Comparison techniques for infinitesimal automorphisms of parabolic geometries

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Srni, January 2012

- This talk reports on joint work in progress with Karin Melnick (University of Maryland).
- We study infinitesimal automorphisms of parabolic geometries such that the corresponding one–parameter group of automorphisms has a higher order fix point.
- This is based on a comparison result by C. Frances, which relates such automorphisms to an automorphism of the homogeneous model of the geometry.
- To exploit this systematically, one can use the algebraic structure which is canonically available on tensor bundles of a parabolic geometry. We will in particular discuss applications to projective and conformal structures, for which the results are known but the proofs are new.

Contents

We consider geometric structures which can be equivalently encoded as *Cartan geometries*, so on a manifold M, we have a principal P–bundle $p : \mathcal{G} \to M$ endowed with a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, which trivializes $T\mathcal{G}$. Here we have fixed a Lie group G with Lie algebra $\mathfrak{g}$ and a closed subgroup $P \subset G$ (determined by the geometry in question).

- Any automorphism of such a geometry uniquely lifts to a principal bundle automorphism $\varphi : \mathcal{G} \to \mathcal{G}$ such that $\varphi^*\omega = \omega$.
- Applying this to local flows, we conclude that any infinitesimal automorphism lifts to a P–invariant vector field $\xi \in \mathfrak{X}(\mathcal{G})$ such that $\mathcal{L}_\xi \omega = 0$.
- P–invariant vector fields on $\mathcal{G}$ are in bijective correspondence with sections of the *adjoint tractor bundle* $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$, and projecting vector fields defines a surjective bundle map $\Pi : \mathcal{A}M \to TM$.

It is easy to see that a principal bundle automorphism $\varphi : \mathcal{G} \to \mathcal{G}$ is locally determined by its value in a single point, so in particular, under mild assumptions the identity map is the only automorphism having a fix point on this level. Likewise, on the level of the Cartan bundle, an infinitesimal automorphism is nowhere vanishing.
On the level of the underlying manifold, fix points of automorphisms are possible:

- $x \in M$ is a fix point, iff for each $u \in p^{-1}(x) \subset \mathcal{G}$, there is an element $b \in P$ such that $\varphi(u) = u \cdot b$, where the dot denotes the principal right action.
- On the infinitesimal level, fix points of the infinitesimal automorphism corresponding to $\sigma \in \Gamma(\mathcal{A}M)$ are points $x \in M$ such that $\Pi(\sigma)(x) = 0$.
- For certain geometries, there is a natural notion of higher order fix points, which will be introduced later.

The basic result we will use is a comparison theorem by C. Frances. An automorphism of a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ with a fix point can be, locally around the fix point, related to an automorphism of the homogeneous model of the geometry.

The homogeneous model

This is the homogeneous space G/P endowed with the principal P–bundle defined by the canonical projection $G \to G/P$ and the left Maurer–Cartan form as the Cartan connection.

- The automorphisms of this geometry are the left translations by elements of G.
- Correspondingly, the infinitesimal automorphisms of the geometry are exactly the *right invariant* vector fields R_X for $X \in \mathfrak{g}$.

Comparison is built on an analog of the exponential mapping for Cartan geometries. Since ω trivializes $T\mathcal{G}$, an element $X \in \mathfrak{g}$ determines the *constant vector field* $\tilde{X} \in \mathfrak{X}(\mathcal{G})$ via $\omega(\tilde{X}) = X$. Fixing a point $u_0 \in \mathcal{G}$ and choosing $X \in \mathfrak{g}$ close enough to zero, the flow $\mathrm{Fl}_t^{\tilde{X}}(u_0)$ is defined up to time one, and we put $\exp_{u_0}(X) := \mathrm{Fl}_1^{\tilde{X}}(u_0)$. A standard argument shows that $\exp_{u_0}$ defines a diffeomorphism from an open neighborhood of zero in $\mathfrak{g}$ onto an open neighborhood of u_0 in $\mathcal{G}$.

Now let φ be an automorphism of $(\mathcal{G} \to M, \omega)$. Having a fix point means that $\varphi(u_0) = u_0 \cdot b_0$ for some $u_0 \in \mathcal{G}$ and $b_0 \in P$. We want to compare this to an automorphism of the homogeneous model $(G \to G/P, \omega^{MC})$, and of course we can do this at $e \in G$. So the right automorphism to compare to is left multiplication by b_0. In particular, we can look at lines in exponential coordinates, which just get reparametrized by this automorphism.

Theorem (C. Frances, K. Melnick)

Let φ be an automorphism of a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ and $u_0 \in \mathcal{G}$ a point such that $\varphi(u_0) = u_0 \cdot b_0$ for some element $b_0 \in P$. Let $X \in \mathfrak{g}$ be an element such that $\exp_{u_0}(sX)$ is defined for s in an interval $I \subset \mathbb{R}$ around zero.
Suppose that there is a diffeomorphism $c : I \to I'$ fixing zero for some interval I' and $b : I \to P$ is a smooth curve with $b(0) = b_0$ such that *in* G we have

$$b_0 e^{sX} = e^{c(s)X} b(s)$$

Then the corresponding equation holds in the curved geometry: $\exp_{u_0}(tX)$ is defined for $t \in I'$ and

$$\varphi(\exp_{u_0}(sX)) = \exp_{u_0}(c(s)X) \cdot b(t).$$

We now specialize to the case that G is semisimple and $P \subset G$ is parabolic in the sense of representation theory. This leads to a natural concept of higher order fix points and to systematic ways to apply the comparison theorem.

A parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ corresponds to a grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ of the Lie algebra $\mathfrak{g}$ such that $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$.

- Defining $\mathfrak{g}^i := \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$, we obtain a filtration $\mathfrak{g} = \mathfrak{g}^{-k} \supset \cdots \supset \mathfrak{g}^k$ of the Lie algebra $\mathfrak{g}$. In particular, the filtration is invariant under the Lie subalgebra $\mathfrak{p} = \mathfrak{g}^0$.
- Thus we obtain an induced filtration $\mathcal{A}M = \mathcal{A}^{-k}M \supset \cdots \supset \mathcal{A}^k M$ of the adjoint tractor bundle by smooth subbundles.
- $\ker(\Pi) = \mathcal{A}^0 M$, so $\mathcal{A}M/\mathcal{A}^0 M \cong TM$ and $\mathcal{A}^1 M \cong T^*M$.

Definition

We say that an infinitesimal automorphism $s \in \Gamma(\mathcal{A}M)$ of a parabolic geometry $(\mathcal{G} \to M, \omega)$ has a higher order fix point in $x_0 \in M$ iff $s(x_0) \in \mathcal{A}^1 M \subset \mathcal{A}M$. In this case, $s(x_0)$ can be viewed as an element of $T^*_{x_0}M$ called the *isotropy* of s at x_0.

Choosing $u_0 \in \mathcal{G}$ with $p(u_0) = x_0$ gives rise to a linear isomorphism $T^*_{x_0}M \to \mathfrak{p}_+ := \mathfrak{g}^1$. Passing to another point in the same fiber changes this linear isomorphism by composition with $\mathrm{Ad}(b)$ for some $b \in P$. Hence the isotropy gives rise to a well defined P–orbit in $\mathfrak{p}_+$, and there is an initial classification of higher order fix points by the space of such P–orbits. This is particularly remarkable in view of a result of E. Vinberg, which says that there are only finitely many different orbits.

Further fix points

Given $Z \in \mathfrak{p}_+$, and a coset in $\mathfrak{g}/\mathfrak{p}$, we may ask whether there is a representative $X \in \mathfrak{g}$ of the coset such that $[[Z, X], X] = 0$ or even $[Z.X] = 0$. These conditions are well behaved with respect to the adjoint action of P. Given $\alpha \in T^*_{x_0}M$, we thus get a subset $\mathcal{F} \subset T_{x_0}M$ from the first condition and a linear subspace $\mathcal{C} \subset \mathcal{F}$ from the second.

- If $[[Z, X], X] = 0$, then $e^{tZ}e^{sX} = e^{sX}e^{tZ}e^{-ts[Z,X]}$ holds in G and $e^{tZ}e^{-ts[Z,X]} \in P$.
- For an infinitesimal automorphism $\sigma \in \Gamma(\mathcal{A}M)$ with a higher order fix point, the element Z comes from the isotropy.
- If X corresponds to a direction in $\mathcal{F}$, then comparison shows that we get a curve consisting of fixed points. If the direction is even in $\mathcal{C}$, then these are higher order fixed points with the same isotropy.

$\mathfrak{sl}_2$–triples

Next we can ask about elements $X \in \mathfrak{g}$ which extend $Z \in \mathfrak{p}_+$ to an $\mathfrak{sl}_2$–triple, i.e. such that $[[Z, X], Z] = 2Z$ and $[[Z, X], X] = -2X$. Such elements always exist by the Jacobson–Morozov theorem and it is easy to see that they never lie in $\mathfrak{p}$. As before, these elements determine a subset (not a subspace in general) $\mathcal{S} \subset T_{x_0}M$.
As a model we compute in $SL(2, \mathbb{R})$:

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} 1+st & t \\ s & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ \frac{s}{1+st} & 1 \end{pmatrix} \begin{pmatrix} 1+st & 0 \\ 0 & (1+st)^{-1} \end{pmatrix} \begin{pmatrix} 1 & \frac{t}{1+st} \\ 0 & 1 \end{pmatrix},$$

so $e^{tZ} e^{sX} = e^{\frac{s}{1+st}X} b_t(s)$, where $b_t(s) = e^{\log(1+st)[Z,X]} e^{\frac{t}{1+st}Z}$.

From the comparison theorem, we now conclude that the curve $s \mapsto p(\exp_{u_0}(sX))$ in M gets projectively reparametrized by the flow φ_t. By construction, one can find such a curve emanating from x_0 in each direction lying in $\mathcal{S} \subset T_{x_0}M$. Let us denote by $\mathcal{S}_0 \subset \mathcal{S}$ the set of those tangent vectors for which u_0 can be chosen in such a way that $X \in \mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$. Then the resulting curve is one of the distinguished curves of the geometry in question, so one obtains more detailed information.

To proceed further, one observes $(\varphi_t)^*\omega = \omega$ implies that $\varphi_t(\exp_u(Y)) = \exp_{\varphi_t(u)}(Y)$. Using this for u on the distinguished curve, one can analyze the behavior close to the initial distinguished curve in certain directions.

Moreover, $b_t(s) = e^{\log(1+st)[Z,X]}e^{\frac{t}{1+st}Z}$ determines the action of the flow φ_t on natural bundles. Since $\lim_{t\to\infty} p(\exp_{u_0}(\frac{s}{1+st}X)) = x_0$, one can deduce restrictions on the curvature, both in the fix point x_0 and along the curves in question.

Projective structures

Here the geometry is given by a projective equivalence class of torsion free linear connections on TM. Equivalently, it can be described by a family of unparametrized curves in M with one curve through each point in each direction, which are the geodesic paths of the connections in the class. The distinguished curves are these paths with a distinguished family of projective parametrizations.

Here $\mathfrak{g}/\mathfrak{p} = \mathbb{R}^n$ and $\mathfrak{g}_1 = \mathbb{R}^{n*}$ with the actions of P given by the standard representation of $GL(n, \mathbb{R})$ and its dual. Hence any two non–zero elements of $\mathfrak{g}/\mathfrak{p}$ respectively of $\mathfrak{g}_1 = \mathfrak{p}_+$ lie in the same P–orbit, so all tangent directions respectively cotangent directions are "equal".

Projective structures II

An infinitesimal automorphism $s \in \Gamma(\mathcal{A}M)$ has a higher order fix point in x_0 iff $\varphi_t(x_0) = x_0$ and $T_{x_0}\varphi_t = \mathrm{id}$ for all t. There is just one type of possible isotropy $\alpha \in T^*_{x_0}M$ for such automorphisms. A simple computation shows that

For $0 \neq \alpha \in T^*_{x_0}M$, we get $\mathcal{F} = \ker(\alpha) \subset T_{x_0}M$, $\mathcal{C} = \{0\}$ and $\mathcal{S} = \mathcal{S}_0 = \{\xi \in T_{x_0}M : \alpha(\xi) = 1\}$.

Our results show that the hyperplane $\ker(\alpha)$ gives rise to a totally geodesic hypersurface fixed by the flow, with x_0 being the only higher order fix point. in any direction transverse to $\ker(\alpha)$, we get a distinguished curve corresponding to an $\mathfrak{sl}_2$–triple, which gets contracted projectively into the fix point. This implies that the geometry is flat locally around the higher order fix point.

Conformal structures

We consider conformal equivalence classes of pseudo–Riemannian metrics of arbitrary signature (p, q) in dimensions $n \geq 3$. This can be described as a parabolic geometry with $\mathfrak{g}/\mathfrak{p} \cong \mathbb{R}^{p,q}$ and $\mathfrak{p}_+ \cong \mathbb{R}^{(p,q)*}$ with the P–actions coming from the natural representations of $CO(p, q)$.
The P–orbits in $\mathfrak{g}_{\pm 1}$ are determined by the sign of $\langle v, v \rangle$ for v in the orbit, but the main distinction is between non–null– and null–directions.
The distinguished curves are conformal circles. For non–null initial directions, they are determined by their two–jet in one point. In null–directions, they are null–geodesics which are conformally invariant up to parametrization.
Higher order fix point again means that φ_t equals the identity to first order, and we have to distinguish between non–null isotropy and null isotropy.

non–null isotropy

If $\alpha \in T^*_{x_0}M$ is non–isotropic, then the hyperplane $\ker(\alpha) \subset T_{x_0}M$ is non–degenerate, so $T_{x_0}M = \ker(\alpha) \oplus \ker(\alpha)^\perp$. A small bit of linear algebra shows that

For non–null α, $\mathcal{F}$ consists of all null–vectors in $\ker(\alpha)$, $\mathcal{C} = \{0\}$, and $\mathcal{S} = \{\xi_0\}$ where ξ_0 is the unique element in $\ker(\alpha)^\perp$ such that $\alpha(\xi_0) = 2$.

We conclude that null geodesics emanating in directions in $\ker(\alpha)$ are point–wise fixed, but contain no further higher order fix points. There is a distinguished conformal circle c with initial tangent vector ξ_0 which gets contracted projectively to the fixed point. This is actually true for an open neighborhood of c, and on this neighborhood the curvature vanishes.

null isotropy

Here $\ker(\alpha) \subset T_{x_0}M$ is degenerate and hence contains the line $\ker(\alpha)^\perp$ dual to α. Again, only linear algebra is needed to show that

$\mathcal{F}$ is consists of all null–vectors in $\ker(\alpha)$ and $\mathcal{C} = \ker(\alpha)^\perp \subset \mathcal{F}$. The subset $\mathcal{S} \subset T_{x_0}M$ consists of all null–vectors ξ such that $\alpha(\xi) = 1$.

So again null–geodesics emanating in directions in $\ker(\alpha)$ consist of fix points, but this time the one emanating in the direction dual to α consist of higher order fix points with null isotropy. Null geodesics emanating in directions transverse to $\ker(\alpha)$ are contracted projectively. This can be extended in some directions leading to an open set on which the curvature has to vanish.