Parabolic geometries and geometric compactifications lecture 1

Andreas Čap

University of Vienna
Faculty of Mathematics

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- This first lecture starts with an introduction to the general concept of a Cartan geometry associated to a homogeneous space.
- In particular, I will outline how Riemannian geometry can be encoded in that way.
- The example of conformal structures shows how Cartan geometries can be used to encode "higher order information" leading to unusual geometric objects.
- The homogeneous model for conformal structures is of rather special type (a generalized flag manifold) and taking more general homogeneous spaces of this type leads to parabolic geometries.

Contents

- In the spirit of F. Klein's Erlangen program, a classical geometry is specified by a homogeneous space G/P.
- If G is a Lie group, there is a definition of an associated geometric structure due to E. Cartan based on the following.

- $p : G \to G/P$ is an P-principal bundle that carries the left Maurer-Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$ with $\mathfrak{g} = \mathrm{Lie}(G)$.
- The left actions of elements of G are exactly the diffeomorphisms of G/P that admit a P-equivariant lift $\Phi : G \to G$ such that $\Phi^*\omega = \omega$.
- Observe that $d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)] = 0$ by the Maurer-Cartan equation.

The definition of a Cartan geometry is obtained by replacing G/P by a manifold M of the same dimension and requiring exactly those properties of the Maurer Cartan form that make sense in the general setting.

Definition

(1) A *Cartan geometry* of type (G, P) on a smooth manifold M is given by a principal P-bundle $p : \mathcal{G} \to M$ and a *Cartan connection* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, i.e.

- each $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism
- $(r^g)^*\omega = \mathrm{Ad}(g)^{-1} \circ \omega$ for all $g \in P$ (equivariancy)
- $\omega(\zeta_X) = X$ for all $X \in \mathfrak{p} \subset \mathfrak{g}$ (fundamental fields)

(2) The *curvature* $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of the geometry $(\mathcal{G}, \omega)$ is defined by $K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$.

- Such geometries exist only for $\dim(M) = \dim(G/P)$.
- There is an obvious notion of morphisms, and morphisms induce local diffeomorphisms between the base spaces.
- The curvature of a Cartan geometry vanishes identically if and only if it is locally isomorphic to its *homogeneous model* G/P.

Example

The nature of the concept of Cartan geometries is illustrated nicely by the example related to Euclidean geometry. Put $G = \mathrm{Euc}(n)$ and $P = O(n)$, so G/P is Euclidean space $\mathbb{E}^n$. Consider an n-manifold M and a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type G/P.

- $\mathfrak{g} = \mathfrak{o}(n) \oplus \mathbb{R}^n$ (semi-direct sum) and splitting $\omega = \gamma \oplus \theta$ accordingly, both components are $O(n)$-equivariant
- θ is equivalent to making $\mathcal{G}$ the orthonormal frame bundle of a Riemannian metric g on M
- γ defines a metric linear connection ∇ on TM
- The curvature K encodes curvature and torsion of ∇.

Existence and uniqueness of the Levi-Civita connection $\iff$ n-dimensional Riemannian manifolds are (categorically) equivalent to Cartan geometries of type (G, P) for which K has values in $\mathfrak{p} \subset \mathfrak{g}$. This similarly works for $G = O(n+1)$ and $G = O(n, 1)$.

General features of Cartan geometries

- K defines a fundamental and complete invariant
- representations of P induce natural vector bundles
- For the representation on $\mathfrak{g}/\mathfrak{p}$ induced by Ad, one obtains $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p}) \cong TM$, so all tensor bundles are associated.
- Starting from distinguished curves in G/P, one obtains general notions of distinguished curves in Cartan geometries.
- Natural notion of infinitesimal automorphisms of a Cartan geometry in $\mathfrak{X}(\mathcal{G})$. Automorphisms of $(\mathcal{G}, \omega)$ form a Lie group of dimension $\leq \dim(G)$ with Lie algebra formed by complete infinitesimal automorphisms.
- Several constructions relating geometries of different type (Correspondence spaces, Fefferman constructions, extension functors).

The conformal sphere

Put $G := SO_0(n+1,1)$ for a Lorentzian inner product on $\mathbb{R}^{n+2}$. Then G acts transitively on S^n, viewed as a space of isotropic rays. Hence $S^n = G/P$, where $P \subset G$ is the stabilizer of one such ray. Elementary arguments show that the action ℓ_g of $g \in G$ on S^n sends the round metric of S^n to a conformally related metric.

- Denoting by $o \in S^n$ the point fixed by P, the map $g \mapsto T_o\ell_g$ defines a surjective homomorphism $P \to G_0 := CO(n)$.
- The kernel of this homomorphism is normal subgroup $P_+ \subset P$ isomorphic to $\mathbb{R}^{n*}$ and $P = G_0 \ltimes P_+$.
- For any $g \in P_+$, ℓ_g coincides with id_{S^n} to first order in o, but for $g \neq e$, they are different on any open neighborhood of o. In particular, there is no G-invariant linear connection on TS^n.

This "higher order issue" will be crucial in what follows.

Let $(p : \mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P). Factoring by the action of $P_+ \subset P$, we obtain $\mathcal{G}_0 := \mathcal{G}/P_+$ and $p_0 : \mathcal{G}_0 \to M$ is a principal bundle with structure group $P/P_+ \cong CO(n)$. Projecting the values of ω to $\mathfrak{g}/\mathfrak{p} \cong \mathbb{R}^n$, the result descends to a strictly horizontal form $\theta \in \Omega^1(\mathcal{G}_0, \mathbb{R}^n)^{G_0}$. Hence we obtain an underlying conformal structure on M (i.e. an inner product up to scale on each tangent space).

Theorem (E. Cartan)

Any conformal structure arises in this way. Imposing a normalization condition on the curvature K makes the inducing Cartan geometry unique up to isomorphism and one obtains an equivalence of categories.

There are two approaches to proving this, which are very different in spirit. Since each of them has interesting advantages, we'll sketch both of them, starting with the classical approach.

Sketch of classical proof

- Starting from a conformal structure $(\mathcal{G}_0, \theta)$, one first observes that there are torsion-free principal connections γ on $\mathcal{G}_0$.
- For each $u_0 \in \mathcal{G}_0$, the values $\gamma(u_0)$ form an n-dimensional affine space. Attaching this to u_0 one constructs a bundle $\mathcal{G} \to M$ and extending the action of G_0 on $\mathcal{G}_0$ defines a principal right action of P on $\mathcal{G}$.
- Using the connection forms of the γ, one defines a natural form $\omega_0 \in \Omega^1(\mathcal{G}, \mathfrak{g}_0)$. For each $u \in \mathcal{G}$ over u_0, $\theta(u_0) \oplus \omega_0(u)$ defines a linear isomorphism $T_{u_0}\mathcal{G}_0 \to \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$.
- The possible lifts to a linear isomorphism $T_u\mathcal{G} \to \mathfrak{g}$ that is compatible with fundamental fields form an affine space and the corresponding curvature K always has values in $\mathfrak{g}_0 \oplus \mathfrak{g}_1$.
- One then shows that there is a unique such lift for which the $\mathfrak{g}_0$-component of K has vanishing Ricci-type contraction.

Sketch of "abstract" proof

- Starting from $(\mathcal{G}_0, \theta)$, define $\mathcal{G} := \mathcal{G}_0 \times_{G_0} P$, so $\mathcal{G}/P_+ \cong \mathcal{G}_0$.
- Choose a principal principal connection on $\mathcal{G}$ and use it and θ to define a Cartan connection $\hat{\omega}$ on $\mathcal{G}$. Then $(\mathcal{G}, \hat{\omega})$ has underlying structure $(\mathcal{G}_0, \theta)$.
- Cartan connections on $\mathcal{G}$ inducing θ form an affine space and there is a concept of homogeneity, which also applies to curvature. The change of curvature in lowest homogeneity is tensorial and induced by a Lie algebra cohomology differential.
- Finding a normalization condition becomes a purely algebraic problem. Having done this, one can normalize $\hat{\omega}$ homogeneity by homogeneity to obtain a normal Cartan connection ω on $\mathcal{G}$.
- Using information on $H^1(\mathbb{R}^n, \mathfrak{g})$ one shows that two normal Cartan connections on $\mathcal{G}$ that induce θ are related by an automorphism covering the identity on $(\mathcal{G}_0, \theta)$.

tractor bundles

We know that for the representation of P on $\mathfrak{g}/\mathfrak{p}$ induced by Ad, we get $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p}) \cong TM$. This representation factors through $P \to P/P_+ \cong G_0$, so to recover higher order information, other constructions are needed:

Via equivariant extension, the Cartan connection ω induces a principal connection $\tilde{\omega}$ on $\tilde{\mathcal{G}} := \mathcal{G} \times_P G$. Taking a representation $\mathbb{V}$ of G and restricting to P, we obtain $\mathcal{V}M := \mathcal{G} \times_P \mathbb{V} = \tilde{\mathcal{G}} \times_G \mathbb{V}$, so this inherits a canonical linear connection. ("tractor bundles and tractor connections")

- Choosing g in the conformal class, its Levi-Civita connection ∇ defines a section $\mathcal{G}_0 \to \mathcal{G}$.
- Using this, one identifies $\mathcal{V}M$ with a bundle associated to $\mathcal{G}_0$ and describes the canonical connection in terms of ∇.
- It can be made explicit how all this changes when rescaling g.

The abstract proof is robust and in particular applies to all pairs (G, P) where G is semisimple and $P \subset G$ is a parabolic subgroup. Here the relevant information on Lie algebra cohomology is provided by Kostant's theorem. Interpretations in the spirit of the classical proof can then be recovered via so-called *Weyl structures*.

Parabolic subgroups are characterized by the fact that there is a Lie algebra grading $\mathfrak{g} = \oplus_{i=-k}^{k} \mathfrak{g}_i$ such that $\mathfrak{p} = \oplus_{i \geq 0} \mathfrak{g}_i$. Putting $\mathfrak{g}^i := \oplus_{j \geq i} \mathfrak{g}_j$ makes $\mathfrak{g}$ into a filtered Lie algebra. Since $\mathfrak{p} = \mathfrak{g}^0$, the filtration is P-invariant and there are natural subgroups $G_0, P_+ \subset P$ corresponding to $\mathfrak{g}_0$ and $\mathfrak{p}_+ := \mathfrak{g}^1$.

Filtrations and associated graded objects are crucial for the theory. Recall that for a filtration by smooth subbundles $TM = T^{-k}M \supset T^{-k+1}M \supset \cdots \supset T^{-1}M$ such that $[\Gamma(T^iM), \Gamma(T^jM)] \subset T^{i+j}M$ the Lie bracket induces a tensorial bracket on $\mathrm{gr}(T_xM) = \oplus_i (T_x^iM / T_x^{i+1}M)$ ("symbol algebra at x").

The underlying structure for parabolic geometries

Given a type (G, P) corresponding to $\mathfrak{g} = \oplus_i \mathfrak{g}_i$, the underlying structure consists of

1. A filtration $TM = T^{-k}M \supset T^{-k+1}M \supset \cdots \supset T^{-1}M$ such that $\mathrm{gr}(TM)$ becomes a locally trivial bundle of Lie algebras modeled on $\mathfrak{g}_- = \oplus_{i<0}\mathfrak{g}_i$.
2. This then has a natural frame bundle with structure group $\mathrm{Aut}_{gr}(\mathfrak{g}_-)$ that contains G_0 as a subgroup and the second ingredient is a reduction to that structure group.

A standard example arises from $G = SU(n+1, 1)$ with P the stabilizer of an isotropic complex line. Here $\mathfrak{g}_-$ is a Heisenberg algebra, so 1 is a contact structure $H \subset TM$. G_0 consists of those automorphisms that are complex linear on $\mathfrak{g}_{-1} \cong \mathbb{C}^n$, so 2 is an almost complex structure on H.

Conformal structures are among the examples in which 1 is vacuous, and one obtains just a G_0-structure ("AHS structures"). There are examples for which 2 is vacuous since $G_0 = \mathrm{Aut}_{gr}(\mathfrak{g}_-)$, e.g. various generic distributions.

Projective structures are one of two examples in which the Cartan geometry is not determined by the underlying structure. Here $G = SL(n+1,\mathbb{R})$ and P is the stabilizer of a ray in $\mathbb{R}^{n+1}$, so $G_0 = GL^+(n,\mathbb{R})$. Then $\mathcal{G}_0 \to M$ is the full oriented frame bundle of M. Any G_0-equivariant section $\mathcal{G}_0 \to \mathcal{G}$ pulls back the $\mathfrak{g}_0$-component of ω to a principal connection on $\mathcal{G}_0$.

Hence there is a class of distinguished connections on TM. It turns out that they all are torsion-free and have the same geodesics up to parametrization. This leads to a "projective equivalence class" of torsion-free connections, which is equivalently encoded by the Cartan geometry.