Projective compactifications

Andreas Čap[1]

University of Vienna
Faculty of Mathematics

Robin Graham 60.5
Tokyo
March 20, 2015

[1] supported by project P27072–N25 of the Austrian Science Fund (FWF)

- This is a survey talk based on joint work with A. Rod Gover (Auckland).
- Attaching a boundary at infinity to complete (pseudo–) Riemannian manifolds plays an important role in several parts of mathematics and mathematical physics, for example in scattering theory, general relativity and AdS/CFT.
- Guided by two classes of examples coming from reductions of projective holonomy, we develop an analog of conformal compactness in the setting of projective differential geometry. This concept is formulated for torsion free affine connections and depends on a parameter, called the *order* of projective compactness, that is related to volume growth.
- Via Levi–Civita connections, the concept is automatically defined for pseudo–Riemannian metrics. In this case, scalar curvature plays a crucial role. This is based on a new interpretation of scalar curvature in projective terms.

Contents

Let $\overline{M}$ be a smooth manifold with boundary ∂M and interior M. Then a conformally compact metric g_0 on M together with its conformal infinity (the induced conformal class on ∂M) can be equivalently described by

- A conformal structure on $\overline{M}$, which contains g_0.
- A section I of the conformal standard tractor bundle, which projects onto a defining 1–density σ for ∂M that is parallel for ∇^{g_0} over M.

If g_0 is Einstein, then I is parallel for the normal tractor connection and one obtains a reduction conformal holonomy.
Together with the tractor–D operator, I defines a degenerate Laplacian. This can be used to develop many elements of boundary calculus, holographic descriptions of GJMS–operators and Q–curvature and so on (Gover–Waldron).

The initial setup for conformal compactness thus is a metric on M, whose conformal class smoothly extends to $\overline{M}$, while the metric itself does not extend (for example, because it is complete).

Apart from the conformal structure defined by a metric, also the projective structure defined by its Levi–Civita connection is a highly interesting weakening.

- This approach automatically emphasizes the role of geodesics.
- Several fundamental operators of Riemannian geometry, for example the Riemannian deformation sequence, admit a projectively invariant formulation.

In that sense many results described here belong to the emerging field of *metric projective geometry*.

Projective differential geometry

Let N be a smooth manifold. *Projective equivalence* of two torsion–free linear connections ∇ and $\hat{\nabla}$ on TN can be equivalently defined as

- same geodesics up to parametrization
- existence of $\Upsilon \in \Omega^1(N)$ such that for all $\xi, \eta \in \mathfrak{X}(N)$ one has $\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$.

We will formally write the second condition as $\hat{\nabla} = \nabla + \Upsilon$.

Proposition

In the situation of $\overline{M} = M \cup \partial M$ and of ∇ on TM, there is a projective modification of ∇ which admits a smooth extension to the boundary if and only if the tracefree parts of the Christoffel symbols in local charts admit smooth extensions to the boundary.

Model examples

Viewed as the space of rays in $\mathbb{R}^{n+2}$, the sphere S^{n+1} carries a projective structure preserved by the action of $SL(n+2,\mathbb{R})$.

1. Choosing a hyperplane $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$ and identifying it with an open hemisphere via *central* projection is a projective diffeomorphism. Hence the projective class of the resulting flat connection on the hemisphere smoothly extends to the closed hemisphere. The boundary sphere is totally geodesic and thus inherits its standard projective structure.
2. Putting a Lorentzian metric on $\mathbb{R}^{n+2}$, one may identify $\mathcal{H}^{n+1}$ with future pointing negative rays, thus obtaining a projective embedding $\mathcal{H}^{n+1} \hookrightarrow S^{n+1}$. The boundary sphere S^n inherits its standard conformal structure from the ambient projective structure ("projective second fundamental form").

Both these examples generalize to holonomy reductions of curved projective structures. The concept of projective compactness is obtained from these examples by analyzing which projective modifications extend.

Definition

Consider $\overline{M} = M \cup \partial M$ and $\alpha \in \mathbb{R}_{>0}$.
A torsion free linear connection ∇ on TM is called *projectively compact of order* α if for each $x \in \partial M$ there is a local defining function ρ for ∂M defined on an open neighborhood U of x in $\overline{M}$ such that the linear connection $\hat{\nabla} := \nabla + \frac{d\rho}{\alpha\rho}$ on $U \cap M$ admits a smooth extension to all of U.

This turns out to be independent of the defining function ρ, but changing α corresponds to replacing ρ by some power of ρ, so α cannot be eliminated. Via the Levi–Civita connection the concept is automatically defined for metrics.

Suppose that ∇ is a *special* affine connection on M (i.e. one that preserves a volume density). Then each of the projective density bundles $\mathcal{E}(w)$ for $w \in \mathbb{R}$ admits non–vanishing sections over M which are parallel for ∇. In this situation, projective compactness of ∇ order α is equivalent to extendability of the projective structure plus a specific rate of volume growth as follows:

Proposition

Given a projective structure on $\overline{M}$ which, over M, contains a connection ∇ as above, the following are equivalent

1. ∇ is projectively compact of order α
2. In terms of a local defining function ρ for ∂M, the parallel volume density ν for ∇ has the form $\rho^{-(n+2)/\alpha}\hat{\nu}$ for a nowhere–vanishing $\hat{\nu}$ on $\overline{M}$.
3. Any non–vanishing section σ of $\mathcal{E}(\alpha)$, which is parallel for ∇ over M, extends by zero to a smooth defining density for ∂M.

One can directly prove existence of many local examples of projectively compact metrics: Suppose that $0 < \alpha \leq 2$ (which ensures that ∂M is "at infinity") and that $2/\alpha \in \mathbb{Z}$.

Theorem

Suppose that g is a pseudo–Riemannian metric on M such that there are local defining functions ρ for ∂M such that g admits an asymptotic form $\boxed{g = C\frac{d\rho^2}{\rho^{4/\alpha}} + \frac{h}{\rho^{2/\alpha}}}$, where h is smooth up to the boundary and non–degenerate on $T\partial M$, and C is a smooth function asymptotic to a non–zero constant on ∂M.
Then g is projectively compact of order α.

For $\alpha = 2$, this form is available (with the same C and conformally related h) for any defining function. For other values of α, it singles out specific defining functions.

The metricity equation

In general, a projective class does not contain any Levi–Civita connections. Existence of such a connection can be equivalently characterized as follows:

- Existence of a non–degenerate solution of the *metricity equation* $tfp(\nabla_i \sigma^{jk}) = 0$, a projectively invariant linear system of PDEs on $\mathcal{E}^{ij}(-2)$.
- Existence of a smooth section of the tractor bundle $S^2\mathcal{T}$, which is parallel for a natural connection on that bundle and satisfies a non–degeneracy condition.

Given $g = g_{ij}$, the solution is $\tau^{-1}g^{ij}$, where $\tau = \mathrm{vol}_g^{-2/(n+2)} \in \Gamma(\mathcal{E}(2))$ and g^{ij} is the inverse metric. The tensor bundle $\mathcal{E}^{(ij)}(-2)$ is the irreducible quotient of $S^2\mathcal{T}$, and the BGG splitting operator provides the second equivalence.

The role of scalar curvature

Suppose that g is a metric on M, such that the projective structure determined by ∇^g extends to $\overline{M}$. Then the tractor bundle $S^2\mathcal{T}$ and the canonical connection on it are defined on all of $\overline{M}$ and g determines a parallel section over M. This can be smoothly extended to all of $\overline{M}$ by parallel transport.

Moreover, a section of $S^2\mathcal{T}$ is a bundle metric on $\mathcal{T}^*$, so it has a well defined determinant. For the case of the metricity solution, one easily sees that this is given by the scalar curvature of g, and one obtains:

Theorem

In this situation, the metricity solution $\tau^{-1}g^{ij}$, the corresponding section of $S^2\mathcal{T}$, and the scalar curvature S of g all admit smooth extensions to all of $\overline{M}$.

The case of non–zero boundary values for S turns out to be surprisingly rigid:

Theorem

Let g be a metric on M whose projective structure smoothly extends to $\overline{M}$, and let $S : \overline{M} \to \mathbb{R}$ be the extended scalar curvature. Then for $x \in \partial M$, the following are equivalent

1. $S(x) \neq 0$
2. g is projectively compact of order $\alpha = 2$ around x.
3. The boundary value of S is a non–zero constant locally around x, and g admits an asymptotic form $g = C\frac{d\rho^2}{\rho^2} + \frac{h}{\rho}$ with a constant C (related to S).

Idea of proof: If $S(x) \neq 0$, the tractor metric determined by $\tau^{-1}g^{ij}$ is non–degenerate around x. Its point–wise inverse is a bundle metric on $\mathcal{T}$. This satisfies a differential equation, showing that τ is a defining density, and projective compactness follows.

More on the proof:

- Assuming projective compactness, volume asymptotics implies that the boundary value of S must be non–zero on a dense open subset in ∂M.
- Knowing explicit connections which extend to the boundary, one can refine the analysis of the inverse tractor metric on that subset to obtain an asymptotic form $g = \frac{-n(n+1)}{4S}\frac{d\rho^2}{\rho^2} + \frac{h}{\rho}$.
- Using geodesics of such a connection to trivialize a neighborhood of the boundary, one verifies directly that the boundary value of S is constant, thus completing the proof.

The basic quantity induced on the boundary is the projective second fundamental form II^p. This is just the restriction to $T\partial M$ of $\hat{\nabla} d\rho$, where ρ is a local defining function and $\hat{\nabla}$ is a connection in the projective class which is smooth up to the boundary. It is a symmetric bilinear form defined up to scale.

Let ∇ be projectively compact of order $\alpha \in (0, 2]$ on M and $\mathsf{P} = \mathsf{P}_{ij}$ is (projective) Schouten tensor. Then $\rho \mathsf{P} + \frac{\alpha - 1}{\alpha^2 \rho} d\rho^2$ admits a smooth extension to the boundary with boundary value representing II^p.

This can also be used to described the asymptotic behavior of the curvature R of ∇:

- For $\alpha = 1$, ρR admits an extension to the boundary, with boundary value the curvature tensor determined by II^p.
- For $\alpha \neq 1$, $\rho^2 R$ admits an extension, with boundary value the curvature tensor determined by $d\rho^2$.

One can also study the boundary geometry directly for the case of metrics having an asymptotic form as discussed before (as sufficient for projective compactness). For $\alpha \neq 2$, the boundary is totally geodesic for such metrics.

For $\alpha = 2$, the bilinear form h showing up in the asymptotic form turns out to represent II^p, which leads to the following asymptotic version of the Einstein property:

Theorem

If g is projectively compact of order $\alpha = 2$ on M, then the trace–free part of the Ricci–tensor admits a smooth extension to the boundary. This leads to a complete description of R up to terms which extend smoothly to the boundary.

Boundary tractors

Assume that ∇ is projectively compact of order $\alpha = 2$, preserves a volume density, and has II^p non–degenerate. Then ∂M inherits a pseudo–Riemannian conformal structure, which can be fully described by the conformal standard tractor bundle and connection.

Theorem

- The natural defining density $\tau \in \Gamma(\mathcal{E}(2))$ determines $L(\tau) \in \Gamma(S^2\mathcal{T}^*)$. The bundle $\mathcal{T}$ with the metric $L(\tau)$ restricts on ∂M to a conformal standard tractor bundle.
- If $\nabla^{S^2\mathcal{T}^*} L(\tau)$ (tractor connection) vanishes on ∂M, then $\nabla^{\mathcal{T}}$ restricts to the conformal tractor connection. (For a metric, vanishing of Ric_0 along ∂M is sufficient.)
- In general, one can describe the conformal tractor connection explicitly in terms of projective data.

Thank you

and

Happy birthday, Robin