A remarkable class of locally conformally symplectic geometries

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- This talk reports on joint work with Tomás Salač (Prague).
- The gradings of simple Lie algebras which give rise to parabolic contact structutres also define subgroups in certain groups of conformal symplectic linear automorphisms. In almost all cases, these are a maximal Lie subalgebras.
- We start by showing that a first order G–sturcuture corresponding to a conformally symplectic group is a locally conformally symplectic structure if and only if it has vanishing intrinsic torsion.
- The main topic of the talk are the geometric structures obtained by adding a reduction of structure group to one of the groups from above to a lcs–structure. Using Kostant's theorem, we prove that all these structures admit canonical compatible linear connections whose torsion satisfies a normalization condition that we describe explicitly.
- This also relates nicely to the theory of special symplectic connections and thus to exotic affine holonomies.

Contents

Let V be a vector space of even dimension $2n$ and let ω be a fixed non–degenerate skew symmetric bilinear form on V. Then we define

$$Sp(V) := \{A \in GL(V) : \omega(Av, Aw) = \omega(v, w)\}$$
$$CSp(V) := \{A \in GL(V) : \omega(Av, Aw) = \lambda\omega(v, w) \text{ for some } \lambda \in \mathbb{R}\}.$$

These are Lie subgroups of $GL(V)$ of dimension $n(2n+1)$ and $n(2n+1)+1$, respectively. The corresponding Lie algebras will be denoted by $\mathfrak{sp}(V)$ and $\mathfrak{csp}(V)$.
For later use, it will be more useful to view $CSp(V)$ as the set of those linear automorphisms of V for which the induced automorphism of $\Lambda^2 V^*$ preserves the line spanned by ω.
It is well known, that $\mathfrak{sp}(V)$ is a simple Lie algebra and that $\mathfrak{sp}(V) \cong S^2 V$ as a representation of $Sp(V)$. Correspondingly, $\mathfrak{csp}(V) = \mathbb{R} \oplus \mathfrak{sp}(V)$ is reductive with 1–dimensional center.

The standard way to study first order structures corresponding to a subgroup $G \subset GL(V)$ with Lie algebra $\mathfrak{g} \subset L(V, V)$ is to consider the linear map $\partial : L(V, \mathfrak{g}) \to L(\Lambda^2 V, V)$ defined by $\partial\Phi(X, Y) = \Phi(X)(Y) - \Phi(Y)(X)$.

Given a reduction of structure group to $CSp(V) \subset GL(V)$ on a manifold of dimension $\dim(V)$ and a compatible connection, this map computes the change of torsion caused by a change of connection.

From this description it is clear that the quotient space $L(\Lambda^2 V, V)/\operatorname{im}(\partial)$ is the target space of a first basic invariant, of such a G–structure, the so–called *intrinsic torsion*.
Structures with vanishing intrinsic torsion are sometimes called *integrable*. By definition, they admit a compatible torsion–free connection.

To deal with the general case, one has to choose a G–invariant complement $\mathcal{N}$ to $\mathrm{im}(\partial)$ in $L(\Lambda^2 V, V)$. This is usually referred to as the choice of a *normalisation condition* on the torsion. Via the G–structure, the subspace $\mathcal{N}$ corresponds to a subbundle in $L(\Lambda^2 TM, TM)$, and one always finds a connection whose torsion has values in that subbundle. Again by construction the value of this torsion is an invariant which equivalently encodes the intrinsic torsion.

The question of uniqueness of connections having this special torsion is related to the *first prolongation* $\mathfrak{g}^{(1)} := \ker(\partial)$ of $\mathfrak{g}$. We will mainly need the fact that if $\mathfrak{g}^{(1)} = \{0\}$, then compatible connections are uniquely determined by their torsion, so having chosen a normalisation condition, one obtains a connection canonically associated to each G–structure of the type in question.

Making this explicit for $\mathfrak{csp}(V)$ is mainly an exercise in representation theory. As representations of $Sp(V)$, we have

$$L(V, \mathfrak{csp}(V)) \cong V \otimes (\mathbb{R} \oplus S^2 V) \cong V \oplus V \oplus S^3 V \oplus W$$
$$L(\Lambda^2 V, V) \cong \Lambda^2 V \otimes V \cong V \oplus V \oplus \Lambda^3_0 V \oplus W.$$

Here W is an irreducible representation which in the first line is realized as the trace free part of the kernel of the symmetrization and in the second line as the tracefree part in the kernel of the alternation, and the subscript 0 denotes the tracfree part.

It is then easy to check directly that ∂ is nonzero on W and injective on the two copies of V, so it is an isomorphism between these components. Thus the first prolongation is isomorphic to $S^3 V$ while the intrinsic torsion has values in $\Lambda^3_0 V$.

To convert this to geometry, we first observe that a first order structure with group $CSp(V)$ on a smooth manifold M of dimension $\dim(V)$ is clearly equivalent to the choice of a line sub–bundle $\ell \subset \Lambda^2 T^*M$ such that each non–zero element in ℓ is non–degenerate.

A linear connection ∇ on TM is compatible with such a reduction if $\nabla\omega \in \Gamma(T^*M \otimes \ell)$ for any $\omega \in \Gamma(\ell) \subset \Omega^2 M$.

To compute the intrinsic torsion of ∇, we have to take its torsion T^a_{bc}, and then form the tracefree part of $\omega_{i[a}T^i_{bc]}$ for a locally non–vanishing $\omega = \omega_{ab} \in \Gamma(\ell)$. But $\omega_{i[a}T^i_{bc]}$ can be computed as $(d\omega)_{abc} - \nabla_{[a}\omega_{bc]}$. Since $\nabla_{[a}\omega_{bc]} \in \Gamma(T^*M \wedge \ell)$ by construction,vanishing of the intrinsic torsion is equivalent to $d\omega \in \Gamma(T^*M \wedge \ell)$ for any section $\omega \in \Gamma(\ell)$. Equivalently, ℓ must locally admit closed sections, i.e. define a *locally conformally symplectic structure*.

Following Cahen and Schwachhöfer, simple Lie algebras lead to a class of special subalgebras of conformally symplectic Lie algebras.

A *contact grading* of a simple Lie algebra $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ (with $\mathfrak{g}_k = 0$ for $|k| > 2$), $\dim(\mathfrak{g}_{-2}) = 1$ and the Lie bracket $[\ ,\ ] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is non–degenerate.

Up to isomorphism, any complex simple Lie algebra admits a unique grading of this type, and this induces a contact grading on most non–compact real forms.

By construction, the bracket defines a non–degenerate line in $\Lambda^2\mathfrak{g}^*_{-1}$, and it is easy to see that the action of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ gives rise to an inclusion $\mathfrak{g}_0 \hookrightarrow \mathfrak{csp}(\mathfrak{g}_{-1})$.

Classification

- Type C_n is exceptional since $\mathfrak{g}_0 = \mathfrak{csp}(\mathfrak{g}_{-1})$ and will not be considered in what follows.
- Type A_n: $\mathfrak{g}_{-1} = \mathbb{R}^n \oplus \mathbb{R}^{n*}$, $\mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{R}) \oplus \mathbb{R}$ ("lc–bi–Lagrangean").
- Type A_n: $\mathfrak{g}_{-1} = \mathbb{C}^n$, $\mathfrak{g}_0 = \mathfrak{cu}(n)$ ("lc–almost–Kähler").
- Type B_n, D_n: $\mathfrak{g}_{-1} = \mathbb{R}^2 \boxtimes \mathbb{R}^{p,q}$, $\mathfrak{g}_0 = \mathfrak{gl}(2, \mathbb{R}) \oplus \mathfrak{o}(p, q)$ ("split–quaternionic").
- Type D_n: $\mathfrak{g}_{-1} = \mathbb{H}^n$, $\mathfrak{g}_0 = \mathfrak{sp}(1) \oplus \mathfrak{cso}^*(2n)$ ("quaternionic").
- Type G_2: $\mathfrak{g}_{-1} = S^3\mathbb{R}^2$, $\mathfrak{g}_0 = \mathfrak{gl}(2, \mathbb{R})$, dim=4.
- Type F_4: $\mathfrak{g}_{-2} = \Lambda^3_0\mathbb{R}^6$, $\mathfrak{g}_0 = \mathfrak{csp}(\mathbb{R}^4)$, dim=14.
- Type E_6: $\mathfrak{g}_{-1} = \Lambda^3\mathbb{R}^6$, $\mathfrak{g}_0 = \mathfrak{gl}(6, \mathbb{R})$ plus two more real forms, dim=20.
- Type E_7: $\mathfrak{g}_{-1} = \mathbb{R}^{32}$, $\mathfrak{g}_0 = \mathfrak{cspin}(12)$, plus two more real forms, dim=32.
- Type E_8: $\mathfrak{g}_{-1} = \mathbb{R}^{56}$, $\mathfrak{g}_0 = \mathfrak{ce}_7$, two real forms, dim=56.

Our main technical tool will be Kostant's theorem, which allows us to explicitly and algorithmically compute the cohomology groups of the nilpotent Lie algebra $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ with coefficients in the representation $\mathfrak{g}$.

The standard complex for computing this cohomology has the form

$$\dots \xrightarrow{\partial_K} L(\Lambda^i \mathfrak{g}_-, \mathfrak{g}) \xrightarrow{\partial_K} L(\Lambda^{i+1} \mathfrak{g}_-, \mathfrak{g}) \xrightarrow{\partial_K} \dots$$

The multilinear maps showing up here can be decomposed according to homogeneity and the the differentials ∂_K preserve homomgeneities, so each $H^i(\mathfrak{g}_- \mathfrak{g})$ splits accordingly.

Kostant defined an algebraic Lapalcian $\square$ on each of the spaces $L(\Lambda^i \mathfrak{g}_-, \mathfrak{g})$ such that $\ker(\square) \cong H^i(\mathfrak{g}_-, \mathfrak{g})$ as a module of $\mathfrak{g}_0$. Elements in this kernel will be called *harmonic*.

Maximality

Theorem

If $\mathfrak{g}$ is not of type A_n, then $\mathfrak{g}_0$ is a maximal subalgebra of $\mathfrak{csp}(\mathfrak{g}_{-1})$.

Proof: It is well known that in these cases, $\mathfrak{g}_{\geq 0}$ is a maximal parabolic subalgebra of $\mathfrak{g}$ ("only one crossed root"). Kostant's theorem then immediately implies that $H^1(\mathfrak{g}_-, \mathfrak{g})$ is an irreducible representation of $\mathfrak{g}_0$.

From the general theory of parabolic geometries, it further follows that for $\mathfrak{g}$ not of type C_n, the cohomology has to sit in homogenity 0 and then $H^1(\mathfrak{g}_-, \mathfrak{g}) = \mathfrak{csp}(\mathfrak{g}_{-1})/\mathfrak{g}_0$ follows from the definition.

Thus any $\mathfrak{g}_0$–invariant subset of $\mathfrak{csp}(\mathfrak{g}_{-1})$ which properly contains $\mathfrak{g}_0$ must be all of $\mathfrak{csp}(\mathfrak{g}_{-1})$. $\square$

In the A_n–case, there are two maximal subalgebras sitting between $\mathfrak{g}_0$ and $\mathfrak{csp}(\mathfrak{g}_{-1})$, namely those preserving just one of the two Lagrangean subspaces.

Each of the inclusions $\mathfrak{g}_0 \hookrightarrow \mathfrak{csp}(\mathfrak{g}_{-1})$ gives rise to a geometric structure on smooth manifolds of dimension $\dim(\mathfrak{g}_{-1})$, which has an underlying non–degenerate line subbundle $\ell \subset \Lambda^2 T^*M$. We will mainly be interested in the case that this line subbundle actually is a lcs–structure. (It turns out that in each case, there is a unique $\mathfrak{g}_0$–invariant line in $\Lambda^2\mathfrak{g}^*_{-1}$.)

Making this explicit is rather easy in each case: In the bi–Lagrangean case, one has a decomposition $TM \cong E \oplus F$ with both E and F isotropic with respect to every element of ℓ. For the other A_n–geometry one needs an almost complex structure on M for which each element of ℓ is Hermitian.

The geometry related to $\mathfrak{so}^*(2n)$ only exists in dimensions divisible by four. Here one has to add an *almost quaternionic structure* on M such that each element of ℓ is Hermitean in the quaternionic sense.

In the split quaternionic cases, it is easiest to describe an *almost split quaternionic structure* as an isomorphism $TM \cong E \otimes F$, where E and F are auxialliary bundles of ranks 2 and n, respectively. The compatibility condition is that ℓ sits inside of $\Lambda^2 E^* \otimes S^2 F^* \subset \Lambda^2(E \otimes F)^*$, so it is equivalent to a symmetric bilinear form on F determined up to scale, and there is a notion of signature in this case.

The structures corresponding to the exceptional Lie algebras can be described analogously in terms of auxiliary bundles. For example, the geometry corresponding to E_6 exists in dimension 20 and is given by an identification $TM \cong \Lambda^3 E$, where E is an auxilliary vector bundle of rank 6. The wedge product determines a skew symmetric bilinear form on $\Lambda^3 E$ with values in the line bundle $\Lambda^6 E$ and thus a line $\ell \subset \Lambda^2(\Lambda^3 E)^*$.

To formulate our main result, observe that given a reduction of structure group to $G_0 \subset GL(\mathfrak{g}_{-1})$, any representation of G_0 gives rise to a natural bundle and any G_0–equivariant map between representations induces a natural bundle map.
In particular, $\ker(\square) \subset \Lambda^2\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$ gives rise to a smooth subbundle of $\Lambda^2 T^*M \otimes TM$, whose elements are called *algebraically harmonic*.
Using this, we can formulate:

Theorem

1. The first prolongation of $\mathfrak{g}_0 \subset \mathfrak{gl}(\mathfrak{g}_{-1})$ vanishes.
2. If M carries a parabolic lcs–structure, then there is a unique linear connection on TM compatible with this structure which has algebraically harmonic torsion.

Proof

Denoting by ∂_S the Spencer differential and by i the inclusion of the trace part, the beginning of the homogeneity 1 bit of the standard complex computing $H^*(\mathfrak{g}_-, \mathfrak{g})$ has the form

$$\begin{tikzcd}
\mathfrak{g}_1 \arrow[r] \arrow[dr, "\cong"'] & \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0 \arrow[r, "\partial_S"] \arrow[dr] & \Lambda^2 \mathfrak{g}^*_{-1} \arrow[r] & \Lambda^3 \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-2} \\
& \mathfrak{g}^*_{-2} \otimes \mathfrak{g}_{-1} \arrow[r, "\cong"] \arrow[ur] & \mathfrak{g}^*_{-1} \wedge \mathfrak{g}^*_{-2} \otimes \mathfrak{g}_{-2} \arrow[ur, "i"'] &
\end{tikzcd}$$

If $\varphi \in \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0$ satisfies $\partial_S \varphi = 0$, then $\partial_K \varphi \in \mathfrak{g}^*_{-2} \otimes \mathfrak{g}_{-1}$. But then $0 = \partial_K \partial_K \varphi$ implies $\partial_K \varphi = 0$.
By Kostant's theorem, there is no first cohomology in homogeneity one, so $\varphi = \partial_K(Z)$ for some $Z \in \mathfrak{g}_1$. But then $\partial_K Z|_{\mathfrak{g}_{-2}} = 0$ implies $Z = 0$ and thus $\varphi = 0$, and the first part follows.

Proof (continued)

$$\begin{tikzcd}
\mathfrak{g}_1 \arrow[rr] \arrow[dr, "\cong"'] & & \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0 \arrow[rr, "\partial_S"] \arrow[dr] & & \Lambda^2 \mathfrak{g}^*_{-1} \arrow[r] & \Lambda^3 \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-2} \\
& \mathfrak{g}^*_{-2} \otimes \mathfrak{g}_{-1} \arrow[rr, "\cong"] \arrow[urrr] & & \mathfrak{g}^*_{-1} \wedge \mathfrak{g}^*_{-2} \otimes \mathfrak{g}_{-2} \arrow[urr, "i"'] & &
\end{tikzcd}$$

For the second part, assume that $\psi \in \Lambda^2\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1}$ represents the torsion of a compatible connection. Then our earlier interpretation of the lcs–condition is equivalent to $\partial_K\psi$ lying in the trace part, and hence to ψ being the first component of an element in $\ker(\partial_K)$. Together with the information on $H^2(\mathfrak{g}_-, \mathfrak{g})$ provided by Kostant, a diagram chase shows that there are elements $\hat\psi \in \ker(\Box) \subset \Lambda^2\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1}$ and $\varphi \in \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0$ such that $\psi = \hat\psi + \partial_S\varphi$. By the first part, φ is uniquely determined which implies the result.

It is easy to make the algebraic harmonicity condition explicit and to see that the torsions of all compatible connections have the same harmonic part.

In the almost bi–Lagrangean case ($TM = E \oplus F$), the only non–vanishing components of an algebraically harmonic torsion are the components $\Lambda^2 E \to F$ and $\Lambda^2 F \to E$, and these are the obstructions to integrability of E and F. For the other geometries, one similarly gets obstructions to integrability of the underlying complex, quaternionic or split–quaternionic structure. (In the last two cases,the results are more surprising since there is no Riemannian metric involved.)

In the case of torsion, one automatically gets a connection with holnomoy being contained in G_0, in the exceptional cases, this implies exotic holnomy.

Thank you for your attention!