# THE HEISENBERG GROUP, $\mathrm{SL}(3,\mathbb{R})$, AND RIGIDITY

ANDREAS ČAP, MICHAEL G. COWLING, FILIPPO DE MARI, MICHAEL EASTWOOD, AND RUPERT MCCALLUM

*Ubi materia, ibi geometria* Johannes Kepler

The second-named author of this paper discussed some of the geometric questions that underlie this paper (Darboux's theorem) with Roger Howe in the 1990s. After that, it took on a life of its own.

## 1. Introduction

A stratified group $N$ is a connected, simply connected nilpotent Lie group whose Lie algebra $\mathfrak{n}$ is stratified, that is

$$n = \mathfrak{n}^1 \oplus \mathfrak{n}^2 \oplus \cdots \oplus \mathfrak{n}^s,$$

where $[\mathfrak{n}^1, \mathfrak{n}^j] = \mathfrak{n}^{j+1}$ when $j = 1, 2, \cdots, s$ (we suppose that $\mathfrak{n}^s \neq \{0\}$ and $\mathfrak{n}^{s+1} = \{0\}$), and carries an inner product for which the $\mathfrak{n}^j$ are orthogonal. The left-invariant vector fields on $N$ that correspond to elements of $\mathfrak{n}^1$ at the identity span a subspace $HN_p$ of the tangent space $TN_p$ of $N$ at each point $p$ in $N$, and the corresponding distribution $HN$ is called the horizontal subbundle of the tangent bundle $TN$. The commutators of length at most $s$ of sections of $HN$ span the tangent space at each point.

Suppose that $U$ and $V$ are connected open subsets of a stratified group $N$, and define

$$\mathrm{Contact}(U,V) = \{f \in \ \mathrm{Diffeo}(U,V) : df(HU_p) = HV_{f(p)} \text{ for all } p \text{ in } U\}$$

(where $\mathrm{Diffeo}(U,V)$ denotes the set of diffeomorphisms from $U$ to a subset of $V$). The group is said to be rigid if $\mathrm{Contact}(U,V)$ is finite-dimensional. If $N$ has additional structure, one might look at spaces of maps preserving this additional structure and ask whether the space of structure preserving maps is finite-dimensional. Much fundamental work on this question

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has been done by N. Tanaka and his collaborators T. Morimoto and K. Yamaguchi (see, for instance, [5, 7, 10]).

Rigidity is a blessing and a curse. It makes life simpler, but it makes it harder to choose coordinates. For smooth maps, much is known about rigidity. If $N$ is the nilradical of a parabolic subgroup of a semisimple Lie group $G$, then, as shown by Yamaguchi, $N$ is rigid (and $\mathrm{Contact}(U,V)$ is essentially a subset of $G$) except in a limited number of cases. If $N$ is the free nilpotent group $N_{g,s}$ of step $s$ on $g$ generators, then the situation is similar: $N_{g,1}$ and $N_{2,2}$ are non-rigid, $N_{2,3}$ and $N_{g,2}$ (where $g \geq 3$) are rigid and $\mathrm{Contact}(U,V)$ is a subset of a semisimple group (see, e.g., [10]; in this case, some contact maps are fractional linear transformations) and otherwise $N_{g,s}$ is rigid and $\mathrm{Contact}(U,V)$ is a subset of the affine group of $N_{g,s}$, that is, the semidirect product $N_{g,s} \rtimes \mathrm{Aut}(N_{g,s})$, where the normal factor $N_{g,s}$ gives us translations and the other factor automorphisms (see, e.g., [9]).

In a fundamental paper, P. Pansu [6] studied quasiconformal and weakly contact maps on Carnot groups (stratified groups with a natural distance function). Weakly contact maps are defined to be those that map rectifiable curves into rectifiable curves (rectifiable curves are defined using the natural distance function, but the rectifiability of a curve is independent of the distance chosen), so weakly contact maps are defined on stratified groups. Pansu showed that the weakly contact mappings of generic step 2 stratified groups are affine, and hence smooth. More recently there has been much work in quasiconformal and weakly contact maps on stratified groups, and it is a question of interest whether weakly contact maps are automatically smooth (if only because if this is true, then there is not much point in developing the theory for non-smooth maps). One result in this direction, due to L. Capogna and Cowling [1], is that 1-quasiconformal maps of Carnot groups are smooth. It then follows from work of N. Tanaka [7] that they form a finite-dimensional group, and so we have rigidity.

## 2. An example

The following example is prototypical of the rigid situation, and we present a simplified version of the known proof of rigidity for smooth maps for this case. Let $N$ be the Heisenberg group of upper diagonal $3 \times 3$ unipotent matrices

$$N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

We define the left-invariant vector fields $X, Y$, and $Z$ by

$$X = \partial/\partial x, \qquad Y = \partial/\partial y + x\partial/\partial z, \qquad Z = \partial/\partial z.$$

Then $[X, Y] = Z$.

Consider $\{f \in \mathrm{Diffeo}(U, V) : df(X) = pX,\ df(Y) = qY\}$, where $p$ and $q$ are arbitrary functions. We might describe these as multicontact maps, as the differential preserves several subbundles of $TN$, not just one. A vector field $M$ on $U$ is called multicontact if $M$ generates a flow of multicontact maps. This boils down to the requirement that $[M, X] = p'X$ and $[M, Y] = q'Y$, where $p'$ and $q'$ are functions. If we write $M$ as $aX + bY + cZ$, where $a, b$ and $c$ are functions, then we find the equations

$$\begin{aligned} &-(Xa)X - (Xb)Y - (Xc)Z - bZ = p'X \\ &-(Ya)X - (Yb)Y - (Yc)Z + aZ = q'Y, \end{aligned}$$

whence

$$\begin{aligned} Xb &= 0 & Xc &= -b \\ Ya &= 0 & Yc &= a. \end{aligned}$$

We see immediately that $c$ determines $a$ and $b$, and that

$$X^2c = Y^2c = 0.$$

These equations imply that $c$ is a polynomial in $X, Y$ and $Z$, and in fact that the Lie algebra of multicontact vector fields is isomorphic to $\mathfrak{sl}(3, \mathbb{R})$. It then follows that the corresponding group of smooth multicontact mappings is a finite extension of $\mathrm{SL}(3, \mathbb{R})$. However, even if care is taken, this argument can only work for mappings that are at least twice differentiable.

Now we consider the non-smooth case. The integrated version of the multicontact equations is quite simple. We denote by $\mathcal{L}$ and $\mathcal{M}$ the sets of integral curves for $X$ and $Y$, i.e.,

$$\begin{aligned} \mathcal{L} &= \{l_{y,z} : y, z \in \mathbb{R}^2\} \\ \mathcal{M} &= \{m_{x,z} : x, z \in \mathbb{R}^2\}. \end{aligned}$$

where $l_{y,z}$ and $m_{x,z}$ are the ranges of the maps

$$s \mapsto \begin{pmatrix} 1 & s & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad t \mapsto \begin{pmatrix} 1 & x & z + xt \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix},$$

and for an open subset $U$ of $N$, we define $\mathcal{L}_U$ to be the set of all connected components of the sets $l \cap U$ as $l$ varies over $\mathcal{L}$, and define $\mathcal{M}_U$ similarly. We may now ask what can be said about maps that send line segments $l$ in $\mathcal{L}_U$ into lines in $\mathcal{L}$ and line segments $m$ in $\mathcal{M}_U$ into lines in $\mathcal{M}$.

We say that points $P_1, \dots, P_4$ in the Heisenberg group $N$ are in general position if neither of the sets $\cup_{i=1}^4 \{(y,z) \in \mathbb{R}^2 : P_i \in l_{y,z}\}$ and $\cup_{i=1}^4 \{(x,z) \in \mathbb{R}^2 : P_i \in m_{x,z}\}$ is contained in a line.

**Theorem 2.1.** *Let $U$ be a connected open subset of $N$, and let $\varphi$ be a map from $U$ into $N$ whose image contains four points in general position, and which maps connected line segments in $\mathcal{L}_U$ into lines in $\mathcal{L}$ and connected line segments in $\mathcal{M}_U$ into lines in $\mathcal{M}$. Then*

(a) *if $U = N$, then $\varphi$ is an affine transformation, i.e., the composition of a translation with a dilation $(x, y, z) \mapsto (sx, ty, stz)$;*

(b) *otherwise, $\varphi$ is a projective transformation.*

The proof will show what we mean by a projective transformation; essentially, $N$ may be identified with a subset of a flag manifold, and projective transformations are those which arise from the action of $\mathrm{SL}(3, \mathbb{R})$ on this manifold.

## 3. Related questions in two dimensions

Before we prove Theorem 2.1, we first discuss related questions in $\mathbb{R}^2$.

**Theorem 3.1.** *Let $U$ be a connected open subset of $\mathbb{R}^2$, and let $\varphi$ be a map from $U$ into $\mathbb{R}^2$ whose image contains three non-collinear points, and which maps connected line segments contained in $U$ into lines. Then*

(a) *if $U = \mathbb{R}^2$ then $\varphi$ is an affine transformation, i.e., the composition of a translation with a linear map;*

(b) *otherwise, $\varphi$ is a projective transformation.*

Theorem 3.1 is a corollary of Theorem 3.2, which we state and prove shortly. But there are some interesting historical remarks to be made before we do this.

We note that (a) is a theorem of Darboux; (b) requires more work. In the special case where $U$ and $\varphi(U)$ are both the unit disc, then (b) amounts to the identification of the geodesic-preserving maps of the hyperbolic plane in the Klein model. The paper [3] by J. Jeffers identifies the geodesic-preserving maps of the hyperbolic plane in the Poincaré model (and observes that this identification was already known). This leads to the consideration of maps of regions in the plane that preserve circular arcs, and in turn leads us to mention a related result of Carathéodory [2], which establishes that maps of regions in the plane that send circular arcs into circular arcs arise from the action of the conformal group.

It is standard that collinearity preserving mappings of the projective plane are projective, i.e., arise from the action of $\mathrm{PGL}(3, \mathbb{R})$ on the projective space $\mathbb{P}^2(\mathbb{R})$. This may be proved by composing the collinearity preserving map with a projective map to ensure that the line at infinity in the projective plane is preserved, then applying Darboux's theorem. Conversely, if one assumes that collinearity preserving mappings of the projective plane are projective, then Darboux's theorem follows once one observes that projective maps of the projective plane that preserve the finite plane are actually affine. Thus Darboux's theorem is essentially equivalent to "the fundamental theorem of projective geometry".

We take a domain $U$ (i.e., a connected open set) in $\mathbb{R}^2$, and we suppose that $\varphi : U \to \mathbb{R}^2$ preserves collinearity for connected line segments in an open set of directions. In the next theorem, we show that we can still control $\varphi$. Recall that three points in $\mathbb{R}^2$ are said to be in general position if they are not collinear. We call a connected line segment in $U$ a chord.

**Theorem 3.2.** *Suppose that $U$ is a domain in $\mathbb{R}^2$, and that the image of $\varphi : U \to \mathbb{R}^2$ contains three points in general position. Suppose that for each $P$ in $U$, there exists a neighbourhood $N_P$ of $P$ and an nonempty open set of slopes $S_P$ such that if $x$, $y$, and $z$ in $N_P$ lie in a line whose slope is in $S_P$, then $\varphi(x), \varphi(y)$ and $\varphi(z)$ are collinear. Then $\varphi$ is a projective transformation.*

*Proof.* It suffices to show that for each $P$, there is a neighbourhood $N'_P$ of $P$ such that $\varphi\big|_{N'_P}$ is projective. For then an analytic continuation argument shows that $\varphi$ is projective on $U$. Choose Cartesian coordinates, and take a small ball $B(P, r)$ with centre $P$ contained in $N_P$.

By composing with a rotation about $P$, we may suppose that $\varphi$ sends chords of $B(P, r)$ with slopes in $[0, \epsilon]$ into lines in $\mathbb{R}^2$, for some small $\epsilon$. By composing with a shear transformation centred at $P$, we may suppose that $\varphi$ sends chords of $B(P, r)$ whose slopes are not in $(-\epsilon, 0)$ into lines in $\mathbb{R}^2$, and by composing with a rotation, we may suppose that $\varphi$ sends chords of $B(P, r)$ with slopes close to 0, $\infty$, $\pm\sqrt{3}$ or $\pm 1/\sqrt{3}$, or in the intervals $(0, 1/\sqrt{3})$ and $(\sqrt{3}, \infty)$, into lines in $\mathbb{R}^2$. By composing in the image space with a projective map, we may suppose that $\varphi$ fixes the vertices and centroid $P$ of an equilateral triangle inside $B(P, r)$ with horizontal base. This triangle will be our set $N'_P$, as we will show that this modified $\varphi$ is the identity thereon, so the original $\varphi$ must have been projective.

Suppose that the triangle is as in Figure 1. Since $\varphi$ sends chords with slopes 0, $\infty$, $\pm\sqrt{3}$ and $\pm 1/\sqrt{3}$ into lines, $\varphi$ preserves the midpoints of the sides. For example, the chord $CP$ is mapped into the line $CP$ and the chord $AB$ is mapped into the line $AB$, and $F = CP \cap AB$, so $\varphi(F) = F$. Similarly, $\varphi$ also preserves $AD \cap EF$ and $BE \cap DF$, that is, the midpoints of $EF$ and $DF$, so the horizontal chord through these points is sent into the line through these points. It follows that the midpoints of $AE$ and $DB$ are also preserved. In this way, we deduce that not only does $\varphi$ preserve the vertices of the four smaller equilateral triangles

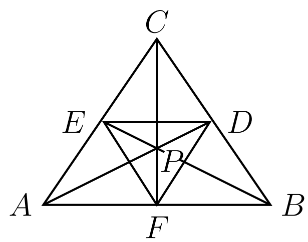


FIGURE 1. Subdivision

$AFE$, $FBD$, $DEF$ and $EDC$, but also $\varphi$ preserves the midpoints of their sides, and their centroids. We can therefore continue to subdivide and find a dense subset of the triangle of points that are fixed by $\varphi$.

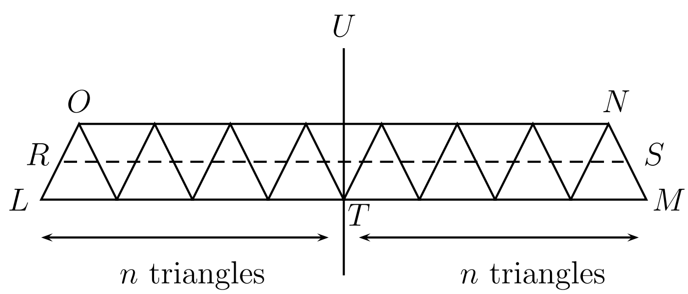


FIGURE 2. Horizontal chords

We now show that all horizontal lines that meet the interior of the triangle are sent into horizontal lines, or more precisely, that a horizontal chord $RS$ (where $R$ and $S$ lie on the edges of the triangle) is sent into a horizontal line $R'S'$ (where $R'$ and $S'$ lie on the edges of the triangle, possibly produced). If the chord is one of those that arises in the subdivision, we are done, by construction. Otherwise, for some arbitrarily large positive integer $n$, we can find a subdivision as shown in Figure 2, so that the chord $RS$ passes through $2n$ congruent equilateral triangles with horizontal bases, and $2n-1$ inverted congruent equilateral triangles with horizontal bases. If $RS$ is as shown, then the absolute values of the slopes of $LS$ and $RM$ are less than $1/(2n-1)$, so if $n$ is large enough, the images of the chords $LS$ and $RM$ lie in lines. Further, no matter what slopes are involved, if $R'$ and $S'$ lie on $AC$ and $CB$, then $R'S'$ is parallel to $LM$ if and only if $LS' \cap R'M$ lies on the vertical line $TU$ through the midpoint of $LM$. Since $TU$ is also preserved, it follows that the image of a horizontal chord is horizontal.

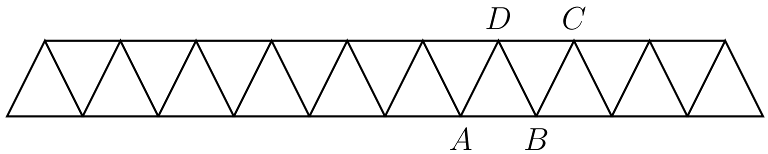


FIGURE 3. A rhombus

We can repeat this argument to deduce that the images of chords with slopes $\pm\sqrt{3}$ that meet the interior of the triangle also have slopes of $\pm\sqrt{3}$.

We now show that $\varphi$ is continuous. To do this, it suffices to show that each small rhombus like $ABCD$ in Figure 3 is preserved; since these can be made arbitrarily small, $\varphi$ is continuous, and hence constant. After a linear transformation centred at $A$, doing this is equivalent to proving the following lemma, and its proof will conclude the proof of Theorem 3.2. □

**Lemma 3.3.** *Let $\Sigma$ denote the square $ABCD$, where the coordinates of $A$, $B$, $C$ and $D$ are $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, and suppose that $\varphi : \Sigma \to \mathbb{R}^2$ preserves the vertices of $\Sigma$, sends chords of $\Sigma$ with slope $0$, $\pm 1$ and $\infty$ into lines of the same slope, and sends chords of $\Sigma$ with slope in $(0,\infty)$ into lines. Then the image of $\varphi$ is contained in the set $\Sigma$.*

*Proof.* Suppose that $0 < y < 1$ and take $s$ in $(0,1)$ so that $y = s^2$. Then $y$ is the ordinate of the point $Q$ constructed as the intersection of $AS$ and $UV$ in the top left-hand diagram below, in which the ordinate of $S$ is $s$, the line $TS$ is horizontal, the line $TU$ has slope $-1$, and the line $UV$ is vertical.

The line $AS$ has gradient in $(0,1)$, so maps into a line. Now $\varphi(A) = A$ and $\varphi(S)$ lies on $BC$ (possibly produced), and so has coordinates $(1,t)$ for some real $t$. Next, the line $\varphi(T)\varphi(S)$ is horizontal and $\varphi(T)$ lies on $AD$ (possibly produced), so $\varphi(T)$ has coordinates $(0,t)$, and the line $\varphi(T)\varphi(U)$ has gradient $-1$ and $\varphi(U)$ lies on $AB$ (possibly produced), and hence $\varphi(U)$ has coordinates $(t,0)$. Now $\varphi(U)\varphi(V)$ is vertical, and so $\varphi(Q)$, which is the intersection of $\varphi(U)\varphi(V)$ and $\varphi(A)\varphi(S)$, has ordinate $t^2$, and lies above $AB$. Since $\varphi$ sends horizontal chords into horizontal lines, the horizontal chord through $Q$ maps into the horizontal line through the image of $\varphi(Q)$, and the square maps into the half-plane above the line $AB$.

Analogous constructions, presented in the three other diagrams in Figure 4, show that the square maps into the half planes below $DC$, left of $AD$ and right of $BC$, together forcing the image to be in the intersection of these four half-planes, that is, $\Sigma$. □

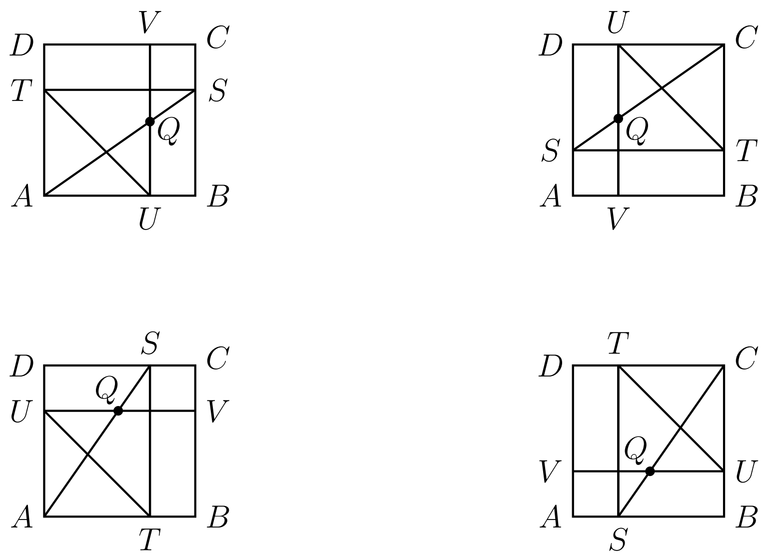


FIGURE 4. Constructions

## 4. Proof of Theorem 2.1

*Proof of (a).* There is a plane at infinity, written as $\{(\infty, y, z) : y, z \in \mathbb{R}^2\}$, that compactifies the $\mathcal{L}$ lines, and a projection $\pi : (x, y, z) \mapsto (\infty, y, z)$ of $N$ onto this plane. Any map $\varphi$ of $N$ to $N$ that maps $\mathcal{L}$ lines into $\mathcal{L}$ lines induces a map $\dot{\varphi}$ of this plane into itself. Any non-vertical line in the plane at infinity is of the form $\pi(m)$ for some $\mathcal{M}$ line $m$, and $\varphi(m)$ is also an $\mathcal{M}$ line. It follows that $\dot{\varphi}$ sends non-vertical lines into non-vertical lines, and hence $\dot{\varphi}$ is a globally defined projective map, that is, an affine map, by Theorem 3.2. One can compactify the $\mathcal{M}$ lines similarly and obtain another affine map $\ddot{\varphi}$ of another plane at infinity. Unravelling things, $\varphi$ must be affine. □

A similar proof works for the case where $U \subset N$.

Theorem 2.1 (where $U = N$) has a projective analogue, which apparently goes back to E. Bertini. The projective completion of $N$ is the flag manifold $\mathbb{F}^3(\mathbb{R})$ whose elements are full flags of subspaces of $\mathbb{R}^3$, i.e., ordered pairs $(V_1, V_2)$ of subspaces of $\mathbb{R}^3$ such that

$$\{0\} \subset V_1 \subset V_2 \subset \mathbb{R}^3$$

(all inclusions are proper, so $\dim(V_1) = 1$ and $\dim(V_2) = 2$). Suppose that $\varphi : \mathbb{F}^3(\mathbb{R}) \to \mathbb{F}^3(\mathbb{R})$ is a bijection. We write $\varphi(V_1, V_2)$ as $(\varphi_1(V_1, V_2),\ \varphi_2(V_1, V_2))$. If $\varphi_1(V_1, V_2)$ depends only on $V_1$, then $\varphi$ induces a map $\dot{\varphi}$ of $\mathbb{P}^2(\mathbb{R}^3)$; similarly if $\varphi_2(V_1, V_2)$ depends only on $V_2$, then $\varphi$ induces a map $\ddot{\varphi}$ of the set of planes in $\mathbb{R}^3$. If both these hypotheses hold, then the map $\dot{\varphi}$ of

$\mathbb{P}^2(\mathbb{R})$ preserves collinearity: indeed, three collinear points in $\mathbb{P}^2(\mathbb{R})$ correspond to three one-dimensional subspaces $V_1, V_1'$ and $V_1''$, of $\mathbb{P}^2(\mathbb{R})$, all contained in a common plane $V_2$, and by hypotheses $\varphi_1(V_1, V_2)$, $\varphi_1(V_1', V_2)$ and $\varphi_1(V_1'', V_2)$ are all subspaces of $\varphi_2(V_1, V_2)$. Conversely, any collinearity-preserving map of $\mathbb{P}^2(\mathbb{R})$ induces a map of $\mathbb{F}^3(\mathbb{R})$. From the affine theorem above, we obtain a corresponding theorem for $\mathbb{F}^3(\mathbb{R})$. Before we state this, let us agree that four points in $\mathbb{F}^3(\mathbb{R})$ are in general position if their one dimensional subspaces are in general position (no three lie in a plane) and their two dimensional subspaces are too (no three meet in a line).

**Theorem 4.1.** *Let $U$ be a connected open subset of $\mathbb{F}^3(\mathbb{R})$, and suppose that $\varphi$ is a map from $U$ into $\mathbb{F}^3(\mathbb{R})$ whose image contains four points in general position. Suppose also that $\varphi_1(V_1, V_2)$ depends only on $V_1$ and $\varphi_2(V_1, V_2)$ depends only on $V_2$. Then $\varphi$ is a projective transformation, i.e., $\varphi$ arises from the action of (a finite extension of)* $\mathrm{PGL}(3, \mathbb{R})$ *on* $\mathbb{F}^3(\mathbb{R})$.

In the case where $U$ is all $\mathbb{F}^3(\mathbb{R})$, then this is a special case of theorem of Tits [8] (also called the fundamental theorem of projective geometry). The point here is that $\mathbb{F}^3(\mathbb{R})$ may be identified with the group $G/P$, where $G = \mathrm{SL}(3, \mathbb{R})$ and $P$ is the subgroup of $G$ of lower triangular matrices. Indeed, $G$ acts on $\mathbb{R}^3$ and $P$ is the stabiliser of the flag $(\mathbb{R}\mathbf{e}_3, \mathbb{R}\mathbf{e}_3 + \mathbb{R}\mathbf{e}_2)$, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis of $\mathbb{R}^3$, and the map $g \mapsto (\mathbb{R}g\mathbf{e}_3, \mathbb{R}g\mathbf{e}_3 + g\mathbf{e}_2)$ identifies $G/P$ with $\mathbb{F}^3(\mathbb{R})$. Note that the stabilisers of the sets $\{(V_1, V_2) \in \mathbb{F}^3(\mathbb{R}) : V_1 = V\}$ and $\{(V_1, V_2) \in \mathbb{F}^3(\mathbb{R}) : V_2 = V\}$ are the parabolic subgroups $P_1$ and $P_2$:

$$P_1 = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \right\} \quad \text{and} \quad P_2 = \left\{ \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \right\}.$$

Maps $\varphi$ of $\mathbb{F}^3(\mathbb{R})$ into itself with the property that $\varphi_1(V_2, V_2)$ depends only on $V_1$ correspond to maps of $G/P$ that pass to maps of $G/P_1$, i.e., maps that preserve the fibration of $G/P$ induced by $P_1$ (the fibres are the sets $xP_1/P$), and maps $\varphi$ such that $\varphi_2(V_1, V_2)$ depends only on $V_2$ correspond to maps of $G/P$ that preserve the fibration of $G/P$ induced by $P_2$.

## 5. Final Remarks

Our original problem (or one of them) was to decide what kind of rigidity results might hold for non-smooth maps of stratified groups (or open subsets thereof). The second-named author makes the following conjecture, which holds in many examples. These include all the nilpotent groups that arise as the Iwasawa $N$ group of a classical group of real rank 2, where the subalgebras $\mathfrak{n}_1$ and $\mathfrak{n}_2$ correspond to the simple restricted root spaces; see [4].

**Conjecture 5.1.** *Suppose that* $\mathfrak{n}_1$ *and* $\mathfrak{n}_2$ *are subalgebras of a stratified algebra* $\mathfrak{n}$*, such that* $\mathfrak{n}^1 = \mathfrak{n}_1^1 \oplus \mathfrak{n}_2^1$*, and for all* $X$ *in* $\mathfrak{n}_1^1$*, there exists* $Y$ *in* $\mathfrak{n}_2^1$ *such that* $[X, Y] \neq 0$*, and vice versa. Let* $U$ *be an open subset of the corresponding group* $N$*, and let* $\varphi : U \to N$ *be a map that maps connected subsets of cosets of* $N_i$ *into cosets of* $N_i$ *when* $i = 1$ *and* $2$*. Then* $\varphi$ *is automatically smooth, and the set of these maps is finite-dimensional.*

Along the lines of the previous section, Tits extended the fundamental theorem of projective geometry to semisimple Lie groups (of rank at least two). For any field (or division ring) $k$, the maps $\varphi$ that preserve lines in $k^2$ are compositions of field (or ring) automorphisms with affine maps, and Tits' theorems are proved in this generality. At the affine level, these extensions include theorems such as a characterisation of maps of the plane preserving circles, a local version of which was proved by Carathéodory [2].

McCallum [4] has proved local versions of "the fundamental theorem of projective geometry" for all the classical groups, and extended these to non-discrete topological fields and division rings, and some other rings, such as the adèles.

There are a number of interesting related results about maps that preserve other geometric structures. As already mentioned, C. Carathéodory [2] studied maps of domains in the plane that send arcs of circles into arcs of circles and showed that these come from restricting the action of the conformal group, and *a fortiori* are all smooth. On the other hand, J. Jeffers [3] observes that there are bijections of the sphere that send great circles into great circles that are not even continuous.

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Fakultät für Mathematik der Universität Wien, Nordbergstrasse 15, 1090 Wien, Austria

School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia

DIPTEM, Universitá di Genova, Piazzale Kennedy — Padiglione D, 16129 Genova, Italy

School of Mathematics, University of Adelaide, Adelaide SA 5005, Australia

School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia