Normal Weyl structures and special solutions of first BGG operators

Andreas Čap

University of Vienna
Faculty of Mathematics

Srni, January 2011

- This talk reports on joint work in progress with Rod Gover (Auckland) and Matthias Hammerl (Vienna).
- Geometric overdetermined systems of PDE play an important ingredient in the study of geometric structures. Prototypical examples of such systems are the infinitesimal automorphism equation for many types of structures, the twistor equation on spinors, the equation describing rescalings to Einstein metrics, and the conformal Killing equations on vectors, differential forms and tensors.
- For some of these equations, strong restrictions on the zero sets of solutions are known classically. Questions on zeros of infinitesimal automorphism and the related questions on essential conformal isometries are intensively studied.
- All these systems can be obtained using the machinery of BGG sequences.

- The machinery of BGG sequences not only provides a uniform construction for such systems intrinsic to a parabolic geometry. It also gives rise to a class of special solutions of these systems ("normal solutions"), which are related to parallel sections of so–called tractor bundles.
- Via normal Weyl structures one obtains a comparison between a general (curved) geometry and the homogeneous model of the geometry, which relates any special solution on the curved geometry to one on the homogeneous model. This in particular leads to a detailed description of the zero sets of special solutions.
- A non–linear analog of this technique can be used to study holonomy reductions, see the talk by Matthias Hammerl.

Contents

1. Parabolic geometries and normal Weyl structures
2. First BGG operators and special solutions
3. The comparison map
4. Examples for projective structures

Let G be a semisimple Lie group and $P \subset G$ a parabolic subgroup. Then a *parabolic geometry* of type (G, P) on a smooth manifold M with $\dim(M) = \dim(G/P)$ by definition consists of a principal fiber bundle $p : \mathcal{G} \to M$ with structure group P and a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G.

Assuming the conditions of regularity and normality, such parabolic geometries provide equivalent ways to encode geometric structures. Well known examples of structures that can be described in this way include classical projective, conformal, and almost quaternionic structures, CR structures of hypersurface type, path geometries, quaternionic contact structures, and several types of generic distributions. The details of this equivalence do not play any role for the purpose of this lecture.

normal Weyl structures

The Cartan connection ω in particular gives rise to a trivialization $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$ of the tangent bundle of $\mathcal{G}$, with the vertical subbundle $V\mathcal{G}$ corresponding to $\mathcal{G} \times \mathfrak{p}$. Here $\mathfrak{p} \subset \mathfrak{g}$ denotes the Lie algebra of the subgroup $P \subset G$. Now there always is a subalgebra $\mathfrak{g}_- \subset \mathfrak{g}$ which forms a complementary subspace to $\mathfrak{p}$.

Fixing a point $u_0 \in \mathcal{G}$ and putting $x_0 = p(u_0) \in M$, we get a canonical local section of $p : \mathcal{G} \to M$ mapping x_0 to u_0: For $X \in \mathfrak{g}_-$, we get the constant vector field $\tilde{X} \in \mathfrak{X}(\mathcal{G})$ characterized by $\omega(\tilde{X}) = X$, and we can consider its flow. Restricting to a suitable open neighborhood V of zero in $\mathfrak{g}_-$, $\varphi(X) := \mathrm{Fl}_1^{\tilde{X}}(u_0)$ defines a smooth map $\varphi : V \to \mathcal{G}$ such that $p \circ \varphi$ is a diffeomorphism onto an open neighborhood U of x_0 in M.

Consequences

- Choosing a basis for $\mathfrak{g}_-$, we obtain local coordinates on U ("normal coordinates").
- We can specify a smooth section of $\mathcal{G}|_U$ by mapping $p(\varphi(X))$ to $\varphi(X)$. This gives rise to a trivialization of $\mathcal{G}|_U$.
- Given any representation $\mathbb{W}$ of P, we can form the the associated bundle $\mathcal{G} \times_P \mathbb{W}$, and we get an induced trivialization of this bundle over U ("normal trivializations"). The resulting trivializations are compatible with any natural bundle map, which comes from a P–equivariant map between the inducing representations.
- Choosing a basis in $\mathbb{W}$, we get a canonical local frame of the associated bundle $\mathcal{G} \times_P \mathbb{W}$ over U ("normal frames"). Again, these are compatible with natural bundle maps.

Note that fixing the point $x_0 \in M$, the family of sections obtained in the above way is parametrized by $P \cong p^{-1}(x_0)$, so there is only a finite dimensional family.

For parabolic geometries, there is a special class of natural bundles. These are called *tractor bundles* and are characterized by the fact that they are induced by restrictions to P of representations of G.

Properties of tractor bundles

- The Cartan connection induces a linear connection ("(normal) tractor connection") on any tractor bundle.
- If $\mathbb{V}$ is an irreducible representation of G, then is is canonically filtered by P–invariant subspaces, and in particular it has a canonical P–irreducible quotient $\mathbb{H}_0$.
- Correspondingly, the tractor bundle $\mathcal{V} := \mathcal{G} \times_P \mathbb{V}$ has a natural quotient bundle $\mathcal{H}_0 := \mathcal{G} \times_P \mathbb{H}_0$. We denote by $\Pi : \mathcal{V} \to \mathcal{H}_0$ the corresponding natural bundle map.

The machinery of BGG–sequences uses the tractor connection and the associated covariant exterior derivative to construct higher order natural differential operators defined on sections of $\mathcal{H}_0$ and other natural subquotients of the bundles of $\mathcal{V}$–valued differential forms. We only need very limited information:

- The first BGG operator D is defined on sections of $\mathcal{H}_0$. The target bundle, the order, and the principal part of the operator can be easily computed from representation theory data of $\mathbb{V}$.
- If $s \in \Gamma(\mathcal{V})$ is a section which is parallel for the tractor connection, then $\Pi(s)$ lies in the kernel of D.
- The map Π is injective on parallel sections, and identifies them with a subspace of solutions of D ("normal solutions").
- On G/P (and geometries locally isomorphic to G/P) the tractor connection is flat and any solution is normal.

The main technical ingredient needed for the further study of normal solutions is easy to prove:

Proposition

Let $s \in \Gamma(\mathcal{V})$ be a section of a tractor bundle, which is parallel for the tractor connection. Then the function $f : U \to \mathbb{V}$ corresponding to s via a normal trivialization is given by $f(p(\varphi(X))) = \exp(-X) \cdot f(x_0)$.

This simple observation has rather strong consequences, however. First, we can get quite detailed information about the structure of potential solutions and for simple tractor bundles, we can even easily give a complete list of all potential solutions:

Polynomiality

Theorem

The coordinate functions of any solution of a first BGG operator with respect to a normal frame are polynomials in the normal coordinates. The degree of these polynomials is bounded by the length of the natural P–invariant filtration of the representation $\mathbb{V}$.

Idea of proof: For a parallel section $s \in \Gamma(\mathcal{V})$ polynomiality of the coordinates in any normal frame follows from the fact that any $X \in \mathfrak{g}_-$ acts by a nilpotent endomorphism on $\mathbb{V}$ with nilpotence index bounded by the filtration length. Thus the result follows for normal frames of $\mathcal{H}_0$ which are obtained by projections from $\mathcal{V}$, and then the general result follows easily.

Lists of potential solutions

To get the flavor, here is the list of potential normal solutions of the equation $\nabla_{(a}\sigma_{bc)} = 0$ on sections of $S^2 T^* M(4)$ on a manifold endowed with a projective structure. Here $\{\varphi_{ij}\}$ is a normal frame of the bundle, the x_i are normal coordinates, and on $\mathbb{R}P^n$ any element in the list is a solution. (The tractor bundle in question consists of tractors with curvature tensor type symmetries.)

φ_{ij}	$i \leq j$
$x_k\varphi_{ij} - x_j\varphi_{ik}$	$i \leq j < k$
$x_k\varphi_{ij} - x_i\varphi_{jk}$	$i < j \leq k$
$x_jx_k\varphi_{ii} - x_ix_k\varphi_{ij} - x_ix_j\varphi_{ik} + {x_i}^2\varphi_{jk}$	$i < j \leq k$
$x_kx_\ell\varphi_{ij} - x_jx_k\varphi_{i\ell} - x_ix_\ell\varphi_{jk} + x_ix_j\varphi_{k\ell}$	$i < j < k \leq \ell$
$x_jx_\ell\varphi_{ik} - x_jx_k\varphi_{i\ell} - x_ix_\ell\varphi_{jk} + x_\ell x_k\varphi_{j\ell}$	$i < j \leq k < \ell$

We can interpret the above results in a slightly different way. Namely, we can combine the inverse of a normal chart for the homogeneous model $G \to G/P$ with a normal chart of a curved geometry $\mathcal{G} \to M$, to obtain a diffeomorphism $\psi : U \to \tilde{U}$ where $U \subset M$ is the domain of a normal chart and $\tilde{U} \subset G/P$ is an appropriate open neighborhood of $o = eP$ in G/P.

Theorem

For any normal solution α of a first BGG operator on M, the diffeomorphism ψ intertwines the coordinate functions of α with respect to a normal frame with the coordinate functions of a normal solution $\tilde{\alpha}$ of the same first BGG operator on G/P.

In particular, ψ intertwines the zero sets of α and $\tilde{\alpha}$ so the zero set of α cannot be more complicated than the one of $\tilde{\alpha}$.

Important points for the applicability

- On G/P tractor bundles are always trivial with flat tractor connections, so parallel sections and hence normal solutions of first BGG operators on G/P are very well understood.
- While the comparison map cannot be an isomorphism of parabolic geometries (since one geometry is flat and the other isn't) it is more than just a diffeomorphism. In particular, one can prove compatibility with certain distinguished curves, which e.g. leads to proofs for zero sets being totally geodesic.
- Normal solutions which are sections of density bundles select (outside of their zero sets) one of the distinguished connections for the geometry. One can carry over properties of this connection related to completeness from G/P to M.

Holonomy reductions

One can describe holonomy reductions of parabolic geometries and of general Cartan geometries as sections of non–linear analogs of tractor bundles which are parallel for an induced fiber bundle connection.

The comparison map can be used to relate holonomy reductions of curved geometries to holonomy reductions of G/P, which again are very well understood. This is the main technical input in the proof of the basic facts on holonomy reductions to be explained in Matthias Hammerl's talk.

Projective structures

- Let M be a smooth manifold of dimension $n \geq 2$. Two torsion free connections on TM are called *projectively equivalent* iff they have the same unparametrized geodesics.
- An equivalence class $[\nabla]$ of such connections is called a *projective structure* on M. This gives rise to a family of distinguished paths, with one path through each point in each direction. The comparison map intertwines paths through the center of the normal chart.
- Projective structures can be equivalently described as normal parabolic geometries of type (G, P), where $G = SL(n+1, \mathbb{R})$ and $P \subset G$ is the stabilizer of a line in $\mathbb{R}^{n+1}$.
- Completely reducible representations of P give rise to tensor bundles. Tractor bundles can be built up from the *standard tractor bundle* $\mathcal{T}$ corresponding to $\mathbb{V} = \mathbb{R}^{n+1}$ and its dual.

To any torsion free connection ∇ on TM, one associates the *Rho–tensor* $\mathsf{P}_{ab} = \frac{-1}{(n-1)(n+1)}(nR_{ab} + R_{ba})$, where R_{ab} is the Ricci curvature of ∇.

Given an irreducible tensor bundle $\mathcal{W}$ and an integer $k \geq 1$, there is a unique first BGG operator of order k defined on a weighted version of $\mathcal{W}$.

The solutions of these operators on the homogeneous model $G/P = \mathbb{R}P^n$ can be nicely described in terms of homogeneous polynomials.

Example 1: Ricci flatness

The first BGG equation corresponding to $\mathcal{T}^* = \mathcal{G} \times_P \mathbb{R}^{(n+1)*}$ turns out to be $\nabla_a \nabla_b \sigma + \mathsf{P}_{ab}\sigma = 0$ on a density σ of weight one. Outside its zero set, a solution σ determines a connection ∇ in the projective class with zero Ricci curvature.

On $\mathbb{R}P^n$, solutions of this equation simply correspond to linear functionals $\lambda \in \mathbb{R}^{(n+1)*}$ and the zero set of the corresponding density is the projective hyperplane corresponding to the kernel of λ. In particular, this is totally geodesic, and we conclude:

If $\sigma \in \Gamma(\mathcal{E}(1))$ satisfies $\nabla_a \nabla_b \sigma + \mathsf{P}_{ab}\sigma = 0$, then its zero set is either empty or a totally geodesic hypersurface of M. On the dense open subset where $\sigma \neq 0$, one obtains a Ricci flat affine connection in the projective class.

Example 2: Klein–Einstein structures

The first BGG equation corresponding to $S^2\mathcal{T}^*$ is the equation

$$\nabla_{(a}\nabla_b\nabla_{c)}\sigma + 4\mathsf{P}_{(ab}\nabla_{c)}\sigma + 2(\nabla_{(a}\mathsf{P}_{bc)})\sigma = 0 \qquad (*)$$

on a density σ of weight two. Outside its zeros, σ determines a connection ∇ in the projective class such that $\nabla_a\mathsf{P}_{bc} = 0$. If P_{ab} is non–degenerate, it defines a pseudo–Riemannian metric, for which ∇ is the Levi–Civita connection and which is automatically Einstein. If M is compact, the closure of such a set can thus be viewed as a compactification of an Einstein manifold.

On $\mathbb{R}P^n$, solutions correspond to symmetric bilinear forms on $\mathbb{R}^{n+1}$ and in the non–degenerate case, the zero set of the corresponding density is a quadric in $\mathbb{R}P^n$, which canonically inherits a conformal structure. Outside the zero set, P_{ab} is non–degenerate and connected components of this open subset are geodesically complete. Thus we obtain:

Let σ be a solution of $(*)$ which corresponds to a non–degenerate bilinear form on $\mathcal{T}$. Then the zero set of σ is either empty or an embedded hypersurface, which inherits a conformal structure from the projective structure on M. On each connected component of of $\{x \in M : \sigma(x) \neq 0\}$, one obtains a pseudo–Riemannian Einstein metric whose Levi–Civita connection lies in the projective class.

If M is compact, these Einstein metrics are geodesically complete. Taking the closure of such a component defines a compactification of the Einstein manifold, in which one adds a boundary at infinity which carries a conformal structure. This is not a conformal compactification, however.