Lie algebra cohomology and overdetermined systems

Andreas Čap

University of Vienna
Faculty of Mathematics

January 2008

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- This can be used as a replacement for Se–Ashi's theory in proofs of Fubini–Griffiths–Harris rigidity in the style of Hwang–Yamaguchi and Landsberg–Robles.
- Via prolongation procedures, it also leads to results on first BGG operators for non–projective AHS–structures.

Contents

1. The basic setup
2. A reduction theorem
3. Applications

Structure

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$|1|$–graded Lie algebras

Consider a semisimple Lie algebra $\mathfrak{g}$ endowed with a grading of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that no simple ideal is contained in $\mathfrak{g}_0$. Then $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a parabolic subalgebra with nilradical $\mathfrak{g}_1$, which leads to a complete classification of such gradings.

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Let G be a Lie group with Lie algebra $\mathfrak{g}$, $P \subset G$ a subgroup corresponding to $\mathfrak{p}$. For $g \in P$, we have $\mathrm{Ad}(g)(\mathfrak{p}) \subset \mathfrak{p}$ and $\mathrm{Ad}(g)(\mathfrak{g}_1) \subset \mathfrak{g}_1$. Let $G_0 \subset P$ be the subgroup consisting of those g for which $\mathrm{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i$ for all $i = -1, 0, 1$.

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It then turns out that $\mathrm{Ad} : G_0 \to GL(\mathfrak{g}_{-1})$ is infinitesimally effective. Hence on manifolds of dimensions $\dim(\mathfrak{g}_{-1})$ the notion of first order G_0–structures makes sense. Further, P is the semidirect product of G_0 and $\mathfrak{g}_1 \cong \mathfrak{g}^*_{-1}$.

AHS–structures

Under a cohomological condition, first order G_0–structures as discussed above are equivalent to normal Cartan geometries of type (G, P). This means that the principal G_0–bundle and the soldering form defining a G_0–structure canonicallly extend to a principal P–bundle and a normal Cartan connection.

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Examples

- $G = SO(p+1, q+1)$, $G_0 = CO(p, q)$ conformal
- $G = PSL(p+q, \mathbb{R})$, $G_0 = S(GL(p, \mathbb{R}) \times GL(q, \mathbb{R}))$ almost Grassmannian
- $G = SL(n+1, \mathbb{H})$, $G_0 = Sp(1)GL(n, \mathbb{H})$ almost quaternionic
- $G = PGL(n+1, \mathbb{R})$, $G_0 = GL(n, \mathbb{R})$ Cartan geometries are equivalent to projective structures, but not to G_0–structures

More setup

Let $\mathbb{V}$ be a representation of G (or simplicity assumed to be effective with values in $SL(\mathbb{V})$), and let $\rho : \mathfrak{g} \to \mathfrak{sl}(\mathbb{V})$ the infinitesmial representation.

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- Via the action of the grading element, $\mathbb{V}$ splits as $\mathbb{V} = \mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_N$ in such a way that $\mathfrak{g}_i \cdot \mathbb{V}_j \subset \mathbb{V}_{i+j}$. This splitting is G_0–invariant.

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- Via ρ, we view $\mathfrak{g}$ as a Lie subalgebra in $\mathfrak{sl}(\mathbb{V})$ and $\mathfrak{g}_i \subset \mathfrak{sl}_i(\mathbb{V})$. This $\mathfrak{g}$–invariant subspace admits a $\mathfrak{g}$–invariant complement $\mathfrak{g}^\perp \subset \mathfrak{sl}(\mathbb{V})$, which decomposes as $\mathfrak{g}^\perp = \oplus \mathfrak{g}_i^\perp$.

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We put $\mathbb{V}^i := \oplus_{j \geq i} \mathbb{V}_j$ and similarly for all the other spaces. This endows each of them with a P–invariant filtration.

For each $i \geq 0$, there is a natural subgroup $SL^i(\mathbb{V}) \subset SL(\mathbb{V})$ with Lie algebra $\mathfrak{sl}^i(\mathbb{V})$. The group $SL^0(\mathbb{V})$ consits of all filtration preserving automorphisms of $\mathbb{V}$. For $i > 0$, the maps in $SL^i(\mathbb{V})$ are congruent to the identity modulo maps which move up in the filtration by i degrees.

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The crucial ingredients for our purposes are subgroups $P^\# \subset G_0^\# \subset SL^0(\mathbb{V})$. We define $G_0^\#$ as the subgroup generated by G_0 and $SL^1(\mathbb{V})$ and $P^\#$ as the subgroup generated by P and $SL^2(\mathbb{V})$. The Lie algebras of these subgroups are $\mathfrak{g}_0 \oplus \mathfrak{sl}^1(\mathbb{V})$ and $\mathfrak{p} \oplus \mathfrak{sl}^2(\mathbb{V})$, respectively.

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Underlying structures

Let M be a smooth manifold of dimension $\dim(\mathfrak{g}_{-1})$ and let $\mathcal{V} \to M$ be a vector bundle modelled on $\mathbb{V}$ with a preferred volume form. Then we naturally get a frame bundle $SL(\mathcal{V})$ for $\mathcal{V}$ with structure group $SL(\mathbb{V})$. Principal connections on this frame bundle are equivalent to volume preserving linear connections on $\mathcal{V}$.

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Proposition

Suppose that γ is a principal connection on $SL(\mathcal{V})$ and that $j : \mathcal{G}_0^\# \to SL(\mathcal{V})$ is a reduction to the structure group $G_0^\#$ such that $j^*\gamma$ is injective on each tangent space and has values in $\mathfrak{g}_{-1} \oplus \mathfrak{sl}^0(\mathbb{V})$.
Then the $\mathfrak{g}_{-1}$–component of $j^*\gamma$ descends to a soldering form on $\mathcal{G}_0 := \mathcal{G}_0^\# / SL^1(\mathbb{V}) \to M$, making it into a first order G_0–structure.

Lie algebra cohomology

In the reduction theorem, we need assumptions on the Lie algebra cohomology group $H^1(\mathfrak{g}_{-1}, \mathfrak{g}^\perp)$. This space is a subquotient of $L(\mathfrak{g}_{-1}, \mathfrak{g}^\perp)$ and hence decomposes according to homogeneous degrees of linear maps. In particular, we get spaces $H^1(\mathfrak{g}_{-1}, \mathfrak{g}^\perp)_i$ and $H^1(\mathfrak{g}_{-1}, \mathfrak{g}^\perp)^i$ for all i.

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Lemma

If none of the simple ideals of $\mathfrak{g}$ is of projective type, then $H^1(\mathfrak{g}_{-1}, \mathfrak{g}^\perp)^2 = 0$. If in addition none of the simple ideals is of conformal type, then also $H^1(\mathfrak{g}_{-1}, \mathfrak{g}^\perp)_1 = 0$.

The reduction theorem

Step 1

Suppose that γ is a principal connection on $SL(\mathcal{V})$ and that $j:\mathcal{G}_0^\# \to SL(\mathcal{V})$ is a reduction to the structure group $G_0^\#$ such that $j^*\gamma$ is injective and has values in $\mathfrak{g}_{-1}\oplus\mathfrak{sl}^0(\mathbb{V})$.
If γ is *flat* and $H^1(\mathfrak{g}_{-1},\mathfrak{g}^\perp)_1=0$, then there is a reduction $\mathcal{P}^\#\hookrightarrow\mathcal{G}_0^\#$ to the structure group $P^\#$ with the same underlying G_0–structure, such that $\tilde{j}^*\gamma$ is injective and has values in $\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{sl}^1(\mathbb{V})$.

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Step 2

Suppose that we have a reduction as in the result of step 1 and that $H^1(\mathfrak{g}_{-1},\mathfrak{g}^\perp)^2=0$. Then there is a reduction $\mathcal{G}\hookrightarrow\mathcal{P}^\#$ to the structure group P with the same underlying G_0–structure, such that $\hat{j}^*\gamma$ has values in $\mathfrak{g}$ and defines a flat Cartan connection on $\mathcal{G}$.

sketch of proof

We sketch the proof of step 1, the second step is proved similarly.

- Let $\mathcal{G}_0 \to M$ be the underlying G_0–structure of $\mathcal{G}_0^\#$. Since $SL^1(\mathbb{V})$ is a contractible group, there is a global G_0–equivariant section $\sigma : \mathcal{G}_0 \to \mathcal{G}_0^\#$.

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- The main aim is to modify σ to a section $\hat{\sigma}$ in such a way that $\hat{\sigma}^* j^* \gamma$ has values in $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{sl}^1(\mathbb{V})$. Since this subspace is normalized by $P^{\#}$ we get a redution with the required properties for $\mathcal{P}^{\#} := \mathcal{G}_0 \times_{G_0} P^{\#}$.

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- To modify σ, we first decompose

$$\sigma^* j^* \gamma = \gamma_{-1} + (\gamma_0 + \gamma_0^\perp) + (\gamma_1 + \gamma_1^\perp) + \gamma_2^\perp + \dots$$

according to the decomposition of $\mathfrak{g}_{-1} \oplus \mathfrak{sl}^0(\mathbb{V})$.

- Since γ is flat, $\sigma^*j^*\gamma$ satisfies the Maurer–Cartan equation. Looking at the component in $\mathfrak{sl}_{-1}(\mathbb{V}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1}^{\perp}$, we get for all $\xi, \eta \in \mathfrak{X}(\mathcal{G}_0)$

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\begin{aligned}
0 &= d\gamma_{-1}(\xi,\eta) + [\gamma_0(\xi), \gamma_{-1}(\eta)] + [\gamma_{-1}(\xi), \gamma_0(\eta)] \\
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- Since $H^1(\mathfrak{g}_{-1}, \mathfrak{g}^{\perp})_1 = 0$, there is an element $Z \in \mathfrak{g}_1^{\perp}$ such that $\varphi = \partial Z$ (easily seen to be unique). A direct computation then shows that $\hat{\sigma}(u) = \sigma(u)\exp(Z(u))$ does the job.

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- The further steps are similar, and the first line in the equation shows that in the end one obtains a flat Cartan connection.

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Fubini–Griffiths–Harris rigidity

Assume that G is complex, none of the simple ideals of $\mathfrak{g}$ is of projective or conformal type and that $\mathbb{V}$ is a complex representation such that $\dim(\mathbb{V}_N)=1$. Then P is the stabilizer of the line $\mathbb{V}_N$ in G, and the representation induces an embedding of the generalized flag variety G/P into the projectivization $\mathbb{PV}$. Via $T(G/P)\cong G\times_P\mathfrak{g}_{-1}$, the kernel of the Fubini form $F_{2,2}$ for this embedding corresponds to a cone in $\mathfrak{g}_{-1}$, and $G_0\subset GL(\mathfrak{g}_{-1})$ is the subgroup of maps preserving this cone.

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Suppose that $X \subset \mathbb{P}\mathbb{V}$ is a subvariety of dimension $\dim(G/P)$ which, in a generic point x, has the same Fubini form $F_{2,2}$ as G/P. Restrict the canonical principal bundle $p : SL(\mathbb{V}) \to \mathbb{P}\mathbb{V}$ to X, and let ω be the Maurer–Cartan form of $SL(\mathbb{V})$. Consider the set of points in this restrictions in which ω has values in $\mathfrak{g}_{-1} \oplus \mathfrak{sl}^0(\mathbb{V})$.

For each such point g we get an induced isomorphism $T_{p(g)}X \to \mathfrak{g}_{-1}$, and we further restrict to those g, for which this isomorphism maps the kernel of the Fubini form $F_{2,2}$ to the distinguished cone. Multiplication from the right defines an action of $G_0^{\#}$ on this subset, which is easily seen to be free and transitive on each fiber.

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Hence we get a principal bundle $\mathcal{G}_0^{\#}$ over a dense open subset of X, which defines a reduction to the structure group $G_0^{\#}$. Denoting by $\tilde{P}$ the stabilizer of $\mathbb{V}_N$ in $SL(\mathbb{V})$, the extension $\tilde{E} := SL(\mathbb{V}) \times_{\tilde{P}} SL(\mathbb{V}) \to \mathbb{P}\mathbb{V}$ carries a flat principal connection induced by ω. Now $\mathcal{G}_0^{\#} \hookrightarrow \tilde{E}$ satisfies the assumptions of the reduction theorem, so we get a reduction to P for which ω pulls back to a flat Cartan connection. This gives a local isomorphism to (the natural embedding of) G/P.

BGG operators with maximal kernel

Let $p_0 : \mathcal{G}_0 \to M$ be a first order G_0–structure. As discussed before, this canonically extends to a normal Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P).

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The representation $\mathbb{V}$ gives rise to natural vector bundle $\mathcal{V} := \mathcal{G} \times_P \mathbb{V} \to M$. Since $\mathbb{V}$ is the restriction of a representation of G, the Cartan connection ω induces a linear connection $\nabla^{\mathcal{V}}$ on $\mathcal{V}$. These are the so–called *tractor bundles* and *tractor connections*.

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The Kostant codifferential induces natural bundle maps $\partial^* : \Lambda^k T^*M \otimes \mathcal{V} \to \Lambda^{k-1} T^*M \otimes \mathcal{V}$ for all k. The subquotients $\ker(\partial^*)/\operatorname{im}(\partial^*)$ turn out to be isomorphic to $\mathcal{G}_0 \times_{G_0} H^k(\mathfrak{g}_{-1}, \mathbb{V}) =: \mathcal{H}^k$ and the cohomologies are computable using Kostant's theorem.

The P–invariant filtration $\{\mathbb{V}^i\}$ of $\mathbb{V}$ induces a filtration $\{\mathcal{V}^i\}$ of $\mathcal{V}$ by smooth subbundles. In particular, $\mathcal{V}/\mathcal{V}^1 \cong \mathcal{G}_0 \times_{G_0} \mathbb{V}_0$. The BGG machinery can be used to construct a differential operator $L : \Gamma(\mathcal{V}/\mathcal{V}^1) \to \Gamma(\mathcal{V})$ such that $\partial^* \circ \nabla^{\mathcal{V}} \circ L = 0$ and hence $\nabla^{\mathcal{V}} \circ L$ induces an invariant operator $D : \Gamma(\mathcal{V}/\mathcal{V}^1) \to \Gamma(\mathcal{H}^1)$, which is always overdetermined. For example, these include all the conformal Killing operators.

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The construction also implies that the obvious projection and L induce inverse bijections

$$\ker(D) \leftrightarrow \{s \in \Gamma(\mathcal{V}) : \nabla^{\mathcal{V}} s \in \Gamma(\mathrm{im}(\partial^*))\}$$

If $\mathcal{G}_0 \to M$ is locally flat, then the connection $\nabla^{\mathcal{V}}$ is flat and one shows that $\nabla^{\mathcal{V}} s \in \Gamma(\mathrm{im}(\partial^*))$ is only possible if $\nabla^{\mathcal{V}} s = 0$. Hence in these cases, the machinery provides a system in closed form which is equivalent to $D(\varphi) = 0$ and $\dim(\ker(D)) = \dim(\mathbb{V})$.

In the general case, one can use prolongation procedures by [BCEG] or [HSSS] to construct a vector bundle map $C : \mathcal{V} \to T^*M \otimes \mathcal{V}$ such that the linear connection $\hat{\nabla} := \nabla^{\mathcal{V}} + C$ has the property that $\hat{\nabla} s = 0$ is equivalent to $\nabla^{\mathcal{V}} s \in \Gamma(\operatorname{im}(\partial^*))$. Thus, $\dim(\ker(D)) \leq \dim(\mathbb{V})$ always holds.

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By construction, the (volume preserving) frame bundle of $\mathcal{V}$ can be written as $SL(\mathcal{V}) := \mathcal{G} \times_P SL(\mathbb{V})$ and the tractor connection $\nabla^{\mathcal{V}}$ is induced by a principal connection γ on $\mathcal{P}$ which under the natural map $\mathcal{G} \to SL(\mathcal{V})$ pulls back to ω. Putting $\mathcal{G}_0^{\#} = \mathcal{G} \times_P G_0^{\#}$ and $\mathcal{P}^{\#} := \mathcal{G} \times_P P^{\#}$, we get a reductions $\mathcal{P}^{\#} \hookrightarrow \mathcal{G}_0^{\#} \hookrightarrow SL(\mathcal{V})$. The pullback of γ under these has values in $\mathfrak{g}_{-1} \oplus \mathfrak{sl}^0(\mathbb{V})$ and $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{sl}^1(\mathbb{V})$, respectively.

The prolongation procedures are set up in such a way that $C(\mathcal{V}^i)$ is always contained in $T^*M \otimes \mathcal{V}^i$ and for torsion free geometries, it is even contained in $T^*M \otimes \mathcal{V}^{i+1}$. Otherwise put, C can be viewed as a one–form on M with values in $\mathfrak{gl}^0(\mathcal{V})$ respectively $\mathfrak{sl}^1(\mathcal{V})$.

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The upshot of this is that the principal connection $\hat{\gamma}$ corresponding to $\hat{\nabla} = \nabla^{\mathcal{V}} + C$ aways admits a reduction to $\mathcal{G}_0^\#$ and in the torsion free case even to $\mathcal{P}^\#$. If $\dim(\ker(D)) = \dim(\mathbb{V})$, then of course the connection $\hat{\nabla}$ must be flat. It is easy to see that replacing C by its tracefree part (which has values in $\mathfrak{sl}^0(\mathcal{V})$) the resulting connection is still flat. If the assumptions of the reduction theorem are satisfied, then we obtain a flat Cartan geometry of type (G, P), whose underlying G_0–structure is the same as for $\mathcal{G}$, so the original structure must have been flat. Hence we conclude

Theorem

Consider a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) corresponding to a $|1|$–grading of $\mathfrak{g}$ such that none of the simple ideals of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$ or of projective type. For a infinitesimally faithful representation $\mathbb{V}$ of $\mathfrak{g}$ assume that the kernel of the corresponding first BGG–operator D has dimension $\dim(\mathbb{V})$. If either none of the simple ideals of $\mathfrak{g}$ is of conformal type or the geometry is torsion free, then $(p : \mathcal{G} \to M, \omega)$ must be locally flat.