Special infinitesimal automorphisms of parabolic geometries

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- This talk reports on joint work arXiv:1208.5510 and arXiv:1211.5477 with Karin Melnick (Univ. of Maryland).
- Apart from results stating that large groups of automorphisms of certain geometric structures are only possible for "simple" geometries, there are also cases in which existence of a *single* (infinitesimal) automorphism of special type implies restrictions on a geometry, c.f. results on essential conformal isometries.
- We study infinitesimal automorphisms whose flow has a higher order fixed point. The basic invariant of such a fixed point is its *isotropy*, which is a covector in the fixed point. To this, one can associate subsets in the tangent space and thus via normal coordinates in the manifold. On some of these subsets, one obtains an explicit description of the flow, on some one gets restrictions on harmonic curvature quantities from an analysis of the dynamics of the flow.

Contents

We consider geometric structures which can be equivalently described by Cartan geometries of some fixed type (G, P), where G is a semisimple Lie group and $P \subset G$ a parabolic subgroup. Such a structure on a manifold M gives rise to a principal P–bundle $p : \mathcal{G} \to M$ and a canonical Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G.

The equivalence to the Cartan geometry in particular implies that automorphisms φ of our geometric structure are in bijective correspondence with principal bundle automorphisms $\Phi : \mathcal{G} \to \mathcal{G}$ such that $\Phi^*\omega = \omega$ via $p \circ \Phi = \varphi \circ p$. General results then imply that the automorphisms Φ form a Lie group $\mathrm{Aut}(\mathcal{G}, \omega)$ whose dimension is $\leq \dim(G)$.

There is a similar equivalence for infinitesimal automorphisms. In the Cartan picture, such an automorphism is described as a P–invariant vector field $\Xi \in \mathfrak{X}(\mathcal{G})$ such that $\mathcal{L}_\Xi \omega = 0$. Via the Cartan connection ω, P–invariant vector fields on $\mathcal{G}$ are identified with sections of the *adjoint tractor bundle* $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$.

A P–invariant vector field Ξ on $\mathcal{G}$ is automatically projectable to some $\xi \in \mathfrak{X}(M)$. Correspondingly, there is a projection $\Pi : \mathcal{A}M \to TM$, so $\xi = \Pi(s)$, where $s \in \Gamma(\mathcal{A}M)$ corresponds to Ξ.

The kernel of Π is a natural subbundle $\mathcal{A}^0M \subset \mathcal{A}M$, and the flow of ξ has a fixed point in $x \in M$ if and only if $s(x) \in \mathcal{A}^0_xM \subset \mathcal{A}_xM$.

In the parabolic case, $\mathcal{A}^0M$ is part of a natural filtration

$$\mathcal{A}M = \mathcal{A}^{-k}M \supset \cdots \supset \mathcal{A}^0M \supset \cdots \supset \mathcal{A}^kM$$

by smooth subbundles, where $k \geq 1$ is determined by the type of geometry.

Definition

(1) An infinitesimal automorphism $s \in \Gamma(\mathcal{A}M)$ of a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ has a *higher order fixed point* in $x \in M$ iff $s(x) \in \mathcal{A}^1_x M \subset \mathcal{A}_x M$.
(2) An infinitesimal automorphism $s \in \Gamma(\mathcal{A}M)$ is called *special* if it has at least one higher order fixed point.

If $k = 1$ then this condition means that the flow equals the identity to first order, for $k > 1$, one has to use weighted order. Any special infinitesimal automorphism is *essential* (as introduced by J. Alt).

It is a general fact on parabolic geometries that $\mathcal{A}^1 M$ is naturally isomorphic to T^*M. Hence if $s \in \Gamma(\mathcal{A}M)$ has a higher order fixed point in x, then we can view $s(x)$ as an element of $T^*_x M$ called the *isotropy* of s at x.

The filtration of $\mathcal{A}M$ actually is induced by a filtration

$$\mathfrak{g} = \mathfrak{g}^{-k} \supset \cdots \supset \mathfrak{g}^0 \supset \cdots \supset \mathfrak{g}^k$$

making the Lie algebra $\mathfrak{g}$ of G into a filtered Lie algebra such that $\mathfrak{g}^0 = \mathfrak{p}$ and $\mathfrak{g}^1 = \mathfrak{p}_+$, the nilradical of $\mathfrak{p}$.

Choosing $u \in \mathcal{G}_x$, the isotropy $s(x) \in \mathcal{A}^1_x M$ corresponds to an element $Z \in \mathfrak{p}_+$. The P–orbit of Z in $\mathfrak{p}_+$ is independent of all choices and thus a fundamental invariant of a higher order fixed point. This is called the *geometric type* of the isotropy.

To deal with an example of a parabolic geometry, the possible geometric types have to be discussed separately. There are usually only very few geometric types. Note that if $k > 1$, then each $\mathfrak{g}^i$ for $i = 2, \dots, k$ is a P–invariant subspace of $\mathfrak{p}_+$. Hence a geometric type is either contained in $\mathfrak{g}^i$ or disjoint from it.

Given $Z \in \mathfrak{p}_+$ corresponding to the isotropy α, we define 3 subsets $C_\mathfrak{g}(Z) \subset F_\mathfrak{g}(Z)$ and $T_\mathfrak{g}(Z)$ of $\mathfrak{g}$ as follows:

$C_\mathfrak{g}(Z) := \{X \in \mathfrak{g} : [X, Z] = 0\}$
$F_\mathfrak{g}(Z) := \{X \in \mathfrak{g} : \mathrm{ad}(X)^\ell(Z) \in \mathfrak{p} \quad \forall \ell\}$
Finally, $T_\mathfrak{g}(Z)$ consists of those $X \in \mathfrak{g}$ which extend Z to an $\mathfrak{sl}_2$–triple, i.e. such that $A = [Z, X]$ satisfies $[A, Z] = 2Z$ and $[A, X] = -2X$.

Notice that $C_\mathfrak{g}(Z)$ is a linear subspace and $F_\mathfrak{g}(Z)$ is closed under multiplication by scalars. While $T_\mathfrak{g}(Z)$ is just some subset, it is always non–empty by the Jacobson–Morozov theorem.

It turns out that via $TM = \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$ these canonically determine subsets $C(\alpha) \subset F(\alpha)$ and $T(\alpha)$ of the tangent space T_xM.

The crucial tool to understand the flow is a comparison theorem by C. Frances and K. Melnick. Take an infinitesimal automorphism $s \in \Gamma(\mathcal{A}M)$ with a higher order fixed point in $x \in M$ and a point $u \in \mathcal{G}_x$ for which the isotropy α corresponds to $Z \in \mathfrak{p}_+$. This is compared to the infinitesimal automorphism of $G \to G/P$ determined by the right invariant vector field R_Z, which has a higher order fixed point at $o = eP \in G/P$ and whose flow at time t is given by left translation by e^{tZ} as follows:

Suppose that $X \in \mathfrak{g}$ is such that the curve $c(r) := e^{rX}P$ in G/P only gets reparametrized by the flow e^{tZ} as $c \circ \psi_t$. Then let $\tilde{c}(r)$ be the flow line in $\mathcal{G}$ of the vector field $\omega^{-1}(X)$ starting in u. Then locally around $r = 0$, the flow of $\xi = \Pi(s)$ also only reparametrizes $p \circ \tilde{c}$ as $p \circ \tilde{c} \circ \psi_t$.

The projections of the flow lines of ω–constant vector fields to M are called *exponential curves* of the geometry. If X lies in a certain subalgebra $\mathfrak{g}_-$ of $\mathfrak{g}$, these curves are even *distinguished curves*, e.g. conformal circles or chains. For X in $C_{\mathfrak{g}}(Z)$ or in $F_{\mathfrak{g}}(Z)$, application of the comparison theorem is very easy:

Proposition

Let $s \in \Gamma(\mathcal{A}M)$ be an infinitesimal isomorphism with a higher order fixed point in $x \in M$ with isotropy $\alpha \in T^*_x M$.
(1) For $\xi \in F(\alpha) \subset T_x M$, there is an exponential curve emanating in x in direction ξ which consists of fixed points of the flow of s.
(2) If $\xi \in C(\alpha) \subset F(\alpha)$ then the curve consists of higher order fixed points with isotropy of the same geometric type as α.

In case that X can even be chosen in $\mathfrak{g}_-$ (which holds in all examples we will discuss) on can replace "exponential curve" by "distinguished curve".

For $X \in T_{\mathfrak{g}}(Z)$, applying the comparison theorem is slightly more complicated. Since Z, $A = [Z, X]$, and X form an $\mathfrak{s}l_2$–triple, they determine a homomorphism $\mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}$ which locally integrates to $SL(2, \mathbb{R})$. Then an explicit computation in $SL(2, \mathbb{R})$ can be used to deduce:

Proposition

Let $s \in \Gamma(\mathcal{A}M)$ be an infinitesimal isomorphism with a higher order fixed point in $x \in M$ with isotropy $\alpha \in T^*_x M$.
For $\xi \in T(\alpha)$, there is an exponential curve c emanating in x in direction ξ such that the flow φ^t of s satisfies $\varphi^t(c(r)) = c(\frac{r}{1+rt})$ for $|r|$ sufficiently small and all $rt > 0$.

Again one obtains a distinguished curve if $X \in \mathfrak{g}_-$. One can also phrase the description of the flow in terms of exponential or even normal coordinates, showing that one gets a full description the intersection of an open neighborhood of 0 with $F(\alpha) \cup (\mathbb{R} \cdot T(\alpha))$.

For the next step, consider $Z \in \mathfrak{p}_+$ and assume that $X \in T_{\mathfrak{g}}(Z)$ is such that $A = [Z, X] \in \mathfrak{g}_0$. Suppose further that $\mathbb{W}$ is a representation of $\mathfrak{g}_0$ on which A acts diagonalizably. Then let $\mathbb{W}_{ss}(A) \subset \mathbb{W}_{st}(A) \subset \mathbb{W}$ be the sum of all eigenspaces with negative respectively non–positive eigenvalues.

If $\mathbb{W}$ actually is a completely reducible representation of P, then it gives rise to a natural bundle $\mathcal{G} \times_P \mathbb{W}$ and sections of such a bundle correspond to P–equivariant functions $\mathcal{G} \to \mathbb{W}$. Via the comparison theorem, one can compute the action of the flow φ^t of s on such a section along the distinguished exponential curve from above. If this section describes a harmonic curvature quantity, then it must be φ^t–invariant, which implies strong restrictions on the values:

Theorem

Let $s \in \Gamma(\mathcal{A}M)$ be an infinitesimal automorphism with a higher order fixed point in $x \in M$ with isotropy $\alpha \in T_x^*M$. Suppose that $u \in \mathcal{G}_x$ is such that the element $Z \in \mathfrak{p}_+$ corresponding to α and $X \in T_{\mathfrak{g}}(Z)$ satisfy the above assumptions. Let $c(r)$ be the flow line of $\omega^{-1}(X)$ starting in u and let $f : \mathcal{G} \to \mathbb{W}$ be the equivariant function corresponding to a harmonic curvature component.

(1) $f(c(r)) \in \mathbb{W}_{st}(A)$ for sufficiently small r.
(2) If $f(u) = 0$, $f(c(r)) \in \mathbb{W}_{ss}(A)$ for sufficiently small r.
(3) If A acts diagonalizablz on $\mathfrak{g}_-$ with non–positive eigenvalues and zero–eigenspace $C_{\mathfrak{g}}(Z) \cap \mathfrak{g}_-$, $f(u) = 0$, and $W_{ss}(A) = 0$, then f vanishes on an open neighborhood of $c((0, \epsilon))$ for some $\epsilon > 0$.

To analyze specific cases, one now only has to do computations in the Lie algebra $\mathfrak{g}$, depending on a fixed element $Z \in \mathfrak{p}_+$ (or rather its geometric type). The simplest possible case is:

Suppose that $Z \in \mathfrak{p}_+$ is such that for each $X \in T_{\mathfrak{g}}(Z)$ we have $W_{st}([Z, X]) = 0$ and that $\mathbb{R} \cdot T(\alpha)$ is dense in some open neighborhood of zero in $T_x M$. Then the geometry has to be locally flat on an open neighborhood of x. Hence the local behavior special infinitesimal automorphisms with isotropy of this geometric type can be fully understood via the homogeneous model.

This happens for projective structures (only one geometric type of isotropy) and for arbitrary parabolic contact structures provided that the isotropy α vanishes on the contact subbundle (i.e. $Z \in \mathfrak{g}_2$). In both cases $T(\alpha)$ is the affine hyperplane $\{\xi : \alpha(\xi) = 1\}$ and $[Z, X]$ is a positive multiple of the grading element.

The next weaker result is local flatness on an open subset which has the higher order fixed point in its closure. (Hence there is no description of the local behavior around the higher order fixed point via the homogeneous model.) Again, there is a rather simple case, but this contains very important examples:

Suppose that $Z \in \mathfrak{p}_+$ is such that for one $X \in T_{\mathfrak{g}}(Z) \cap \mathfrak{g}_-$ and $A = [Z, X]$ we have $\mathbb{W}_{st}(A) = 0$ and the conditions on the action of A on $\mathfrak{g}_-$ are satisfied. Then for the distinguished exponential curve c determined by the element $\xi \in T(\alpha)$ corresponding to X, the geometry is locally flat on an open neighborhood of $c((0, \epsilon))$.

This holds for non–null isotropy on conformal structures and on partially integrable almost CR structures. In both cases, $T(\alpha)$ consists of a single element and $[Z, X]$ is a positive multiple of the grading element.

In case that $\mathbb{W}_{st}([Z,X]) \neq 0$ there still is a way to prove local flatness results. Given $Z \in \mathfrak{p}_+$ we have to require that $\cap_{X \in T_{\mathfrak{g}}(Z) \cap \mathfrak{g}_-} \mathbb{W}_{st}([Z,X]) = 0$ to deduce vanishing of the harmonic curvature component in the fixed point. If we then find at least one such X such that $A = [Z,X]$ satisfies $\mathbb{W}_{ss}(A) = 0$ and the condition on the action on $\mathfrak{g}_-$, then one gets local flatness on an open subset with the higher order fixed point in its closure as before. This applies in the following cases:

- null isotropy on conformal structures
- arbitrary isotropy on almost quaternionic structures
- rank two isotropy on almost Grassmannian structures
- rank one isotropy on (torsion free) Grassmannian structures

In the case of rank one isotropy on almost Grassmannian structures, we cannot deduce vanishing of the torsion on an open subset with the higher order fixed point in its closure, since for the corresponding representation $\mathbb{W}$ one always has $\mathbb{W}_{ss}([Z, X]) \neq 0$. In this case and for null isotropy on almost CR structures we only get vanishing on the set of higher order fixed points (which has positive dimension in these two cases).