure group of the linear frame bundle, see Section 2.5. Indeed, suppose that we have given an abstract principal $O(n)$-bundle $p : P \to M$ together with a one-form $\theta \in \Omega^1(M, \mathbb{R}^n)$. Suppose further that θ is equivariant in the above sense, i.e. satisfies $(r^A)^*\theta = A^{-1} \circ \theta$ for all $A \in O(n)$ and *strictly horizontal* in the sense that $\ker(\theta(u)) = \ker(T_up)$ for each $u \in P$. Then for each $u \in P$, the linear map $\theta(u) : T_uP \to \mathbb{R}^n$ descends to a injection $T_uP/\ker(T_up) \to \mathbb{R}^n$, which must be a linear isomorphism, since both spaces have the same dimension. Since $T_uP/\ker(T_up) \cong T_{p(u)}M$, the inverse of this linear isomorphism can be viewed as an element $\iota(u)$ in the fiber $\mathcal{P}_xM$ of the linear frame bundle at $x := p(u)$.

We can view this construction as defining a map $\iota : P \to \mathcal{P}M$ and equivariancy of θ exactly says that $\iota(u \cdot A) = \iota(u) \circ A$. This shows that ι maps fibers to fibers and is injective on each fiber. One easily verifies that ι is smooth and thus a principal bundle map, which makes P into a reduction of structure group of $\mathcal{P}M$. By Proposition 2.5, this such a reduction is the orthonormal frame bundle of a Riemannian metric.

The soldering form can be used to give a nice description of the equivariant smooth function associated to a vector field on M:

PROPOSITION A.1. *Let (M, g) be a Riemannian manifold with orthonormal frame bundle $\mathcal{O}M$ and let $\theta \in \Omega^1(M, \mathbb{R}^n)$ be the soldering form. Suppose that $\xi \in \mathfrak{X}(M)$ is a (local) smooth vector field and $\tilde{\xi} \in \mathfrak{X}(\mathcal{O}M)$ is a local smooth lift of ξ, i.e. that $T_u p \cdot \tilde{\xi}(u) = \xi(p(u))$ for all u. Then $\theta(\tilde{\xi}) : \mathcal{O}M \to \mathbb{R}^n$ is the P-equivariant smooth function corresponding to ξ via Proposition 2.8.*

PROOF. By definition, $\theta_u(\tilde{\xi}(u)) = u^{-1}(T_u p \cdot \tilde{\xi}(u)) = u^{-1}(\xi(p(u))) \in \mathbb{R}^n$. But this is exactly the element $v \in \mathbb{R}^n$ such that $[(u, v)] = \xi(p(u))$ so we exactly get the function constructed in Proposition 2.8. □

To see that this result is useful, we have to observe that smooth lifts of vector fields exist both locally and globally. Indeed, we can even find smooth lifts which are $O(n)$-invariant in the sense that $(r^A)^*\tilde{\xi} = \tilde{\xi}$ for each $A \in O(n)$. For a principal bundle chart $\varphi : p^{-1}(U) \to U \times O(n)$ and a vector field $\xi \in \mathfrak{X}(U)$, we can define such a lift via $\tilde{\xi}(u) := T_u\varphi^{-1}(\xi(p(u)), 0)$, where we identify $T(U \times O(n))$ with $TU \times TO(n)$ as usual. Otherwise put, $\tilde{\xi} = \varphi^*(\xi, 0)$ and since $\varphi \circ r^A = (\mathrm{id}, \rho^A) \circ \varphi$, this readily implies that $(r^A)^*\tilde{\xi} = \tilde{\xi}$. Globally, we can start from $\xi \in \mathfrak{X}(M)$, a principal bundle atlas $\{(U_i, \varphi_i) : i \in I\}$ for $\mathcal{O}M$ and a partition $\{f_i : i \in I\}$ of unity subordinate to the covering $\{U_i : i \in I\}$ of M. Constructing $\tilde{\xi}_i \in \mathfrak{X}(p^{-1}(U_i))$ for each i from $\tilde{\xi}|_{U_i}$ as above, we can define $\tilde{\xi} := \sum_{i \in I}(f_i \circ p)\tilde{\xi}_i$. This clearly is a lift of ξ and since each $f_i \circ p$ is constant along the fibers of $\mathcal{O}M$, $\tilde{\xi}$ is again $O(n)$-invariant.

A.2. The Levi-Civita connection. Having constructed the soldering form $\theta \in \Omega^1(\mathcal{O}M, \mathbb{R}^n)$, we can form the exterior derivative $d\theta \in \Omega^2(\mathcal{O}M, \mathbb{R}^n)$. This can be either done by viewing θ as an n-tuple $(\theta^1, \dots, \theta^n)$ of real valued forms and defining $d\theta$ as $(d\theta_1, \dots, d\theta_n)$, or by directly using the global formula $d\theta(\tilde{\xi}, \tilde{\eta}) = \tilde{\xi} \cdot \theta(\tilde{\eta}) - \tilde{\eta} \cdot \theta(\tilde{\xi}) - \theta([\tilde{\xi}, \tilde{\eta}])$ for vector fields $\tilde{\xi}, \tilde{\eta} \in \mathfrak{X}(\mathcal{O}M)$.

Now suppose that $\gamma \in \Omega^1(\mathcal{O}M, \mathfrak{o}(n))$ is a principal connection form on $\mathcal{O}M$ as discussed in Section 2.8. This means that for a fundamental vector field ζ_X with $X \in \mathfrak{o}(n)$, we get $\gamma(\zeta_X) = X$ and that γ is $O(n)$-equivariant in the sense that $(r^A)^*\gamma = \mathrm{Ad}(A^{-1}) \circ \gamma$. Here the adjoint action just reduces to conjugation, i.e. $\mathrm{Ad}(A^{-1})(X) = A^{-1}XA$ for $A \in O(n)$ and $X \in \mathfrak{o}(n)$. Then we can look at the combination

$$T(\tilde{\xi}, \tilde{\eta}) := d\theta(\tilde{\xi}, \tilde{\eta}) + \gamma(\tilde{\xi})(\theta(\tilde{\eta})) - \gamma(\tilde{\eta})(\theta(\tilde{\xi})), \tag{A.1}$$

which by construction defines a two-form on $\mathcal{O}M$ with values in $\mathbb{R}^n$. Our main result will be that there is a unique connection form γ such that the two-form T defined by (A.1) vanishes identically. First we have to clarify some properties of this expression.

LEMMA A.2. *For any principal connection γ on $\mathcal{O}M$, the two-from T defined by* (A.1) *vanishes if one of its entries is vertical. This implies that for each $u \in \mathcal{O}M$, there is a bilinear, skew-symmetric map $\tau(u) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $T(\tilde{\xi}, \tilde{\eta})(u) = \tau(u)(\theta(\tilde{\xi}(u)), \theta(\tilde{\eta}(u))$. This map satisfies $\tau(u \circ A)(v, w) = A^{-1}\tau(u)(Av, Aw)$ for each $A \in O(n)$ and $v, w \in \mathbb{R}^n$. In particular, if T vanishes in a point of $\mathcal{O}M$ then it vanishes on the whole fiber through that point.*

Sketch of Proof. One first has to prove that $d\theta(\zeta_X,\tilde\eta)=-X\theta(\tilde\eta)$ for each $X\in\mathfrak{o}(n)$ and $\tilde\eta\in\mathfrak{X}(M)$, which is an infinitesimal version of equivariancy of θ. This uses the facts that the flow of ζ_X is $r^{\exp(tX)}$, and that the derivative at $t=0$ of $(\mathrm{Fl}^{\zeta_X}_t)^*\theta$ is the Lie derivative $\mathcal{L}_{\zeta_X}\theta$. Finally, since $\theta(\zeta_X)=0$, the Lie derivative can be written as $d\theta(\zeta_X,\_)$, which implies the claimed equation.

This exactly says that $T(\zeta_X,\tilde\eta)=0$ for each $\tilde\eta\in\mathfrak{X}(M)$ and since any vertical tangent vector in a point can be written as a value of a fundamental vector field, this shows that T vanishes if one of its entries lies $\ker(Tp)$. But this implies that for each $u\in\mathcal{O}M$, the value $T(u)$ descends to a bilinear map $(T_u\mathcal{O}M/\ker(T_uP))^2\to\mathbb{R}^n$, which is skew symmetric by definition. Since $\theta(u)$ induces a linear isomorphism $T_u\mathcal{O}M/\ker(T_uP)\to\mathbb{R}^n$ we see that there is a map $\tau(u)$ as claimed.

For the last part, we use that equivariancy of θ says $(r^A)^*\theta=A^{-1}\circ\theta$ for each $A\in O(n)$, which implies that also $(r^A)^*d\theta=A^{-1}\circ d\theta$. On the other hand, we also know equivariancy of γ, and putting all these together, one easily verifies that

$$T(u\circ A)(T_ur^A\cdot\tilde\xi(u),T_ur^A\cdot\tilde\eta(u))=A^{-1}T(u)(\tilde\xi(u),\tilde\eta(u)).$$

Since $\theta(u\circ A)(T_ur^A\cdot\tilde\xi(u))=A^{-1}(\theta(u)(\tilde\xi(u)))$ and similarly for $\tilde\eta$, this shows that $\tau(u\circ A)(A^{-1}v,A^{-1}w)=A^{-1}\tau(u)(v,w)$, which is equivalent to the claimed condition. □

Remark A.2. For those having background on Riemannian geometry, we observe that the form T defined in (A.1) exactly expresses the torsion of the covariant derivative induced by γ. Fixing γ we can work with the horizontal lift ξ^{hor} for a vector field $\xi\in\mathfrak{X}(M)$. This is characterized by $\gamma(\xi^{\mathrm{hor}})=0$ and $T_up\cdot\xi^{\mathrm{hor}}(u)=\xi(p(u))$. Evaluating (A.1) on ξ^{hor} and η^{hor}, the terms involving γ vanish and we are left with

$$T(\xi^{\mathrm{hor}},\eta^{\mathrm{hor}})=d\theta(\xi^{\mathrm{hor}},\eta^{\mathrm{hor}})=\xi^{\mathrm{hor}}\cdot\theta(\eta^{\mathrm{hor}})-\eta^{\mathrm{hor}}\cdot\theta(\xi^{\mathrm{hor}})-\theta([\xi^{\mathrm{hor}},\eta^{\mathrm{hor}}]).$$

Now by Proposition A.1, $\theta(\eta^{\mathrm{hor}})$ is the equivariant function corresponding to η, so $\xi^{\mathrm{hor}}\cdot\theta(\eta^{\mathrm{hor}})$ is the equivariant function representing the covariant derivative $\nabla_\xi\eta$. In the same way, the second summand represents $-\nabla_\eta\xi$. For the last summand, it is well known that $[\xi^{\mathrm{hor}},\eta^{\mathrm{hor}}]$ is a lift of the Lie bracket $[\xi,\eta]$ so again by Proposition A.1, $-\theta([\xi^{\mathrm{hor}},\eta^{\mathrm{hor}}])$ is the function corresponding to $-[\xi,\eta]$. Hence the whole expression (A.1) represents $\nabla_\xi\eta-\nabla_\eta\xi-[\xi,\eta]$ which is the usual definition of the torsion evaluated on ξ and η.

Now the fundamental theorem on existence and uniqueness of the Levi-Civita connection can be formulated as follows:

Theorem A.2. *Let (M,g) be a Riemannian manifold with orthonormal frame bundle $\mathcal{O}M\to M$ and let $\theta\in\Omega^1(\mathcal{O}M,\mathbb{R}^n)$ be the soldering form. Then there is a unique principal connection form $\gamma\in\Omega^1(\mathcal{O}M,\mathfrak{o}(n))$ on $\mathcal{O}M$ such that the two-form T defined by* (A.1) *vanishes identically.*

Proof. One first has to prove that there is some principal connection $\hat\gamma$ on $\mathcal{O}M$ (which is a general fact about principal bundles). Let us start with a principal bundle chart (U,φ) for $\mathcal{O}M$. Identifying $T(U\times G)$ with $TU\times TG$, we can send a tangent vector $(\xi_1,\xi_2)\in T_xU\times T_AG$ to $T_A\lambda_{A^{-1}}\cdot\xi_2\in T_{\mathbb{I}}O(n)=\mathfrak{o}(n)$ and pull back the resulting form with φ to a $\mathfrak{o}(n)$-valued one-form on $p^{-1}(U)$. It is easy to see that this defines a principal connection on $p^{-1}(U)\to U$. Do this for a principal bundle atlas $\{(U_i,\varphi_i):i\in I\}$ and denote by $\hat\gamma_i$ the resulting form on $p^{-1}(U_i)$. Then take a partition $\{f_i\}$ of unity subordinate to the covering $\{U_i\}$ of M and define $\hat\gamma\in\Omega^1(\mathcal{O}M,\mathfrak{o}(n))$ by $\hat\gamma(\tilde\xi):=\sum_i(f_i\circ p)\gamma_i(\tilde\xi)$. It is easy to verify that this indeed defines a principal connection on $\mathcal{O}M\to M$.

Now we need surprising input from linear algebra. Namely, for a linear map $\alpha : \mathbb{R}^n \to \mathfrak{o}(n)$ we can consider the map $\partial\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $\partial\alpha(v,w) := \alpha(v)w - \alpha(w)v$. This is clearly skew symmetric and bilinear, so we have defined a linear map $\partial : L(\mathbb{R}^n, \mathfrak{o}(n)) \to L(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$, and we claim that this is a linear isomorphism. Since both spaces have dimension $n\binom{n}{2}$, it suffices to prove that ∂ has trivial kernel. Thus suppose that for all $v, w, z \in \mathbb{R}^n$, we have

$$0 = \langle \partial\alpha(v,w), z\rangle = \langle \alpha(v)w, z\rangle - \langle \alpha(w)v, z\rangle.$$

This says that the trilinear map $\Phi(v,w,z) := \langle \alpha(v)w, z\rangle$ is symmetric in the first two entries. But since α is in $\mathfrak{o}(n)$, we get $\Phi(v,w,z) = -\langle w, \alpha(v)z\rangle = -\Phi(v,z,w)$, so Φ is skew symmetric in the last two entries. But this already implies formally that $\Phi = 0$ and hence $\alpha = 0$: We just have to compute

$$\begin{aligned}\Phi(v,w,z) &= -\Phi(v,z,w) = -\Phi(z,v,w) = \Phi(z,w,v)\\ &= \Phi(w,z,v) = -\Phi(w,v,z) = -\Phi(v,w,z).\end{aligned}$$

Now let $U \subset M$ be an open subset such that there is a smooth local section $\sigma : U \to \mathcal{O}M$ of $\mathcal{O}M$ defined on U. Consider the smooth function $\hat{\tau}$ associated to $\hat{\gamma}$ as in Lemma A.2 and define $\alpha : U \to L(\mathbb{R}^n, \mathfrak{o}(n))$ as $\partial^{-1} \circ \hat{\tau} \circ \sigma$. Finally, for each $x \in U$ and $\tilde{\xi} \in T_{\sigma(x)}\mathcal{O}M$ define

$$\gamma(\sigma(x))(\tilde{\xi}) := \hat{\gamma}(\sigma(x))(\tilde{\xi}) - \alpha(\sigma(x))(\theta(\sigma(x))(\tilde{\xi})).$$

A vertical vector $\zeta_X(\sigma(x))$ is mapped to X by $\hat{\gamma}$ and to 0 by θ, so $\gamma(\sigma(x))(\zeta_X) = X$. It is easy to see that γ can be uniquely extended to an $O(n)$-equivariant form on $p^{-1}(U)$ which still reproduces the generators of fundamental vector fields and thus defines a principal connection on this subset. In a point $\sigma(x)$, the definition in (A.1) readily implies that the function τ associated to γ is given by $\tau(\sigma(x)) = \hat{\tau}(\sigma(x)) - \partial\alpha(\sigma(x)) = 0$. Lemma A.2 then implies that T vanishes on all of $p^{-1}(U)$.

Now suppose that we have done this construction over two open subsets U_1 and U_2 such that $U_{12} = U_1 \cap U_2 \neq \emptyset$, getting principal connections γ_1 and γ_2. For a point $u \in p^{-1}(U_{12})$ consider $\gamma_1(u) - \gamma_2(u) : T_u\mathcal{O}M \to \mathfrak{o}(n)$. By definition this vanishes on $\zeta_X(u)$ for all $X \in \mathfrak{o}(n)$, so there has to be a linear map $\alpha : \mathbb{R}^n \to \mathfrak{o}(n)$ such that $(\gamma_1(u) - \gamma_2(u))(\tilde{\xi}) = \alpha(\theta(u)(\tilde{\xi}))$. But then the definition in (A.1) readily implies that the functions τ_1 and τ_2 associated to the two connections satisfy $\tau_2(u) = \tau_1(u) + \partial\alpha$. Since both these functions vanish identically, we get $\partial\alpha = 0$ and hence $\alpha = 0$ by injectivity of ∂. Thus $\gamma_1|_{p^{-1}(U_{12})} = \gamma_2|_{p^{-1}(U_{12})}$ and hence the local principal connections we have constructed fit together to define a unique global one. $\square$

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