Parabolic geometries and geometric compactifications lecture 2

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Srni, January 2020

Recap / program

- In the first lecture, we have discussed the description of conformal structures as Cartan geometries and the generalization to parabolic geometries.
- Today's lecture will start with a fundamental example of geometric compactifications. Starting from the example of hyperbolic space, I will introduce the concept of conformally compact metrics and of Poincaré-Einstein metrics, which are of interest in a broad variety of slightly different settings.
- We then show an efficient description of such metrics via the standard tractor bundle associated to the conformal Cartan geometry. This relates Poincaré-Einstein metrics to parallel tractors and hence to reductions of conformal holonomy.
- I'll briefly outline work of R. Gover and A. Waldron on a resulting boundary calculus and generalizations of the Willmore energy.

Contents

The model example for a geometric compactification is adding the sphere S^n as a boundary at infinity to hyperbolic space $\mathcal{H}^{n+1}$. Let $\overline{M} \subset \mathbb{R}^{n+1}$ be the closed unit ball, $\mathcal{H}^{n+1}$ its interior endowed with the *hyperbolic metric* $g := \frac{4}{(1-r^2)^2} g_{Euc}$ and S^n its boundary.

The function $\rho := 1 - r^2$ is an example of a *defining function* for the boundary $S^n \subset \overline{M}$. This means that $\rho : \overline{M} \to \mathbb{R}$ is smooth with zero set S^n and $d\rho|_{S^n}$ is nowhere vanishing. Any other defining function is of the form $f\rho$, where $f : \overline{M} \to \mathbb{R}$ is smooth and nowhere vanishing (locally around S^n).

Turning things around, g has the property that $\rho^2 g$ admits a smooth extension to all of $\overline{M}$ with the boundary values defining a Riemannian metric on S^n (the round one). This then holds for any defining function, but one obtains a metric on S^n conformal to the round one. Then $Isom(\mathcal{H}^{n+1}) \cong Conf(S^n)$. Observe that $\rho^\alpha g$ does not extend for $\alpha < 2$, while for $\alpha > 2$ it extends, but the boundary values are zero.

There is a general concept of local defining functions (and sections of line bundles) for arbitrary hypersurfaces $\Sigma \subset M$. In particular, this applies to the boundary in any manifold with boundary. The crucial feature of those is that any smooth function f such that $f|_\Sigma = 0$ can be written as ρh for a smooth function h. This leads to a notion of *order of vanishing* on Σ and of *growth* towards Σ.

Definition

Let $\overline{M}$ be a smooth manifold with boundary ∂M and interior M. A Riemannian metric g on M is called *conformally compact* if for any local defining function ρ for ∂M, the metric $\rho^2 g$ admits a smooth extension to all of $\overline{M}$, whose restrict to $T\partial M$ is non-degenerate. If g in addition is Einstein with negative scalar curvature, then it is called a *Poincaré-Einstein metric* (PE metric).

This leads to a well defined conformal structure $[\rho^2 g|_{T\partial M}]$ on ∂M, the *conformal infinity* of g. One is led to a variety of interesting problems in different settings:

- Starting from (M, g), one studies asymptotic aspects of Riemannian geometry, in particular in the PE case.
- Looking for asymptotic invariants of metrics that are asymptotic to the hyperbolic metric leads to a hyperbolic version of mass. (Here the PE case is trivial.)
- Given a conformal structure on ∂M, one can try to "fill in" a PE metric on M. This is interesting both on a formal level (Fefferman-Graham) and on an analytical level.
- The picture is the setup for the AdS/CFT correspondence and various versions of holography in physics.
- This is the model for compactifications of symmetric spaces. In general, the boundary structure is much more involved and it is difficult to endow boundary components with reasonable geometric structures.

We will next describe the setup from the point of view of conformal geometry.

densities

- A metric g on N defines a volume density $\mathrm{vol}_g = \sqrt{\det(g_{ij})}$.
- Forming powers and duals of the resulting line bundle, one obtains a family $\mathcal{E}[w] \to N$ of line bundles for $w \in \mathbb{R}$. The standard convention is that $\mathrm{vol}_g \in \Gamma(\mathcal{E}[-n])$.
- $\mathcal{E}[w]$ is associated to $\mathcal{G}_0$ via a representation of $Z(CO(n))$.

For a choice of metric g, $(\mathrm{vol}_g)^{-w/n}$ is a nowhere vanishing section of $\mathcal{E}[w]$, thus identifying $\Gamma(\mathcal{E}[w])$ with $C^\infty(N, \mathbb{R})$. Changing from g to $\hat{g} = f^2 g$, this identification changes as $\hat{\sigma} = f^{-w}\sigma$, which explains the convention.

Conversely, for $w \neq 0$, any nowhere vanishing $\sigma \in \Gamma(\mathcal{E}[w])$ determines a unique metric g in the class such that σ is parallel for the connection induced by the Levi-Civita connection ∇^g.

We will use abstract indices, so $\mathcal{E}^a = TN$, $\mathcal{E}_a = T^*N$ and so on. Adding $[w]$ indicates a tensor product with $\mathcal{E}[w]$.

The conformal class spans a line subbundle of $\mathcal{E}_{(ab)}$ isomorphic to $\mathcal{E}[-2]$. This defines a tautological section $\mathbf{g}_{ab} \in \Gamma(\mathcal{E}_{(ab)}[2])$ ("conformal metric"). This has an inverse $\mathbf{g}^{ab} \in \Gamma(\mathcal{E}^{(ab)}[-2])$. Hence we may raise and lower indices at the expense of a weight.

We next describe the *standard tractor bundle* $\mathcal{E}^A$ which is an equivalent encoding of the Cartan geometry associated to a conformal structure. Recall that this has type (G, P), where $G = SO_0(n+1,1)$. Restricting the standard representation gives a representation of P on $\mathbb{V} := \mathbb{R}^{n+2}$ and $\mathcal{E}^A = \mathcal{G} \times_P \mathbb{V}$ and we get:

- A Lorentzian bundle metric h_{AB} with inverse h^{AB}.
- A line subbundle $\cong \mathcal{E}[-1]$ (isotropic for h) whose inclusion defines $X^A \in \Gamma(\mathcal{E}^A[1])$.
- A surjection $\mathcal{E}^A \to \mathcal{E}[1]$ given by $X_A = h_{AB}X^B$.

The natural line subbundle in $\mathcal{E}^A$ is isotropic, thus contained in its orthocomplement and defining a filtration. We write this as a composition series $\mathcal{E}^A = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$. Choosing a metric in the conformal class defines a splitting of the filtration, thus identifying $\mathcal{E}^A$ with a direct sum, which we denote by vectors.

Changing from g to $\hat{g} = f^2 g$, we put $\Upsilon_a = f^{-1}df$ and the splitting changes as $\begin{pmatrix} \hat{\sigma} \\ \hat{\mu}_a \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} \sigma \\ \mu_a + \Upsilon_a \sigma \\ \rho - \mathbf{g}^{ab}(\Upsilon_a \mu_b + \frac{1}{2}\Upsilon_a \Upsilon_b \sigma) \end{pmatrix}$. $\mathcal{E}^A$ carries the canonical tractor connection. In the splitting for g this is given in terms of $\nabla = \nabla^g$ and the Schouten tensor P_{ab} of g as $\nabla_a \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + \mathbf{g}_{ab}\rho + \mathsf{P}_{ab}\sigma \\ \nabla_a \rho - \mathbf{g}^{ij}\mathsf{P}_{ai}\mu_j \end{pmatrix}$. Next, $D^A \tau := \begin{pmatrix} w(n+2w-2)\tau \\ (n+2w-2)\nabla_a \tau \\ -\mathbf{g}^{ij}(\nabla_i \nabla_j + \mathsf{P}_{ij})\tau \end{pmatrix}$ defines a natural operator $D^A : \Gamma(\mathcal{E}[w]) \to \Gamma(\mathcal{E}^A[w-1])$.

We will mainly use this on $\mathcal{E}[1]$ and put $I^A := \frac{1}{n} D^A \sigma$, which has σ as its top component ("BGG splitting operator"). Of course, we can then form $|I|^2 := h_{AB} I^A I^B$, which is a smooth function.

To interpret $|I|^2$, we first look at $U := \{x : \sigma(x) \neq 0\}$. The metric $g_{ab} := (1/\sigma^2)\mathbf{g}_{ab}$ on U satisfies $\nabla_a\sigma = 0$, and in this scale, it is evident that $|I|^2$ is a negative multiple of $Scal(g)$. For $x \notin U$, $\nabla_a\sigma(x)$ is independent of the choice of metric and outside of U, we get $|I|^2 = \mathbf{g}^{ij}(\nabla_i\sigma)(\nabla_j\sigma)$.

Parallel sections of $\mathcal{E}^A$ are closely related to Einstein metrics in the conformal class:

- Any parallel section is of the form I^A as above. (Determined by the top component.)
- For U and g as above, $\nabla_a I^A|_U = 0$ is equivalent to the Schouten tensor P_{ab} of g being proportional to g_{ab} and hence to g being Einstein.
- If I^A is parallel, then $|I|^2$ is constant and on U is a negative multiple of the Einstein constant of g.

Consider a conformal manifold $\overline{M} = M \cup \partial M$ with boundary, let g be a metric in the class on M and take $\sigma := (\mathrm{vol}_g)^{-1/n} \in \Gamma(\mathcal{E}[1])$.

Then g is conformally compact iff σ extends by zero to a defining density for ∂M.

Proof: For a local defining function ρ for ∂M put $\hat{g} := \rho^2 g$. If g is conformally compact, $\hat{g}$ is a metric in our class defined on all of $\overline{M}$. Thus $\hat{\sigma}$ is nowhere vanishing. But $\mathrm{vol}_{\hat{g}} = \rho^n \mathrm{vol}_g$ and hence $\sigma = \rho\hat{\sigma}$ on M, which shows that σ extends as required. Conversely, if σ extends to a defining density, then $\rho^{-1}\sigma$ smoothly extends to $\overline{M}$ and the metric it determines coincides with $\rho^2 g$ on M.

Theorem

For $\overline{M} = M \cup \partial M$ let g be a negative Einstein metric on M such that the conformal class $[g]$ smoothly extends to $\overline{M}$, but g itself does not admit a smooth extension to any neighborhood of a boundary point (e.g. because g is complete).
Then g is conformally compact and hence Poincaré-Einstein.

- $\mathcal{E}^A$ and the tractor connection are defined on $\overline{M}$.
- The tractor I^A determined by g is parallel over M hence can be smoothly extended to a parallel tractor on $\overline{M}$.
- Projecting I^A to $\Gamma(\mathcal{E}[1])$ provides a (unique) smooth extension of σ to all of $\overline{M}$.
- If $\sigma(x) \neq 0$ for some $x \in \partial M$, one obtains a smooth extension of g to a neighborhood of x, so all boundary values are zero.
- Since $|I|^2$ is constant on $\overline{M}$ and nonzero on M, σ is a defining density.

The setup described here is the starting point for a detailed analysis in several articles of R. Gover and A. Waldron: Given $\overline{M} = M \cup \partial M$ and a conformally compact metric g on M, take the corresponding conformal structure on $\overline{M}$, the defining density $\sigma \in \Gamma(\mathcal{E}[1])$ for ∂M selected by g and put $I^A := \frac{1}{n} D^A \sigma$. We assume that $|I|^2$ is nowhere vanishing.

Consider $I \cdot D : \Gamma(\mathcal{E}[w]) \to \Gamma(\mathcal{E}[w-1])$, $\tau \mapsto h_{AB} I^A D^B \tau$. This naturally extends to sections of weighted tractor bundles.

- On M, this is a Yamabe type operator associated to g.
- If $|I|^2 \equiv 1$ close to ∂M, then it restricts to the conformally invariant Robin operator on a neighborhood of ∂M.
- Together with multiplication by σ and a weight operator, $I \cdot D$ forms an $\mathfrak{sl}_2$-triple. This allows for very efficient computations (analysis of eigenfunctions, problems of harmonic extension, operators acting tangentially, etc.)

There are very interesting applications to the study of (oriented) hypersurfaces Σ in a conformal manifold $(N, [g])$. The natural question here is whether one can find a defining density $\sigma \in \Gamma(\mathcal{E}[1])$ for $\Sigma \subset N$ such that the corresponding tractor $I^A = \frac{1}{n} D^A \sigma$ satisfies $|I|^2 \equiv 1$ (singular Yamabe problem).

Starting from any defining density σ_0 for Σ the problem can be studied formally along Σ:

- If $n = \dim(N)$, there exists σ (unique up to $\mathcal{O}(\sigma^{n+1})$) such that $|I|^2 = 1 + \mathcal{O}(\sigma^n)$.
- For this σ, $\sigma^{-n}(|I|^2 - 1)$ is a smooth section of $\mathcal{E}[-n]$ defined locally around Σ, whose restriction to Σ is an invariant of $(N, [g], \Sigma)$.
- for $n = 3$, this produces the Willmore energy, so one obtains a natural family of higher order Willmore energies and invariants.