On (systems of) ODEs of C–class

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- This talk reports on joint work with Boris Doubrov (Minsk) and Dennis The (Tromsø) which is available as arXiv:1709.01130, based on a general construction of Cartan connections from arXiv:1707.05627.
- Equations of C–class is a concept of É. Cartan aiming at finding classes of equations in which generic members can be solved without integration. Technically, we will study the question whether a Cartan geometry associated to a system of equations descends to the space of solutions.
- For low orders, one deals with parabolic geometries and the question of descending is answered by general results on correspondence spaces and twistor spaces.
- For higher orders, the geometries are not parabolic. Still, the techniques used to characterize descending are inspired by recent versions of the machinery of BGG sequences.

Contents

Consider the space $J^\ell_m := J^\ell(\mathbb{R}, \mathbb{R}^m)$ of ℓ–jets of curves in $\mathbb{R}^m$. This carries a natural EDS which is, in usual jet coordinates (t, u^j_i) with $i = 0, \dots, \ell$, $j = 1, \dots, m$, spanned by the forms $du^j_i - u^j_{i+1}dt$. The joint kernel of these forms is the *contact subbundle* $T^{-1}J^\ell_m \subset TJ^\ell_m$ of rank $m + 1$.

The weak derived flag of this subbundle grows by m dimensions in each step and has the form

$$T^{-1}J^\ell_m \subset T^{-2}J^\ell_m \subset \dots \subset T^{-\ell}J^\ell_m \subset T^{-\ell-1}J^\ell_m = TJ^\ell_m.$$

It is also well known that smooth curves in J^ℓ_m which are tangent to $T^{-1}J^\ell_m$ in each point are exactly the ℓ–jet prolongations of smooth curves in $\mathbb{R}^m$. Moreover, diffeomorphisms of J^ℓ_m preserving $T^{-1}J^\ell_m$ are exactly the prolongations of contactomorphisms on J^1_1 if $m = 1$ and of diffeomorphisms on J^0_m if $m > 1$.

Suppose now that we have given a system of m ODEs on a curve u in $\mathbb{R}^m$ in the form $u^{(n+1)}(t) = f(t, u(t), u'(t), ..., u^{(n)}(t))$ for a smooth function f. This defines a submanifold $\mathcal{E} \subset J^{n+1}_m$ such that the natural projection $\pi^{n+1}_n : J^{n+1}_m \to J^n_m$ restricts to a local diffeomorphism on $\mathcal{E}$.

Using this diffeomorphism, we can carry over the natural filtration of the tangent bundle TJ^n_m to $T\mathcal{E}$ to get a filtration $T^{-1}\mathcal{E} \subset T^{-2}\mathcal{E} \subset \cdots \subset T^{-n-1}\mathcal{E} = T\mathcal{E}$.
This makes $\mathcal{E}$ into a filtered manifold locally isomorphic to J^n_m.

Since we have $\mathcal{E} \subset J^{n+1}_m$, we can also restrict the "highest order" contact forms $du^j_n - u^j_{n+1}dt$ to $T\mathcal{E}$. Their joint kernel is a line subbundle $E \subset T^{-1}M$ which encodes the actual equation, and has the property that $T^{-1}\mathcal{E} = E \oplus F^{-1}$, where F^{-1} comes from the smallest vertical subbundle $\ker(T\pi^n_{n-1}) \subset TJ^n_m$.

The Lie bracket of vector fields induces a tensorial bracket on the associated graded $\mathrm{gr}(T_x\mathcal{E}) = \oplus_i(T_x^i\mathcal{E}/T_x^{i+1}\mathcal{E})$, thus making it into a nilpotent graded Lie algebra. This turns out to be independent of the point x and of the equation $\mathcal{E}$. The information on the equation is encoded into the direct sum decomposition of $T^{-1}\mathcal{E} = \mathrm{gr}_{-1}(T\mathcal{E})$ as $E \oplus F^{-1}$, which can be interpreted as a reduction of the structure group of $\mathrm{gr}(TM)$.

Thus we associate to any ODE–system a filtered G_0–structure with structure group $GL_1 \times GL_m$ reflecting the direct sum decomposition. It follows from the construction, that this structure equivalently encodes the system up to contact transformations for $m = 1$ and up to point transformations for $m > 1$.

Not all filtered geometric structures of this type come from systems of ODEs. The involutive subbundles in $T\mathcal{E}$ obtained from $\ker(T\pi^n_k) \subset TJ^n_m$ for $k < n - 1$ lead to better compatibility of the Lie bracket with the filtration than required in general.

By Doubrov–Komrakov–Morimoto, there are canonical Cartan connections associated to the filtered G_0–structures we consider. We are working with a variant of their construction, which uses a manifestly invariant normalization condition and allows stronger uniqueness results. As usual, we start from a homogeneous model.

This model comes from the natural trivialization of $J^n(\mathcal{O}(n)^m)$ of jets of sections of $\mathcal{O}(n)^m \to \mathbb{R}P^1$ via homogeneous polynomials. Put $Q = SL_2 \times GL_m$, $P \subset Q$ the product of the Borel subgroup of SL_2 with GL_m, $V_n^m := S^n\mathbb{R}^2 \boxtimes \mathbb{R}^m$ and $G := Q \ltimes V_n^m$ (semi–direct product). Then $J^n(\mathcal{O}(n)^m) \cong G/P$ encodes the trivial system $u_i^{(n+1)} = 0$.

The Lie algebra $\mathfrak{g}$ of G carries a natural grading of the form $\mathfrak{g}_{-n-1} \oplus \cdots \oplus \mathfrak{g}_1$, which comes from the $|1|$–grading of $\mathfrak{sl}_2$ and the weight decomposition of $S^n\mathbb{R}^2$ (with the highest and lowest weight spaces having degrees -1 and $-n-1$, respectively).

To apply the general construction of canonical Cartan connections from arXiv:1707.05627, one first has to verify that, for the given grading, $\mathfrak{g}$ is the full Tanaka prolongation of its non–positive part. This can be done by verifying that $H^1(\mathfrak{g}_-, \mathfrak{g})$ is concentrated in non–positive homogeneities.

This is the case if either $n \geq 3$ or $m, n \geq 2$. For $n = 1$, the full prolongation is $\mathfrak{sl}_{m+2}$, while for for $m = 1$ and $n = 2$ it is $\mathfrak{sp}_4$. This leads to the description of 2^{nd} order systems via path geometries and of 3rd order ODEs as parabolic geometries.

The second ingredient needed to apply the general theory is an appropriate choice of normalization condition. In the most general form, such a condition is a linear subspace $\mathcal{N} \subset L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$. This space is naturally a P–module and one requires $\mathcal{N}$ to be P–invariant to ensure that normality has a geometric meaning.

To express the second main requirement on $\mathcal{N}$, we observe that $L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$ carries a P–invariant filtration (coming from homogeneity of maps) and that the associated graded space is isomorphic to $L(\Lambda^2\mathfrak{g}_-, \mathrm{gr}(\mathfrak{g}))$. Here the grading is by homogeneity and in our case $\mathrm{gr}(\mathfrak{g}) \cong \mathfrak{g}$. A Lie algebra cohomology differential $\partial_{\mathfrak{g}_-}$ acts on that space and we require:

Mapping the the intersection of $\mathcal{N}$ with any positive filtration component of $L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$ to the associated graded, one has to obtain a linear complement to $\ker(\partial_{\mathfrak{g}_-})$ in the given homogeneity.

One way to construct normalization conditions is via P–equivariant codifferentials ∂^*, which map $A^k := L(\Lambda^k(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$ to A^{k-1} for $k = 2, 3$. Under appropriate assumptions, one shows that $\mathcal{N} = \ker(\partial^*)$ is a normalization condition and projecting to $\ker(\partial^*)/\operatorname{im}(\partial^*)$ leads to an analog of harmonic curvature.

For our choice of $(\mathfrak{g}, P)$, we construct a codifferential as follows. We view each A^k as a P–submodule of $C^k := L(\Lambda^k\mathfrak{g}, \mathfrak{g})$, on which there is a $\mathfrak{g}$–equivariant Lie algebra cohomology differential $\partial_{\mathfrak{g}}$. On $\mathfrak{q}$ and its representation V_n^m there are inner products with a nice compatibility to the $\mathfrak{q}$–action in terms of a Cartan–involution. These can then be used to define an inner product on each of the spaces C^k.

Proposition

The adjoints of the maps $\partial_{\mathfrak{g}}$ with respect to that inner product restrict to P–homomorphisms $\partial^* : A^k \to A^{k-1}$, which for $k = 2, 3$ have all necessary properties for a codifferential to give rise to a normalization condition.

One also proves that the representation of P on $\ker(\partial^*)/\operatorname{im}(\partial^*)$ is completely reducible, which implies that the analog harmonic curvature is a simple geometric object.

By general results, the principal bundle describing the filtered G_0–structure on an equation $\mathcal{E}$ can be extended to a principal P–bundle $p : \mathcal{G} \to \mathcal{E}$. There is a regular Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, which is normal in the sense that its curvature function κ has values in $\ker(\partial^*) \subset A^2$. The pair $(\mathcal{G}, \omega)$ is uniquely determined up to isomorphism by the latter property.

Solutions of the system are the integral submanifolds of the line subbundle $E \subset T^{-1}\mathcal{E}$, which corresponds to $\mathfrak{q}/\mathfrak{p} \subset \mathfrak{g}/\mathfrak{p}$. On the homogeneous model, the space of solutions is G/Q. So we may ask in general whether the Cartan geometry $(p : \mathcal{G} \to \mathcal{E}, \omega)$ descends to local leaf–spaces of the foliation determined by $E \subset T\mathcal{E}$.

If this is the case, then local invariants descend and hence are constant along each solution. In sufficiently generic situations, this allows solving the system without integration. This idea is closely related to E. Cartan's concept of a "C–class of equations".

Descending the geometry amounts to extending the P–action on $\mathcal{G}$ to an action of Q and verifying that ω is Q–equivariant. There is a general theorem stating that this is equivalent to that fact that all values of κ vanish upon insertion of one element of $\mathfrak{q}/\mathfrak{p} \subset \mathfrak{g}/\mathfrak{p}$. The latter condition defines a P–submodule $\mathbb{E} \subset L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$.

If the geometry does descend, then the tangent bundle to local leaf spaces is the associated bundle corresponding to the representation $\mathfrak{g}/\mathfrak{q} \cong S^n\mathbb{R}^2 \boxtimes \mathbb{R}^m$ of Q. Thus any local leaf space inherits a corresponding first–order Q–structure, which is called a Segré–structure or, for $m = 1$, a GL_2–structure.

This relates to results of B. Doubrov on (generalized) Wilczynski invariants. These are invariants obtained via the linearization of the system around a solution. They can be interpreted as obstructions to descending of the filtered G_0–structure on $\mathcal{E}$ to a Segré structure on local spaces of solutions.

The latter description can be connected to the canonical Cartan geometry, and one proves:

- The Wilczynski invariants of $\mathcal{E}$ can be recovered from the projection of the curvature function κ to $\ker(\partial^*)/\operatorname{im}(\partial^*)$.
- Vanishing of all Wilczynski invariants is equivalent to κ having values in the P–submodule $\mathbb{E} + \operatorname{im}(\partial^*)^1$ (where the superscript indicates filtration-homogeneity).

Our main result is that the latter condition is equivalent to κ having values in $\mathbb{E}$ thus proving

Theorem

The canonical Cartan geometry $(p : \mathcal{G} \to \mathcal{E}, \omega)$ descends to local spaces of solutions iff all Wilczynski invariants of $\mathcal{E}$ vanish.

One crucial ingredient for the proof is the by P–equivariancy, ∂^* induces a tensorial operator on horizontal, equivariant $\mathfrak{g}$–valued forms on $\mathcal{G}$.

The second crucial ingredient is the *covariant exterior derivative* d^ω on $\Omega^*(\mathcal{G},\mathfrak{g})$ obtained by defining $d^\omega\varphi(\xi_0,\dots,\xi_k)$ as

$$d\varphi(\xi_0,\dots,\xi_k)+\textstyle\sum_{i=0}^k(-1)^i[\omega(\xi_i),\varphi(\xi_0,\dots,\widehat{\xi_i},\dots,\xi_k)].$$

This has the following properties:

- It preserves the subspaces of horizontal, equivariant forms and is compatible with the natural filtration on these spaces.
- The resulting operator on the associated graded is tensorial and induced by $\partial_{\mathfrak{g}_-}$.
- Bianchi–identity: The curvature K of ω satisfies $d^\omega K=0$.

One then has to verify directly, that if a horizontal, equivariant form $\varphi\in\Omega^2(\mathcal{G},\mathfrak{g})$ corresponds to an $\mathbb{E}$–valued function, then the same holds for $\partial^*d^\omega\varphi$. Using this, the theorem can be proved by the following inductive procedure.

Assume that we can write $K = K_1 + K_2$ such that κ_1 is $\mathbb{E}$–valued and κ_2 has values in $\mathrm{im}(\partial^*)$ and is homogeneous of degree $\geq \ell$ for some $\ell \geq 1$. (For $\ell = 1$, this follows from Wilczynski–flatness.)

By the Bianchi–identity, we get $\partial^* d^\omega K_2 = -\partial^* d^\omega K_1$, so this corresponds to a an $\mathbb{E}$–valued function.

Studying the action of $\partial^* d^\omega$ on the associated graded, one finds that there is a universal polynomial p such that
$p(\partial^* d^\omega)(\partial^* d^\omega K_2) \equiv K_2$ modulo elements of homogeneity $\geq \ell + 1$, and again the left hand side is $\mathbb{E}$–valued.
Adding this to K_1 and subtracting it from K_2 we get a new decomposition of the same type, for which the second component is homogeneous of degree $\ell + 1$. Iterating this, we see that κ is $\mathbb{E}$–valued, which implies descending of the geometry.