Holonomy reductions of Cartan geometries and applications

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- This talk starts with an outline of the general theory of holonomy reductions of Cartan geometries developed in joint work with R. Gover and M. Hammerl.
- The crucial feature of such reductions is that they come with a decomposition of the underlying manifold into "curved orbits" of different dimension that inherit geometric structures of different types.
- This provides a path towards new applications of Cartan geometries to the study of various kinds of geometric compactifications. In particular, conformal and projective compactness of Einstein metrics are examples.
- In the second part of the talk, I will sketch how these ideas lead to applications of parabolic geometry methods to compactifications of homogeneous spaces.

Contents

Let $(p : \mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P). Then ω trivializes $T\mathcal{G}$, so there are no horizontal curves, and hence no holonomy in a naive sense. But one can naturally extend ω to a principal connection on a bigger bundle.

Consider the principal G-bundle $\tilde{\mathcal{G}} := \mathcal{G} \times_P G$. This comes with a canonical inclusion $i : \mathcal{G} \to \tilde{\mathcal{G}}$ and there is a unique principal connection $\tilde{\omega}$ on $\tilde{\mathcal{G}}$ such that $i^*\tilde{\omega} = \omega$.

The basic idea is to define the holonomy of ω as the holonomy of $\tilde{\omega}$. This came up in the early 2000's for the canonical Cartan connections associated to conformal and projective structures under the name "conformal (projective) holonomy". There were some early results, including classification results on possible holonomy groups (S. Armstrong), but some basic geometric features remained unnoticed for a longer time.

An immediate consequence of the approach is that bundles that are associated to $\tilde{\mathcal{G}}$ and thus to actions of G ("tractor bundles") play an important role for holonomy. Any action (representation) of G can be restricted to P, and then $\mathcal{G} \times_P S \cong \tilde{\mathcal{G}} \times_G S$. Hence $\tilde{\omega}$ induces a connection on such bundles.

The general definition of a holonomy reduction is best formulated via a parallel section of $\mathcal{G} \times_P \mathcal{O}$, where $\mathcal{O}$ is a homogeneous space of G (viewed as a P-space). For most cases of interest, one can work with a parallel section of $\mathcal{G} \times_P \mathbb{V}$ for a representation $\mathbb{V}$ of G. Then $\mathcal{O}$ is one of the G-orbits in $\mathbb{V}$.

For the homogeneous model $G \to G/P$, one gets $G \times_P G \cong G/P \times G$ via $(g, \tilde{g}) \mapsto (gP, g\tilde{g})$. The connection $\tilde{\omega}$ is the corresponding flat connection. Hence a holonomy reduction is given by the choice of a subgroup $H \subset G$ with $\mathcal{O} = G/H$. Then G/P decomposes into H-orbits (of different dimensions). Each orbit can be viewed as $H/(H \cap \tilde{P})$, where $\tilde{P} \subset G$ is conjugate to P.

This picture persists in the curved case: Decompose $\mathcal{O} = \sqcup \mathcal{O}_i$ into P-orbits (indexed by $i \in H \backslash G/P$). Then a holonomy reduction of $(p : \mathcal{G} \to M, \omega)$ is defined by a P-equivariant function $\mathcal{G} \to \mathcal{O}$ and its values decompose $M = \sqcup M_i$ ("curved orbits"). (Some M_i may be empty.)

Via an analog of development, one proves that locally around each $x \in M$, there is a diffeomorphism to the homogeneous model which is compatible with the (curved) orbit decompositions. Further, each M_i canonically inherits a Cartan geometry of type $(H, H \cap \tilde{P}_i)$.

There are several immediate consequences:

- Each $M_i \subset M$ is an initial submanifold.
- If an H-orbit in G/P is an embedded submanifold, then also any corresponding curved orbit is an embedded submanifold.
- Normalization conditions on the curvature of ω often imply restrictions on the curvatures of the induced Cartan geometries.

Example: Put $G = SL(n+1, \mathbb{R})$, let $P \subset G$ be the stabilizer of a line in $\mathbb{R}^{n+1}$, so $G/P = \mathbb{R}P^n$. Take $0 < p, q$ with $p + q = n + 1$ and put $H = SO(p, q) \subset G$, so $\mathcal{O}$ is the space of inner products of signature (p, q) on $\mathbb{R}^{n+1}$. Under P, $\mathcal{O} = \mathcal{O}_- \sqcup \mathcal{O}_0 \sqcup \mathcal{O}_+$ according to the restriction of the inner product to the line stabilized by P.

A normal Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) is equivalent to a projective structure on M. A holonomy reduction of type G/H is given by a parallel bundle metric of signature (p, q) on the standard (co-)tractor bundle and

- $M = M_- \sqcup M_0 \sqcup M_+$ with $M_\pm$ open and M_0 a separating embedded hypersurface
- The induced Cartan geometries are equivalent to Einstein metrics of signature $(p - 1, q)$ respectively $(p, q - 1)$ on $M_\pm$ and a conformal structure of signature $(p - 1, q - 1)$ on M_0, respectively.

For G/P, $(G/P)_\pm$ are hyperbolic spaces (of appropriate signature). Hence the metrics are complete and from inside these orbits the hypersurface $(G/P)_0$ is at infinity. Again this persists in the curved case, and the behavior of the metrics towards M_0 can be analyzed precisely. This also shows how they induce a conformal structure on M_0.

This is the model case of metrics that are *projectively compact* (of order 2). These are mainly studied in the case of a manifold $\overline{M} = M \sqcup \partial M$ such that $M = \overline{M}_+$ and $\partial M = \overline{M}_0$.

There are parallel studies for Sasaki- and 3-Sasaki-Einstein metrics (Gover-Neusser-Willse) and for Kähler-Einstein metrics ("c-projective compactness", Č-Gover). Also, the (more popular) concept of *conformally compact* Einstein metrics admits a description in terms of reductions of conformal holonomy.

Returning to the homogeneous model, let us assume that G/P is compact, e.g. a generalized flag variety. Each H-orbit in G/P gives rise to an embedding $H/K \to G/P$, where $K = H \cap \tilde{P}$ as before. The closure of the image defines a *homogeneous compactification* of H/K, which is a union of H-orbits. This is particularly interesting if the initial H-orbit is open, since then it inherits a flat geometry of type (G, P) which extends to the closure. There are many examples of this situation in the parabolic setting:

Theorem (J. Wolf, 1976)

Let G be a real simple Lie group and θ an involutive automorphism of G with fixed point group $H \subset G$. Then for each parabolic subgroup $P \subset G$, H acts on G/P with finitely many orbits.

This implies that there are open orbits and their compactification is obtained by adding finitely many orbits as a "boundary". Examples include $SO \subset SL$, $SU \subset SL$, $Sp \subset SL$ (symplectic and quaternionic), $SL(n, \mathbb{C}) \subset SL(2n, \mathbb{R})$ and many others.

Example

Let us return to $H = SO(p,q) \subset SL(p+q,\mathbb{R}) = G$ with $p \leq q$ but now take G/P to be the Grassmannian $Gr(p,\mathbb{R}^{p+q})$. By linear algebra, H-orbits are classified by rank and signature of the restriction of the inner product defining H to subspaces and an orbit is open iff this restriction is non-degenerate.

So one of the open orbits is the Riemannian symmetric space $SO(p,q)/S(O(p) \times O(q))$ and its closure consists of the $p+1$ orbits $\mathcal{O}_i$ with positive semi-definite restrictions of rank i. For other open orbits and other flag varieties, one obtains compactifications of more complicated (non-symmetric) homogeneous spaces.

$\mathcal{O}_0$ is the unique closed orbit, an isotropic Grassmannian. Lie theory shows that the other additional orbits fiber over smaller isotropic Grassmannians with Riemannian symmetric fibers and leads to an infinitesimal transversal to each of the orbits.

Applications of the BGG machinery

Any G-representation $\mathbb{V}$ carries a P-invariant filtration, which in particular leads to a P-irreducible quotient. Correspondingly, there is a natural quotient bundle $\mathcal{H}$ of the corresponding tractor bundle $\mathcal{V}$. The BGG machinery gives rise to a geometric overdetermined operator ("first BGG operator") D on $\Gamma(\mathcal{H})$ as well as a splitting operator $S : \Gamma(\mathcal{H}) \to \Gamma(\mathcal{V})$.

On G/P, S and the quotient projection induce inverse bijections between $\ker(D)$ and and space of parallel sections of $\mathcal{V}$, which is isomorphic to $\mathbb{V}$. One can also extract finer jet information on $\sigma \in \Gamma(\mathcal{H})$ from $S(\sigma)$.

In our setting, the bilinear form defining H gives rise to a section $b = b_1 \in \Gamma(\mathcal{V}_1)$ for $\mathbb{V}_1 = S^2\mathbb{R}^{(p+q)*}$. For $k > 1$, we can next consider $b_k := b \wedge \cdots \wedge b \in \Lambda^k(\mathbb{V}_1)$. This turns out to lie in a certain irreducible component $\mathbb{V}_k \cong \circledcirc^2(\Lambda^k\mathbb{R}^{(p+q)*})$.

Let $E \to G/P$ be the tautological bundle. Then $\mathcal{H}_1 = S^2E^*$ and the quotient projection maps b_1 to $\sigma_1 \in \Gamma(\mathcal{H}_1)$, which associates to each subspace the restriction of b to that subspace. Each b_k projects to $\sigma_k \in \Gamma(\mathcal{H}_k)$ in a similar way. This then leads to:

Theorem

For each $k = 1, \dots, p$, σ_k is a smooth section of a natural vector bundle associated to an irreducible representation of P which is a solution to a geometric overdetermined system. Restricting to $\overline{\mathcal{O}_p}$, the zero set of σ_k equals the orbit closure $\overline{\mathcal{O}_{k-1}} = \cup_{i<k}\mathcal{O}_i$.

A finer analysis can be based on local defining sections constructed from σ_1. This shows that

- each $\mathcal{O}_i$ is an embedded submanifold
- slice theorem: Locally, a neighborhood of $\mathcal{O}_i$ looks like the product of $\mathcal{O}_i$ with positive semi-definite matrices of size $n-i$; orbit structure corresponds to stratification by rank.

A similarly analysis can be carried out for the Riemannian symmetric space $SO(n,\mathbb{C})/SO(n)$. This is based on the embedding of $SO(n,\mathbb{C})$ into $SO(n,n)$ obtained by forgetting the complex structure. Surprisingly, positive semi-definite symmetric matrices get replaced by positive semi-definite Hermitian matrices here. In particular, the stratification of these matrices by rank features in the slice theorem.