K–theory for convenient algebras

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1. Smooth spaces

In this chapter we study basic properties of the category of smooth spaces and natural topologies on smooth spaces.

1.1. Definition. A *smooth space* is a set X together with a set of curves $\mathcal{C}_X \subset X^{\mathbb{R}}$ and a set of functions $\mathcal{F}_X \subset \mathbb{R}^X$ such that
(1): For any $c \in \mathcal{C}_X$ and any $f \in \mathcal{F}_X$ we have $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$.
(2): The curves and functions determine each other in the following sense: If $c \in X^{\mathbb{R}}$ is such that $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for any $f \in \mathcal{F}_X$ then $c \in \mathcal{C}_X$ and if $f \in \mathbb{R}^X$ is such that $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for any $c \in \mathcal{C}_X$ then $f \in \mathcal{F}_X$.

A map $f : X \to Y$ between smooth spaces is called smooth iff it satisfies one of the following equivalent conditions:
(1): $f \circ c \in \mathcal{C}_Y$ for all $c \in \mathcal{C}_X$
(2): $\varphi \circ f \in \mathcal{F}_X$ for all $\varphi \in \mathcal{F}_Y$
(3): $\varphi \circ f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $\varphi \in \mathcal{F}_Y$ and all $c \in \mathcal{C}_X$

Let $\underline{C}^\infty$ denote the category of smooth spaces and smooth maps.

1.2. Smooth structures generated by a family of curves or functions. Let X be a set, $\mathcal{C} \subset X^{\mathbb{R}}$ an arbitrary set of curves. Then we can define a smooth structure on X as follows: Let $\mathcal{F}_X$ be the set of all $f \in \mathbb{R}^X$ such that $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $c \in \mathcal{C}$ and let $\mathcal{C}_X$ be the set of all $c \in X^{\mathbb{R}}$ such that $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $f \in \mathcal{F}_X$. Then one easily verifies that $(\mathcal{C}_X, \mathcal{F}_X)$ defines a smooth structure on X, called the structure generated by the family $\mathcal{C}$ of curves. Moreover one easily verifies that a map $f : X \to Y$ into an arbitrary smooth space Y is smooth for this structure if and only if $f \circ c \in \mathcal{C}_Y$ for all $c \in \mathcal{C}$.

Dually we define the smooth structure on X generated by a set of functions $\mathcal{F} \subset \mathbb{R}^X$. This structure has then the property that a map $f : Y \to X$ from an arbitrary smooth space Y to X is smooth for this structure if and only if $\varphi \circ f \in \mathcal{F}_Y$ for all $\varphi \in \mathcal{F}$.

1.3. Lemma. *The category $\underline{C}^\infty$ has initial and final structures with respect to the forgetful functor to the category of sets.*

Proof. We give an explicit description of the structures. Let X be a set, $g_\alpha : X \to X_\alpha$ a family of maps into smooth spaces X_α. One easily verifies that the smooth structure

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generated by the set of functions $\{f \circ g_\alpha : f \in \mathcal{F}_{X_\alpha}\}$ is initial. (It suffices to take f in a subset of $\mathcal{F}_{X_\alpha}$ which generates the smooth structure of X_α.) In particular a curve $c : \mathbb{R} \to X$ is smooth iff $g_\alpha \circ c$ is a smooth curve into X_α for all α.

On the other hand if $g_\alpha : X_\alpha \to X$ is a family of smooth maps then one shows that the structure generated by the set of curves $\{g_\alpha \circ c : c \in \mathcal{C}_{X_\alpha}\}$ is final. (Again it suffices to take c in a subset which generates the smooth structure.) In particular a function $f : X \to \mathbb{R}$ is smooth iff $f \circ g_\alpha$ is smooth for all α. □

1.4. Corollary. *The category of smooth spaces is complete and cocomplete, i.e. all categorical limits and colimits can be formed.*

Proof. It is a general result of category theory that limits (colimits) can be constructed by forming the limit (colimit) of the underlying sets and then putting the initial (final) smooth structure on them. □

1.5. Examples of smooth spaces. Let (E, E') be a dual pair, i.e. E is a real vector space and E' is a point separating linear subspace of the algebraic dual of E. Then on E we consider the smooth structure generated by E'. In particular on a locally convex space we consider the smooth structure generated by all continuous linear functionals.

Then it turns out that on any Banach space the smooth curves in this sense are exactly the usual smooth curves. Moreover it can be shown that the smooth functions between Banach spaces in the sense of 1.1 are exactly the usual smooth functions (c.f. [F–K, 4.3.16]). From this one easily deduces that for maps between finite dimensional smooth manifolds (or more generally smooth Banach manifolds) with the smooth structure given by the usual smooth curves and real valued functions definition 1.1 gives the usual notion of smoothness.

1.6. Smooth structures on sets of smooth maps. Let X and Y be smooth spaces, $C^\infty(X, Y)$ the set of all smooth functions from X to Y. Let $\mathcal{C}_{X,Y}$ be the set of all curves $c : \mathbb{R} \to C^\infty(X, Y)$ such that $\hat{c} : \mathbb{R} \times X \to Y$ is smooth where $\hat{c}$ is defined by $\hat{c}(t, x) := c(t)(x)$ and consider the smooth structure generated by this set of curves.

Then it turns out that a curve into $C^\infty(X, Y)$ is smooth for this structure if and only if it belongs to $\mathcal{C}_{X,Y}$. From this one deduces the following theorem:

1.7. Theorem. *The category of smooth spaces and smooth maps is cartesian closed, i.e. for any smooth spaces X, Y and Z there is a natural isomorphism (which is even a diffeomorphism):*

$$C^\infty(X, C^\infty(Y, Z)) \cong C^\infty(X \times Y, Z)$$

Proof. [F–K, 1.1.7 and 1.4.3] □

1.8. Corollary. *Let X, Y, and Z be smooth spaces. Then the following canonical mappings are smooth:*
$ev : C^\infty(X, Y) \times X \to Y,\ ev(f, x) := f(x)$
$ins : X \to C^\infty(Y, X \times Y),\ ins(x)(y) = (x, y)$
$comp : C^\infty(Y, Z) \times C^\infty(X, Y) \to C^\infty(X, Z),\ (g, f) \mapsto g \circ f$
$f_* : C^\infty(X, Y) \to C^\infty(X, Z),\ f_*(g) := f \circ g$ *where* $f : Y \to Z$ *is a smooth map*
$g^* : C^\infty(Z, Y) \to C^\infty(X, Y),\ g^*(f) = f \circ g$ *where* $g : X \to Z$ *is a smooth map*

Proof. ev and ins are associated to identity maps via cartesian closedness, the other three maps can be built up from suitably chosen evaluation and insertion maps. □

1.9. Natural topologies on smooth spaces. On a smooth space X there are two obvious natural topologies: First there is the final topology with respect to all smooth curves in X, which we denote by $\tau_\mathcal{C}$ and second there is the initial topology with respect to all smooth real valued functions on X, which will be denoted by $\tau_\mathcal{F}$. For locally convex vector spaces the $\tau_{\mathcal{C}}$ topology coincides with the Mackey closure topology ([F–K, 2.2]) and is also called c^∞–topology. For Fréchet spaces this topology coincides with the given one ([F–K, 6.1.4]).

By definition the topology $\tau_{\mathcal{C}}$ is always finer than $\tau_{\mathcal{F}}$. A smooth space on which the two natural topologies coincide will be called *balanced*. We will use topological notions mainly for balanced spaces.

A smooth space X is called *Hausdorff* iff the smooth real valued functions on X are point separating, i.e. iff the $\tau_{\mathcal{F}}$ topology (and thus also the $\tau_{\mathcal{C}}$ topology) on X is Hausdorff.

A smooth space is called a *base space* iff it is balanced compact and Hausdorff.

1.10. Examples of balanced smooth spaces. It can be shown that a Banach space is balanced if and only if there is a nonzero smooth real valued function on it which has bounded support ([Bonic–Frampton, 1966]). Moreover any nuclear Fréchet space as well as any function space $C^\infty(M, E)$ where M is a finite dimensional smooth manifold and E is a nuclear Fréchet space is balanced ([Michor, 1983], [F–K, 4.4.44]).

For finite dimensional smooth manifolds balancedness follows immediately from the existence of smooth partitions of unity, which also shows that the $\tau_{\mathcal{F}}$ topology coincides with the usual one.

1.11. Lemma. *Let X, Y be smooth spaces, $U \subset X$ a $\tau_{\mathcal{C}}$–open subset, $f : X \to Y$ a function. Then the following conditions are equivalent:*
(1): For any smooth curve $c : \mathbb{R} \to X$ with $c(\mathbb{R}) \subset U$ the curve $f \circ c$ is smooth.
(2): For any smooth curve $c : \mathbb{R} \to X$ the curve $f \circ c : c^{-1}(U) \to Y$ is smooth.

Proof. Obviously (2) implies (1).
By composing with smooth real valued functions on Y we may without loss of generality assume that $Y = \mathbb{R}$. Let $c : \mathbb{R} \to X$ be an arbitrary smooth curve and let $t_0 \in \mathbb{R}$ be such that $c(t_0) \in U$. As U is $\tau_{\mathcal{C}}$–open there is a closed interval, say $V := [t_0 - \delta, t_0 + \delta]$ such that $c(V) \subset U$. Now let $h \in C^\infty(\mathbb{R}, \mathbb{R})$ be a smooth function such that $h(\mathbb{R}) \subset V$ and $h(t) = t$ locally around t_0. Then by (1) $f \circ c \circ h$ is smooth and hence $f \circ c$ is smooth in t_0. $\square$

1.12. Lemma. *Let X and Y be smooth spaces, $f : X \to Y$ a map.*
(1): If f is smooth then it is continuous for the $\tau_{\mathcal{C}}$ topologies as well as for the $\tau_{\mathcal{F}}$ topologies.
(2): If $(U_\alpha)_{\alpha \in A}$ is a $\tau_{\mathcal{C}}$–open covering of X such that for any α the restriction of f to U_α is smooth then f is smooth.

Proof. (1) follows immediately from the definitions.
(2): Composing with smooth real valued functions we may assume that $Y = \mathbb{R}$. Now let $c : \mathbb{R} \to X$ be a smooth curve. By 1.11 for any α the function $f \circ c \restriction c^{-1}(U_\alpha) : c^{-1}(U_\alpha) \to \mathbb{R}$ is smooth. By assumption $(c^{-1}(U_\alpha))_{\alpha \in A}$ is an open covering of $\mathbb{R}$ and thus $f \circ c$ is smooth. $\square$

1.13. Lemma. *For any smooth space X the $\tau_{\mathcal{F}}$ topology of X is smoothly completely regular, i.e. if $A \subset X$ is $\tau_{\mathcal{F}}$–closed and x is a point of $X \setminus A$ then there is a smooth function $f : X \to [0, 1]$ with $f(x) = 1$ and $f(A) = 0$. The function f can even be chosen such that its support (the $\tau_{\mathcal{F}}$–closure of the set of all points where f is nonzero) is contained in $X \setminus A$. (On $[0, 1]$ we consider the smooth structure induced by the embedding into $\mathbb{R}$.)*

Proof. As $X \setminus A$ is $\tau_{\mathcal{F}}$–open there is a smooth function $g : X \to \mathbb{R}$ such that $x \in g^{-1}((0, 1)) \subseteq X \setminus A$. Now let $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ be a smooth function such that φ has support in $(0, 1)$ and $\varphi(g(x)) = 1$. Then $f := \varphi \circ g$ is obviously smooth and has the desired property. $\square$

1.14. Lemma. *Let X be a smooth space, $U \subset X$.*
(1): If U is $\tau_{\mathcal{C}}$–open then the trace topology of the $\tau_{\mathcal{C}}$ topology of X equals the $\tau_{\mathcal{C}}$ topology on U.
(2): If U is $\tau_{\mathcal{F}}$–open then the trace topology of the $\tau_{\mathcal{F}}$ topology of X equals the $\tau_{\mathcal{F}}$ topology on U.

Proof. (1): The inclusion $U \hookrightarrow X$ is by definition smooth and thus continuous, so the trace topology is coarser than the $\tau_{\mathcal{C}}$ topology. To prove the converse let $V \subset U$ be $\tau_{\mathcal{C}}$–open. Clearly it suffices to show that V is $\tau_{\mathcal{C}}$–open in X. So let $c : \mathbb{R} \to X$ be a smooth curve and let $t \in \mathbb{R}$ be such that $c(t) \in V$. As U is $\tau_{\mathcal{C}}$–open in X there is a $\delta > 0$ such that $[t - 2\delta, t + 2\delta] \subset c^{-1}(U)$. Now let $h \in C^\infty(\mathbb{R}, \mathbb{R})$ be a smooth function such that $h(\mathbb{R}) \subset [t - 2\delta, t + 2\delta]$ and $h(s) = s$ for $s \in [t - \delta, t + \delta]$. Then $c \circ h$ is a smooth curve into U and $c \circ h(t) \in V$ and thus there exists an open neighborhood of t in $\mathbb{R}$ which is contained in $(c \circ h)^{-1}(V)$. By construction the intersection of this open neighborhood with $(t - \delta, t + \delta)$ is contained in $c^{-1}(V)$ and thus V is $\tau_{\mathcal{C}}$–open in X.
(2): As in (1) one concludes that the trace topology is coarser. Conversely let V be $\tau_{\mathcal{F}}$–open in U and take $x \in V$. By 1.13 we can find functions $f \in C^\infty(X, \mathbb{R})$ and $g \in C^\infty(U, \mathbb{R})$ which both have values in $[0, 1]$ such that $f(x) = g(x) = 1$, the support of f is contained in U and g is identically zero on $U \setminus V$. Then the function $f \cdot g$ is well defined on X since f is zero locally around points in which g is not defined. Moreover it is smooth by 1.12(2) since it is clearly smooth on the open subsets U and $X \setminus supp(f)$. Clearly we have $x \in (f \cdot g)^{-1}(0, \infty) \subset V$ and thus V is $\tau_{\mathcal{F}}$–open in X. □

1.15. Lemma. *Let X and Y be smooth spaces, $p : X \to Y$ a final morphism in the category of smooth spaces, i.e. p is a smooth map and a map $f : Y \to Z$ to an arbitrary smooth space Z is smooth if and only if $f \circ p : X \to Z$ is smooth. Then for any $\tau_{\mathcal{F}}$–open subset U of Y the restriction $p : p^{-1}(U) \to U$ is a final morphism.*

Proof. Clearly the restriction is smooth so let $f : U \to Z$ be a map into a smooth space Z such that $f \circ p : p^{-1}(U) \to Z$ is smooth. Composing with all smooth real valued functions on Z we see that it suffices to consider the case $Z = \mathbb{R}$. Take an arbitrary point $x \in U$. By 1.13 there is a smooth function $g : Y \to [0, 1]$ with support in U such that $g(x) = 1$. Composing with a smooth function $\varphi \in C^\infty(\mathbb{R}, [0, 1])$ which satisfies $\varphi(t) = 0$ for $t \leq 0$ and $\varphi(t) = 1$ for $t \geq 1/2$ we see that without loss of generality we may assume that there is a $\tau_{\mathcal{F}}$–open neighborhood V of x in U such that g is identically one on V.

Now consider the function $h : Y \to \mathbb{R}$ defined by $h(y) = g(y) \cdot f(y)$. Then h is well defined as $g(y) = 0$ if $f(y)$ is not defined. Moreover by 1.12(2) $h \circ p : X \to \mathbb{R}$ is smooth since it is obviously smooth on the $\tau_{\mathcal{F}}$–open sets $p^{-1}(U)$ and $p^{-1}(Y \setminus supp(g))$ and thus by assumption h is smooth. But on V the two maps h and f coincide and thus the restriction of f to V is smooth and so again by 1.12(2) f is smooth. □

1.16. Proposition. *If X and Y are base spaces then $X \times Y$ is a base space and the topology on $X \times Y$ induced by the smooth structure equals the product topology.*

Proof. Let us first show that $X \times \mathbb{R}$ is a balanced and that the topology induced by the smooth structure equals the product topology. By definition the product topology is coarser than the $\tau_{\mathcal{F}}$ topology which is in turn coarser than the $\tau_{\mathcal{C}}$ topology. Thus it suffices to show that the product topology is finer than the $\tau_{\mathcal{C}}$ topology. So let $U \subset X \times \mathbb{R}$ be a $\tau_{\mathcal{C}}$ open subset and let (x_0, t_0) be a point in U. As for any smooth curve $c : \mathbb{R} \to \mathbb{R}$ the map $t \mapsto (x_0, c(t))$ is a smooth curve into $X \times \mathbb{R}$ the set of all $t \in \mathbb{R}$ such that $(x_0, t) \in U$ is open in $\mathbb{R}$ and thus contains a compact neighborhood V of t_0.

Now let $W \subset X$ be the set of all x such that $\{x\} \times V \subset U$. We want to show that W is open in X. If not then there is a smooth curve $c : \mathbb{R} \to X$ such that $c^{-1}(W)$ is not open in $\mathbb{R}$. Consider the smooth function $\varphi : \mathbb{R}^2 \to X \times \mathbb{R}$ defined by $(t, s) \mapsto (c(t), s)$. As φ is smooth and thus continuous for the $\tau_{\mathcal{C}}$ topology the set $\varphi^{-1}(U)$ is open in $\mathbb{R}^2$. (Recall that the topology on $\mathbb{R}^2$ induced by the smooth structure is the usual one.) Now for any $t \in c^{-1}(W)$ the compact set $t \times V$ is contained in $\varphi^{-1}(U)$ and thus there is an open neighborhood W' of t in $\mathbb{R}$ such that $W' \times V \subset \varphi^{-1}(U)$. Then $c(W') \subset W$ and thus t is an inner point of $c^{-1}(W)$ which is a contradiction.

Now from the fact that the topology of $X \times \mathbb{R}$ is the product topology one immediately concludes that if U is an open subset of $X \times \mathbb{R}$ and $K \subset X$ is compact and there is a $t \in \mathbb{R}$ such that $K \times \{t\} \subset U$ then there is an open neighborhood V of t such that $K \times V \subset U$.

So let us turn to the general case: For arbitrary base spaces X and Y one concludes as above that it suffices to show that the product topology on $X \times Y$ is finer than the $\tau_{\mathcal{C}}$ topology. Let $U \subset X \times Y$ be $\tau_{\mathcal{C}}$ open and let (x_0, y_0) be a point in U. As above one shows that the set V' of all $x \in X$ such that $(x, y_0) \in U$ is open in X. As X is balanced there is a smooth function $f \in C^\infty(X, [0,1])$ such that $f(x_0) = 1$ and $f^{-1}((0,1]) \subset V'$. Then $V := f^{-1}([1/2, 1])$ is a closed and thus compact neighborhood of x_0 in X. Now let W be the set of all $y \in Y$ such that $V \times \{y\} \subset U$. If W is not open in Y then we get a contradiction as above by considering the smooth function $\varphi : X \times \mathbb{R} \to X \times Y$ defined by $\varphi(x,t) := (x, c(t))$, where c is a smooth curve into Y such that $c^{-1}(W)$ is not open in $\mathbb{R}$. □

1.17. Lemma. *Any base space X is smoothly normal, i.e. if A and B are closed subsets of X then there is a smooth function $f \in C^\infty(X, \mathbb{R})$ such that f is identically one on A and identically zero on B.*

Proof. By 1.13 for any $x \in A$ there is a smooth function $f_x : X \to [0,1]$ such that $f_x(x) = 1$ which vanishes identically on B. The sets $f_x^{-1}((0,1])$ form an open covering of A and since A is closed and thus compact in the Hausdorff space X there are points $x_1, \ldots, x_n \in X$ such that the sets $f_{x_i}^{-1}((0,1])$ cover A. Then $f := \sum_{i=1}^n f_{x_i}$ is a smooth function $X \to \mathbb{R}$ which vanishes on B and is strictly positive on A. As f is smooth and thus continuous it attains a minimum value $a > 0$ on A. Now let $\varphi \in C^\infty(\mathbb{R}, [0,1])$ be a smooth function such that $\varphi(0) = 0$ and $\varphi(t) = 1$ for all $t \geq a$. Then $\varphi \circ f$ has the desired property. □

1.18. Theorem. *Let X, Y and Z be base spaces, $\varphi : Z \to X$ an arbitrary smooth map and $\psi : Z \to Y$ an injective smooth map such that for any smooth real valued function $f \in C^\infty(X, \mathbb{R})$ there is a $g \in C^\infty(Y, \mathbb{R})$ with $g \circ \psi = f \circ \varphi$. Consider the push out:*

$$\begin{tikzcd} Z \arrow[r, "\psi"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi_*\varphi"] \\ X \arrow[r, "\varphi_*\psi"] & X \cup_Z Y \end{tikzcd}$$

Then $X \cup_Z Y$ is a base space and the topology induced by the smooth structure equals the push out topology.

Proof. Let us first show that the $\tau_{\mathcal{F}}$–topology on $X \cup_Z Y$ is Hausdorff: So let a and b be distinct points of $X \cup_Z Y$. By definition as a set $X \cup_Z Y$ equals the quotient of the disjoint union of X and Y by the equivalence relation generated by $\varphi(z) \sim \psi(z)$ for all $z \in Z$. Thus $X \cup_Z Y$ is the disjoint union of $\varphi_*\psi(X)$ and $\psi_*\varphi(Y \setminus \psi(Z))$ and both maps are injective on the indicated subsets. If both points lie in $\varphi_*\psi(X)$ then there is a smooth function $f \in C^\infty(X, \mathbb{R})$ which separates the points $(\psi_*\varphi)^{-1}(a)$ and $(\psi_*\varphi)^{-1}(b)$. Then $f \circ \varphi : Z \to \mathbb{R}$ is smooth and thus by assumption there is a smooth map $g \in C^\infty(Y, \mathbb{R})$ such that $g \circ \psi = f \circ \varphi$ and by the universal property of the pushout these two maps define a smooth map $X \cup_Z Y \to \mathbb{R}$ which obviously separates a and b.

So let us assume that a lies in $\psi_*\varphi(Y \setminus \psi(Z))$. As ψ is continuous the set $\psi(Z)$ is compact and thus closed in Y. By 1.13 there is an $f \in C^\infty(Y, [0,1])$ such that $f((\psi_*\varphi)^{-1}(a)) = 1$ which vanishes on $\psi(Z)$. If b is also in $\psi_*\varphi(Y \setminus \psi(Z))$ then f can be chosen such that it vanishes on $(\psi_*\varphi)^{-1}(b)$, too. In both cases f and the zero function on X define a smooth function on $X \cup_Z Y$ which separates a and b.

By definition the push out topology is finer than the $\tau_{\mathcal{C}}$ topology which in turn is finer than the $\tau_{\mathcal{F}}$ topology, so the identity is continuous from the push out topology to the $\tau_{\mathcal{F}}$–topology and since the push out topology is compact and the $\tau_{\mathcal{F}}$–topology is Hausdorff it is a homeomorphism. □

1.19. Cells. Put $E^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ with the smooth structure induced by the inclusion $E^n \hookrightarrow \mathbb{R}^n$. By D^n we denote the set of points of norm less then one and by S^{n-1} the set of points of norm one. Let us consider the smooth structures induced on D^n and S^{n-1} by the inclusions into E^n. By definition a curve into one of these spaces is smooth if and only if it is smooth as a curve into $\mathbb{R}^n$ and has values in the space. Thus in both cases we get the 'usual' smooth curves and since the smooth curves determine the smooth structure these two spaces have their usual smooth structures (S^{n-1} as a smooth manifold and D^n as an open subset of R^n).

1.20. Proposition. *For any n the space E^n is balanced and the topology induced by the smooth structure is the trace topology.*

Proof. By definition the inclusion $E^n \hookrightarrow \mathbb{R}^n$ is smooth and thus continuous for the topologies induced by the smooth structure, so the trace topology is coarser than both of them. Thus it suffices to show that the $\tau_{\mathcal{C}}$ topology is coarser than the trace topology. So let $U \subset E^n$ be $\tau_{\mathcal{C}}$ open. Obviously $U \cap D^n$ is $\tau_{\mathcal{C}}$ open in D^n and thus open in $\mathbb{R}^n$ since D^n is $\tau_{\mathcal{C}}$ open in R^n (c.f. 1.14). On the other hand $U \cap S^{n-1}$ is open in S^{n-1}. So for $x \in U \cap S^{n-1}$ there is an $\varepsilon > 0$ such that the intersection of the ball with radius ε around x with S^{n-1} lies in U. We claim that ε can be chosen such that even the intersection of the ball with E^n lies in U. If not, then for each $n \in \mathbb{N}$ we can choose a point $a_n \in E_n \setminus U$ such that $\|x - a_n\| < 1/n^n$ for any n. By the special curve lemma ([F–K, 2.3.4]) there is a smooth curve $c : \mathbb{R} \to \mathbb{R}^n$ such that $c(t) = x$ for $t \leq 0$, $c(1/2^n) = a_n$ for all n and $c(t) = a_0$ for $t \geq 1$ such that the image of c is the polygon through the points a_n. Thus $c(\mathbb{R}) \subset E^n$ and so c is a smooth curve into E^n. But this leads to a contradiction since $c^{-1}(U)$ must be open and contain zero but does not contain the points $1/2^n$. $\square$

1.21. Attaching cells. Let X be a smooth space, $\varphi : S^{n-1} \to X$ a smooth map. Then we define the smooth space $E^n \cup_\varphi X$ to be the pushout

$$\begin{tikzcd} S^{n-1} \arrow[r, "incl"] \arrow[d, "\varphi"'] & E^n \arrow[d, "j"] \\ X \arrow[r, "i"] & E^n \cup_\varphi X \end{tikzcd}$$

and call it the space obtained by attaching an n–cell to X along φ. From the description of colimits in 1.4 we see that the underlying set of $E^n \cup_\varphi X$ is the quotient of the disjoint union of X and E^n by the equivalence relation generated by $y \sim \varphi(y)$, while the smooth structure is the final one with respect to the canonical maps $i : X \to E^n \cup_\varphi X$ and $j : E^n \to E^n \cup_\varphi X$. Thus a map $f : E^n \cup_\varphi X \to Y$ into an arbitrary smooth space Y is smooth if and only if $f \circ i$ and $f \circ j$ are smooth.

If X is a base space then so is $E^n \cup_\varphi X$ by 1.18 since clearly any real valued smooth function on S^n has a smooth extension to E^n.

1.22. Definition. We define smooth cell complexes inductively:
A zero dimensional smooth cell complex is a finite discrete set, where discrete means that the only smooth curves are the constant ones.
A n–dimensional smooth cell complex is a smooth space which is obtained by attaching finitely many n–cells to an $n-1$–dimensional smooth cell complex along smooth maps.

1.23. Theorem. *(1): Any finite dimensional smooth cell complex is a base space.*
(2): With its natural topology any finite dimensional smooth cell complex is a finite CW–complex

Proof. This follows immediately by induction using 1.18 as finite discrete sets obviously have these properties. $\square$

1.24 Pointed smooth spaces. Let $\underline{C_0^\infty}$ denote the category of pointed smooth spaces and base point preserving smooth maps. We denote the space of all base point preserving smooth maps between two pointed smooth spaces X and Y by $C_0^\infty(X,Y)$. On this space we put the initial smooth structure with respect to the inclusion into $C^\infty(X,Y)$. Moreover the space $C_0^\infty(X,Y)$ has a natural base point, namely the map which sends the whole space X into y_0.

One easily checks that the product in $\underline{C_0^\infty}$ is the same as the product in the category of smooth spaces and the base point of a product is the point which has as 'coordinates' the base points of the factors. The coproduct in this category, however, differs from the one in $\underline{C^\infty}$. We denote the coproduct of two pointed spaces (X,x_0) and (Y,y_0) in $\underline{C_0^\infty}$ by $X\vee Y$. One easily checks that this coproduct is given as the push out (in $\underline{C^\infty}$):

$$\begin{array}{ccc} \{x_0\}\cup\{y_0\} & \longrightarrow & X\cup Y \\ \downarrow & & \downarrow \\ pt & \longrightarrow & X\vee Y \end{array}$$

where $X\cup Y$ denotes the coproduct (disjoint union) of X and Y in $\underline{C^\infty}$. The space $X\vee Y$ is also called the *wedge* of X and Y.

Let us now assume that X and Y are pointed base spaces. Then one easily shows that $X\cup Y$ is a base space and thus from 1.18 one immediately concludes that $X\vee Y$ is a base space.

1.25. The category of pointed smooth spaces is not cartesian closed in the sense of 1.7 since if $f: X\to C_0^\infty(Y,Z)$ is a base point preserving smooth map then the smooth map $\hat f: X\times Y\to Z$ which is associated to f via cartesian closedness of $\underline{C^\infty}$ must satisfy $\hat f(x_0,y)=z_0$ for all $y\in Y$ and thus we cannot get all base point preserving maps in this way. But it is possible to get an isomorphism of this type by replacing the product by another functor denoted by $\wedge$ and called the *smash product*.

In order to construct $X\wedge Y$ for pointed smooth spaces X and Y we proceed as follows: The smooth maps $x\mapsto(x,y_0)$ and $y\mapsto(x_0,y)$ define a smooth map from $X\cup Y$ to $X\times Y$ and together with the map $pt\mapsto(x_0,y_0)$ this defines an injective smooth map $i: X\vee Y\to X\times Y$. Now we define $X\wedge Y$ to be the following push out in $\underline{C^\infty}$:

$$\begin{array}{ccc} X\vee Y & \xrightarrow{i} & X\times Y \\ \downarrow & & \downarrow \\ pt & \longrightarrow & X\wedge Y \end{array}$$

From 1.18 it follows immediately that if X and Y are pointed base spaces then $X\wedge Y$ is a pointed base space. Functoriality of the wedge product follows immediately from the fact that the product is functorial.

1.26. Theorem. *For any pointed smooth spaces X, Y and Z there is a natural isomorphism:* $C_0^\infty(X,C_0^\infty(Y,Z))\cong C_0^\infty(X\wedge Y,Z)$

Proof. Let $f: X\to C_0^\infty(Y,Z)$ be a base point preserving smooth map. Then by definition f is a smooth map into $C^\infty(Y,Z)$ which has values in $C_0^\infty(Y,Z)$. By cartesian closedness the associated map $\hat f: X\times Y\to Z$ defined by $\hat f(x,y):=f(x)(y)$ is smooth. Now as f is base point preserving we have $\hat f(x_0,y)=z_0$ and since f has values in the space of base point preserving maps we have $\hat f(x,y_0)=z_0$. Thus $\hat f\circ i$ is the constant map z_0 and $\hat f$ induces a smooth map $X\wedge Y\to Z$.

Conversely if $g : X \wedge Y \to Z$ is a smooth base point preserving map then composing it with the natural map $X \times Y \to X \wedge Y$ we get a smooth map $g' : X \times Y \to Z$. By cartesian closedness the associated map $\check{g}' : X \to C^\infty(Y,Z)$ is smooth and from the fact that g' sends $X \vee Y$ into z_0 one immediately concludes that $\check{g}'$ has values in the space of base point preserving maps and is base point preserving. $\square$

1.27. Definition. Let $I = [0,1]$ denote the unit interval in $\mathbb{R}$ with the initial smooth structure. Let X and Y be smooth spaces, $f, g : X \to Y$ smooth functions. f and g are called smoothly homotopic iff there is a smooth map $H : X \times I \to Y$ such that $H \circ ins_0 = f$ and $H \circ ins_1 = g$ where for $t \in I$ we define $ins_t : X \to X \times I$ as $ins_t(x) := (x,t)$. Being smoothly homotopic is an equivalence relation since smooth homotopies can be pieced together as follows: Let $\varphi \in C^\infty(\mathbb{R}, [0,1])$ be an increasing function such that $\varphi(t) = 0$ for all $t \le 0$, $\varphi(t) = 1/2$ locally around $1/2$ and $\varphi(t) = 1$ for all $t \ge 1$. Then if H_1 and H_2 are smooth homotopies such that $H_1 \circ ins_1 = H_2 \circ ins_0$ we can define a smooth homotopy between $H_1 \circ ins_0$ and $H_2 \circ ins_1$ by $H(x,t) = H_1(x, 2\varphi(t))$ for $t \le 1/2$ and $H_2(x, 2\varphi(t) - 1)$ for $t \ge 1/2$.

By $[X,Y]$ we denote the set of homotopy classes of smooth maps from X to Y. Obviously $[\quad,\quad]$ is a bifunctor from the category of smooth spaces to the category of sets which is contravariant in the first and covariant in the second variable.

Now suppose that X and Y are pointed, i.e. in each space there is a distinguished base point. Then we denote by $[X,Y]_0$ the set of homotopy classes of base point preserving maps. (Here homotopic means homotopic through base point preserving maps.)

The spaces X and Y are called smoothly homotopy equivalent iff there are smooth maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is smoothly homotopy equivalent to Id_X and $f \circ g$ is smoothly homotopic to Id_Y.

1.28. Proposition. *For smooth spaces X and Y the following is equivalent:*
(1): X and Y are smoothly homotopy equivalent.
(2): There is a natural equivalence of the functors $[X,\quad]$ and $[Y,\quad]$.
(3): There is a natural equivalence of the functors $[\quad,X]$ and $[\quad,Y]$.

Proof. We only show that (1) and (2) are equivalent, the proof that (3) is equivalent to (1) is similar.

Let X and Y be smoothly homotopy equivalent via maps $f : X \to Y$ and $g : Y \to X$. Then obviously the maps $g^* : [X,Z] \to [Y,Z]$ constitute a natural equivalence with inverses $f^* : [Y,Z] \to [X,Z]$.

On the other hand if φ is a natural transformation between the functors $[X,\quad]$ and $[Y,\quad]$ such that each $\varphi_Z : [X,Z] \to [Y,Z]$ is bijective then one easily shows that any map in the class $\varphi_Y^{-1}([Id_Y]) \in [X,Y]$ is a homotopy equivalence with homotopy inverse any map in the class $\varphi_X([Id_X]) \in [Y,X]$. $\square$

Clearly the obvious analog of this proposition for the category of pointed smooth spaces also holds (with the same proof).

2. Classifying spaces of smooth groups

2.1. Definition. A smooth group is a smooth space G together with two smooth maps $\mu : G \times G \to G$ and $\nu : G \to G$ such that G is a group with multiplication μ and inversion ν.

2.2. Examples. (1): Any finite dimensional Lie–group is a smooth group.
(2): Let X be an arbitrary smooth space, $\mathrm{Diff}(X)$ the subset of all maps $f \in C^\infty(X,X)$ which have a smooth inverse. On $\mathrm{Diff}(X)$ put the initial smooth structure with respect to

the maps $i, j : \mathrm{Diff}(X) \to C^\infty(X, X)$, where i is the inclusion and $j(f) = f^{-1}$. We claim that with this smooth structure $\mathrm{Diff}(X)$ is a smooth group. The inversion map ν is obviously smooth as $i \circ \nu = j$ and $j \circ \nu = i$. For the multiplication we have $i \circ \mu = comp \circ (i \times i)$ where $comp$ is the composition map on $C^\infty(X, X)$ which is smooth by cartesian closedness (1.8), and $j \circ \mu = comp \circ \varphi \circ (j \times j)$, where φ is the map $(f, g) \mapsto (g, f)$ which is obviously smooth. Thus the multiplication is smooth, too.
(3): In chapter 4 we will see that for a large class of locally convex algebras the set of all invertible elements with a smooth structure like the one in (2) forms a smooth group.

2.3. Smooth principal bundles. Let X be a smooth space, G a smooth group. A smooth principal bundle with group G over X is a smooth map $p : P \to X$ from a smooth space P to X such that there is a $\tau_{\mathcal{F}}$–open covering (U_α) of X and there are diffeomorphisms $\psi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times G$ with $pr_1 \circ \psi_\alpha = p$, such that for any α, β with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ the diffeomorphism $\psi_\alpha \circ \psi_\beta^{-1} : U_{\alpha\beta} \times G \to U_{\alpha\beta} \times G$ is given as $(x, g) \mapsto (x, \psi_{\alpha\beta}(x) \cdot g)$, where $\psi_{\alpha\beta} : U_{\alpha\beta} \to G$ is a smooth map.

A smooth principal bundle $p : P \to X$ is called trivial iff P is isomorphic to $X \times G$.

If $p : P \to X$ is a smooth principal bundle with group G over X we define a right action of G on P by $\psi_\alpha((\psi_\alpha^{-1}(x, g)) \cdot g') = (x, g \cdot g')$. Clearly this is well defined. Moreover it is obviously smooth as a map $p^{-1}(U_\alpha) \times G \to P$ and since $(p^{-1}(U_\alpha) \times G)_{\alpha \in A}$ is a $\tau_{\mathcal{F}}$ open covering of $P \times G$ the action is smooth by 1.12.

We call this action the principal action of G on P.

We can also define a map which is somehow an inverse to the principal action as follows: Let $P \times_X P$ be the fibered product (pullback). As a set this is given by $P \times_X P = \{(u, v) \in P \times P : p(u) = p(v)\}$ and the smooth structure is the initial one with respect to the two canonical projections to P. Obviously we can define a map $\tau : P \times_X P \to G$ implicitly by $v = u \cdot \tau(u, v)$. From the pullback we get a canonical smooth map $\pi : P \times_X P \to X$. As a set $\pi^{-1}(U_\alpha)$ equals $p^{-1}(U_\alpha) \times_{U_\alpha} p^{-1}(U_\alpha)$ and using the universal properties of the pullbacks one easily sees that this equality also holds for the smooth structures. But on $p^{-1}(U_\alpha) \times_{U_\alpha} p^{-1}(U_\alpha)$ the map τ is given by $\tau(u, v) = \mu(\nu \circ pr_2 \circ \psi_\alpha(u), pr_2 \circ \psi_\alpha(v))$, where μ and ν are the multiplication and inversion maps of G, and thus is smooth. As the sets $\pi^{-1}(U_\alpha)$ form a $\tau_{\mathcal{F}}$ open covering of $P \times_X P$ the map τ is smooth.

2.4. Lemma. *Let $p : P \to X$ be a smooth map, G a smooth group. Suppose we have given:*
(1): A smooth free right action $\rho : P \times G \to P$ which is fiber respecting, i.e. $p(\rho(u, g)) = p(u)$ for all u and g.
(2): A smooth map $\tau : P \times_X P \to G$ such that $\rho(u, \tau(u, v)) = v$ for all u, v with $p(u) = p(v)$.
(3): A $\tau_{\mathcal{F}}$–open covering U_α of X and smooth maps $\sigma_\alpha : U_\alpha \to p^{-1}(U_\alpha)$ such that $p \circ \sigma_\alpha = Id_{U_\alpha}$.
Then $p : P \to X$ is a smooth principal bundle with group G.

Proof. Let $i_\alpha : p^{-1}(U_\alpha) \hookrightarrow P$ be the inclusion. The maps $\sigma_\alpha \circ p$ and i_α define a smooth map $j_\alpha : p^{-1}(U_\alpha) \to P \times_X P$ and we define $\psi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times G$ by $\psi_\alpha(u) := (p(u), \tau(j_\alpha(u)))$ which is obviously smooth. Moreover using the freeness of the action ρ one easily shows that the smooth map $(x, g) \mapsto \rho(\sigma_\alpha(x), g)$ is inverse to ψ_α and thus ψ_α is a diffeomorphism. Now suppose that α and β are such that $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$. Then $\psi_\alpha \circ \psi_\beta^{-1}$ is given by $(x, g) \mapsto (x, \tau(\sigma_\beta(x), \sigma_\alpha(x)) \cdot g)$ and thus has the required form. $\square$

2.5. Definition. A smooth space X is called smoothly paracompact iff it is Hausdorff and every $\tau_{\mathcal{F}}$–open covering $(U_\alpha)_{\alpha \in A}$ of X has a subordinate smooth partition of unity, i.e. there is a family of smooth functions $(f_\beta : X \to [0, 1])_{\beta \in B}$ such that:
(1): $(supp(f_\beta))_{\beta \in B}$ is a locally finite covering of X.
(2): For any $\beta \in B$ there is an $\alpha \in A$ such that $supp(f_\beta) \subset U_\alpha$.
(3): For any $x \in X$ we have $\sum_{\beta \in B} f_\beta(x) = 1$.
As before $supp(f)$ is the $\tau_{\mathcal{F}}$ closure of the set of all points where f is nonzero.

It is well known that any finite dimensional smooth manifold satisfies this condition. Moreover one easily sees that any Hausdorff smooth space which is compact for the $\tau_{\mathcal{F}}$–topology is smoothly paracompact. In particular any base space is smoothly paracompact. I do not know whether paracompactness (in the topological sense) of the $\tau_{\mathcal{F}}$–topology of a Hausdorff smooth space implies that the space is smoothly paracompact.

2.6. Lemma (Pullbacks). *Let $p : P \to Y$ be a smooth principal bundle with group G, X a smooth space.*
(1): For a smooth map $f : X \to Y$ consider the pullback (in the category of smooth spaces)

$$\begin{array}{ccc} f^*P & \xrightarrow{p^*f} & P \\ {\scriptstyle f^*p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y \end{array}$$

Then $f^*p : f^*P \to X$ is a smooth principal bundle with group G called the pullback of P.
(2): Let $p' : P' \to X$ be a smooth principal bundle with group G and let $F : P' \to P$ be a smooth map which is equivariant with respect to the principal G actions. Then there is a smooth map $f : X \to Y$ such that we have a commutative diagram:

$$\begin{array}{ccc} P' & \xrightarrow{F} & P \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y \end{array}$$

and P' is isomorphic to f^*P.

Proof. (1): As a set f^*P is given as $\{(x,z) \in X \times P : f(x) = p(z)\}$. Now let (U_α, ψ_α) be an atlas for P. As f is $\tau_{\mathcal{F}}$ continuous the sets $\bar{U}_\alpha := f^{-1}(U_\alpha)$ form a $\tau_{\mathcal{F}}$ open covering of X. Moreover by construction we have $(f^*p)^{-1}(f^{-1}(U_\alpha)) = (p^*f)^{-1}(p^{-1}(U_\alpha))$. Define $\bar{\psi}_\alpha : (f^*p)^{-1}(\bar{U}_\alpha) \to \bar{U}_\alpha \times G$ as $\bar{\psi}_\alpha(u) := (f^*p(u), pr_2 \circ \psi_\alpha \circ p^*f(u))$. These maps are clearly bijective and smooth. For the inverses we get $p^*f \circ (\bar{\psi}_\alpha)^{-1} = \psi_\alpha^{-1}$ and $f^*p \circ (\bar{\psi}_\alpha)^{-1} = Id$ and thus they are smooth since f^*P has by definition the initial smooth structure with respect to f^*p and p^*f. Finally one easily sees that $\bar{\psi}_{\alpha\beta}(x,g) = (x, (\psi_{\alpha\beta} \circ f)(x) \cdot g)$.
(2): From the equivariancy of F one immediately sees that for $u, v \in P'$ with $p'(u) = p'(v)$ we have $p(F(u)) = p(F(v))$ and thus there is a map f making the diagram commutative. Now let $(U'_\alpha, \psi'_\alpha)$ be an atlas for P'. Define $\sigma_\alpha : U_\alpha \to {p'}^{-1}(U_\alpha)$ by $\sigma_\alpha(x) := (\psi_\alpha)^{-1}(x, e)$, where e denotes the unit element of G. Obviously this defines a smooth section. On U'_α we have $f = p \circ F \circ \sigma_\alpha$ and so f is smooth on U'_α and hence f is smooth.

By the universal property of the pullback we get a unique smooth map $\Phi : P' \to f^*P$ with $f^*p \circ \Phi = p'$ and $p^*f \circ \Phi = F$. From (1) one sees that p^*f is equivariant for the principal G actions and thus bijective on each fiber and as by assumption F also has these properties one easily concludes that Φ has them, too. So Φ is a bijective homomorphism of principal bundles and it suffices to show that its inverse is smooth. By 1.12 it suffices to show that $\Phi^{-1} : (f^*p)^{-1}(U_\alpha) \to {p'}^{-1}(U_\alpha)$ is smooth where U_α is a $\tau_{\mathcal{F}}$ open subset of X such that (U_α, ψ'_α) is a chart for P' and $(U_\alpha, \bar{\psi}_\alpha)$ is a chart for f^*P. Using equivariancy of Φ one easily shows that $\psi'_\alpha \circ \Phi \circ \bar{\psi}_\alpha^{-1}$ can be written as $(x,g) \mapsto (x, \varphi_\alpha(x) \cdot g)$ where $\varphi_\alpha : X \to G$ is a smooth function. Thus $\bar{\psi}_\alpha \circ \Phi^{-1} \circ (\psi'_\alpha)^{-1}$ is the map $(x,g) \mapsto (x, (\nu \circ \varphi_\alpha(x)) \cdot g)$ where ν is the inversion map of G and consequently Φ^{-1} is smooth. $\square$

2.7. Next we want to show that the isomorphism class of the pullback of a smooth principle bundle to a smoothly paracompact space depends only on the homotopy class of the map. This needs some preparation:

Lemma. *Let $I := [0,1] \subset \mathbb{R}$ denote the unit interval. Suppose that $f \in C^\infty(I,\mathbb{R})$ is a smooth function and $t_0 \in I$ is a point such that $f(t_0) = 1$. Then there is a smooth function $F : C^\infty(I,\mathbb{R}) \to \mathbb{R}$ and a real number $\delta > 0$ such that $F(f) = 1$ and for any g with $F(g) > 0$ we have $g(t) > 0$ for all t with $|t_0 - t| < \delta$. In particular this shows that the $\tau_{\mathcal{F}}$ topology on $C^\infty(I,\mathbb{R})$ is finer than the compact open topology.*

Proof. In [Kriegl, 1990] it is proved that for a convex subset K of $\mathbb{R}^n$ with nonempty interior a function $f : K \to \mathbb{R}$ is smooth for the initial smooth structure if and only if f is smooth (in the usual sense) in the interior of K and all derivatives on the interior extend to continuous maps on K. Thus for any $g \in C^\infty(I,\mathbb{R})$ and any $h \in C^\infty(\mathbb{R},\mathbb{R})$ the integral $\int_0^1 (h \circ g')(s)ds$ is well defined. Moreover if c is a smooth curve into $C^\infty(I,\mathbb{R})$, i.e. $\hat{c} : \mathbb{R} \times I \to \mathbb{R}$ is smooth then using this argument it is clear that $t \mapsto \int_0^1 h(\frac{d}{ds}\hat{c}(t,s))ds$ is smooth and thus for any h as above $g \mapsto \int_0^1 (h \circ g')(s)ds$ is a smooth real valued function on $C^\infty(I,\mathbb{R})$. Now the proof of [F–K, 4.7.1] can be applied:

Let us first assume that $f = 1$. Let $h_0 \in C^\infty(\mathbb{R},\mathbb{R})$ be a function with $h_0(t) \geq |t|$ for all t and $h_0(0) = 1/3$, and let $h_1 : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that $h_1(1, 2/3) = 1$ and $h_1(t,s) \neq 0$ implies $t > 1/2$ and $s > 1/2$. For any $g \in C^\infty(I,\mathbb{R})$ and any $s \in I$ we have $|g(s) - g(0)| = |\int_0^s g'(t)dt| \leq \int_0^1 h_0(g'(t))dt$. Now define $\varphi : C^\infty(I,\mathbb{R}) \to \mathbb{R}$ by $\varphi(g) := h_1(g(0), 1 - \int_0^1 h_0(g'(t))dt)$. Then φ is obviously smooth, $\varphi(f) = h_1(1, 1 - h_0(0)) = 1$ and $\varphi(g) \neq 0$ implies that $g(0) > 1/2$ and $|g(s) - g(0)| < 1/2$ for all $s \in I$ and thus $g(s) > 0$ for all $s \in I$.

In the general case since f is smooth and thus continuous there is a $\delta > 0$ such that $|t - t_0| < \delta$ implies $f(t) > 1/2$. Define $h : I \to I$ by $h(t) := t_0 + (2t - 1)\delta$ which is obviously smooth and let $\tilde{h} \in C^\infty(\mathbb{R},\mathbb{R})$ be such that $\tilde{h}(t) = 0$ for $t \leq 0$ and $\tilde{h}(t) = 1$ for $t \geq 1/2$. By 1.8 the map $h_* \circ \tilde{h}^* : C^\infty(I,\mathbb{R}) \to C^\infty(I,\mathbb{R})$ given by $g \mapsto \tilde{h} \circ g \circ h$ is smooth and the composition of the map constructed above with this one has the desired property. □

2.8. Proposition. *Let X be a smooth space, I the unit interval. Then the $\tau_{\mathcal{F}}$ topology on $X \times I$ is the product of the $\tau_{\mathcal{F}}$ topology of X and the topology of I.*

Proof. By definition the product topology is coarser. So let $U \subset X \times I$ be $\tau_{\mathcal{F}}$ open and let (x, t_0) be a point in U. By 1.13 there is a smooth function $f : X \times I \to \mathbb{R}$ with values in I such that $f((x,t_0)) = 1$ and f vanishes outside of U. By cartesian closedness the function $\check{f} : X \to C^\infty(I,\mathbb{R})$ is smooth and thus $\tau_{\mathcal{F}}$ continuous. Now $\check{f}(x)(t_0) = 1$ and thus by 2.7 there is a smooth function $F : C^\infty(I,\mathbb{R}) \to \mathbb{R}$ and a $\delta > 0$ such that $F(\check{f}(x)) = 1$ and $F(g) \neq 0$ implies that $g(t) > 0$ whenever $|t_0 - t| \leq \delta$. The set $W := F^{-1}((0,\infty)) \subset C^\infty(I,\mathbb{R})$ is $\tau_{\mathcal{F}}$ open and thus $V := (\check{f})^{-1}(W)$ is $\tau_{\mathcal{F}}$ open in X and $V \times (t_0 - \delta, t_0 + \delta)$ is a product neighborhood of x which is contained in U. □

2.9. Lemma. *Let X be a smooth space, $a < b < c < d \in \mathbb{R}$ and let $i_1 : B_1 := X \times [a,c) \hookrightarrow X \times [a,d] =: B$ and $i_2 : B_2 := X \times (b,d] \hookrightarrow X \times [a,d]$ be the inclusions. Let $p : P \to X \times [a,d]$ be a smooth principal bundle. If $i_1^* P$ and $i_2^* P$ are trivial then P is trivial.*

Proof. Let $u_1 : B_1 \times G \to i_1^* P$ and $u_2 : B_2 \times G \to i_2^* P$ be isomorphisms and let $v_i := u_i \restriction (X \times (b,c)) \times G$ for $i = 1, 2$. Then $v_2^{-1} v_1 : (X \times (b,c)) \times G \to (X \times (b,c)) \times G$ is an isomorphism of principal bundles and thus it is of the form $(x,t,g) \mapsto (x,t,\varphi(x,t) \cdot g)$ where $\varphi : X \times (b,c) \to G$ is a smooth map. Now let $h \in C^\infty(\mathbb{R},\mathbb{R})$ be a map such that $h(t) = t$ for all $t \leq \frac{2b+c}{3}$ and $h(t) = \frac{b+2c}{3}$ for all $t \geq \frac{b+2c}{3}$ and define $w : B_2 \times G \to B_2 \times G$ by $w(x,t,g) := (x,t,\varphi(x,h(t)) \cdot g)$. Then w is an isomorphism of trivial principal bundles, and the isomorphisms u_1 and $u_2 \circ w$ coincide on the open set $X \times (b, \frac{2b+c}{3})$. Thus they define a map $P \to B \times G$ which is smooth by 1.12 as it is obviously smooth over the open subsets $X \times [a, \frac{2b+c}{3})$ and $X \times (b,d]$. A similar construction works for the inverse of this map so it is indeed an isomorphism of principal bundles. □

2.10. Lemma. *Let X be a smooth space, $p : P \to X \times I$ a smooth principal bundle. Then there is a cover (U_α) of X such that P is trivial over $U_\alpha \times I$ for any α.*

Proof. For any $(x,t) \in X \times I$ there is by 2.8 a neighborhood $U(x,t)$ of x in X and a neighborhood $V(x,t)$ of t in I such that P is trivial over $U(x,t) \times V(x,t)$. By compactness of I there are $t_1, \dots, t_n \in I$ such that $(V(x,t_i))_{i=1,\dots,n}$ covers I. Put $U(x) := \cap_{i=1}^n U(x,t_i)$. Then the cover $(U(x))_{x\in X}$ has the desired property by 2.9. $\square$

2.11. Theorem. *Let X be smoothly paracompact, $p : P \to Y$ a smooth principal bundle over an arbitrary smooth space Y. Suppose that $\tilde{H} : X \times \mathbb{R} \to Y$ is a smooth map such that $\tilde{H}(x,t) = \tilde{H}(x,1)$ for all $t > 1$. Define $H := \tilde{H}|_{X\times I}$ and $f := H|_{X\times\{0\}}$. Then there is an isomorphism $\Phi : f^*P \times I \to H^*P$ of principal bundles such that $\Phi|_{f^*P\times\{0\}}$ is the natural inclusion.*

Proof. Define $\bar{H} := \tilde{H}|_{X\times[0,2]}$ and identify H^*P and f^*P with the appropriate subbundles of $\bar{H}^*P$. By 2.10 there is an open cover (U_α) of X with a subordinate partition of unity (f_α) such that the bundle $\bar{H}^*P$ is trivial over the sets $U_\alpha \times [0,2]$. Thus there are isomorphisms of principal bundles $h_\alpha : (U_\alpha \times [0,2]) \times G \to (\bar{H}^*p)^{-1}(U_\alpha \times [0,2])$. Now for any α we define $\varphi_\alpha : X \times I \times I \to X \times [0,2]$ by $\varphi_\alpha(x,t,s) := (x, t + sf_\alpha(x))$, which is clearly smooth. Next we define $\Phi_\alpha : H^*P \times I \to \bar{H}^*P$ as follows: Φ_α is the inclusion outside of $H^*p^{-1}(U_\alpha \times I)$ and $\Phi_\alpha(h_\alpha(x,t,g),s) := h_\alpha(x, t+sf_\alpha(x), g)$. Then Φ_α is smooth since the support of f_α is contained in U_α and by construction it is a isomorphism of principal bundles covering φ_α.

Now let $\Phi : f^*P \times I \to H^*P$ be the composition of all maps Φ_α in some fixed succession given by a well ordering of the index set. This composition makes sense since (f_α) is a partition of unity. Moreover since the covering U_α is locally finite any point in $f^*P \times I$ has a neighborhood on which Φ equals the composition of finitely many smooth homomorphisms of principal bundles and thus Φ itself is a smooth homomorphism of principal bundles. By construction Φ covers the identity and thus is an isomorphism (c.f. 2.6) and by construction $\Phi|_{f^*P\times\{0\}}$ is the inclusion. $\square$

2.12. Corollary. *Let X be smoothly paracompact, $p : P \to Y$ a smooth principal bundle over an arbitrary smooth space Y. Suppose that $f_0, f_1 : X \to Y$ are smoothly homotopic smooth maps. Then the smooth principal bundles f_0^*P and f_1^*P are isomorphic.*

Proof. If f_0 and f_1 are smoothly homotopic then there clearly is a homotopy H between them which satisfies the requirements of 2.11, and as above we can identify f_1^*P with a subbundle of H^*P. By 2.11 there is an isomorphism $f_0^*P \times I \to H^*P$ and its restricton to $f_0^*P \times \{1\}$ is an isomorphism between f_0^*P and f_1^*P. $\square$

2.13. Our next aim is to show that the construction of universal bundles of [Milnor, 1963] can be adapted to the smooth setting. Let us consider the set of all sequences (t_i, g_i) where $t_i \in [0,1]$ and $g_i \in G$ such that only finitely many t_i are nonzero and $\sum_{i\in\mathbb{N}} t_i = 1$. On this set we define an equivalence relation by $(t_i, g_i) \sim (t_i', g_i')$ if and only if $t_i = t_i'$ for all i and $g_i = g_i'$ for those i for which t_i is nonzero. Let EG denote the set of equivalence classes.

Let $c : \mathbb{R} \to EG$ be a curve. Then $c(t) = (c_i(t), \tilde{c}_i(t))$ where the c_i are curves into $[0,1]$ and the $\tilde{c}_i$ are curves into G. We say that $c \in \mathcal{C}$ if and only if for any i the curve c_i is a smooth curve into $[0,1]$ and the restriction $\tilde{c}_i \restriction c_i^{-1}((0,1]) : c_i^{-1}((0,1]) \to G$ is smooth.

Now we put on EG the smooth structure generated by $\mathcal{C}$, so by definition a real valued function on EG is smooth if and only if the composite with any $c \in \mathcal{C}$ is smooth and a curve into EG is smooth if and only if the composite with each real valued smooth function is smooth.

2.14. Lemma. *A curve into EG is smooth if and only if it belongs to $\mathcal{C}$.*

Proof. By definition each element of $\mathcal{C}$ is smooth so let us assume that $c : \mathbb{R} \to EG$, $c(t) = (c_i(t), \tilde{c}_i(t))$ is a curve such that for any smooth real valued function f on EG the composite $f \circ c$ is smooth. Define $f_k : EG \to \mathbb{R}$ by $f_k(t_i, g_i) := t_k$. Then for all $k \in \mathbb{N}$ the function f_k is well defined and smooth. Thus for any k the curve c_k has to be smooth.

Now let $\varphi \in C^\infty(\mathbb{R}, [0,1])$ be a smooth function which is zero locally around zero and equal to 1 for all $t \geq 1$. Let $t_0 \in \mathbb{R}$ be such that $c_k(t_0) > 0$. Then there is an $N \in \mathbb{N}$ and a neighborhood U of t_0 in $\mathbb{R}$ such that $c_k(t) > 1/N$ for all $t \in U$. Now let $h \in C^\infty(G, \mathbb{R})$ be an arbitrary smooth function on G. Consider the function $f_{h,k,N} : EG \to \mathbb{R}$ defined by $(t_i, g_i) \mapsto \varphi(N t_k) h(g_k)$. Then this is a well defined smooth function as φ is zero locally around zero. But $f_{h,k,N} \circ c \restriction U = h \circ \tilde{c}_k \restriction U$ as for all $t \in U$ we have $\varphi(N c_k(t)) = 1$ and thus $\tilde{c}_k \restriction U : U \to G$ is smooth. So for any $t \in c_k^{-1}((0,1])$ we get a neighborhood of t on which $\tilde{c}_k$ is smooth and thus by 1.12 $\tilde{c}_k : c_k^{-1}((0,1]) \to G$ is smooth $\square$

2.15. We define a right action of G on EG by $(t_i, g_i) \cdot g := (t_i, g_i \cdot g)$. This action is obviously well defined, smooth and free. Now let BG be the space of orbits of this action with the final smooth structure with respect to the canonical projection $p : EG \to EG/G =: BG$. BG is called the *classifying space* of the smooth group G.

2.16. Proposition. *If G is Hausdorff then so are EG and BG.*

Proof. Recall that a smooth space is called Hausdorff iff the real valued smooth functions on it are point separating.

Let us first consider EG: Suppose we have $x, y \in EG$, $x \neq y$, $x = (t_i, g_i)$, $y = (t'_i, g'_i)$. If for some k we have $t_k \neq t'_k$ then we can separate the two points via the smooth function $(t_i, g_i) \mapsto t_k$. So let us assume that $t_i = t'_i$ for all i. As $x \neq y$ there is a k such that $t_k > 0$ and $g_k \neq g'_k$. Now choose a function $f \in C^\infty(G, \mathbb{R})$ such that $f(g_k) \neq f(g'_k)$ and let $\varphi \in C^\infty([0,1],[0,1])$ be a smooth function which is zero locally around zero but such that $\varphi(t_k) > 0$. Define $\psi : EG \to \mathbb{R}$ by $\psi((t_i, g_i)) := \varphi(t_k) f(g_k)$. Then ψ is easily seen to be well defined and smooth and clearly $\psi(x) \neq \psi(y)$.

So let us turn to BG. We write $[(t_i, g_i)]$ for the orbit through the equivalence class of the sequence (t_i, g_i). As G acts only on the g's and not on the t's, the maps $(t_i, g_i) \mapsto t_k$ factor through BG and are by definition smooth there, so we can separate points which differ in one t coordinate. Thus let us assume we have $x = [(t_i, g_i)]$, $y = [(t_i, g'_i)]$ with $x \neq y$. Then there must be $k, \ell \in \mathbb{N}$ such that t_k and t_ℓ are nonzero and $g_k^{-1} g'_k \neq g_\ell^{-1} g'_\ell$ or equivalently $g_k g_\ell^{-1} \neq g'_k {g'_\ell}^{-1}$. Now choose $f \in C^\infty(G, \mathbb{R})$ which separates these two elements and $\varphi \in C^\infty([0,1],[0,1])$ which is zero locally around zero and nonzero at t_k and t_ℓ. Define $\psi : BG \to \mathbb{R}$ by $\psi[(t_i, g_i)] := \varphi(t_k)\varphi(t_\ell) f(g_k g_\ell^{-1})$. This map is easily seen to be well defined and smooth and clearly it separates x and y. $\square$

2.17. Theorem. *$p : EG \to BG$ is a smooth principal bundle with group G.*

Proof. In the proof of 2.16 we saw that the functions $f_k : [(t_i, g_i)] \mapsto t_k$ are smooth real valued functions on BG. Put $U_k := f_k^{-1}((0,1])$. Then $(U_i)_{i \in \mathbb{N}}$ is a point finite $\tau_{\mathcal{F}}$ open covering of BG. Now define $\psi_k : p^{-1}(U_k) \to U_k \times G$ by $\psi_k((t_i, g_i)) := (p(t_i, g_i), g_k)$. Then ψ_k is obviously well defined and smooth. To show that it is a diffeomorphism we construct a smooth section of the bundle EG over U_k.

Define $\sigma_k : U_k \to p^{-1}(U_k)$ by $\sigma_k[(t_i, g_i)] := (t_i, g_i g_k^{-1})$. If (t_i, g'_i) is in the orbit of (t_i, g_i) then there is a $g \in G$ such that $g'_i = g_i g$ for all i, and thus $g'_i {g'_k}^{-1} = g_i g_k^{-1}$ for all i and σ_k is well defined. To show that σ_k is smooth it suffices by 1.15 to show that $\sigma_k \circ p : p^{-1}(U_k) \to p^{-1}(U_k)$ is smooth. But this is obvious since by definition of the smooth structure of EG the map $(t_i, g_i) \mapsto g_k$ is smooth on $p^{-1}(U_k)$ and $\sigma_k \circ p$ is just the composition of the principal action with the inversion on G and this map.

Now one easily sees that the inverse to ψ_k is given by $([(t_i, g_i)], g) \mapsto \sigma_k([(t_i, g_i)]) \cdot g$ which is obviously smooth. Thus each ψ_k is a diffeomorphism.

Next we have $(\psi_\ell \circ \psi_k^{-1})([(t_i, g_i)], g) = ([(t_i, g_i)], g_\ell g_k^{-1} g)$ on $U_k \cap U_\ell$. The function $[(t_i, g_i)] \mapsto g_\ell g_k^{-1}$ is easily seen to be well defined as a map $U_k \cap U_\ell \to G$, and using 1.15 one immediately sees that it is smooth. $\square$

2.18. Lemma. *Let X be a smoothly paracompact smooth space, $p : P \to X$ a smooth principal bundle. Then there is a countable locally finite covering $(U_i)_{i\in\mathbb{N}}$ of X such that P is trivial over each U_i.*

Proof. As X is smoothly paracompact there is a partition of unity $(f_\alpha)_{\alpha\in A}$ on X such that P is trivial over any of the sets $U_\alpha := f_\alpha^{-1}((0,1])$. For any $x \in X$ let $S(x) \subset A$ be the finite set of all α such that $x \in U_\alpha$. For each finite subset $S \subset A$ let $W(S) \subset X$ be the $\tau_\mathcal{F}$ open subset of all $x \in X$ for which $f_\alpha(x) > f_\beta(x)$ for any $\alpha \in S$ and any $\beta \notin S$.

If $S, S' \subset A$ are distinct and have the same number of elements then by construction $W(S) \cap W(S') = \emptyset$. Now for any $n \in \mathbb{N}$ let W_n be the union of all sets $W(S(x))$ such that $S(x)$ has exactly n elements. By construction $\alpha \in S(x)$ implies $W(S(x)) \subset U_\alpha$ so P is trivial over any of the sets $W(S(x))$ and as we saw above W_n is a disjoint union of such sets so P is trivial over W_n for any n. Any $x \in X$ obviously lies in $W(S(x))$ and thus $(W_n)_{n\in\mathbb{N}}$ is a covering of X. Finally as $\{f_\alpha\}$ is a partition of unity $x \in X$ has a neighborhood which intersects only finitely many, say N, of the sets $supp(f_\alpha)$. Then clearly this neighborhood cannot intersect W_n for $n > N$ and thus the covering $\{W_n\}$ is locally finite. $\square$

2.19. Lemma. *Let X be a smoothly paracompact smooth space, $\pi : P \to X$ a smooth principal bundle. Then there is a smooth map $f : X \to BG$ such that f^*EG is isomorphic to P.*

Proof. By 2.18 we may assume that we have a locally finite atlas $(U_i, \psi_i)_{i\in\mathbb{N}}$ for P, and as X is smoothly paracompact there is a smooth partition of unity $\{f_\alpha\}_{\alpha\in A}$ subordinate to the covering $\{U_i\}$. Now let $\varphi : A \to \mathbb{N}$ be a map such that for any $\alpha \in A$ we have $supp(f_\alpha) \subset U_{\varphi(\alpha)}$ and define $f_i : X \to [0,1]$ by $f_i(x) := \sum_{\alpha:\varphi(\alpha)=i} f_\alpha(x)$ for $i \in \mathbb{N}$. Then one easily shows that $\{f_i\}$ is a partition of unity subordinate to the covering $\{U_i\}$. Define $F : P \to EG$ by $F(z) := (f_i(p(z)), pr_2 \circ \psi_i(z))$. This is well defined, as for those z for which $\psi_i(z)$ is not defined $f(p(z))$ must be zero, and it is obviously smooth. Moreover by definition of the principal action F is G–equivariant, and thus it gives rise to a well defined map $f : X \to BG$ such that $f \circ \pi = p \circ F$. Now consider the restriction of f to U_k. There f can be written as $x \mapsto p \circ F \circ \psi_k^{-1}(x, e)$ where e denotes the unit element of G, so f is smooth on U_k for each k and thus by 1.12 f is smooth. The lemma now follows from 2.6(2). $\square$

2.20. Lemma. *The space EG is smoothly contractible, so any two smooth maps from an arbitrary smooth space X into EG are smoothly homotopic.*

Proof. First we define a homotopy $A : EG \times [0,1] \to EG$ as follows (c.f. [Ramadas, 1982]): Let $\varphi \in C^\infty(\mathbb{R}, [0,1])$ be a smooth function such that $\varphi(t) = 0$ for all $t \leq \varepsilon$, where ε is some small positive number, and $\varphi(t) = 1$ for all $t \geq 1$. For $n \in \mathbb{N}$ define $\varphi_n(t) := \varphi(n((n+1)t - 1))$. Write $A(x,t) := A_t(x)$. Then we define $A_0 := Id$ and for $1/(n+1) \leq t \leq 1/n$:

$$\begin{aligned} A_t((t_i, g_i)) := (&t_0, g_0, \dots, t_{n-2}, g_{n-2}, t_{n-1}(1 - \varphi_n(t)), g_{n-1}, \\ &t_{n-1}\varphi_n(t), g_{n-1}, t_n(1 - \varphi_n(t)), g_n, t_n\varphi_n(t), g_n, \dots \\ &\dots, t_i(1 - \varphi_n(t)), g_i, t_i\varphi_n(t), g_i, \dots) \end{aligned}$$

Obviously $A_t((t_i, g_i))$ is in EG for any t. To show that A is smooth let $c : \mathbb{R} \to EG \times [0,1]$ be a smooth curve. Then $c(s)$ can be written as $((c_i(s), \bar{c}_i(s)), \tilde{c}(s))$, where each c_i and $\tilde{c}$ is a smooth curve into $[0,1]$ and the restriction of any $\bar{c}_i$ to $c_i^{-1}((0,1])$ is a smooth curve into G. Write $A_{\tilde{c}(s)}((c_i(s), \bar{c}_i(s))) = (\gamma_i(s), \bar{\gamma}_i(s))$.

Now let $k \in \mathbb{N}$ be fixed and consider the curve γ_k. Let $U_{k+2} \subset \mathbb{R}$ be the set of those s for which $\tilde{c}(s) < \frac{1}{k+2}(1+\frac{\varepsilon}{k+1})$, so $\varphi_{k+1}(\tilde{c}(s)) = 0$ for all $s \in U_{k+2}$. Then we have $\gamma_k(s) = c_k(s)$ for all $s \in U_{k+2}$, so γ_k is smooth on U_{k+2}. Next for $1 \leq n \leq k+1$ put $U_n \subset \mathbb{R}$ the set of all s such that $\frac{1}{n+1} < \tilde{c}(s) < \frac{1}{n}(1+\frac{\varepsilon}{n-1})$. Then on each U_n the curve γ_k is obviously smooth while on $U_n \cap U_{n+1}$ it is either identically zero or equal to some c_i. As $(U_n)_{n=1,\dots,k+2}$ is an open covering of $\mathbb{R}$ the curve γ_k is smooth.

Similar arguments show that the curves $\bar{\gamma}_k : \mathbb{R} \to G$ are smooth, and thus the map A is smooth. For $t = 1$ we have:

$$A_1((t_i, g_i)) = (0, e, t_0, g_0, 0, e, t_1, g_1, \dots, 0, e, t_k, g_k, 0, e, \dots).$$

So we define a homotopy $H : EG \times [0,1] \to EG$ by $H_t = A_{2\psi(t)}$ for $t \leq 1/2$ and

$$H_t((t_i, g_i)) = (2\psi(t) - 1, e, (2 - 2\psi(t))t_0, g_0, 0, e, \dots \\ , 0, e, (2 - 2\psi(t))t_k, g_k, 0, e, \dots),$$

where $\psi \in C^\infty(\mathbb{R}, [0,1])$ is a smooth increasing map such that $\psi = 0$ locally around zero, $\psi = 1/2$ locally around $1/2$ and $\psi = 1$ locally around one. Then H is obviously a smooth homotopy from the identity to a constant map. $\square$

2.21. Let us investigate the homotopy A constructed in the proof of 2.20 more closely. Let EG^{odd} denote the set of all $(t_i, g_i) \in EG$ such that $t_i = 0$ for i even and EG^{even} the set of those with $t_i = 0$ for i odd. With BG^{odd} and BG^{even} we denote the images under p of EG^{odd} and EG^{even}, respectively. In the proof we saw that $A_1(EG) = EG^{odd}$ and similarly one sees that $A_{1/2}(EG) = EG^{even}$. Consider the map $p \times Id : EG \times I \to BG \times I$. Then this is obviously a smooth principal bundle and from the definition of A one immediately sees that it is a homomorphism of principal bundles $EG \times I \to EG$. Thus it covers a smooth map $H : BG \times I \to BG$ such that $H_0 = Id$, $H_{1/2}(BG) \subset BG^{even}$ and $H_1(BG) \subset BG^{odd}$.

2.22. Lemma. *Let G be a smooth group, $f, g : Y \to BG$ smooth mappings and assume that $\varphi : f^*EG \to g^*EG$ is an isomorphism of principal bundles. Then there is a homomorphism $\Phi : f^*EG \times I \to EG$ of principal bundles such that $\Phi|_{f^*EG\times\{0\}} = p^*f$ and $\Phi|_{f^*EG\times\{1\}} = p^* \circ \varphi$.*

Proof. By 2.21 f is smoothly homotopic to a map f_e which satisfies $f_e(Y) \subset BG^{even}$ and g is smoothly homotopic to a map g_o such that $g_o(Y) \subset BG^{odd}$. We chose homotopies $H_e : Y \times [0, 2/3] \to BG$ and $H_o : Y \times [1/3, 1] \to BG$ such that $H_e(y, 0) = f(y)$, $H_e(y, t) = f_e(y)$ for $t > 1/3 - \varepsilon$, $H_o(y, t) = g_o(y)$ for $t < 2/3 + \varepsilon$ and $H_o(y, 1) = g(y)$, where ε is some small positive number. By 2.11 there are isomorphisms $\Phi_e : f^*EG \times [0, 2/3] \to H_e^*EG$ and $\Phi_o : g^*EG \times [1/3, 1] \to H_o^*EG$ such that the restrictions of Φ_e to $f^*EG \times \{0\}$ and of Φ_o to $g^*EG \times \{1\}$ are the natural inclusions.

Next consider the mapping $p^*H_e \circ \Phi_e : f^*EG \times [0, 2/3] \to EG$. For $t > 1/3 - \varepsilon$ this map has values in EG^{even} and thus we can write it as $(z, t) \mapsto (t_0(z, t), g_0(z, t), 0, id, t_2(z, t), g_2(z, t), 0, id, \dots)$. On the other hand for $t < 2/3 + \varepsilon$ the map $p^*H_o \circ \Phi_o \circ (\varphi \times id) : f^*EG \times [1/3, 1] \to EG$ has values in EG^{odd} and we write it as $(z, t) \mapsto (0, id, t_1(z, t), g_1(z, t), 0, id, t_3(z, t), g_3(z, t), \dots)$.

Now let $\gamma \in C^\infty(\mathbb{R}, I)$ be a smooth increasing map such that $\gamma(t) = 0$ for $t < \varepsilon$ and $\gamma(t) = 1$ for $t > 1 - \varepsilon$ and define $\Phi : f^*EG \times I \to EG$ by $\Phi(z, t) = (p^*H_e \circ \Phi_e)(z, t)$ for $t \leq 1/3$, $\Phi(z, t) = (p^*H_o \circ \Phi_o \circ (\varphi \times id))(z, t)$ for $t \geq 2/3$ and by

$$(z, t) \mapsto ((1 - \gamma(3t - 1))t_0(z, t), g_0(z, t), \gamma(3t - 1)t_1(z, t), g_1(z, t), \dots)$$

for $1/3 \leq t \leq 2/3$. Then one immediately checks that Φ is a smooth homomorphism of principal bundles and has the required properties. $\square$

2.23. Theorem. *For a smoothly paracompact space X let $[X, BG]$ denote the set of homotopy classes of smooth maps from X to BG and let $\mathcal{P}_G(X)$ denote the set of isomorphism classes of principal bundles with group G over X. Then the map $[X, BG] \to \mathcal{P}_G(X)$ induced by $f \mapsto f^*EG$ is a bijection.*

Proof. The map is well defined by 2.12 and surjective by 2.19. Morover if $f, g : X \to BG$ are smooth maps such that $f^*EG \cong g^*EG$ then taking any isomorphism φ we get by 2.22 a homomorphism $\Phi : f^*EG \times I \to EG$ of principal bundles which restricts to p^*f on $f^*EG \times \{0\}$ and to $p^*g \circ \varphi$ on $f^*EG \times \{1\}$. This homomorphism covers a smooth map $X \times I \to BG$ and since p^*f covers f, p^*g covers g and φ covers the identity this map is a smooth homotopy between f and g. $\square$

2.24. Definition. Let X and F be smooth spaces. A smooth map $\pi : E \to X$ from a smooth space E to X is called a smooth fiber bundle with fiber F iff there is a $\tau_{\mathcal{F}}$–open covering $\{U_\alpha\}$ of X and there are diffeomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ with $pr_1 \circ \varphi_\alpha = p$.

If U_α and U_β are two sets such that $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ then we can consider the smooth map $\varphi_\alpha \circ \varphi_\beta^{-1} : U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F$. By definition it is of the form $(x, f) \mapsto (x, \psi(x, f))$, where $\psi : U_{\alpha\beta} \times F \to F$ is a smooth map. The bundle E is said to have *structure group* G (where G is a smooth group) iff there is a smooth left action $\lambda : G \times F \to F$ of G on F and there are smooth functions $\varphi_{\alpha\beta} : U_{\alpha\beta} \to G$ such that $(\varphi_\alpha \circ \varphi_\beta^{-1})(x, f) = (x, \lambda(\varphi_{\alpha\beta}(x), f))$ for all α, β. In this terminology a smooth principal bundle with group G is a smooth fiber bundle with fiber G and structure group G and the left multiplication as the action.

Let us now show that any smooth fiber bundle with fiber F has structure group $\mathrm{Diff}(F)$: Let $\varphi_{\alpha\beta} : X \to C^\infty(F, F)$ be the smooth map which is associated to the smooth map $\psi : X \times F \to F$ from above via cartesian closedness. Obviously then $\varphi_{\beta\alpha}(x) = (\varphi_{\alpha\beta}(x))^{-1}$ for any $x \in U_{\alpha\beta}$ and thus each $\varphi_{\alpha\beta}$ has values in $\mathrm{Diff}(F)$ and is even smooth as a map to this space with the smooth structure from 2.2.

If $p : E \to X$ and $p' : E' \to X$ are smooth fiber bundles over X with fiber F and structure group G for the same action λ then an isomorphism between the bundles is a diffeomorphism $g : E \to E'$ such that $p' \circ g = p$ and if $(U_\alpha, \varphi_\alpha)$ and (U'_i, φ'_i) are charts of E and E' such that $U_{i\alpha} := U_\alpha \cap U'_i \neq \emptyset$ then the map $\varphi'_i \circ g \circ \varphi_\alpha^{-1} : U_{i\alpha} \times F \to U_{i\alpha} \times F$ has the form $(x, f) \mapsto (x, \lambda(g_{i\alpha}(x), f))$ where $g_{i\alpha} : U_{i\alpha} \to G$ is a smooth map.

2.25. Associated bundles. Let G be a smooth group, $p : P \to X$ a smooth principal bundle over a smooth space X and F an arbitrary smooth space. Let $\lambda : G \times F \to F$ be a smooth left action of G on F. By cartesian closedness this is equivalent to the fact that $\check{\lambda} : G \to \mathrm{Diff}(F) \subset C^\infty(F, F)$ is a homomorphism of smooth groups, where $\mathrm{Diff}(F)$ carries the smooth structure introduced in 2.2. Let us denote by $\rho : P \times G \to P$ the principal action of G on P. Now consider the map $r : (P \times F) \times G \to P \times F$ given by $r(z, f, g) := (\rho(z, g), \lambda(g^{-1}, f))$. Then obviously r is a smooth right action of G on the smooth space $P \times F$. By $P \times_G F$ or $P[F]$ we denote the space of orbits of this action equipped with the final smooth structure with respect to the canonical map $q : P \times F \to P \times_G F$. The space $P[F]$ is called the bundle associated to P with fiber F.

2.26. Theorem. *In the setting of 2.25 we have:*
(1): $q : P \times F \to P[F]$ is a smooth principal bundle with group G and principal action r.
(2): $P[F]$ is a smooth fiber bundle over X with fiber F and structure group G.
(3): If $p' : P' \to X$ is a principal bundle isomorphic to P then the fiber bundles $P[F]$ and $P'[F]$ are isomorphic.

Proof. The map $P \times F \to X$, $(z, f) \mapsto p(z)$ is obviously smooth and invariant under the action r and thus gives rise to a well defined map $\pi : P[F] \to X$ which is smooth by definition of the smooth structure of $P[F]$.
(1): The smooth action r of G on $P \times F$ is clearly fiber respecting and it is free since the

principal action on P is free. The composition of the first projection $P \times F \to P$ with the canonical maps $(P \times F) \times_{P[F]} (P \times F) \to P \times F$ induces by the universal property of the pullback $P \times_X P$ a smooth map $g : (P \times F) \times_{P[F]} (P \times F) \to P \times_X P$, and we define $\tau : (P \times F) \times_{P[F]} (P \times F) \to G$ by $\tau = \tau_P \circ g$, where $\tau_P : P \times_X P \to G$ is the smooth map constructed in 2.3 for the bundle P. Now let (z_1, f_1) and (z_2, f_2) be in one fiber of q. Then by construction we have $\rho(z_1, \tau((z_1, f_1), (z_2, f_2))) = z_2$ and since the two points are in the same orbit we must have $\lambda(\tau((z_1, f_1), (z_2, f_2))^{-1}, f_1) = f_2$ by freeness of the action ρ. Thus τ has the right property and to finish the proof of (1) by 2.4 we only have to construct local sections of q.

So let $(U_\alpha, \omega_\alpha)$ be an atlas for the bundle P and define $\sigma'_\alpha : U_\alpha \to p^{-1}(U_\alpha)$ by $\sigma'_\alpha(x) := \omega_\alpha^{-1}(x, e)$, where e denotes the unit element of G. Then the maps σ'_α are local smooth sections of the bundle P. By construction we have $q^{-1}(\pi^{-1}(U_\alpha)) = p^{-1}(U_\alpha) \times F$.

Now consider the map $p^{-1}(U_\alpha) \times F \to p^{-1}(U_\alpha) \times F$ given by

$$(z, f) \mapsto (\sigma'_\alpha(p(z)), \lambda(\tau_P(\sigma'_\alpha(p(z)), z), f)).$$

This is obviously smooth and it is easily seen to be invariant for the action r and thus gives rise to a well defined map $\sigma_\alpha : \pi^{-1}(U_\alpha) \to p^{-1}(U_\alpha) \times F$ which is smooth by 1.15. Moreover one immediately sees that the map above sends (z, f) to an element in the orbit of (z, f) and thus σ_α is indeed a smooth section of q.

(2): We want to show that $\pi : P[F] \to X$ is a smooth fiber bundle with fiber F and structure group G. Consider the map $p^{-1}(U_\alpha) \times F \to U_\alpha \times F$ defined by $(z, f) \mapsto (p(z), \lambda(\tau_P(\sigma'_\alpha(p(z)), z), f))$. This is obviously smooth and it is easily seen to be invariant under the action r, and thus it gives rise to a well defined map $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ which is smooth by 1.15 and satisfies $pr_1 \circ \varphi_\alpha = \pi$. Moreover a short computation shows that the smooth map $U_\alpha \times F \to \pi^{-1}(U_\alpha)$ defined by $(x, f) \mapsto q(\sigma'_\alpha(x), f)$ is inverse to φ_α.

If $U_{\alpha\beta} \neq \emptyset$ then a short computation shows that we have

$$(\varphi_\alpha \circ \varphi_\beta^{-1})(x, f) = (x, \lambda(\tau_P(\sigma'_\alpha(x), \sigma'_\beta(x)), f))$$

and thus the bundle $P[F]$ has indeed structure group G.

(3): Let $\Phi : P' \to P$ be an isomorphism. Then $q \circ (\Phi \times Id) : P' \times F \to P[F]$ is smooth and G invariant and thus induces a smooth map $\bar{\Phi} : P'[F] \to P[F]$. The same works for Φ^{-1} and one easily sees that it gives an inverse to $\bar{\Phi}$. $\square$

2.27. Theorem. *If $\pi : E \to X$ is a smooth fiber bundle with fiber F and structure group G over a smooth space X then there is a (up to isomorphism) unique smooth principal bundle $p : P \to X$ with group G such that E is isomorphic to $P[F]$. The isomorphism class of P depends only on the isomorphism class of F.*

Proof. Let $(U_\alpha, \varphi_\alpha)$ be an atlas for E which satisfies the conditions of 2.24. Thus we have the smooth transition functions $\varphi_{\alpha\beta} : U_{\alpha\beta} \to G$, which satisfy $(\varphi_\alpha \circ \varphi_\beta^{-1})(x, f) = (x, \lambda(\varphi_{\alpha\beta}(x), f))$ Let $\tilde{P}$ be the disjoint union (coproduct) of the smooth spaces $U_\alpha \times G$. The first projections $U_\alpha \times G \to U_\alpha$ define a smooth map $\tilde{p} : \tilde{P} \to X$. Then define an equivalence relation on $\tilde{P}$ by declaring $(x, g) \in U_\alpha \times G$ to be equivalent to $(x', g') \in U_\beta \times G$ if and only if $x = x'$ and $g = \varphi_{\alpha\beta}(x) \cdot g'$. (That this is indeed an equivalence relation can be proved easily by showing that the maps $\varphi_{\alpha\beta}$ satisfy the usual cocycle conditions.) By P we denote the quotient of $\tilde{P}$ with respect to this equivalence relation and the final smooth structure with respect to the canonical map $\gamma : \tilde{P} \to P$. Then obviously $\tilde{p}$ factors to a map $p : P \to X$ which is smooth by definition of the smooth structure. We claim that $p : P \to X$ is a smooth principal bundle with group G.

Consider the subset $p^{-1}(U_\alpha) \subset P$. Then this is exactly the set of those equivalence classes which have a representative in $U_\alpha \times G$ and such a representative is necessarily unique since

$\varphi_{\alpha\alpha}(x) = e$ for all x. Let $\psi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times G$ be the map which assigns to each equivalence class its unique representative in $U_\alpha \times G$. To show that this map is smooth it suffices by 1.15 to show that its composition with γ is smooth as a map $\gamma^{-1}(p^{-1}(U_\alpha)) \to U_\alpha \times G$ and again by 1.15 for this it suffices that for any β the composition with i_β is smooth as a map on $i_\beta^{-1}(\gamma^{-1}(p^{-1}(U_\alpha)))$ where $i_\beta : U_\beta \times G \to \tilde{P}$ is the canonical map. But $i_\beta^{-1}(\gamma^{-1}(p^{-1}(U_\alpha))) = U_{\alpha\beta} \times G \subset U_\beta \times G$ and there the map is given by $(x, g) \mapsto (x, \varphi_{\alpha\beta}(x) \cdot g)$ which is obviously smooth. Moreover obviously ψ_α^{-1} is the canonical map $U_\alpha \times G \to \tilde{P} \to P$ which is smooth, so each ψ_α is a diffeomorphism. This also shows that for $U_{\alpha\beta} \neq \emptyset$ one gets $(\psi_\alpha \circ \psi_\beta^{-1})(x, g) = (x, \varphi_{\alpha\beta}(x) \cdot g)$ and thus $p : P \to X$ is a smooth principal bundle.

Now let us show that $P[F] \cong E$. First we claim that the smooth map $\gamma \times Id : \tilde{P} \times F \to P \times F$ is a final morphism. So let $f : P \times F \to Z$ be a map into an arbitrary smooth space such that $f \circ (\gamma \times Id)$ is smooth. Then by cartesian closedness the map $(f \circ (\gamma \times Id))^\vee = \check{f} \circ \gamma : \tilde{P} \to C^\infty(F, Z)$ is smooth. Since γ is final the map $\check{f}$ is smooth and thus again by cartesian closedness f is smooth.

Another consequence of cartesian closedness is that the functor $\cdot \times F$ has a right adjoint, hence commutes with colimits, and thus $\tilde{P} \times F$ is the coproduct of the smooth spaces $U_\alpha \times G \times F$.

We define maps $h_\alpha : U_\alpha \times G \times F \to E$ by $h_\alpha(x, g, f) := \varphi_\alpha^{-1}(x, \lambda(g, f))$. Then these maps induce a smooth map $\tilde{P} \times F \to E$ and a short computation shows that this map factors over $\gamma \times Id$ and thus defines a smooth map $h : P \times F \to E$. By construction this map is invariant for the G action on $P \times F$ and thus defines a smooth map $\bar{h} : P[F] \to E$ which is immediately seen to be an isomorphism of fiber bundles with structure group G.

So it remains to show that the construction is independent of the choice of the atlas $(U_\alpha, \varphi_\alpha)$. First let us a assume that we have a trivialization (V_i, φ_i) where the cover (V_i) is a refinement of the cover (U_α) and the functions φ_i are appropriate restrictions of the functions φ_α. Then one easily sees that the construction leads to an isomorphic principal bundle. To prove the rest let us assume that $p' : P' \to X$ is a smooth principal bundle with group G such that $P'[F] \cong E$ as a bundle with structure group G and let $p : P \to X$ be the principal bundle constructed from E as above. By the argument above we may assume that we have a cover (U_α) of X such that both principal bundles (and thus both associated bundles) are trivial over any of the sets U_α. Let $\psi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times G$ and $\psi'_\alpha : p'^{-1}(U_\alpha) \to U_\alpha \times G$ be the corresponding atlases for the principal bundles and let $\psi_{\alpha\beta}, \psi'_{\alpha\beta} : U_{\alpha\beta} \to G$ be the corresponding transition functions. From the proof of 2.26 we see that the transition functions of an associated bundle are the same as the transition functions of the corresponding principal bundle and using this fact one easily shows that from the isomorphism of the associated bundles we get smooth functions $\omega_\alpha : U_\alpha \to G$ such that $\omega_\alpha(x) \cdot \psi_{\alpha\beta}(x) = \psi'_{\alpha\beta}(x) \cdot \omega_\beta(x)$ for any $x \in U_{\alpha\beta}$. Now we define maps $\Phi_\alpha : U_\alpha \times G \to P'$ by $\Phi_\alpha(x, g) := (\psi'_\alpha)^{-1}(x, \omega_\alpha(x) \cdot g)$. Then these maps define a smooth map $\tilde{\Phi} : \tilde{P} \to P'$ and a short computation shows that this factors to a map $\Phi : P \to P'$ which is immediately seen to be a smooth homomorphism of principal bundles which covers the identity. Using 2.6(2) it is clear that Φ is an isomorphism of principal bundles. □

2.28. Corollary. *For a smoothly paracompact space X let $\mathcal{B}_G F(X)$ be the set of all isomorphism classes of smooth fiber bundles with fiber F and structure group G over X. Then the map $[X, BG] \to \mathcal{B}_G F(X)$ induced by $f \mapsto (f^* EG)[F]$ is bijective. In particular if $\mathcal{B}F(X)$ denotes the set of all isomorphism classes of smooth fiber bundles over X with fiber F and without structure group (i.e. with structure group $\mathrm{Diff}(F)$) then there is a bijection between the sets $\mathcal{B}F(X)$ and $[X, B\,\mathrm{Diff}(F)]$.*

Proof. This is clear by 2.23, 2.26 and 2.27. □

3. Smooth fibrations and cofibrations

Fibrations and cofibrations are an important concept of homotopy theory since they lead to long exact sequences of sets or groups of homotopy classes. It turns out that the obvious analogs of the classical definitions of fibration and cofibration do not lead to a reasonable theory. This can be seen as follows:

3.1. Counter example. It is well known that if X is a topological space and Y is a closed deformation retract of X then the inclusion $Y \hookrightarrow X$ is a cofibration. In fact these are rather trivial examples, so at least this should also hold in the smooth category. Now if one takes the analog of the classical definition of cofibration in the smooth category then one immediately proves (as in topology) that if X is a space with a subspace Y, the inclusion of which is a cofibration then $X \times \{0\} \cup Y \times I$ is a retract of $X \times I$, where I denotes the unit interval. We claim that this already fails in the case $X = I$ and $Y = \{0\}$: Suppose that $I \times \{0\} \cup \{0\} \times I$ is a smooth retract of $I \times I$. Obviously $I \times I$ and thus also $I \times \{0\} \cup \{0\} \times I$ has the initial smooth structure with respect to the inclusion into $\mathbb{R}^2$ so we have a smooth map $f : I \times I \to \mathbb{R}^2$ which has values in $I \times \{0\} \cup \{0\} \times I$. From [Kriegl, 1990] one concludes that this map has a smooth extension (also denoted by f):$\mathbb{R}^2 \to \mathbb{R}^2$. Since f restricts to the identity on $I \times \{0\} \cup \{0\} \times I$, we clearly have $df(0) = Id_{\mathbb{R}^2}$. But now consider the smooth curve $c : \mathbb{R} \to \mathbb{R}^2$ defined by $c(t) = (t, t)$. Then $f \circ c : \mathbb{R} \to \mathbb{R}^2$ is a smooth function and by the chain rule we have $d(f \circ c)(0) = (1, 1)$ but for $0 \leq t \leq 1$ we have $(f \circ c)(t) \in I \times \{0\} \cup \{0\} \times I$ which is a contradiction.

Thus for the analog of the classical definition of cofibration not even $\{0\} \hookrightarrow I$ would be a smooth cofibration. But we also see that the problem can be circumvented if one weakens the notion of cofibration in a way such that one does not get a retraction $I \times I \to I \times \{0\} \cup \{0\} \times I$ but a map which is deformed a bit from the identity on at least one copy of I. Moreover it turns out that several other canonical examples of fibrations and cofibrations lead to the same problem and the same solution. So we are led to the following definition (in 3.27 we will see that in fact our definition is quite near to the classical one).

3.2. Definition. (1): Let X and Y be smooth spaces. A smooth map $i : Y \to X$ is called a *smooth cofibration* iff it has the following homotopy extension property: If Z is an arbitrary smooth space and $H : Y \times I \to Z$ and $f : X \to Z$ are smooth maps such that $f \circ i := H|_{Y\times\{0\}}$ then there is a smooth map $\tilde{H} : X \times I \to Z$ such that $\tilde{H} \circ (i \times Id) = H$ and $\tilde{H}|_{X\times\{0\}}$ is smoothly homotopic to f relative to Y, i.e. there is a smooth homotopy $h : X \times I \to Z$ such that $h(x, 0) = \tilde{H}(x, 0)$, $h(x, 1) = f(x)$ and $h(i(y), t) = f(i(y))$ for all $x \in X$, $y \in Y$ and $t \in I$.
(2): For a smooth cofibration $i : Y \to X$ we define the *cofiber* of the cofibration to be the quotient space $X/i(Y)$ with the final smooth structure with respect to the canonical projection $X \to X/i(Y)$. The space $X/i(Y)$ can obviously be interpreted as the push out of the map i and the unique smooth map $Y \to pt$, where pt denotes the smooth space consisting of a single point.

3.3. Proposition. *If $i : Y \to X$ is a smooth cofibration then i is an initial morphism in the category of smooth spaces. If in addition Y is Hausdorff then i is injective, so in this case Y can be viewed as a subspace of X.*

Proof. Let us first show that for any smooth function $f : Y \to \mathbb{R}$ there is a smooth function $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f} \circ i = f$. Consider the smooth map $H : Y \times I \to \mathbb{R}$ defined by $H(y, t) := t \cdot f(y)$. The constant map $0 : X \to \mathbb{R}$ is an extension of $H|_{Y\times\{0\}}$ and thus there is a smooth map $\tilde{H} : X \times I \to \mathbb{R}$ such that $\tilde{H} \circ (i \times Id) = H$. Then clearly $\tilde{f} := \tilde{H}|_{X\times\{1\}}$ has the desired property.

To show that i is initial it suffices to show that if $c : \mathbb{R} \to Y$ is a curve such that $i \circ c : \mathbb{R} \to X$ is smooth then c is smooth, i.e. for any smooth function $f : Y \to \mathbb{R}$ the map $f \circ c$ is smooth. But this is clear as the function $\tilde{f} : X \to \mathbb{R}$ is smooth and the curve $i \circ c : \mathbb{R} \to X$ is smooth and thus $\tilde{f} \circ i \circ c = f \circ c$ is smooth.

Finally if Y is Hausdorff and $a, b \in Y$ are distinct points then there is a smooth function $f : Y \to \mathbb{R}$ such that $f(a) \neq f(b)$ and thus $\tilde{f}(i(a)) \neq \tilde{f}(i(b))$ and hence $i(a) \neq i(b)$. □

3.4. Lemma. *(1): If $i : Y \to X$ and $j : X \to U$ are smooth cofibrations then $j \circ i : Y \to U$ is a smooth cofibration.*
(2): If $i : Y \to X$ is a smooth cofibration and U is an arbitrary smooth space then $i \times Id_U : Y \times U \to X \times U$ is a smooth cofibration.

Proof. Let Z be an arbitrary smooth space, $H : Y \times I \to Z$ and $f : U \to Z$ smooth maps such that $f \circ j \circ i = H|_{Y \times \{0\}}$. Since i is a smooth cofibration we get a smooth map $\bar{H} : X \times I \to Z$ such that $\bar{H} \circ (i \times Id) = H$ and $\bar{H}|_{X \times \{0\}}$ is smoothly homotopic to $f \circ j$ relative to Y. Thus there is a smooth map $\bar{h} : X \times I \to Z$ such that $\bar{h}|_{X \times \{0\}} = f \circ j$ and $\bar{h}|_{X \times \{1\}} = \bar{H}|_{X \times \{0\}}$ and $\bar{h} \circ (i \times Id) = f \circ j \circ i \circ pr_1$.

Applying the cofibration property of j to $\bar{h}$ and f we get a smooth homotopy $H' : U \times I \to Z$ such that $H' \circ (j \times Id) = \bar{h}$ and $H'|_{U \times \{0\}}$ is smoothly homotopic to f relative to X. Now for $\tilde{f} := H'|_{U \times \{1\}} : U \to Z$ we have $\tilde{f} \circ j = \bar{H}|_{X \times \{0\}}$ and since j is a cofibration we get a smooth map $\tilde{H} : U \times I \to Z$ such that $\tilde{H} \circ (j \times Id) = \bar{H}$ and thus $\tilde{H} \circ ((j \circ i) \times Id) = H$. So it remains to show that $\tilde{H}|_{U \times \{0\}}$ is smoothly homotopic to f relative to Y.

By the cofibration property $\tilde{H}|_{U \times \{0\}}$ is homotopic to $\tilde{f} = H'|_{U \times \{1\}}$ relative to X. By construction H' is a homotopy relative to Y and thus $H'|_{U \times \{1\}}$ is homotopic to $H'|_{U \times \{0\}}$ relative to Y. But by construction of H' the map $H'|_{U \times \{0\}}$ is homotopic to f relative to X and putting all these homotopies together we get the result.
(2): Let $H : Y \times U \times I \to Z$ and $f : X \times U \to Z$ be smooth maps such that $f \circ (i \times Id_U) = H|_{Y \times U \times \{0\}}$. By cartesian closedness the associated maps $\check{H} : Y \times I \to C^\infty(U, Z)$ and $\check{f} : X \to C^\infty(U, Z)$ are smooth and obviously $\check{f} \circ (i \times Id_I) = \check{H}|_{Y \times \{0\}}$. Applying the cofibration property to these two maps and using cartesian closedness again one easily shows the result. □

3.5. Lemma. *Let $i : Y \to X$ be a smooth cofibration and let*

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i_*g} \\ V & \xrightarrow{g_*i} & U \end{array}$$

be a push out. Then $g_*i : V \to U$ is a smooth cofibration.
Moreover if X, Y and V are base spaces then U is a base space. In particular this implies that for a smooth cofibration between base spaces the cofiber is a base space.

Proof. Let Z be a smooth space, $H : V \times I \to Z$ and $f : U \to Z$ smooth maps such that $f \circ g_*i = H|_{V \times \{0\}}$. Then we have $f \circ i_*g \circ i = f \circ g_*i \circ g = (H \circ (g \times Id))|_{Y \times \{0\}}$. Applying the cofibration property of i to $H \circ (g \times Id)$ and $f \circ i_*g$ we get a smooth map $\bar{H} : X \times I \to Z$ such that $\bar{H} \circ (i \times Id) = H \circ (g \times Id)$ and $\bar{H}|_{X \times \{0\}}$ is smoothly homotopic to $f \circ i_*g$ relative to Y.

By cartesian closedness the functor $. \times I$ has a right adjoint and thus commutes with colimits so the diagram

$$\begin{array}{ccc} Y \times I & \xrightarrow{i \times Id} & X \times I \\ {\scriptstyle g\times Id}\downarrow & & \downarrow{\scriptstyle i_* g \times Id} \\ V \times I & \xrightarrow{g_* i \times Id} & U \times I \end{array}$$

is a push out. By the universal property of this push out the maps $\bar{H} : X \times I \to Z$ and $H : V \times I \to Z$ induce a unique smooth map $\tilde{H} : U \times I \to Z$ such that $\tilde{H} \circ (i_* g \times Id) = \bar{H}$ and $\tilde{H} \circ (g_* i \times Id) = H$. So it remains to show that $\tilde{H}|_{U\times\{0\}}$ is smoothly homotopic to f relative to Y.

Since i is a cofibration we get a smooth homotopy $\bar{h} : X \times I \to Z$ such that $\bar{h}|_{X\times\{0\}} = \bar{H}|_{X\times\{0\}}$, $\bar{h}|_{X\times\{1\}} = f \circ i_* g$ and $\bar{h} \circ (i \times Id) = f \circ i_* g \circ i \circ pr_1 = f \circ g_* i \circ g \circ pr_1$. By the universal property of the push out the maps $\bar{h} : X \times I \to Z$ and $f \circ g_* i \circ pr_1 : V \times I \to Z$ induce a unique smooth map $h : U \times I \to Z$ such that $h \circ (i_* g \times Id) = \bar{h}$ and $h \circ (g_* i \times Id) = f \circ g_* i \circ pr_1$, so h is a homotopy relative to V. Moreover by construction of $\bar{h}$ we have $h|_{U\times\{0\}} \circ i_* g = \bar{H}|_{X\times\{0\}}$ and $h|_{U\times\{0\}} \circ g_* i = f \circ g_* i = H|_{V\times\{0\}}$ and thus $h|_{U\times\{0\}} = \tilde{H}|_{U\times\{0\}}$, and $h|_{U\times\{1\}} \circ i_* g = f \circ i_* g$ and $h|_{U\times\{1\}} \circ g_* i = f \circ g_* i$ and thus $h|_{U\times\{1\}} = f$.

Let us now assume that X, Y and V are base spaces. Then i is injective by 3.3. Moreover from the proof of 3.3 we see that any real valued smooth function on Y has a smooth extension to X and thus U is a base space by 1.18. □

3.6. Proposition. *The inclusion $i : S^{n-1} \hookrightarrow E^n$ is a smooth cofibration.*

Proof. Consider E^n as being embedded into $\mathbb{R}^n$ as usual and assume we have given smooth maps $H : S^{n-1} \times I \to Z$ and $f : E^n \to Z$ into some smooth space Z such that $f|_{S^{n-1}} = H|_{S^{n-1}\times\{0\}}$. Let $\varphi \in C^\infty(\mathbb{R}, I)$ be an increasing map such that $\varphi(t) = 0$ for $t \le \varepsilon$ and $\varphi(t) = 1$ for $t \ge 1 - \varepsilon$, where ε is some small positive number and let $\|\ \ \|$ be the usual norm on $\mathbb{R}^n$. Then $\|\ \ \|$ is a smooth real valued function on $\mathbb{R}^n \setminus \{0\}$ and obviously $x \mapsto \frac{\varphi(2\|x\|)}{\|x\|}$ defines a smooth real valued function on $\mathbb{R}^n$. Now define $\tilde{H} : E^n \times I \to Z$ by:

$$\tilde{H}(x,t) := \begin{cases} f(x \cdot \frac{\varphi(2\|x\|)}{\|x\|}) & \|x\| \le 1/2 \\ H(\frac{x}{\|x\|}, t \cdot \varphi(2\|x\| - 1)) & \|x\| \ge 1/2 \end{cases}$$

Since $H(\frac{x}{\|x\|}, 0) = f(\frac{x}{\|x\|})$ for all $x \ne 0$ the map $\tilde{H}$ is smooth since it is obviously smooth on the open disk of radius $(1+\varepsilon)/2$ and on the ring formed by all x with $(1-\varepsilon)/2 < \|x\| \le 1$. By definition we have $\tilde{H}(x,t) = H(x,t)$ for all $x \in S^{n-1}$. Moreover we have $\tilde{H}(x,0) = f(x \cdot \frac{\varphi(2\|x\|)}{\|x\|})$ for all $X \in E^n$. Now define $h : E^n \times I \to Z$ by $h(x,t) := f(x \cdot (t + (1-t)\frac{\varphi(2\|x\|)}{\|x\|}))$. This is well defined as $(t + (1-t)\frac{\varphi(2\|x\|)}{\|x\|}) \le \frac{1}{\|x\|}$ and is obviously smooth. Clearly $h(x,0) = \tilde{H}(x,0)$ and $h(x,1) = f(x)$ and for $x \in S^{n-1}$ we have $h(x,t) = f(x)$ for all t. □

3.7. Corollary. *If A is any smooth space and X is a smooth space which is obtained from A by attaching finitely many cells, then the inclusion of A into X is a smooth cofibration. In particular this applies if X is a smooth cell complex and A is a subcomplex.*

Proof. Induction on the number of cells using 3.6, 3.5 and 3.4(1). □

3.8. Mapping cylinders. Let X and Y be arbitrary smooth spaces, $g : X \to Y$ a smooth map. We define the *mapping cylinder* M_g of g to be the push out:

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & M_g \end{array}$$

The inclusion of $X \times \{1\}$ into $X \times I$ induces a smooth map $i : X \to M_g$.

As a special case we define the *cone over* X, CX to be the mapping cylinder of the unique smooth map $X \to pt$, where pt denotes the smooth space consisting of a single point.

3.9. Proposition. *Let M_g be the mapping cylinder of a smooth map $g : X \to Y$. Then we have:*
(1): The map $i : X \to M_g$ is a smooth cofibration.
(2): The natural map $j : Y \to M_g$ is a homotopy equivalence.
(3): If X and Y are base spaces then M_g is a base space.

Proof. (1): Let Z be a smooth space, $H : X \times I \to Z$ and $f : M_g \to Z$ smooth maps such that $f \circ i = H|_{X\times\{0\}}$. By cartesian closedness the following diagram is a push out:

$$\begin{array}{ccc} X \times \{0\} \times I & \xrightarrow{g\times Id} & Y \times I \\ \big\downarrow & & \big\downarrow{\scriptstyle j\times Id} \\ X \times I \times I & \xrightarrow{k\times Id} & M_g \times I \end{array}$$

Now let $\varphi \in C^\infty(\mathbb{R}, I)$ be an increasing map such that $\varphi(t) = 0$ for $t \leq \varepsilon$ and $\varphi(t) = 1$ for $t \geq 1 - \varepsilon$, where ε is some small positive number, and define $\bar{H} : X \times I \times I \to Z$ by

$$\bar{H}(x,s,t) := \begin{cases} f(k(x, \varphi(2s))) & s \leq 1/2 \\ H(x, t\varphi(2s-1)) & s \geq 1/2 \end{cases}.$$

This map is smooth since by assumption $f(k(x,1)) = f(i(x)) = H(x,0)$ for all x and thus it is obviously smooth on the open subsets $X \times [0,(1+\varepsilon)/2) \times I$ and $X \times ((1-\varepsilon)/2, 1] \times I$. Moreover we have $\bar{H}(x,0,t) = f(k(x,0)) = f(j(g(x)))$. Thus by the universal property of the push out the maps $\bar{H}$ and $f \circ j \circ pr_1 : Y \times I \to Z$ induce a smooth map $\tilde{H} : M_g \times I \to Z$ such that $\tilde{H} \circ (k \times Id) = \bar{H}$ and $\tilde{H} \circ (j \times Id) = f \circ j \circ pr_1$. In particular this implies that $\tilde{H}(i(x),t) = \tilde{H}(k(x,1),t) = \bar{H}(x,1,t) = H(x,t)$. So it remains to show that $\tilde{H}|_{M_g\times\{0\}}$ is smoothly homotopic to f relative to X.

Consider the smooth map $\bar{h} : X \times I \times I \to Z$ defined by $\bar{h}(x,s,t) := f(k(x,(1-t)s + t\varphi(2s)))$. This map satisfies $\bar{h}(x,0,t) = f(k(x,0)) = f(j(g(x)))$ and thus together with the map $f \circ j \circ pr_1 : Y \times I \to Z$ the map $\bar{h}$ induces a smooth map $h : M_g \times I \to Z$ such that $h \circ (k \times Id) = \bar{h}$ and $h \circ (j \times Id) = f \circ j \circ pr_1$. Now we have $(h|_{M_g\times\{1\}} \circ k)(x,s) = f(k(x,\varphi(2s))) = \bar{H}(x,s,0)$ and $h|_{M_g\times\{1\}} \circ j = f \circ j$ and thus $h|_{M_g\times\{1\}} = \tilde{H}|_{M_g\times\{0\}}$ and $h|_{M_g\times\{0\}} \circ k = f \circ k$ and $h|_{M_g\times\{0\}} \circ j = f \circ j$, so $h|_{M_g\times\{0\}} = f$. Finally $h(i(x),t) = h(k(x,1),t) = \bar{h}(x,1,t) = f(k(x,1)) = f(i(x))$ and thus h is a homotopy relative to X and the proof of (1) is complete.
(2): The maps $g \circ pr_1 : X \times I \to Y$ and $Id_Y : Y \to Y$ induce by the universal property of the push out a smooth map $h : M_g \to Y$ such that $h \circ k = g \circ pr_1$ and $h \circ j = Id_Y$. So we only have to show that $j \circ h : M_g \to M_g$ is homotopic to the identity. Define $H' : X \times I \times I \to M_g$ by $H'(x,t,s) = k(x,ts)$. Then $H'(x,0,s) = k(x,0) = j(g(x))$ and thus together with $j \circ pr_1 : Y \times I \to M_g$ the map H' induces a smooth map $H : M_g \times I \to M_g$ such that $H \circ (k \times Id) = H'$ and $H \circ (j \times Id) = j \circ pr_1$. Now we have $(H|_{M_g\times\{0\}} \circ k)(x,t) = k(x,0) = j(g(x))$ and $H|_{M_g\times\{0\}} \circ j = j$ and thus $H|_{M_g\times\{0\}} = j \circ h$ and $H|_{M_g\times\{1\}} \circ k = k$ and $H|_{M_g\times\{1\}} \circ j = j$, so $H|_{M_g\times\{1\}} = Id_{M_g}$.
(3): By 1.16 $X \times I$ is a base space and since $\{0\} \to I$ is easily seen to be a smooth cofibration so is $X \times \{0\} \to X \times I$ by 3.4(2). Thus the result follows from 3.5. $\square$

3.10. Corollary. *Let X and Y be arbitrary smooth spaces, $f : X \to Y$ a smooth map. Then there is a cofibration $i : X \to Y'$ and a homotopy equivalence $h : Y' \to Y$ such that $f = h \circ i$.*

Proof. Put $Y' = M_f$, i and h as in 3.9. Then obviously $h \circ i = f$. □

3.11. Mapping cones. Let $g : X \to Y$ be a smooth map. We define the *mapping cone* or *homotopy cofiber* C_g of g to be the cofiber of the cofibration $i : X \to M_g$.

Proposition. *Let C_g be the mapping cone of a smooth map $g : X \to Y$. Then C_g is diffeomorphic to the smooth space $CX \cup_g Y$ which is the push out:*

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ CX & \longrightarrow & CX \cup_g Y \end{array}$$

Moreover the natural map $Y \to C_g$ is a smooth cofibration.

Proof. We keep the notation of the proof of 3.9. Recall that CX is defined as the mapping cylinder of the unique smooth map $X \to pt$. Thus there is a natural smooth map $\alpha : X \times I \to CX$. Together with the canonical map $Y \to CX \cup_g Y$ the composition of the map $CX \to CX \cup_g Y$ with the map $(x, t) \mapsto \alpha(x, 1-t)$ induces a smooth map $\tilde{\varphi} : M_g \to CX \cup_g Y$. By construction this map sends $i(X)$ to a single point and thus it factors to a smooth map $\varphi : M_g/i(X) =: C_g \to CX \cup_g Y$.

On the other hand the composition of the natural map $M_g \to C_g$ with the map $X \times I \to M_g$ given by $(x, t) \mapsto k(x, 1-t)$ induces a smooth map $CX \to C_g$ and one easily checks that together with the natural map $Y \to C_g$ this map induces a smooth map $\psi : CX \cup_g Y \to C_g$ and a diagram chase shows that ψ is inverse to φ.

By 3.9(1) the map $X \to CX$ is a smooth cofibration, so by 3.5 the map $Y \to CX \cup_g Y$ is a smooth cofibration. By construction the map $Y \to C_g$ is the composition of ψ and the natural map $Y \to CX \cup_g Y$ and since diffeomorphisms are clearly smooth cofibrations the second claim follows from 3.4(1). □

3.12. Corollary. *If $g : X \to Y$ is a smooth map between base spaces then the homotopy cofiber C_g of g is a base space.*

Proof. Since $X \to CX$ is a smooth cofibration the space $CX \cup_g Y$ is a base space by 3.5. Thus the result follows from 3.11. □

3.13. Proposition. *Let $i : Y \to X$ be a smooth cofibration and assume that Y is smoothly contractible. Then the natural projection $p : X \to X/i(Y)$ is a smooth homotopy equivalence.*

Proof. Let $H : Y \times I \to Y$ be a smooth map such that $H(y, 0) = y$ and $H(y, 1) = y_0$ for some $y_0 \in Y$. Then the identity on X satisfies $Id_X \circ i = i \circ (H|_{Y \times \{0\}})$, so there is a smooth map $\tilde{H} : X \times I \to X$ such that $\tilde{H} \circ (i \times Id) = i \circ H$ and $\tilde{H}|_{X \times \{0\}}$ is smoothly homotopic to Id_X relative to Y. Thus $\tilde{H}|_{X \times \{1\}}$ maps $i(Y)$ to the point $i(y_0)$, so it induces a smooth map $j : X/i(Y) \to X$. By construction we have $j \circ p = \tilde{H}|_{X \times \{1\}}$ which is smoothly homotopic to $\tilde{H}|_{X \times \{0\}}$ and thus to the identity on X.

As $\tilde{H}|_{X \times \{0\}}$ is smoothly homotopic to the identity relative to Y, there is a homotopy h between $j \circ p$ and the identity on X which maps $i(Y) \times I$ to $i(Y)$, so $p \circ h : X \times I \to X/i(Y)$ is a homotopy between $p \circ j \circ p$ and p which map $i(Y) \times I$ to a single point and thus it induces a smooth homotopy $\bar{h} : X/i(Y) \times I \to X/i(Y)$ which by construction connects $p \circ j$ and the identity. Thus j is a homotopy inverse to p. □

3.14. Corollary. *If $i : X \to Y$ is a smooth cofibration then the cofiber and the homotopy cofiber of i are smoothly homotopy equivalent.*

Proof. By 3.11 we may consider the homotopy cofiber to be the space $CX \cup_i Y$. Now since i is a smooth cofibration we conclude from 3.5 that the natural map $\alpha : CX \to CX \cup_i Y$ is a smooth cofibration. As CX is clearly smoothly contractible the homotopy cofiber of i is smoothly homotopy equivalent to $(CX \cup_i Y)/\alpha(CX)$, so we only have to show that this space is smoothly homotopy equivalent to $Y/i(X)$.

Consider the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i"] & CX \arrow[r] \arrow[d, "\alpha"] & pt \arrow[d] \\ Y \arrow[r] & CX \cup_i Y \arrow[r] & (CX \cup_i Y)/\alpha(CX) \end{tikzcd}$$

By definiton the left hand square as well as the right hand square is a push out and thus also the outer square is a push out. But this is just the push out which defines the space $Y/i(X)$ so this space is even diffeomorphic to $(CX \cup_i Y)/\alpha(CX)$. $\square$

3.15. The constructions of cylinders and cones done so far are not well adapted to pointed spaces and base point preserving maps. In fact consider the definiton of the mapping cylinder in 3.8 and assume that (X, x_0) and (Y, y_0) are pointed spaces. Then there would be two canonical choices for the base point of M_f, namely $j(y_0)$ and $i(x_0)$ and clearly these two points never coincide. Thus only one of the two canonical maps i and j can be base point preserving. So we have to modify the constructions for pointed spaces. Recall that for pointed spaces we have the wedge and smash product (c.f. 1.24 and 1.25). Finally note that there is an obvious functor $X \mapsto X^+$ from the category of smooth spaces to the category of pointed smooth spaces which maps the space X to the disjoint union of X and a single point x^+, which is then considered as the base point.

3.16. Definition. Let (X, x_0) and (Y, y_0) be pointed smooth spaces, $g : X \to Y$ a base point preserving smooth map. We define the *reduced mapping cylinder* M'_g of g to be the push out:

$$\begin{tikzcd} I \arrow[r] \arrow[d] & M_g \arrow[d] \\ pt \arrow[r] & M'_g \end{tikzcd}$$

where the map $I \to M_g$ is the composition of the natural map $X \times I \to M_g$ with the map $t \mapsto (x_0, t)$. Then the maps $i : X \to M_g$ and $j : Y \to M_g$ induce maps $i : X \to M'_g$ and $j : Y \to M'_g$, and we have $i(x_0) = j(y_0)$, so we define this point to be the base point of M'_g. Now assume that (Z, z_0) is a pointed smooth space and we have given maps $H : X \times I \to Z$ and $f : M'_g \to Z$ such that $H(x_0, t) = z_0$ and $f \circ i = H|_{X \times \{0\}}$. Going through the proof of 3.9(1) we see that we get a smooth map $\tilde{H} : M'_g \to Z$ such that $\tilde{H} \circ i = H$ and such that $\tilde{H}|_{M'_g \times \{0\}}$ is smoothly homotopic to f relative to X. Thus $i : X \to M'_g$ is a smooth cofibration in the category $\underline{C_0^\infty}$. Moreover the proof of 3.9(2) shows that the map $j : Y \to M'_g$ is a smooth homotopy equivalence of pointed smooth spaces and thus the analogue of corollary 3.10 in the category $\underline{C_0^\infty}$ also holds.

As before we define the *reduced cone* $C'X$ of a pointed smooth space X to be the reduced mapping cylinder of the unique smooth map $X \to pt$.

Finally we define the *reduced mapping cone* C'_g of the map g to be the cofiber of the cofibration $i : X \to M'_g$. As in the proof of 3.11 one shows that C'_g is diffeomorphic to the space $C'X \cup_g Y$.

3.17. Definition. Let X and Y be smooth spaces. A smooth map $p : X \to Y$ is called a *smooth fibration* iff it satisfies the following homotopy lifting property: If Z is an arbitrary smooth space and $H : Z \times I \to Y$ and $f : Z \to X$ are smooth maps such that $p \circ f = H|_{Z\times\{0\}}$, then there is a smooth map $\tilde{H} : Z \times I \to X$ such that $p \circ \tilde{H} = H$ and such that $\tilde{H}|_{Z\times\{0\}}$ is homotopic to f via a fiber preserving homotopy, i.e. there is a smooth map $h : Z \times I \to X$ such that $h|_{Z\times\{0\}} = \tilde{H}|_{Z\times\{0\}}$, $h|_{Z\times\{1\}} = f$ and $p \circ h = p \circ f \circ pr_1$.

3.18. Proposition. *Let X and Y be smooth spaces and assume that Y is smoothly path connected, i.e. for every pair of points in Y there is a smooth curve passing through the two points, and let $p : X \to Y$ be a smooth fibration. Then p is surjective and a final morphism in the category of smooth spaces.*

Proof. Let y_0 be a point in Y which is contained in the image of X and let x_0 be such that $p(x_0) = y_0$. First we show that any smooth curve $c : \mathbb{R} \to Y$ with $c(0) = y_0$ has a lift to X, i.e. there is a smooth curve $\tilde{c} : \mathbb{R} \to X$ such that $p \circ \tilde{c} = c$: Consider the smooth map $H : \mathbb{R} \times I \to Y$ defined by $H(t, s) := c(ts)$. The constant curve $x_0 : \mathbb{R} \to X$ is a lift of $H|_{\mathbb{R}\times\{0\}}$, so there is a smooth map $\tilde{H} : \mathbb{R} \times I \to X$ such that $p \circ \tilde{H} = H$ and obviously $\tilde{c} := \tilde{H}|_{\mathbb{R}\times\{1\}}$ has the required property. Since Y is smoothly path connected this immediately implies the surjectivity of p, and this in turn clearly implies that any smooth curve in Y has a lift to X.

To show that p is final it suffices to show that any function $f : Y \to \mathbb{R}$ is smooth provided that $f \circ p$ is smooth. For this we have to show that the composition of f with any smooth curve in Y is smooth. But this immediately follows from the existence of smooth liftings of smooth curves. □

3.19. Lemma. *(1): If $p : X \to Y$ and $q : Y \to U$ are smooth fibrations then so is $q \circ p : X \to U$.*
(2): If $p : X \to Y$ is a smooth fibration and U is an arbitrary smooth space then $p_* : C^\infty(U, X) \to C^\infty(U, Y)$ is a smooth fibration.

Proof. (1): Let Z be an arbitrary smooth space and let $H : Z \times I \to U$ and $f : Z \to X$ be smooth maps such that $q \circ p \circ f = H|_{Z\times\{0\}}$. As q is a smooth fibration we get a smooth map $\bar{H} : Z \times I \to Y$ such that $q \circ \bar{H} = H$ and $\bar{H}|_{Z\times\{0\}}$ is homotopic to $p \circ f$ via a fiber preserving homotopy. Let $\bar{h} : Z \times I \to Y$ be such a homotopy, so $\bar{h}|_{Z\times\{0\}} = p \circ f$, $\bar{h}|_{Z\times\{1\}} = \bar{H}|_{Z\times\{0\}}$ and $q \circ \bar{h} = q \circ p \circ f \circ pr_1$. Applying the fibration property of p to $\bar{h}$ and f we get a smooth map $H' : Z \times I \to X$ such that $p \circ H' = \bar{h}$ and $H'|_{Z\times\{0\}}$ is homotopic to f via a fiber preserving homotopy. Now for $\tilde{f} := H'|_{Z\times\{1\}}$ we have $p \circ \tilde{f} = \bar{h}|_{Z\times\{1\}} = \bar{H}|_{Z\times\{0\}}$ and by the fibration property of p we thus get a smooth map $\tilde{H} : Z \times I \to X$ such that $p \circ \tilde{H} = \bar{H}$ and thus $q \circ p \circ \tilde{H} = H$. So it remains to show that $\tilde{H}|_{Z\times\{0\}}$ is homotopic to f via a homotopy which respects the fibers of $q \circ p$.

By construction $\tilde{H}|_{Z\times\{0\}}$ is homotopic to $\tilde{f} = H'|_{Z\times\{1\}}$ via a homotopy which preserves the fibers of p. As $q \circ p \circ H' = q \circ \bar{h}$ the homotopy H' preserves the fibers of $q \circ p$ and finally by construction $H'|_{Z\times\{0\}}$ is homotopic to f via a homotopy which preserves the fibers of p. Putting these homotopies together we get the result.
(2): Let Z be an arbitrary smooth space and assume we have given maps $H : Z \times I \to C^\infty(U, Y)$ and $f : Z \to C^\infty(U, X)$ such that $p_* \circ f = H|_{Z\times\{0\}}$. By cartesian closedness the associated mappings $\hat{H} : Z \times U \times I \to Y$ and $\hat{f} : Z \times U \to X$ are smooth and clearly we have $p \circ \hat{f} = \hat{H}|_{Z\times\{0\}}$. Applying the fibration property of p to these two maps and using cartesian closedness again one easily proves the result. □

3.20. Lemma. *Let $p : X \to Y$ be a smooth fibration and let*

$$\begin{tikzcd} U \arrow[r, "p^*g"] \arrow[d, "g^*p"'] & X \arrow[d, "p"] \\ V \arrow[r, "g"] & Y \end{tikzcd}$$

be a pullback. Then $g^*p : U \to V$ is a smooth fibration.

Proof. Let Z be a smooth space, $H : Z \times I \to V$ and $f : Z \to U$ be smooth maps such that $g^*p \circ f = H|_{Z\times\{0\}}$. Then we have $p \circ p^*g \circ f = g \circ g^*p \circ f = (g \circ H)|_{Z\times\{0\}}$, so we get a smooth map $\bar{H} : Z \times I \to X$ such that $p \circ \bar{H} = g \circ H$ and thus by the universal property of the pullback a smooth map $\tilde{H} : Z \times I \to U$ such that $g^*p \circ \tilde{H} = H$ and $p^*g \circ \tilde{H} = \bar{H}$. So we only have to show that $\tilde{H}|_{Z\times\{0\}}$ is homotopic to f via a homotopy which preserves the fibers of g^*p. By construction $\bar{H}|_{Z\times\{0\}}$ is homotopic to $p^*g \circ f$ via a homotopy which preserves the fibers of p. So there is a homotopy $\bar{h} : Z \times I \to X$ such that $\bar{h}|_{Z\times\{0\}} = \bar{H}|_{Z\times\{0\}}$ and $\bar{h}|_{Z\times\{1\}} = p^*g \circ f$ and $p \circ \bar{h} = p \circ p^*g \circ f \circ pr_1 = g \circ g^*p \circ f \circ pr_1$. Thus together with the map $g^*p \circ f \circ pr_1 : Z \times I \to V$ the map $\bar{h}$ induces by the universal property of the pullback a smooth map $h : Z \times I \to U$ such that $p^*g \circ h = \bar{h}$ and $g^*p \circ h = g^*p \circ f \circ pr_1$. Thus h is a homotopy which preserves the fibers of g^*p and one easily shows that it is a homotopy between f and $\tilde{H}|_{Z\times\{0\}}$. $\square$

3.21. Mapping cocylinders and mapping cocones. Let X and Y be smooth spaces, $g : X \to Y$ and arbitrary smooth map. We define the *mapping cocylinder* M^g of g to be the pullback:

$$\begin{tikzcd} M^g \arrow[r] \arrow[d] & C^\infty(I, Y) \arrow[d, "ev_0"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

The composition of the evaluation at 1 and the canonical map $M^g \to C^\infty(I, Y)$ defines a smooth map $p : M^g \to Y$.

As a special case we define for a pointed smooth space (X, x_0) the *path space* PX over X to be the mapping cocylinder of the inclusion of x_0 into X.

For a base point preserving map $g : X \to Y$ between pointed smooth spaces we define the *mapping cocone* or *homotopy fiber* C^g of g to be the pullback:

$$\begin{tikzcd} C^g \arrow[r] \arrow[d] & PY \arrow[d, "p"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

3.22. Proposition. *Let M^g be the mapping cocylinder of a smooth map $g : X \to Y$. Then we have:*
(1): The map $p : M^g \to Y$ is a smooth fibration.
(2): The natural map $q : M^g \to X$ is a smooth homotopy equivalence.

Proof. (1): Let Z be an arbitrary smooth space, $H : Z \times I \to Y$ and $f : Z \to M^g$ smooth maps such that $p \circ f = H|_{Z\times\{0\}}$. Composing the natural map $M^g \to C^\infty(I, Y)$ with f and taking the map associated via cartesian closedness we get a smooth map $\tilde{f} : Z \times I \to Y$ such that $\tilde{f}(z, 1) = H(z, 0)$. Now let $\varphi \in C^\infty(\mathbb{R}, I)$ be an increasing map such that $\varphi(t) = 0$

for $t \leq \varepsilon$ and $\varphi(t) = 1$ for $t \geq 1 - \varepsilon$, where ε is some small positive number, and define $\bar{H} : Z \times I \times I \to Y$ by

$$\bar{H}(z,t,s) := \begin{cases} \tilde{f}(z, \varphi(2s)) & s \leq 1/2 \\ H(z, t \cdot \varphi(2s-1)) & s \geq 1/2 \end{cases}$$

This map is smooth since $\tilde{f}(z,1) = H(z,0)$ and thus it is obviously smooth on the open subsets $Z \times I \times [0,(1+\varepsilon)/2)$ and $Z \times I \times ((1-\varepsilon)/2, 1]$. Thus also the associated map $\bar{H}^\vee : Z \times I \to C^\infty(I,Y)$ given by $(\bar{H}^\vee(z,t))(s) = \bar{H}(z,t,s)$ is smooth and we have $\bar{H}^\vee(z,t)(0) = \tilde{f}(z,0)$ and thus $ev_0 \circ \bar{H}^\vee = g \circ q \circ f$, where $q : M^g \to X$ denotes the natural map. By the universal property of the pullback we get a smooth map $\tilde{H} : Z \times I \to M^g$ induced by $\bar{H}^\vee$ and $q \circ f$. Now $(p \circ \tilde{H})(z,t) = \bar{H}^\vee(z,t,1) = H(z,t)$ and thus we only have to show that $\tilde{H}|_{Z\times\{0\}}$ is smoothly homotopic to f via a fiber preserving homotopy.

Consider the smooth map $\bar{h} : Z \times I \times I \to Y$ defined by $\bar{h}(z,t,s) := \tilde{f}(z,(1-t)s + t\varphi(2s))$. By catesian clesedness the associated map $\bar{h}^\vee : Z \times I \to C^\infty(I,Y)$ is smooth and we have $(\bar{h}^\vee(z,t))(0) = \tilde{f}(z,0) = g(q(f(z)))$ and thus together with $q \circ f \circ pr_1 : Z \times I \to X$ the map $\bar{h}^\vee$ induces a smooth homotopy $h : Z \times I \to M^g$. In particular this implies that $q \circ h = q \circ f \circ pr_1$, so the homotopy h respects the fibers of q, and one easily checks that it is indeed a homotopy between $\tilde{H}|_{Z\times\{0\}}$ and f.
(2): Together with the map $X \to C^\infty(I,Y)$ which assigns to $x \in X$ the constant map $t \mapsto g(x)$ the identity on X induces a smooth map $h : X \to M^g$ such that $q \circ h = Id_X$. So it remains to show that $h \circ q : M^g \to M^g$ is homotopic to the identity. Define $\bar{H} : M^g \times I \to C^\infty(I,Y)$ by $(\bar{H}(u,t))(s) := (k(u))(ts)$, where $k : M^g \to C^\infty(I,Y)$ is the natural map. Then $ev_0 \circ \bar{H} = ev_0 \circ k = g \circ q$ and thus togehter with $q \circ pr_1 : M^g \times I \to X$ the map $\bar{H}$ induces a smooth homotopy $H : M^g \times I \to M^g$, and one immediately checks that this is a homotopy between $h \circ q$ and the identity. $\square$

3.23. Corollary. *Let X and Y be smooth spaces, $f : X \to Y$ a smooth map. Then there is a homotopy equivalence $h : X \to X'$ and a fibration $p : X' \to Y$ such that $f = p \circ h$.*

Proof. Put $X' = M^f$, p and h as in the proof of 3.22. Then clearly $p \circ h = f$. $\square$

3.24. Corollary. *Let X and Y be pointed smooth spaces, C^g the mapping cocone of a base point preserving smooth map $g : X \to Y$. Then the natural map $p : C^g \to X$ is a smooth fibration.*

Proof. $PY \to Y$ is a fibration by 3.22(1). Thus the result follows from 3.20. $\square$

3.25. Proposition. *Let X and Y be pointed smooth spaces, C^g the mapping cocone of a base point preserving smooth map $g : X \to Y$. Then C^g is diffeomorphic to the fiber over y_0 of the smooth fibration $p : M^g \to Y$. Moreover the diffeomorphism can be chosen such that its composition with the natural map $M^g \to X$ euqals the natural map $C^g \to X$.*

Proof. Recall that C^g is the pullback of the path fibration PY over Y along g and that PY is the set of all smooth functions $c : I \to Y$ such that $c(0) = y_0$ while the projection to Y is given by the evaluation at 1. Now let $\varphi : PY \to C^\infty(I,Y)$ be the map defined by $\varphi(c)(t) := c(1-t)$ which is obviously smooth by cartesian closedness. Then the composition of this map with the natural map $C^g \to PY$ together with the natural map $C^g \to X$ induce by the universal property of the pullback defining M^g a smooth map $\Phi : C^g \to M^g$ and one immediately checks that the composition $p \circ \Phi$ is the constant map y_0.

On the other hand one easily checks that the fiber over y_0 of the fibration $p : M^g \to Y$ is just the pullback of the diagram

$$\begin{array}{ccc} & & \tilde{P}Y \\ & & \Big\downarrow{\scriptstyle ev_0} \\ X & \xrightarrow{g} & Y \end{array}$$

where $\tilde{P}Y$ is the space of all smooth maps $c : I \to Y$ such that $c(1) = y_0$ with the initial smooth structure with respect to the inclusion into $C^\infty(I, Y)$, and the obvious diffeomorphism $\tilde{P}Y \to PY$ given by the same formula as the map φ above induces and inverse to Φ. $\square$

3.26. Before we can continue the study of smooth fibrations we have to do some more work on cofibrations. First we give a characterization of inclusion mappings which are cofibrations. This characterization also shows that the notion of cofibration we use is quite near to the classical notion (c.f. [Whitehead, 1978, I.5.1.]).

Definiton[1]**.** Let X be a smooth space, $A \subset X$ a subspace. The pair (X, A) is called a *smooth NDR–pair* iff there are smooth maps $u : X \to I$ and $h : X \times I \to X$ such that:
(i) $u(A) = 0$
(ii) $h|_{X\times\{0\}} = Id_X$
(iii) $h|_{A\times I} = pr_1$, so h is a homotopy relative to A.
(iv) $h(x, 1) \in A$ for all $x \in X$ such that $u(x) < 1$.

There is one subtle point in this context: Let $i : A \to X$ be the inclusion of a subspace and consider the smooth space $(A \times I) \cup_i X$ which is defined as the push out

$$\begin{tikzcd} A\times\{0\} \arrow[r] \arrow[d, "i"'] & A\times I \arrow[d] \\ X \arrow[r] & (A\times I)\cup_i X \end{tikzcd}$$

Then the maps $x \mapsto (x, 0)$ and $i \times id_I : A \times I \to X \times I$ induce a smooth map $j : (A \times I) \cup_i X \to X \times I$ which is clearly a bijection onto the subspace $X \times \{0\} \cup A \times I$. But it is not clear at all that the two smooth structures coincide, so we have to distinguish between these two spaces.

3.27. Proposition. *Let X be a smooth space, $A \subset X$ a subspace. Then the following conditions are equivalent:*
(1): (X, A) is a smooth NDR–pair.
(2): There is a smooth map $H : X \times I \times I \to X \times I$ such that $H|_{X\times I\times\{0\}} = Id_{X\times I}$, $H(X\times\{0\}\times I) \subset X\times\{0\}$, H is a homotopy relative to $A\times I$, $H(X\times I\times\{1\}) \subset X\times\{0\}\cup A\times I$ and $H|_{X\times I\times\{1\}}$ is smooth as a map to $(A \times I) \cup_i X$.
(3): There is a smooth map $f : X \times I \to (A \times I) \cup_i X$ such that the composition of f with the map j defined above is homotopic to the identity relative to $A \times I$.
(4): The inclusion map of A into X is a smooth cofibration.

Proof. (1) $\Rightarrow$ (2): Let u and h be the maps ocurring in the definition of an NDR–pair and let $\varphi \in C^\infty(\mathbb{R}, I)$ be an increasing map such that $\varphi(t) = 0$ for all $t \le \varepsilon$ and $\varphi(t) = 1$ for all $t \ge 1 - \varepsilon$, where ε is some small positive number. Then define $H : X \times I \times I \to X \times I$ by $H(x, t, s) := (h(x, s), t(1 - s\varphi(u(x))))$. Then obviously $H(x, t, 0) = (x, t)$ and $H(x, 0, s) \in X \times \{0\}$ and since $h(a, t) = a$ and $u(a) = 0$ for all $a \in A$ we have $H(a, t, s) = (a, t)$. Finally if x is such that $u(x) = 1$ then clearly $H(x, t, 1) \in X \times \{0\}$ and if $u(x) < 1$ then $h(x, 1) \in A$ and thus $H(x, t, 1) \in A \times I$. So it suffices to show that $H|_{X\times I\times\{1\}}$ is smooth as a map to $(A \times I) \cup_i X$. By definition of the smooth structure on a push out a map $(A \times I) \cup_i X \to \mathbb{R}$ is smooth if and only if it is induced by smooth map $g_1 : X \to \mathbb{R}$ and $g_2 : A \times I \to \mathbb{R}$ which satisfy $g_1|_A = g_2|_{A\times\{0\}}$. Now by construction of H the point $H(x, t, 1)$ lies in $X \times \{0\}$ if $u(x) > 1-\varepsilon$ and in $A\times I$ if $u(x) < 1$. Thus the composition of a smooth real valued function

[1]Note that this definition is also a little weaker than the usual definition in topology, since there one only deals with closed subspaces and requires that the subspace A is exactly the zero set of the function u.

on $(A \times I) \cup_i X$ as above with $H|_{X\times I\times\{1\}}$ is given by $g_1(h(x,1))$ if $u(x) > 1-\varepsilon$ and by $g_2(h(x,1), t(1-\varphi(u(x))))$ if $u(x) < 1$ and from this description one immediately concludes that it is smooth.
(2) $\Rightarrow$ (3): Put $f := H|_{X\times I\times\{1\}} : X \times I \to (A \times I) \cup_i X$. By cartesian closedness the diagram

$$\begin{tikzcd} A \times \{0\} \times I \arrow[r] \arrow[d, "i\times id_I"'] & A \times I \times I \arrow[d] \\ X \times I \arrow[r] & ((A \times I) \cup_i X) \times I \end{tikzcd}$$

is a push out. Moreover the inclusion of $X \times \{0\}$ into $X \times I$ is obviously an initial morphism and thus $H|_{X\times\{0\}\times I} : X \times I \to X$ is a smooth map and one immediately checks that the composition of the natural map $X \to (A \times I) \cup_i X$ with this map together with the composition of the natural map $A\times I \to (A\times I)\cup_i X$ with the projection $A\times I\times I \to A\times I$ onto the first two factors induces the desired homotopy.
(3) $\Rightarrow$ (4): Let Z be an arbitrary smooth space and let $H : A \times I \to Z$ and $g : X \to Z$ be smooth maps such that $g|_A = H_{A\times\{0\}}$. Then these maps define a smooth map $(A\times I)\cup_i X \to Z$ and composing this map with f we get a smooth map $\tilde{H} : X \times I \to Z$. Clearly we have $\tilde{H}|_{A\times I} = H$ and from the properties of f one immediately deduces that $\tilde{H}|_{X\times\{0\}}$ is smoothly homotopic to g relative to A.
(4) $\Rightarrow$ (3): Applying the cofibration property to the canonical maps $X \to (A \times I) \cup_i X$ and $A \times I \to (A \times I) \cup_i X$ we get a smooth map $f : X \times I \to (A \times I) \cup_i X$ such that $f|_{A\times I}$ is the canonical map $A \times I \to (A \times I) \cup_i X$ and such that $f|_{X\times\{0\}}$ is smoothly homotopic to the natural map $X \to (A \times I) \cup_i X$ relative to A. Together with the composition of the natural map $A \times I \to (A \times I) \cup_i X$ with the projection $A \times I \times I \to A \times I$ onto the first two factors such a homotopy induces a smooth homotopy between the identity and $f \circ j$.
(3) $\Rightarrow$ (1): Together with the constant map $1 : X \to I$ the map $A \times I \to I$ given by $(a, t) \mapsto 1 - t$ induces a smooth map $v : (A \times I) \cup_i X \to I$ and we define $u : X \to \mathbb{R}$ as $v \circ f|_{X\times\{1\}}$. Then by construction $u(A) = 0$. Let $p : (A \times I) \cup_i X \to X$ be the map induced by the identity on X and the first projection $A \times I \to X$ and let $H : ((A \times I) \cup_i X) \times I \to (A \times I) \cup_i X$ be a smooth homotopy relative to $A \times I$ from the identity to $f \circ j$. Then we can view the composition of p with the smooth map $X \times I \to (A \times I) \cup_i X$ induced by H as a homotopy relative to A between the identity on X and the map $p \circ (f|_{X\times\{0\}})$. Next $p \circ f$ is just a homotopy relative to A between $p \circ (f|_{X\times\{0\}})$ and $p \circ (f|_{X\times\{1\}})$. Piecing theses two homotopies together smoothly we get a smooth homotopy relative to A, $h : X \times I \to X$ from the identity to $p\circ(f|_{X\times\{1\}})$. But by construction of u if $u(x) < 1$ then $(p\circ(f|_{X\times\{0\}}))(x) \in A$ and thus h satisfies all conditions of 3.26. $\square$

3.28. Proposition. *Let $p : X \to Y$ be an arbitrary smooth map and let (Z, A) be a smooth NDR–pair. Suppose we have given smooth maps $H : Z \times I \to Y$, $H_1 : A \times I \to X$ and $f : Z \to X$ such that $p \circ f = H|_{Z\times\{0\}}$, $p \circ H_1 = H|_{A\times I}$ and $f|_A = H_1|_{A\times\{0\}}$.*

Then there is a smooth map $\tilde{H} : Z \times I \to X$ such that $\tilde{H}|_{A\times I} = H_1$, $p \circ \tilde{H}$ is smoothly homotopic to H relative to $A \times I$ and $\tilde{H}|_{Z\times\{0\}}$ is smoothly homotopic to f relative to A.

Proof. By 3.27(2) there is a smooth homotopy $\Phi : Z \times I \times I \to Z \times I$ relative to $A \times I$ with $\Phi|_{Z\times I\times\{0\}} = Id_{Z\times I}$ such that the map $\varphi := \Phi|_{Z\times I\times\{1\}}$ has values in $Z \times \{0\} \cup A \times I$. Moreover the restriction of Φ to the subspace $Z \times \{0\} \times I$ has values in $Z \times \{0\}$. As in the proof of 3.27 we see that the maps H_1 and f define a smooth map $g : Z \times \{0\} \cup A \times I \to X$ and by assumption $p\circ g = H \circ j$, where $j : Z\times\{0\}\cup A\times I \to Z\times I$ is the inclusion. Consider the map $\tilde{H} := g \circ \varphi : Z \times I \to X$. As Φ is a homotopy relative to $A \times I$ we obviously have $\tilde{H}|_{A\times I} = g|_{A\times I} = H_1$. Next we have $p\circ\tilde{H} = H\circ j\circ\varphi$. Now the homotopy Φ passed backwards is a homotopy relative to $A \times I$ between $j \circ \varphi$ and the identity on $Z \times I$ and thus $p \circ \tilde{H}$ is smoothly homotopic to H relative to $A\times I$. Finally $\tilde H|_{Z\times\{0\}} = g\circ(\varphi|_{Z\times\{0\}}) = f\circ(\varphi|_{Z\times\{0\}})$. Now the restriction of Φ to the subspace $Z\times\{0\}\times I$ defines a smooth homotopy relative to A between $\varphi|_{Z\times\{0\}}$ and the identity on Z and thus the maps $\tilde H|_{Z\times\{0\}}$ and f are smoothly homotopic relative to A. □

3.29. Fibers of a fibration. Let $p : X \to Y$ be a smooth fibration and let y_0 be a point in Y. We define the *fiber* over y_0 to be the set $p^{-1}(y_0)$ together with the initial smooth structure with respect to the inclusion into X.

Now let $u : I \to Y$ be a path in Y and put $y_i := u(i)$ and let F_i be the fiber over y_i for $i = 0, 1$. Let Z be an arbitrary smooth space and let $f : Z \to F_0$ be a smooth map. Then we define a smooth map $H : Z\times I \to Y$ by $H(z,t) := u(t)$. Viewing f as a map to X we see that $p\circ f = H|_{Z\times\{0\}}$, so as p is a smooth fibration there is a smooth map $\tilde H : Z\times I \to X$ such that $p\circ\tilde H = H$ and $\tilde H|_{Z\times\{0\}}$ is smoothly homotopic to f via a fiber respecting homotopy, which just means that the maps $\tilde H|_{Z\times\{0\}}$ and f are smoothly homotopic as maps from Z to F_0. For the restriction $g := \tilde H|_{Z\times\{1\}}$ we have $p\circ g = y_1$ and thus g is a smooth map from Z to F_1.

3.30. Lemma. *Let $p : X \to Y$ be a smooth fibration, $y_0, y_1 \in Y$ and let $u, v : I \to Y$ be maps with $u(i) = v(i) = y_i$ for $i = 0, 1$ which are homotopic relative to $\{0,1\}$, so there is a smooth map $h : I\times I \to Y$ such that $h(0,t) = u(t)$, $h(1,t) = v(t)$ and $h(s,i) = y_i$ for $i = 0, 1$ and all $s \in I$. Moreover let Z be an arbitrary smooth space and let $U, V : Z\times I \to X$ be smooth maps such that $p(U(z,t)) = u(t)$ and $p(V(z,t)) = v(t)$.*

If the maps $U|_{Z\times\{0\}}$ and $V|_{Z\times\{0\}}$ are smoothly homotopic as maps from Z to $F_0 := p^{-1}(y_0)$ then the maps $U|_{Z\times\{1\}}$ and $V|_{Z\times\{1\}}$ are smoothly homotopic as maps from Z to $F_1 := p^{-1}(y_1)$.

Proof. Define $H_1 : Z\times\{0,1\}\times I \to X$ by $H_1(z,0,t) := U(z,t)$ and $H_1(z,1,t) = V(z,t)$ and define $H : Z\times I\times I \to Y$ by $H(z,s,t) := h(s,t)$. Let $f : Z\times I \to X$ be a smooth homotopy between $U|_{Z\times\{0\}}$ and $V|_{Z\times\{0\}}$, viewed as a function with values in X. Then by construction we have $p\circ f = y_0 = H|_{Z\times I\times\{0\}}$, $p\circ H_1 = H|_{Z\times\{0,1\}\times I}$ and $f|_{Z\times\{0,1\}} = H_1|_{Z\times\{0,1\}\times\{0\}}$. Now $(I,\{0,1\})$ is easily seen to be a smooth NDR–pair and thus by 3.4(2) and 3.27 $(Z\times I, Z\times\{0,1\})$ is a smooth NDR–pair. Thus we can apply proposition 3.28 to get a smooth map $\tilde H : Z\times I\times I \to X$ such that $\tilde H|_{Z\times\{0,1\}\times I} = H_1$ and such that $p\circ\tilde H$ is smoothly homotopic to H relative to $Z\times\{0,1\}\times I$.
Thus there is a homotopy $\Phi : Z\times I\times I\times I \to Y$ such that $\Phi|_{Z\times I\times I\times\{0\}} = p\circ\tilde H$ and $\Phi|_{Z\times I\times I\times\{1\}} = H$ and $\Phi(z,i,t,s) = H(z,i,t)$ for $i = 0, 1$. Applying the homotopy lifting property of the fibration p to the maps Φ and $\tilde H$ we get a smooth map $\tilde\Phi : Z\times I\times I\times I \to X$ such that $p\circ\tilde\Phi = \Phi$ and such that $\tilde\Phi|_{Z\times I\times I\times\{0\}}$ is smoothly homotopic to $\tilde H$ via a fiber preserving homotopy. Now consider the map $\Psi := \tilde\Phi|_{Z\times I\times\{1\}\times\{1\}}$. We have $p\circ\Psi = \Phi|_{Z\times I\times\{1\}\times\{1\}} = H|_{Z\times I\times\{1\}} = y_1$, so Ψ can be considered as a smooth homotopy between the maps $\Psi|_{Z\times\{0\}}$ and $\Psi|_{Z\times\{1\}} : Z \to F_1$.

Next we have $p\circ\tilde\Phi|_{Z\times\{i\}\times\{1\}\times I} = H|_{Z\times\{i\}\times\{1\}} = y_1$ for $i = 0, 1$, so $\tilde\Phi|_{Z\times\{i\}\times\{1\}\times I}$ defines smooth homotopies between the maps $\Psi|_{Z\times\{i\}}$ and $\tilde\Phi|_{Z\times\{i\}\times\{1\}\times\{0\}}$ as maps from Z to F_1 for $i = 0, 1$. By construction the map $\tilde\Phi|_{Z\times I\times I\times\{0\}}$ is smoothly homotopic to $\tilde H$ via a fiber preserving homotopy $Z\times I\times I\times I \to X$. Restricting this homotopy to $Z\times\{i\}\times\{1\}\times I$ we get smooth homotopies between the maps $\tilde\Phi|_{Z\times\{i\}\times\{1\}\times\{0\}}$ and $\tilde H|_{Z\times\{i\}\times\{1\}}$ as maps from Z to F_1 for $i = 0, 1$ and we have $\tilde H(z,0,1) = U(z,1)$ and $\tilde H(z,1,1) = V(z,1)$. Piecing all these homotopies together smoothly we see that $U|_{Z\times\{1\}}$ and $V|_{Z\times\{1\}}$ are smoothly homotopic as maps from Z to F_1. □

3.31. Theorem. *Let $p : X \to Y$ be a smooth fibration and assume that Y is smoothly path connected. Then for any two points $y_0, y_1 \in Y$ the fiber F_0 over y_0 is smoothly homotopy* *equivalent to the fiber F_1 over y_1.*

Proof. Let $u : I \to Y$ be a path in Y such that $u(i) = y_i$ for $i = 0, 1$. Now from the identity on F_0 we construct as in 3.29 a smooth homotopy $H_1 : F_0 \times I \to X$ such that $p(H_1(x,t)) = u(t)$ and such that $H_1|_{F_0\times\{0\}}$ is smoothly homotopic to the identity as a map from F_0 to F_0, and we put $f := H_1|_{F_0\times\{1\}} : F_0 \to F_1$. In the same way replacing the path u by the path $\bar{u}$ defined by $\bar{u}(t) = u(1-t)$ we get a homotopy $H_2 : F_1 \times I \to X$ such that $p(H_2(x,t)) = u(1-t)$ and such that $H_2|_{F_0\times\{0\}}$ is smoothly homotopic to the identity, and a smooth map $g := H_2|_{F_1\times\{1\}}$.

As $H_2|_{F_1\times\{0\}}$ is smoothly homotopic to the identity there is a smooth map $H : F_1 \times I \to X$ such that $p \circ H = y_1$ and $H(x,0) = x$ and $H(x,1) = H_2(x,0)$. Now let $\varphi \in C^\infty(\mathbb{R}, I)$ be an increasing map such that $\varphi(t) = 0$ for $t \leq \varepsilon$ and $\varphi(t) = 1$ for $t \geq 1-\varepsilon$, where ε is some small positive number and consider the map $U : F_0 \times I \to X$ defined by

$$U(x,t) := \begin{cases} H_1(x, \varphi(3t)) & t \leq 1/3 \\ H(f(x), \varphi(3t-1)) & 1/3 \leq t \leq 2/3 \\ H_2(f(x), \varphi(3t-2)) & t \geq 2/3 \end{cases}$$

The map U is smooth as it is by construction obviously smooth on the open subsets $F_0 \times [0, (1+\varepsilon)/3)$, $F_0 \times ((1-\varepsilon)/3, (2+\varepsilon)/3)$ and $F_0 \times ((2-\varepsilon)/3, 1]$. By construction $U|_{F_0\times\{0\}}$ is smoothly homotopic to the identity and $U|_{F_0\times\{1\}} = g \circ f$. The map $p \circ U$ is a path $\tilde{u} : I \to Y$ given by

$$\tilde{u}(t) = \begin{cases} u(\varphi(3t)) & t \leq 1/3 \\ y_1 & 1/3 \leq t \leq 2/3 \\ u(1-\varphi(3t-2)) & t \geq 2/3 \end{cases}$$

Now the path $\tilde{u}$ is smoothly homotopic to the constant path y_0 relative to $\{0,1\}$ via the homotopy $h : I \times I \to Y$ defined by

$$h(s,t) := \begin{cases} u(s\varphi(3t)) & t \leq 1/3 \\ u(s) & 1/3 \leq t \leq 2/3 \\ u(s(1-\varphi(3t-2))) & t \geq 2/3 \end{cases}$$

Applying lemma 3.30 to $\tilde{u}$ and the constant path y_0 and to U and $V : F_0 \times I \to X$ given by $V(x,t) = x$ we see that $g \circ f : F_0 \to F_0$ is smoothly homotopic to the identity. In the same way one shows that $f \circ g : F_1 \to F_1$ is homotopic to the identity and thus F_0 and F_1 are smoothly homotopy equivalent. □

3.32. Definition. Let $p : X \to Y$ and $p' : X' \to Y'$ be smooth fibrations with Y and Y' smoothly path connected. A smooth map $F : X \to X'$ is called a *fibered morphism* iff for any $x_1, x_2 \in X$ with $p(x_1) = p(x_2)$ we have $p'(F(x_1)) = p'(F(x_2))$. Thus we get a well defined map $f : Y \to Y'$ such that $f \circ p = p' \circ F$. Since p is a final morphism in the category of smooth spaces by 3.18 the map f is smooth.

Now assume that $Y = Y'$, so wee have two fibrations over the same base. Then p and p' are said to have the same *fiber homotopy type* iff there are fibered morphisms $F : X \to X'$ and $G : X' \to X$ such that $f = g = Id_Y$, i.e. $p' \circ F = p$ and $p \circ G = p'$, such that $G \circ F$ and $F \circ G$ are homotopic to Id_X and $Id_{X'}$ respectively via fiber preserving homotopies.

If p and p' have the same fiber homotopy type then in particular for any point $y \in Y$ the fibers $p^{-1}(y)$ and ${p'}^{-1}(y)$ are smoothly homotopy equivalent.

3.33. Proposition. *Let $f : X \to Y$ be a smooth fibration with Y smoothly path connected and let M^f be the mapping cocylinder of f. Then the fibration $p : M^f \to Y$ constructed in 3.21 has the same fiber homotopy type as f.*

Proof. Recall that M^f is defined as the pullback

$$\begin{tikzcd} M^f \arrow[r, "j"] \arrow[d, "q"'] & C^\infty(I,Y) \arrow[d, "ev_0"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and that $p : M^f \to Y$ is defined as the composition of the evaluation at 1 with the natural map $j : M^f \to C^\infty(I,Y)$. In the proof of 3.22 we constructed a smooth map $h : X \to M^f$ such that $q \circ h = Id_X$, $j(h(x))$ is the constant curve $t \mapsto f(x)$ and $p \circ h = f$. Consider the smooth map $H : M^f \times I \to Y$ defined by $H(z,t) := j(z)(t)$. Clearly $f \circ q = H|_{M^f \times \{0\}}$, so as f is a smooth fibration we get a smooth map $\tilde{H} : M^f \times I \to X$ such that $f \circ \tilde{H} = H$ and such that $\tilde{H}|_{M^f \times \{0\}}$ is smoothly homotopic to q via a homotopy which preserves the fibers of f. Put $g := \tilde{H}|_{M^f \times \{1\}} : M^f \to X$. Then $f \circ g = H|_{M^f \times \{1\}} = ev_1 \circ j = p$.

Consider the map $g \circ h : X \to X$. By construction $j(h(x))$ is the constant curve $t \mapsto f(x)$ and thus $f \circ \tilde{H} \circ (h \times Id) = f \circ pr_1$, so $\tilde{H} \circ (h \times Id)$ is a fiber preserving homotopy between $g \circ h$ and $\tilde{H}|_{M^f \times \{0\}} \circ h$. By construction this map in turn is homotopic to $q \circ h = Id_X$ via a fiber preserving homotopy.

Next define $\omega : C^\infty(I,Y) \times I \to C^\infty(I,Y)$ by $\omega(u,t)(s) := u(s+t-st)$. This is well defined as $s+t-st \le 1$ and it is immediately seen to be smooth using cartesian closedness. By construction we have $(ev_0 \circ \omega)(j(z),t) = j(z)(t) = H(z,t) = (f \circ \tilde{H})(z,t)$ for all $z \in M^f$, so by the universal property of the pullback there is a unique map $\bar{H} : M^f \times I \to M^f$ such that $j \circ \bar{H} = \omega \circ (j \times Id)$ and $q \circ \bar{H} = \tilde{H}$. In particular this implies that $p \circ \bar{H} = ev_1 \circ \omega \circ (j \times Id) = p \circ pr_1$ by construction of ω, so the homotopy $\bar{H}$ respects the fibers of p. Now $q \circ \bar{H}|_{M^f \times \{1\}} = g = q \circ h \circ g$ and $(j \circ \bar{H}|_{M^f \times \{1\}})(z)$ is the constant curve $t \mapsto j(z)(1) = p(z) = f(g(z))$ and this is just $j(h(g(z)))$, so $\bar{H}|_{M^f \times \{1\}} = h \circ g$.

So $h \circ g$ is smoothly homotopic to $\bar{H}|_{M^f \times \{0\}}$ via a homotopy which preserves the fibers of p and we have $j \circ \bar{H}|_{M^f \times \{0\}} = j$ and $q \circ \bar{H}|_{M^f \times \{0\}} = \tilde{H}|_{M^f \times \{0\}}$. By construction of $\tilde{H}$ there is a smooth homotopy $\Phi : M^f \times I \to X$ such that $\Phi|_{M^f \times \{0\}} = \tilde{H}|_{M^f \times \{0\}}$ and $\Phi|_{M^f \times \{1\}} = q$ and $f \circ \Phi = ev_0 \circ j \circ pr_1$. By the universal property of the pullback the map Φ together with the map $j \circ pr_1 : M^f \times I \to C^\infty(I,Y)$ induces a smooth map $\tilde{\Phi} : M^f \times I \to M^f$ such that $q \circ \tilde{\Phi} = \Phi$ and $j \circ \tilde{\Phi} = j \circ pr_1$, so in particular $p \circ \tilde{\Phi} = p \circ pr_1$ and thus $\tilde{\Phi}$ respects the fibers of p, and one immediately checks that $\tilde{\Phi}$ is a smooth homotopy between $\bar{H}|_{M^f \times \{0\}}$ and the identity on M^f. □

3.34. Our next task is to study exact sequences of sets of homotopy classes. We begin with the case of free homotopy classes. Let W be a smoothly path connected smooth space. Then for any smooth space X the set of homotopy classes $[X,W]$ has a natural base point, namely the homotopy class of the map which maps the whole of X to a single point. (As W is smoothly path connected all smooth maps which map X to a single point are smoothly homotopic.) Now let $f : X \to Y$ and $g : Y \to Z$ be smooth maps. Then the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is said to be *right exact* if and only if for any smoothly path connected smooth space W the sequence $[Z,W] \xrightarrow{g^*} [Y,W] \xrightarrow{f^*} [X,W]$ is an exact sequence of pointed sets, i.e. a homotopy class in $[Y,W]$ is mapped by f^* to the base point of $[X,W]$ if and only if it lies in the image of g^*.

3.35. Proposition. *If $i : X \to Y$ is a smooth cofibration with cofiber $Y/i(X)$ then the sequence $X \xrightarrow{i} Y \xrightarrow{p} Y/i(X)$ is right exact.*

Proof. As $p \circ i$ maps the whole of X to a point it is clear that $i^* \circ p^*$ maps $[Y/i(X), W]$ to the base point of $[X, W]$. So let us conversely assume that $f : Y \to W$ is a smooth map such that $i^*([f])$ is the base point of $[X, W]$. This just means that there is a smooth homotopy $H : X \times I \to W$ such that $H(x, 0) = (f \circ i)(x)$ and $H(x, 1) = w_0$ for some $w_0 \in W$. Now by the cofibration property this implies that there is a smooth homotopy $\tilde{H} : Y \times I \to W$ such that $\tilde{H} \circ (i \times Id) = H$ and such that $\tilde{H}|_{Y \times \{0\}}$ is smoothly homotopic to f relative to X. Put $\tilde{f} := \tilde{H}|_{Y \times \{1\}}$. Then by construction $\tilde{f}$ is smoothly homotopic to f and maps $i(X)$ to w_0. Thus it factors to a smooth map on $Y/i(X)$ and hence the homotopy class of $\tilde{f}$ which equals the class of f lies in the image of p^*. □

3.36. Corollary. *Let $f : X \to Y$ be an arbitrary smooth map with homotopy cofiber C_f. Then the sequence $X \xrightarrow{f} Y \to C_f$ is right exact.*

Proof. Let M_f be the mapping cylinder of f. Then by 3.9 and 3.10 we get a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, equal] \arrow[d, "j"] & Y \arrow[d, "k"] \\ X \arrow[r, "i"] & M_f \arrow[r, "p"] & C_f \end{tikzcd}$$

which is commutative up to homotopy and in which i, j and k are inclusions, i is a smooth cofibration and j is a homotopy equivalence. Thus the result follows from 3.35. □

3.37. Now we can iterate the above procedure. Let $f_0 : X_0 \to X_1$ be a smooth map and let $f_1 : X_1 \to X_2$ be the homotopy cofiber of f_0. Then for any smoothly path connected space W there is a long exact sequence of pointed sets

$$\dots \xrightarrow{f_n^*} [X_n, W] \xrightarrow{f_{n-1}^*} [X_{n-1}, W] \xrightarrow{f_{n2}^*} \dots \xrightarrow{f_1^*} [X_1, W] \xrightarrow{f_0^*} [X_0, W]$$

in which $f_n : X_n \to X_{n+1}$ is the homotopy cofiber of f_{n-1}. We will give an explicit description of the spaces X_n later on.

3.38. Now we consider the case of exact sequences of homotopy classes of base point preserving smooth maps. For arbitrary pointed smooth spaces (X, x_0) and (Y, y_0) the set $[X, Y]_0$ of homotopy classes of base point preserving smooth maps has a natural base point, namely the homotopy class of the map which maps the whole space X into y_0. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of base point preserving smooth maps between pointed smooth spaces is called right exact if and only if for any pointed smooth space (W, w_0) the sequence $[Z, W]_0 \xrightarrow{g^*} [Y, W]_0 \xrightarrow{f^*} [X, W]_0$ is exact.

3.39. Proposition. *(1): Let (X, x_0) be a pointed smooth space and let $i : X \to Y$ be a smooth cofibration in the category $\underline{C_0^\infty}$ (c.f. 3.16) and consider $i(x_0)$ as the basepoint of Y. Then the sequence $X \xrightarrow{i} Y \xrightarrow{p} Y/i(X)$ is right exact.*
(2): Let $f : X \to Y$ be a base point preserving smooth map between pointed smooth spaces X and Y and let C'_f be the reduced mapping cone of f. Then the sequence $X \xrightarrow{f} Y \to C'_f$ is right exact.
(3): Let $f_0 : X_0 \to X_1$ be a base point preserving smooth map between pointed smooth spaces. Then for any pointed smooth space W there is a long exact sequence of pointed sets

$$\dots \xrightarrow{f_n^*} [X_n, W]_0 \xrightarrow{f_{n-1}^*} [X_{n-1}, W]_0 \xrightarrow{f_{n2}^*} \dots \xrightarrow{f_1^*} [X_1, W]_0 \xrightarrow{f_0^*} [X_0, W]_0$$

in which $f_n : X_n \to X_{n+1}$ *is the reduced mapping cone of* f_{n-1}.

Proof. The proof is similar to 3.35, 3.36 and 3.37. $\square$

3.40. Our next task is to give a more explicit interpretation of the long exact sequence constructed in 3.37. For this we need some definitions. Recall that for a smooth space X the cone CX over X was defined as the mapping cylinder of the unique smooth map $X \to pt$. Thus there is a smooth cofibration $X \to CX$ and we define the (unreduced) *suspension* SX over X to be the cofiber of this cofibration, i.e. $SX = CX/X$. By 3.5 the suspension over a base space is again a base space. Note that the cone has obvious functorial properties, so for a smooth map $f : X \to Y$ we get a smooth map $C(f) : CX \to CY$ which is induced by $f \times Id : X \times I \to Y \times I$. This map in turn now induces a smooth map $S(f) : SX \to SY$. Moreover note that by 3.14 the space SX is smoothly homotopy equivalent to the homotopy cofiber of the cofibration $X \to CX$ which is the space $CX \cup_{Id_X} CX$. For later use we also need the anti suspension operator $-S(f) : SX \to SY$ induced by a smooth map $f : X \to Y$. This is the smooth map induced by $(x,t) \mapsto (f(x), 1-t)$.

For later use we also define the reduced suspension $S'X$ of a pointed smooth space X as $S'X := C'X/X$. In the same way as above we define $S'^n X$ and the maps $S'^n(f) : S'^n X \to S'^n Y$ for a base point preserving smooth map $f : X \to Y$.

3.41. Lemma. *Let* $f : X \to Y$ *be an arbitrary smooth map with homotopy cofiber* $g : Y \to C_f$*, and let* $h : C_f \to C_g$ *be the homotopy cofiber of* g*. Then there is a commutative diagram*

$$\begin{array}{ccccc} Y & \xrightarrow{g} & C_f & \xrightarrow{h} & C_g \\ & & \| & & \downarrow{\scriptstyle q_1} \\ & & C_f & \xrightarrow{q} & SX \end{array}$$

where q_1 *is a homotopy equivalence and* q *is induced by the natural projection* $C_f \to C_f/g(Y)$.

Proof. By 3.11 the map g is a smooth cofibration and from 3.14 we see that there is a smooth homotopy equivalence $\tilde{q}_1 : C_g \to C_f/g(Y)$ such that $\tilde{q}_1 \circ h$ is the natural projection. We claim that the space $C_f/g(Y)$ is diffeomorphic to SX. Recall that $C_f = CX \cup_f Y$. Now in the diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & pt \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow \\ CX & \longrightarrow & C_f & \longrightarrow & C_f/g(Y) \end{array}$$

the left hand square and the right hand square are by definition push outs and thus the outer square is also a push out. But this is just the push out which defines SX, so we get the claimed diffeomorphism. $\square$

3.42. Applying the above lemma to the next step in the sequence we get a commutative diagram

$$\begin{array}{ccccc} C_f & \xrightarrow{h} & C_g & \xrightarrow{i} & C_h \\ & & \| & & \downarrow{\scriptstyle q_2} \\ & & C_g & \xrightarrow{q'} & SY \end{array}$$

where q_2 arises as $C(C_f) \cup_h C_g \sim C_g/h(C_f) \cong SY$.

Lemma. *The diagram*

$$\begin{tikzcd} C_g \arrow[r, "i"] \arrow[d, "q_1"] & C_h \arrow[d, "q_2"] \\ SX \arrow[r, "-S(f)"] & SY \end{tikzcd}$$

is commutative up to homotopy.

Proof. It suffices to show that $-S(f) \circ q_1$ is smoothly homotopic to q'. Let us temporarily denote by $\tilde{S}Y$ the space $CY \cup_{Id_Y} CY$ and to distinguish the two cones let us denote them by C_1Y and C_2Y. Now we have two homotopy equivalences $p_1, p_2 : \tilde{S}Y \to SY$ induced by the canonical projections $\tilde{S}Y \to \tilde{S}Y/C_1Y$ and $\tilde{S}Y \to \tilde{S}Y/C_2Y$, respectively and we want to clarify the relation between these two maps. As an intermediate step we define a third map $\varphi : \tilde{S}Y \to SY$ as follows: Consider the map $Y \times I \to Y \times I$ given by $(y,t) \mapsto (y, t/2)$. This map obviously induces a smooth map $CY \to CY$ and composing it with the natural projection $CY \to SY$ we get a smooth map $\varphi_1 : CY \to SY$. On the other hand the composition of the natural map $Y \times I \to CY \to SY$ with the map $(y,t) \mapsto (y, 1-t/2)$ maps $Y \times \{0\}$ to a single point and thus induces a smooth map $\varphi_2 : CY \to SY$. Now one easily sees that the maps φ_1 on C_1Y and φ_2 on C_2Y induce a smooth map $\varphi : \tilde{S}Y \to SY$.

Next the map $Y \times I \times I \to Y \times I$ defined by $(y,t,s) \mapsto (y, st/2)$ induces a smooth map $\Phi_1 : CY \times I \to SY$ and the map given by $(y,t,s) \mapsto (y, 1 - t(1 - s/2))$ induces a smooth map $\Phi_2 : CY \times I \to SY$. By cartesian closedness of the category of smooth spaces the diagram

$$\begin{tikzcd} Y \times I \arrow[r] \arrow[d] & C_1Y \times I \arrow[d] \\ C_2Y \times I \arrow[r] & \tilde{S}Y \times I \end{tikzcd}$$

is a push out and the maps Φ_1 on $C_1Y \times I$ and Φ_2 on $C_2Y \times I$ induce a smooth homotopy $\Phi : \tilde{S}Y \times I \to SY$. From the definition it immediately follows that $\Phi|_{\tilde{S}Y \times \{0\}}$ is induced by the maps $C_1 \mapsto pt$ and $(y,t) \mapsto (y, 1-t)$ on C_2Y and thus $\Phi|_{\tilde{S}Y \times \{0\}} = -S(Id_Y) \circ p_2$ while obviously $\Phi|_{\tilde{S}Y \times \{1\}} = \varphi$. Similarly one constructs a homotopy between φ and p_1 using the maps $(y,t,s) \mapsto (y, t(1-s/2))$ and $(y,t,s) \mapsto (y, 1 - st/2)$ and thus the map p_1 is smoothly homotopic to $-S(Id_Y) \circ p_2$.

Recall that in the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"] & CY \arrow[d] \\ CX \arrow[r] & C_f \arrow[r] & C_g \end{tikzcd}$$

the two squares are push outs and thus the outer rectangle is a push out, too. Hence the identification of CY with C_1Y together with the map $C(f) : CX \to C_2Y$ induces a smooth map $\Psi : C_g \to \tilde{S}Y$. Now $p_2 \circ \Psi : C_g \to SY$ maps the space C_f to a single point while it induces the natural map $CY \to SY$ on CY and thus $p_2 \circ \Psi = q'$. On the other hand $p_1 \circ \Psi$ maps CY to a single point while on C_f it is given by mapping Y to a point and the induced map $C_f/g(Y) \cong SX \to SY$ is clearly equal to $S(f)$ and thus $p_1 \circ \Psi = S(f) \circ q_1$. Thus we see that $S(f) \circ q_1$ is smoothly homotopic to $-S(Id_Y) \circ q'$ and since clearly $-S(Id_Y) \circ -S(Id_Y) = Id_{SY}$ and $-S(Id_Y) \circ S(f) = -S(f)$ the lemma follows. $\square$

3.43. We are now ready to formulate the final version of the long exact sequence 3.37. For a smooth space X we define inductively the n–fold (unreduced) suspension S^nX over X by $S^1X := SX$ and $S^nX := S(S^{n-1}X)$, and for a smooth map $f : X \to Y$ we define $S^1(f) := S(f)$ and $S^n(f) := S(S^{n-1}(f)) : S^nX \to S^nY$.

Theorem. *Let $f : X \to Y$ be a smooth map between arbitrary smooth spaces with homotopy cofiber $g : Y \to C_f$, and let $q : C_f \to SX$ be the map constructed in 3.41. Then for any smoothly path connected smooth space W the sequence*

$$\begin{aligned} \dots \to [S^{n+1}X, W] \xrightarrow{S^n(q)^*} [S^nC_f, W] \xrightarrow{S^n(g)^*} \\ \to [S^nY, W] \xrightarrow{S^n(f)^*} [S^nX, W] \to \dots \\ \dots \to [SX, W] \xrightarrow{q^*} [C_f, W] \xrightarrow{g^*} [Y, W] \xrightarrow{f^*} [X, W] \end{aligned}$$

of pointed sets is exact.

Proof. The exactness of the sequence with all maps $S^n(\ :\ )^*$ replaced by $(-S)^n(\ :\ )^*$ follows via induction from 3.41 and 3.42. Clearly changing the maps from $(-S)^n(\quad)^*$ to $S^n(\quad)^*$ does not destroy the exactness. □

3.44. Our final task in this section is to establish the dual version of the exact sequence derived in 3.43. As before we are mainly interrested in the case of sets of free homotopy classes. If (X, x_0) is a pointed smooth space then for any smooth space W the set $[W, X]$ of free homotopy classes of smooth maps from W to X has a natural base point, namely the homotopy class of the map which maps the whole space W to x_0.

Now let $f : X \to Y$ and $g : Y \to Z$ be base point preserving smooth maps between pointed smooth spaces. Then the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is said to be *left exact* if and only if for any smooth space W the sequence $[W, X] \xrightarrow{f_*} [W, Y] \xrightarrow{g_*} [W, Z]$ is an exact sequence of pointed sets.

3.45. Proposition. *Let $p : X \to Y$ be a smooth fibration, $x_0 \in X$ a point, $y_0 := p(x_0)$ and let $i : F \hookrightarrow X$ be the inclusion of the fiber $F := p^{-1}(y_0)$. Then with the natural base point $f_0 := i^{-1}(x_0) \in F$ the sequence $F \xrightarrow{i} X \xrightarrow{p} Y$ is left exact.*

Proof. Let W be any smooth space. By definition $p \circ i$ maps the whole space F to the point y_0 and thus clearly $p_* \circ i_*$ maps the whole set $[W, F]$ to the base point of $[W, Y]$. So let us assume that $f : W \to X$ is a smooth map such that $p_*([f])$ is the base point of $[W, Y]$. This means that there is a homotopy $H : W \times I \to Y$ such that $H|_{W\times\{0\}} = p \circ f$ and $H|_{W\times\{1\}}$ is the constant map y_0. Applying the fibration property of p to the maps H and f we get a smooth map $\tilde{H} : W \times I \to X$ such that $p \circ \tilde{H} = H$ and $\tilde{H}|_{W\times\{0\}}$ is smoothly homotopic to f (even via a fiber preserving homotopy). Thus f is smoothly homotopic to the map $\tilde{H}|_{W\times\{1\}}$ which by construction has values in $i(F)$ and thus $[f]$ lies in the image of i_*. □

3.46. Corollary. *Let $f : X \to Y$ be an arbitrary base point preserving smooth map with homotopy fiber C^f. Then the sequence $C^f \to X \xrightarrow{f} Y$ is left exact.*

Proof. Let M^f be the mapping cocylinder of f. Then by 3.22 and 3.25 we get a diagram

$$\begin{array}{ccccc} F & \longrightarrow & M^f & \xrightarrow{p} & Y \\ \uparrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle q} & & \| \\ C^f & \longrightarrow & X & \xrightarrow{f} & Y \end{array}$$

in which F is the fiber over y_0 of the fibration p, Φ is a diffeomorphism, q is a homotopy equivalence, the left hand square is commutative and the right hand square is commutative up to homotopy since the evaluation at 0 and the evaluation at 1 are smoothly homotopic. Now M^f has a natural base point, namely the one which is sent by the canonical maps to x_0 and the constant map y_0, respectively and this also defines the base point of C^f. With this conventions all maps in the above diagram are base point preserving. Thus the result follows from 3.45. $\square$

3.47. As in the case of the dual sequence we can now iterate the above procedure to get a long exact sequence as follows: Let $f^0 : X^1 \to X^0$ be a base point preserving smooth map and let $f^1 : X^2 \to X^1$ be the homotopy fiber of f^0. Then for any smooth space W there is a long exact sequence of pointed sets

$$\dots \xrightarrow{f^n_*} [W, X^n] \xrightarrow{f^{n-1}_*} [W, X^{n-1}] \xrightarrow{f^{n-2}_*} \dots \xrightarrow{f^1_*} [W, X^1] \xrightarrow{f^0_*} [W, X^0]$$

in which $f^n : X^{n+1} \to X^n$ is the homotopy fiber of f^{n-1}. Our next task will be to describe the spaces X^n more explicitely.

3.48. Recall that for any pointed smooth space X we defined the path fibration $PX \to X$ as the mapping cocylinder of the inclusion of the base point into X. Now we define the *loop space* ΩX of X to be the fiber over x_0 of this fibration. Thus ΩX is the set of all smooth maps $c : I \to X$ such that $c(0) = c(1) = x_0$ with the initial smooth structure with respect to the inclusion into $C^\infty(I, X)$. Note that the loop space is obviously funtorial: Any base point preserving smooth map $f : X \to Y$ induces a smooth map $f_* : C^\infty(I, X) \to C^\infty(I, Y)$ which restricts to a smooth map $\Omega(f) : \Omega X \to \Omega Y$. For later use we also define the *anti loop operator* $-\Omega(f) : \Omega X \to \Omega Y$ to be the smooth map defined by $-\Omega(f)(c)(t) := f(c(1-t))$.

3.49. Proposition. *Put $\mathfrak{S} := I/\{0,1\}$ with the final smooth structure. Then for any pointed smooth space X there is a canonical diffeomorphism $\Omega X \cong C^\infty_0(\mathfrak{S}, X)$.*

Proof. Obviously the two underlying sets coincide, so we only have to consider the smooth structures. Let $p : I \to \mathfrak{S}$ be the canonical projection. Since $C^\infty_0(\mathfrak{S}, X)$ has by definition the initial smooth structure with respect to the inclusion into $C^\infty(\mathfrak{S}, X)$ we only have to show that the smooth map $p^* : C^\infty(\mathfrak{S}, X) \to C^\infty(I, X)$ is an initial morphism.

First we show that for any smooth space Y the morphism $p \times Id_Y : I \times Y \to \mathfrak{S}^1 \times Y$ is final. So let $f : \mathfrak{S} \times Y \to \mathbb{R}$ be a function such that $f \circ (p \times Id_Y)$ is smooth. Then by cartesian closedness $(f \circ (p \times Id_Y))^\vee : I \to C^\infty(Y, \mathbb{R})$ is smooth and one immediately verifies that $(f \circ (p \times Id_Y))^\vee = \check{f} \circ p$. Since p is final this means that $\check{f}$ is smooth and thus by cartesian closedness f is smooth.

So let us turn to the initiality of p^*. Let $c : \mathbb{R} \to C^\infty(\mathfrak{S}^1, X)$ be a curve such that $p^* \circ c$ is smooth. Then by cartesian closedness $(p^* \circ c)\hat{}: \mathbb{R} \times I \to X$ smooth. Now one immediately verifies that $(p^* \circ c)\hat{} = \hat{c} \circ (Id_\mathbb{R} \times p)$ and since $Id_\mathbb{R} \times p$ is final we see that $\hat{c}$ is smooth and thus by cartesian closedness c is smooth. $\square$

3.50. Proposition. *Let X be a pointed smooth space, $S'X$ the reduced suspension of X (c.f. 3.40). Then there is a natural diffeomorphism $S'X \cong \mathfrak{S} \wedge X$, where $\wedge$ denotes the smash product (c.f. 1.25). Thus for any pointed smooth space Y there is a natural diffeomorphism $C^\infty_0(S'X, Y) \to C^\infty_0(X, \Omega Y)$, so the functors S' and Ω form an adjoint pair. Moreover this diffeomorphism induces a bijection $[S'X, Y]_0 \cong [X, \Omega Y]_0$.*

Proof. By definition the smash product is the quotient space $(\mathfrak{S} \times X)/(\mathfrak{S} \vee X)$. On the other hand the reduced suspension is given as $S'X = (X \times I)/(X \times \{0,1\} cup \{x_0\} \times I)$. From the proof of 3.49 we see that $\mathfrak{S} \times X \cong (X \times I)/(X \times \{0,1\})$. Thus the identity on $X \times I$ induces a smooth map $\mathfrak{S} \times X \to S'X$ and since in $S'X$ the fiber over the base point is contracted this map factors to a smooth map $\mathfrak{S} \wedge X \to S'X$. In the same way the identity on $X \times I$ induces a smooth map $X \times I \to \mathfrak{S} \wedge X$ and clearly this map factors to a smooth map $S'X \to \mathfrak{S} \wedge X$ which is obviously inverse to the map constructed above.

Now the diffeomorphism between the spaces of base point preserving smooth functions immediately follows from 1.26. Finally one immediately checks the compatibility of this diffeomorphism with homotopies. $\square$

3.51. Lemma. *Let X and Y be pointed smooth spaces, $f : X \to Y$ a base point preserving smooth map with homotopy fiber $g : C^f \to X$, and let $h : C^g \to C^f$ be the homotpy fiber of g. Then there is a commutative diagram*

$$\begin{tikzcd} \Omega Y \arrow[r, equal] \arrow[d, "k_1"] & \Omega Y \arrow[d, "k"] & \\ C^g \arrow[r, "h"] & C^f \arrow[r, "g"] & X \end{tikzcd}$$

in which k_1 is a smooth homotopy equivalence and k is induced by the inclusion of the fiber of g over x_0.

Proof. Recall that by 3.24 the map $g : C^f \to X$ is a smooth fibration. By the universal property of the pullback defining C^f the inlusion of ΩY into PY together with the constant map x_0 induces a smooth map $k : \Omega Y \to C^f$ and one immediately verifies that k is a diffeomorphism onto the fiber over x_0 of g. Now let M^g be the mapping cocylinder of g. Then from the proof of 3.33 we see that there is a smooth fiber homotopy equivalence $\tilde{H} : C^f \to M^g$ (which is called h in 3.33) with the following properties:
(1): $p \circ \tilde{H} = g$, where $p : M^g \to X$ is the natural map.
(2): $q \circ \tilde{H} = Id$ where $q : M^g \to C^f$ is the natural map.
Let H denote the restriction of $\tilde{H}$ to the fibers over x_0 which is then a homotopy equivalence, too. Finally let Ψ be the diffeomorphism between the fiber over x_0 of p which is inverse to the one constructed in 3.25. So by construction we have $h \circ \Psi = q$. Now put $k_1 := \Psi \circ H \circ k$. Then we get $h \circ k_1 = h \circ \Psi \circ H \circ k = q \circ H \circ k = k$. From the construction one easily verifies that the map $k_1 : \Omega Y \to C^g$ is induced by $k : \Omega Y \to C^f$ and the map which sends the whole space ΩY to the constant path x_0 in PX. $\square$

3.52. Applying the above lemma to the next step of the sequence we get a commutative diagram

$$\begin{tikzcd} \Omega X \arrow[r, equal] \arrow[d, "k_2"] & \Omega X \arrow[d, "k'"] & \\ C^h \arrow[r, "i"] & C^g \arrow[r, "h"] & C^f \end{tikzcd}$$

in which k_2 is a smooth homotopy equivalence and k' is induced by the inclusion of the fiber of h over the base point of C^f.

Lemma. *The diagram*

$$\begin{tikzcd} \Omega X \arrow[r, "-\Omega(f)"] \arrow[d, "k_2"] & \Omega Y \arrow[d, "k_1"] \\ C^h \arrow[r, "i"] & C^g \end{tikzcd}$$

is commutative up to homotopy.

Proof. Clearly it suffices to show that $k_1 \circ -\Omega(f)$ is smoothly homotopic to k'. Let us temporarily denote by $\tilde{\Omega}X$ the homotopy fiber of the path fibration over X. Explicitely this space can be realized as the set of pairs (c_1, c_2) with $c_i \in PX$ such that $c_1(1) = c_2(1)$ with the initial smooth structure with respect to the two projections to $C^\infty(I, X)$. There are two obvious maps $j_1, j_2 : \Omega X \to \tilde{\Omega} X$ defined by $j_1(c) := (c, x_0)$ and $j_2(c) = (x_0, c)$, where by x_0 we also denote the constant map. First we clarify the relation between these two maps: Define $H : \Omega X \times I \to \tilde{\Omega} X$ as $H(\omega, t) = (c_1(\omega, t), c_2(\omega, t))$ where $c_1(\omega, t)(s) := \omega(ts)$ and $c_2(\omega, t)(s) := \omega(1 - (1 - t)s)$. Then this is obviously smooth and one easily checks that it indeed has values in $\tilde{\Omega} X$. Thus H defines a smooth homotopy between j_1 and $j_2 \circ -\Omega(Id)$.

In the diagram

$$\begin{array}{ccccc} C^g & \longrightarrow & C^f & \longrightarrow & PY \\ \big\downarrow & & \big\downarrow{\scriptstyle g} & & \big\downarrow \\ PX & \longrightarrow & X & \xrightarrow{f} & Y \end{array}$$

both squares are pull backs and thus the outer rectangle is a pullback, too. Hence the projection $\tilde{\Omega} X \to PX$ onto the first factor together with the composition of the map $P(f) : PX \to PY$ with the projection onto the second factor induces a smooth map $\Phi : \tilde{\Omega} X \to C^g$.

Now the composition $\Phi \circ j_1$ is given by the inclusion $\Omega X \to PX$ and the constant map to the base point of C^f and thus we have $\Phi \circ j_1 = k'$. On the other hand $\Phi \circ j_2$ is induced by the constant map to the base point of PX and a map $\Omega X \to C^f$ which is in turn induced by the constant map to the base point of X and the composition of the inclusion $\Omega Y \to PY$ with $\Omega(f)$. From these data one easily verifies that $\Phi \circ j_2 = k_1 \circ \Omega(f)$. Thus from above we conclude that k' is smoothly homotopic to $k_1 \circ \Omega(f) \circ -\Omega(Id)$ and since clearly $\Omega(f) \circ -\Omega(Id) = -\Omega(f)$ this completes the proof. $\square$

3.53. Now we can formulate the final versions of the exact sequence constructed in 3.47. For a pointed smooth space X we define inductively the n–fold loop space $\Omega^n X$ over X by $\Omega^1 X := \Omega X$ and $\Omega^n X := \Omega(\Omega^{n-1} X)$ and for a base point preserving smooth map $f : X \to Y$ we define $\Omega^1(f) := \Omega(f)$ and $\Omega^n(f) := \Omega(\Omega^{n-1}(f)) : \Omega^n X \to \Omega^n Y$.

Theorem. *Let $f : X \to Y$ be a base point preserving smooth map between pointed smooth spaces with homotopy fiber $g : C^f \to X$, and let $k : \Omega Y \to C^f$ be the smooth map constructed in 3.51. Then for any smooth space W the sequence*

$$\cdots \to [W, \Omega^{n+1} Y] \xrightarrow{\Omega^n(k)_*} [W, \Omega^n C^f] \xrightarrow{\Omega^n(g)_*}$$
$$\xrightarrow{\Omega^n(g)_*} [W, \Omega^n X] \xrightarrow{\Omega^n(f)_*} [W, \Omega^n Y] \xrightarrow{\Omega^{n-1}(k)_*} \ldots$$
$$\ldots \xrightarrow{\Omega(f)_*} [W, \Omega Y] \xrightarrow{k_*} [W, C^f] \xrightarrow{g_*} [W, X] \xrightarrow{f_*} [W, Y]$$

is an exact sequence of pointed sets.

Proof. The exactness of the sequence with all maps $\Omega^i(\ :\ )_*$ replaced by $(-\Omega)^i(\ :\ )_*$ follows by induction from 3.47, 3.51 and 3.52. Clearly changing the maps from $\Omega^i(\quad)_*$ to $(-\Omega)^i(\quad)_*$ does not destroy the exactness. $\square$

4. Convenient vector spaces, algebras and modules

4.1. Definition. (1): A smooth vector space is a smooth space E which is also a real vector space such that the addition $E \times E \to E$ and the scalar multiplication $\mathbb{R} \times E \to E$ are smooth maps.

(2): A smooth vector space is called *preconvenient* iff its smooth structure is generated by some set of real valued linear functionals.

(3): For smooth vector spaces E and F we write $L(E, F)$ for the vector space of all smooth linear maps from E to F and we write E' for $L(E, \mathbb{R})$.

4.2. On a preconvenient vector space E there is a canonical locally convex topology, namely the finest one for which E' becomes the topological dual of E. That this topology exists is shown in [Ja, p. 58, p. 61] and in [F–K, 2.1.9]. In fact this so called Mackey topology defines a functor α from the category of preconvenient vector spaces and smooth linear maps to the category of locally convex vector spaces and continuous linear maps.

On the other hand on any locally convex vector space there is a canonical smooth structure namely the one generated by the topological dual. This defines a functor β from the category of locally convex vector spaces and continuous linear maps to the category of preconvenient vector spaces and smooth linear maps.

Obviously the composition $\beta \circ \alpha$ is the identity, but this is not the case for $\alpha \circ \beta$. Our next aim is to describe this composition.

4.3. Definition. Let E be a locally convex vector space.
(1): A subset $B \subset E$ is called bounded iff for any 0–neighborhood U in E there is a real number t such that $B \subset t \cdot U$.
(2): A linear map between locally convex vector spaces is called bounded iff the image of any bounded set is bounded.
(3): The *bornologification* of E is the vector space E together with the topology induced by taking as a basis of 0–neighborhoods all absolutely convex sets U such that for any bounded subset $B \subset E$ there is a real number t such that $B \subset tU$. That this defines a unique locally convex topology on E is shown in [Ja, p. 33].
(4): For an absolutely convex subset B of a locally convex vector space E we denote by E_B the linear span of B in E with the topology induced by the so called Minkowsky functional $\|x\|_B := inf\{t > 0 : x \in tB\}$, which is a seminorm.

4.4. Proposition. *For any locally convex vector space E the space $\beta\alpha E$ is the bornologification of E. The bornologification can also be described as follows: Let $\mathcal{B}_0$ be a basis of the bornology of E, i.e. a family of bounded set such that any bounded set is contained in a set which belongs to $\mathcal{B}_0$, which consists of absolutely convex sets. (Such a basis always exists.) Then the inclusion makes $\mathcal{B}_0$ a directed set and for $B_1 \subseteq B_2$ the inclusion $E_{B_1} \hookrightarrow E_{B_2}$ is continuous and the bornologification of E is the colimit in the category of locally convex spaces of the so obtained inductive system.*

Proof. [F–K, 2.4.1, 2.4.3 and 2.1.19] □

4.5. As by definition a preconvenient vector space is a smooth space it also carries the canonical topologies defined in 1.9 and in particular the $\tau_{\mathcal{C}}$–topology which is also called c^∞–topology or Mackey–closure topology. (We will use the notation c^∞–closed for closed in this topology etc.) The c^∞–topology is also the final topology with respect to the inclusions of the seminormed spaces E_B for B in a basis of the bornology of E consisting of absolutely convex sets. (c.f. [F–K, 2.2.23])

In general the c^∞–topology on a preconvenient vector space is not a vector space topology since the addition is only partially continuous. It turns out that the (bornological) locally convex topology αE is the finest locally convex topology which is coarser than the c^∞–topology. A condition which ensures that the two topologies coincide is that the locally convex topology is metrizable.

The relation between the $\tau_{\mathcal{F}}$–topology of a preconvenient vector space and its locally convex topology seems to be much more complicated.

4.6. It turns out that the smooth linear mappings between two locally convex vector spaces are exactly the bounded ones. We will always view preconvenient vector spaces as locally convex spaces and follow the convention that if we have given a preconvenient vector space as a locally convex space then we consider on it the given locally convex topology (and not the bornologification). On the other hand if we directly construct preconvenient vector spaces then we always equip them with the bornological topology.

Note that two locally convex spaces can be isomorphic as preconvenient vector spaces without being isomorphic as locally convex spaces. (For instance any locally convex spaces is isomorphic to its bornologification as a preconvenient vector space.)

4.7. Definition. A preconvenient vector space E is called *convenient* iff (1): E' is point separating and
(2): For any smooth curve $c : \mathbb{R} \to E$ there is a smooth curve $\dot{c} : \mathbb{R} \to E$ such that for all $\lambda \in E'$ we have $\lambda(\dot{c}(0)) = \frac{\partial}{\partial t}\big|_0 (\lambda \circ c)(t)$.

4.8. Remarks. There are several equivalent ways to express the separation condition (1) and the completeness condition (2) of 4.7 (c.f. [F–K, 2.5.2 and 2.6.2]). We only list two important conditions which are equivalent to (2) (assuming (1)):
(2'): Every sequence (x_n) with $x_n \in E$ such that there are positive reals $t_{m,n}$ with $\lim_{m,n\to\infty} t_{m,n} = \infty$ such that the set of all $t_{m,n}(x_m - x_n)$ is bounded in E (such a sequence is called a Mackey–Cauchy sequence) converges (weakly).
(2"): Any bounded subset of E is contained in an absolutely convex bounded subset B such that E_B is a Banach space.
From (2') it is obvious that the completion condition is quite weak.

4.9. Proposition (Completion). *For any preconvenient vector space E there is a (up to isomorphism) unique convenient vector space $\tilde{E}$ called the completion of E with a bounded linear map $i : E \to \tilde{E}$ such that for any bounded linear map $f : E \to F$ into a convenient vector space F there is a unique bounded linear map $\tilde{f} : \tilde{E} \to F$ such that $\tilde{f} \circ i = f$. This completion defines a functor which is left adjoint to the inclusion of the category of convenient vector spaces into the category of preconvenient vector spaces.*

Proof. [F–K, 2.6.5]. Explicitly $\tilde{E}$ can be constructed as the closure in the $\tau_{\mathcal{C}}$–topology of the image of E under the canonical map $E \to \prod_{E'} \mathbb{R}$. □

4.10. Theorem. *Let $\underline{Pre}$ be the category of preconvenient vector spaces and bounded linear maps, $\underline{Con}$ the full subcategory of convenient vector spaces.*
(1): The category $\underline{Pre}$ has initial and final structures with respect to the forgetful functor to the category of vector spaces.
(2): The category $\underline{Pre}$ is complete and cocomplete.
(3): The category $\underline{Con}$ is complete and cocomplete.

Proof. (1): The initial smooth structure as described in 1.3 is obviously linearly generated. For the final structure with respect to a family of linear maps $f_\alpha : E_\alpha \to E$ one takes the smooth structure generated by all linear maps $\lambda : E \to \mathbb{R}$ such that $\lambda \circ f_\alpha \in E'_\alpha$ for all α. One easily verifies directly that this defines initial and final structures.
(2): This follows from pure category theory since the category of vector spaces is complete and cocomplete. Limits (colimits) are formed by forming the limit (colimit) of the underlying vector spaces and then putting on it the initial (final) structure.
(3): This also follows from pure category theory since the completion functor is left adjoint to the inclusion. Limits are formed by forming them in $\underline{Pre}$ and colimits are formed by applying the completion functor to the colimit in $\underline{Pre}$.

Explicit descriptions of some limits and colimits in the categories $\underline{Pre}$ and $\underline{Con}$ can be found in [F–K, 3.3–3.5]. □

4.11. A more delicate question is the existence of initial and final $\underline{Con}$ structures, i.e. the question whether subspaces and quotients of convenient vector spaces are again convenient. One can show that c^∞–closed subspaces of convenient vector spaces are again convenient ([F–K, 3.2.1]). On the other hand a quotient of a convenient vector space is convenient iff the kernel is closed in the locally convex topology and the final locally convex topology satisfies the completeness condition 4.8(2') ([F–K, 3.2.4]).

4.12. Our next task is to show that spaces of bounded linear and multilinear mappings between preconvenient vector spaces are preconvenient vector spaces and that they are convenient under certain assumptions. To do this we first have to consider spaces of smooth functions from smooth spaces to preconvenient vector spaces which are obviously vector spaces with the pointwise operations.

Theorem. *Let X be a smooth space, E a preconvenient vector space. Then the smooth structure on the space $C^\infty(X,E)$ defined in 1.6 is linearly generated and thus $C^\infty(X,E)$ is a preconvenient vector space. Moreover if E is convenient then so is $C^\infty(X,E)$.*

Proof. [F–K, 4.4.12] □

4.13. Let E and F be preconvenient vector spaces. On $L(E,F)$, the space of all smooth linear maps from E to F we put the initial $\underline{Pre}$ structure with respect to the inclusion $L(E,F) \hookrightarrow C^\infty(E,F)$. This structure exists since $L(E,F)$ is obviously a linear subspace of $C^\infty(E,F)$. Moreover by 1.8 for any $x \in E$ the map $ev_x : C^\infty(E,F) \to F$ is smooth and thus continuous for the c^∞ topologies. Now $L(E,F)$ is the intersection of the kernels of all maps $ev_x + t \cdot ev_y - ev_{x+ty}$ and thus it is c^∞–closed in $C^\infty(E,F)$. Thus by 4.11 and 4.12 if F is convenient so is $L(E,F)$.

For preconvenient vector spaces $E_1,\dots,E_m$ and F denote by $L(E_1,\dots,E_m;F)$ set of all m–linear smooth maps from $E_1 \times \dots \times E_m$ to F. On this space we put the initial $\underline{Pre}$ structure with respect to the inclusion into $C^\infty(E_1 \times \dots \times E_m, F)$. As above one easily shows that $L(E_1,\dots,E_m;F)$ is c^∞–closed in $C^\infty(E_1 \times \dots \times E_m, F)$ and thus is convenient if F is.

4.14. Proposition. *For preconvenient vector spaces $E_1,\dots,E_m$ and F there is a natural isomorphism of preconvenient vector spaces*

$$L(E_1,\dots,E_m;F) \cong L(E_1,\dots,E_k;L(E_{k+1},\dots,E_m;F))$$

Proof. By cartesian closedness of the category of smooth spaces (1.7) there is a natural diffeomorphism $C^\infty(E_1 \times \dots \times E_m, F) \to C^\infty(E_1 \times \dots \times E_k, C^\infty(E_{k+1} \times \dots \times E_m, F))$. One easily checks that this diffeomorphism is linear and restricts to the subspaces of multilinear maps. □

4.15. Theorem (Multilinear uniform boundedness principle).
Let $E_1,\dots,E_m$ be convenient vector spaces, F a preconvenient vector space. Then for a subset B of $L(E_1,\dots,E_m;F)$ the following conditions are equivalent:
(1): B is bounded, i.e. $B(A) \subset F$ is bounded for all bounded $A \subset E_1 \times \dots \times E_m$.
(2): $B(A_1 \times \dots \times A_m)$ is bounded for all bounded $A_i \subset E_i$.
(3): $B(x_1,\dots,x_m)$ is bounded for all $x_i \in E_i$.

This implies that the $\underline{Pre}$ structure on $L(E_1,\dots,E_m;F)$ is the initial one with respect to all evaluation maps $ev_{x_1,\dots,x_m} : L(E_1,\dots,E_m;F) \to F$.
Moreover a m–linear map $\ell : E_1 \times \dots \times E_m \to F$ is bounded and thus smooth iff it is partially bounded.

Proof. [F–K, 3.6.4, 3.7.4 and 3.7.5] □

4.16. Tensor products. Let E and F be preconvenient vector spaces, $E \otimes F$ the algebraic tensor product. By $b : E \times F \to E \otimes F$ we denote the canonical bilinear map. On $E \otimes F$ we put the smooth structure generated by all linear maps $h : E \otimes F \to \mathbb{R}$ such that $h \circ b : E \times F \to \mathbb{R}$ is smooth. With this structure $E \otimes F$ is by definition a preconvenient vector space. If E and F are convenient then we denote by $E \tilde{\otimes} F$ the completion of $E \otimes F$.

Theorem. *(1): With the tensor product $\otimes$ the category $\underline{Pre}$ is symmetric monoidally closed, i.e. $\otimes$ is a functor and there are natural isomorphisms (of preconvenient vector spaces): $L(E_1, L(E_2, E_3)) \cong L(E_1 \otimes E_2, E_3)$, $E_1 \otimes E_2 \cong E_2 \otimes E_1$, $E_1 \otimes (E_2 \otimes E_3) \cong (E_1 \otimes E_2) \otimes E_3$ and $E \otimes \mathbb{R} \cong E$.*
(2): With the tensor product $\tilde{\otimes}$ the category $\underline{Con}$ is symmetric monoidally closed, i.e. $\tilde{\otimes}$ is a functor and there are natural isomorphisms (of convenient vector spaces): $L(E_1, L(E_2, E_3)) \cong L(E_1 \tilde{\otimes} E_2, E_3)$, $E_1 \tilde{\otimes} E_2 \cong E_2 \tilde{\otimes} E_1$, $E_1 \tilde{\otimes} (E_2 \tilde{\otimes} E_3) \cong (E_1 \tilde{\otimes} E_2) \tilde{\otimes} E_3$ and $E \tilde{\otimes} \mathbb{R} \cong E$.

Proof. [F–K, 3.8.1 and 3.8.4] $\square$

4.17. Corollary. *Let E_j, F be (pre)convenient vector spaces. Then there are natural isomorphisms of (pre)convenient vector spaces:*
(1): $L(F, \prod_{j \in J} E_j) \cong \prod_{j \in J} L(F, E_j)$
(2): $L(\coprod_{j \in J} E_j, F) \cong \prod_{j \in J} L(E_j, F)$

Proof. ([F–K, 3.8.5]) (1): By existence of the tensor product the functor $L(F, \quad)$ has a left adjoint and thus commutes with limits and in particular with products.
(2): The existence of a natural isomorphism

$$L(E_1, L(E_2, F)) \cong L(E_2, L(E_1, F))$$

can be expressed as the fact that the functor $L(\quad, F) : \underline{Pre}^{op} \to \underline{Pre}$ has a left adjoint, namely $L(\quad, F) : \underline{Pre} \to \underline{Pre}^{op}$ and thus commutes with limits, i.e. transforms limits in $\underline{Pre}$ to colimits in $\underline{Pre}$. In particular it transforms coproducts into products. $\square$

4.18. Definition. A *convenient algebra* is a convenient vector space A together with a bilinear bounded map $\mu : A \times A \to A$ such that A is an associative algebra with multiplication μ. We will always assume that the algebra A has a multiplicative unit element and that all homomorphisms preserve the unit elements.

For a convenient algebra A we denote by A^{op} the opposite algebra to A, i.e. $A^{op} = A$ as a vector space but the multiplication in A^{op} is given by $\mu^{op}(a, b) := \mu(b, a)$. Obviously A^{op} is also a convenient algebra.

4.19. Examples. (1): Let E be a convenient vector space. Then the composition map $\circ : L(E, E) \times L(E, E) \to L(E, E)$ is smooth by cartesian closedness of the category of smooth spaces (1.8). Thus $(L(E, E), \circ)$ is a convenient algebra.
(2): Let X be a smooth space, A a convenient algebra and consider the space $C^\infty(X, A)$ which is a convenient vector space by 4.12. By cartesian closedness the functor $C^\infty(X, \cdot)$ has a left adjoint and thus commutes with limits and in particular with products, so $C^\infty(X, A) \times C^\infty(X, A) \cong C^\infty(X, A \times A)$. Composing this isomorphism with the map $\mu_* : C^\infty(X, A \times A) \to C^\infty(X, A)$ which is smooth by 1.8 we see that the pointwise multiplication in $C^\infty(X, A)$ is smooth and thus $C^\infty(X, A)$ with the pointwise operations is a convenient algebra. In particular this applies to $C^\infty(X, \mathbb{R})$.

4.20. Lemma. *Let A be a convenient algebra. By A^* we denote the set of those elements of A which have a multiplicative inverse. Let $i : A^* \to A$ be the inclusion and let $\nu : A^* \to A$ be defined by $\nu(a) := a^{-1}$. Then A^* with the initial smooth structure with respect to the maps i and ν is a smooth group.*

Proof. Obviously A^* is a group with multiplication μ induced by the multiplication of A and inversion ν, so we only have to show that these maps are smooth: For the inversion ν we have $i \circ \nu = \nu$ and $\nu \circ \nu = i$ and thus ν is smooth. On the other hand $i \circ \mu$ is the composition of the multiplication in A with the map $i \times i$ and $j \circ \mu$ is the composition of the multiplication in A with $(j \times j) \circ \varphi$ where $\varphi(a, b) := (b, a)$ and thus μ is smooth. $\square$

4.21. Definition. (1): A *convenient category* is a category $\mathcal{C}$ such that for each pair of objects E and F of $\mathcal{C}$ the set of morphisms $\mathcal{C}(E,F)$ is a convenient vector space and such that all composition maps $\mathcal{C}(F,G)\times\mathcal{C}(E,F)\to\mathcal{C}(E,G)$ are bounded and bilinear. Note that a convenient algebra A can be viewed as a convenient category with a single object and morphisms corresponding to the elements of A.
(2):Let $\mathcal{C}$ and $\mathcal{C}'$ be convenient categories. A functor $\varphi:\mathcal{C}\to\mathcal{C}'$ is called *convenient* iff for any pair of objects E and F of $\mathcal{C}$ the map $\mathcal{C}(E,F)\to\mathcal{C}'(\varphi(E),\varphi(F))$ induced by φ is a bounded linear map.
(3): Let A be a convenient algebra, $\mathcal{C}$ a convenient category. A *convenient left A–module in $\mathcal{C}$* is a covariant convenient functor from A viewed as a convenient category to $\mathcal{C}$, so it is just an object E of $\mathcal{C}$ together with a bounded algebra homomorphism $\lambda:A\to\mathcal{C}(E,E)$. A *convenient right A–module in $\mathcal{C}$* is a convenient left A^{op}–module in $\mathcal{C}$.
(4): Let (E,λ_E) and (F,λ_F) be two convenient left A–modules in a convenient category $\mathcal{C}$. A homomorphism of A–modules between E and F is a natural transformation between the corresponding functors, so it is a morphism $f\in\mathcal{C}(E,F)$ such that $f\circ\lambda_E(a)=\lambda_F(a)\circ f$ for all $a\in A$. This also defines homomorphisms between convenient right A–modules.
(5): By $\mathcal{C}-\mathrm{Mod}_A$ we denote the category of convenient left A–modules in $\mathcal{C}$ and module homomorphisms and we write $\mathcal{C}-\mathrm{Mod}^A$ for right modules and homomorphisms. Moreover we write simply Mod_A and Mod^A instead of $\underline{Con}-\mathrm{Mod}_A$ and $\underline{Con}-\mathrm{Mod}^A$ and we write Hom_A and Hom^A for the morphisms in these categories.

4.22. Proposition. *Let $\mathcal{C}$ be a convenient category, $\mathcal{D}$ and arbitrary small category. Then the category $\mathcal{C}^{\mathcal{D}}$ of all (covariant) functors from $\mathcal{D}$ to $\mathcal{C}$ is a convenient category.*
If the category $\mathcal{D}$ is also convenient then the convenient functors form a full subcategory and thus also a convenient category. In particular this implies that for any convenient category $\mathcal{C}$ and any convenient algebra A the categories $\mathcal{C}-\mathrm{Mod}_A$ and $\mathcal{C}-\mathrm{Mod}^A$ are convenient categories.

Proof. The morphisms in $\mathcal{C}^{\mathcal{D}}$ are by definition the natural transformations. Let $I=\{\dots,i,j,\dots\}$ denote the set of objects of $\mathcal{D}$. Then a natural transformation between two functors $\varphi,\psi:\mathcal{D}\to\mathcal{C}$ consist of morphisms $\alpha_i\in\mathcal{C}(\varphi(i),\psi(i))$ for each $i\in I$ such that for each morphism $f_{ij}\in\mathcal{D}(i,j)$ we have $\psi(f_{ij})\circ\alpha_i=\alpha_j\circ\varphi(f_{ij})$. This induces an injective map $\mathcal{C}^{\Delta}(\varphi,\psi)\to\prod_I\mathcal{C}(\varphi(i),\psi(i))$. Moreover the image of this map is a c^∞–closed linear subspace since it is exactly the intersection of all kernels of the bounded linear maps $\prod_I\mathcal{C}(\varphi(i),\psi(i))\to\mathcal{C}(\varphi(j),\psi(k))$ given by $x\mapsto\psi(f_{jk})\circ pr_j(x)-pr_k(x)\circ\varphi(f_{jk})$ for all pairs (j,k) of objects of $\mathcal{D}$ and all $f_{jk}\in\mathcal{D}(j,k)$. Thus all morphism sets in $\mathcal{C}^{\mathcal{D}}$ have canonical structures of convenient vector spaces. Moreover the composition of morphisms is induced by the composition in $\mathcal{C}$ and using this it is easily seen to be bounded and bilinear. $\square$

4.23. Proposition. *If A and B are convenient algebras and $\mathcal{C}$ is a convenient category then the categories $(\mathcal{C}-\mathrm{Mod}_B)-\mathrm{Mod}_A$ and $(\mathcal{C}-\mathrm{Mod}_A)-\mathrm{Mod}_B$ are naturally isomorphic.*

Proof. This is clear since both categories are isomorphic to the category of convenient bifunctors from $A\times B$ (product of categories) to $\mathcal{C}$. $\square$

4.24. Now we can define bi– and multimodules in an appropriate way. We write $\mathcal{C}-\mathrm{Mod}_{A-B}$ for the category $(\mathcal{C}-\mathrm{Mod}_B)-\mathrm{Mod}_A$ and $\mathcal{C}-\mathrm{Mod}_A^B$ for $(\mathcal{C}-\mathrm{Mod}^B)-\mathrm{Mod}_A$ and so on. Inductively we define the categories $\mathcal{C}-\mathrm{Mod}_{A_1-\dots-A_n}^{B_1-\dots-B_m}$. By 4.22 all these categories are convenient categories. From the proof of 4.23 we see that an object of $\mathcal{C}-\mathrm{Mod}_{A_1-\dots-A_n}^{B_1-\dots-B_m}$ is just an object E of $\mathcal{C}$ together with bounded algebra homomorphisms $\lambda_i:A_i\to\mathcal{C}(E,E)$ and $\rho_i:B_i^{op}\to\mathcal{C}(E,E)$ such that each morphism in the image of λ_i commutes with all morphisms in the images of λ_j for $j\neq i$ and all morphisms in the images of all ρ_k and so on. The morphisms in these categories are just the morphisms in $\mathcal{C}$ which commute with the actions of all algebras.

As before we use the convention that we write Mod for $\underline{Con}$ $-$ Mod and we write $\mathrm{Hom}^{B_1-\cdots-B_m}_{A_1-\cdots-A_n}$ for the morphism sets in $\mathrm{Mod}^{B_1-\cdots-B_m}_{A_1-\cdots-A_n}$.

4.25. Lemma. *Let $\mathcal{C}$ and $\mathcal{C}'$ be convenient categories, $\varphi : \mathcal{C} \to \mathcal{C}'$ a covariant convenient functor and A a convenient algebra. Then φ lifts canonically to a smooth functor $\tilde{\varphi}$: $\mathcal{C} - \mathrm{Mod}_A \to \mathcal{C}' - \mathrm{Mod}_A$ such that the following diagram commutes*

$$\begin{tikzcd}
\mathcal{C} - \mathrm{Mod}_A \arrow[r, "\tilde{\varphi}"] \arrow[d] & \mathcal{C}' - \mathrm{Mod}_A \arrow[d] \\
\mathcal{C} \arrow[r, "\varphi"] & \mathcal{C}'
\end{tikzcd}$$

where the vertical arrows are the forgetful functors.

If φ is contravariant then the same holds with $\mathcal{C}' - \mathrm{Mod}_A$ replaced by $\mathcal{C}' - \mathrm{Mod}^A$.

In non categorical language this means that for any left A–module E in $\mathcal{C}$ we get a natural left (if φ is covariant) respectively right (if φ is contravariant) A–module structure on $\varphi(E)$.

Proof. The lift $\tilde{\varphi}$ is just the restriction to the subcategory of convenient functors $A \to \mathcal{C}$ of the natural lift $\varphi_* : \mathcal{C}^A \to \mathcal{C}'^A$ of φ to the categories of functors. Since for a functor $\psi : A \to \mathcal{C}$ we have $\varphi_*(\psi) = \varphi \circ \psi$ the functor φ_* maps smooth functors to smooth functors and thus $\tilde{\varphi}$ makes sense. Moreover φ_* acts on a natural transformation by acting on each morphism of the transformation via φ and from this it follows immediately that $\tilde{\varphi}$ is a smooth functor. $\square$

4.26. Proposition. *The following are smooth functors:*
(1): $L(E,.) : \underline{Con} \to \underline{Con}$ for any convenient vector space E.
(2): $\mathrm{Hom}^{B_1-\cdots-B_j}_{A_1-\cdots-A_i}(E,.) : \mathrm{Mod}^{B_1-\cdots-B_m}_{A_1-\cdots-A_n} \to \mathrm{Mod}^{B_{j+1}-\cdots-B_m}_{A_{i+1}-\cdots-A_n}$ for any object E of $\mathrm{Mod}^{B_1-\cdots-B_m}_{A_1-\cdots-A_n}$.

(3): $L(.,E) : \underline{Con} \to \underline{Con}$ for any convenient vector space E.
(4): $\mathrm{Hom}^{B_1-\cdots-B_j}_{A_1-\cdots-A_i}(.,E) : \mathrm{Mod}^{B_1-\cdots-B_m}_{A_1-\cdots-A_n} \to \mathrm{Mod}^{A_{i+1}-\cdots-A_n}_{B_{j+1}-\cdots-B_m}$ for any object E of $\mathrm{Mod}^{B_1-\cdots-B_m}_{A_1-\cdots-A_n}$.

(5): $E\tilde{\otimes}. : \underline{Con} \to \underline{Con}$ for any convenient vector space E.

Proof. (1): The map $f_* = L(E,f) : L(E,F) \to L(E,G)$ induced by $f \in L(F,G)$ is given by $g \mapsto f \circ g$. We have to show that the map $f \mapsto f_*$ is a smooth linear map $L(F,G) \to L(L(E,F), L(E,G))$. Linearity is obvious and to show smoothness it suffices in view of cartesian closedness of the category of smooth spaces (c.f. 1.7) and the fact that the spaces of bounded linear maps are closed linear subspaces of the corresponding spaces of smooth maps to show that the associated map $L(F,G) \times L(E,F) \to L(E,G)$ is smooth. But this is just the composition mapping which is smooth by 1.8.
(2): In the case $n = i$ and $m = j$ the functor has values in $\underline{Con}$ and the induced map is just the restriction of the map considered in (1) to appropriate subspaces, so smoothness follows as in (1). The rest follows by induction using 4.25.
(3): The map $f^* : L(f,E) : L(G,E) \to L(F,E)$ induced by $f \in L(F,G)$ is given by $g \mapsto g \circ f$ and we have to consider the map $f \mapsto f^*$ which is a map $L(F,G) \to L(L(G,E), L(F,E))$. As in (1) we can now pass to the associated map which is again the composition map.
(4): This follows from (3) in the same way as (2) is deduced from (1).
(5): The map $E\tilde{\otimes}f : E\tilde{\otimes}F \to E\tilde{\otimes}G$ induced by $f \in L(F,G)$ is the map induced by the bilinear bounded map $(a,b) \mapsto a \otimes f(b)$, $E \times F \to E\tilde{\otimes}G$. We have to consider the map $f \mapsto E\tilde{\otimes}f$ as a map $L(F,G) \to L(E\tilde{\otimes}F, E\tilde{\otimes}G)$. This map can be written as:

$$L(F,G) \xrightarrow{Id_E \times .} L(E \times F, E \times G) \xrightarrow{b_*}$$

$$\to L(E \times F, E\tilde{\otimes}G) \xrightarrow{\Phi} L(E\tilde{\otimes}F, E\tilde{\otimes}G)$$

where $b : E \times G \to E\tilde{\otimes}G$ is the canonical bilinear map and Φ is the isomorphism constructed in 4.16. Since all the maps in the composition are easily seen to be bounded an linear the proof is complete. $\square$

4.27. Now we can use smooth functors to show the existence of limits and colimts in categories of modules. Let $\mathcal{D}$ be an arbitrary small category and assume that every diagram of type $\mathcal{D}$ in $\mathcal{C}$ (i.e. every functor in $\mathcal{C}^{\mathcal{D}}$) has a limit. Then we get a functor $\lim : \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$ as follows: For a functor $\varphi : \mathcal{D} \to \mathcal{C}$, $\lim(\varphi)$ is the limit of φ and for a natural transformation $\alpha : \varphi \to \psi$ we get by the universal property of the limit a unique morphism $\lim(\alpha) : \lim(\varphi) \to \lim(\psi)$.

In the same way if we assume that all diagrams of type $\mathcal{D}$ in $\mathcal{C}$ have a colimit we get a functor $\operatorname{colim} : \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$.

4.28. Proposition. *For arbitrary convenient algebras $A_1, \dots, A_n$ and $B_1, \dots, B_m$ the category $\mathrm{Mod}^{B_1-\dots-B_m}_{A_1-\dots-A_n}$ is a complete and cocomplete additive category.*

Proof. As the category $\underline{Con}$ of convenient vector spaces is complete and cocomplete by 4.10 we have for any small category $\mathcal{D}$ the functors lim and colim : $\underline{Con}^{\mathcal{D}} \to \underline{Con}$. We claim that these are smooth functors. Let us start with the functor lim: For two functors $\varphi, \psi : \mathcal{D} \to \underline{Con}$ we have to show that the map $\underline{Con}^{\mathcal{D}}(\varphi, \psi) \to L(\lim(\varphi), \lim(\psi))$ induced by lim is bounded and linear. Linearity is clear by the universal property of the limit. To show boundedness recall that $\underline{Con}^{\mathcal{D}}(\varphi, \psi)$ is a closed linear subspace of $\prod_I L(\varphi(i), \psi(i))$ where I is the set of objects of $\mathcal{D}$ and that the limits of φ and ψ can be realized as closed linear subspaces of $\prod_I \varphi(i)$ and $\prod_I \psi(i)$. Then it is obvious that the map under consideration is just the restriction to $\underline{Con}^{\mathcal{D}}(\varphi, \psi)$ of the map $\prod_I L(\varphi(i), \psi(i)) \to L(\prod_I \varphi(i), \prod_I \psi(i))$ which sends $(f_i)_{i\in I}$ to the map $(x_i)_{i\in I} \mapsto (f_i(x_i))_{i\in I}$ and thus it suffices to show that this map is smooth. Using cartesian closedness one concludes that it suffices to show that the associated map $\prod_I L(\varphi(i), \psi(i)) \times \prod_I \varphi(i) \to \prod_I \psi(i)$ is smooth. But this is just a product of evaluation maps which are smooth by 1.8.

The proof of smoothness of the functor colim is similar.

Since spaces of module homomorphisms are just c^∞ closed linear subspaces of the corresponding spaces of bounded linear maps the above proofs show also smoothness of the functors lim and colim on categories of modules in $\underline{Con}$.

Now let $\varphi : \mathcal{D} \to \mathrm{Mod}^{B_1-\dots-B_m}_{A_1-\dots-A_n}$ be a diagram. Then φ is a functor to the category of smooth functors from $A_1 \times \dots \times B_m$ (product as categories) to $\underline{Con}$, so it corresponds to a bifunctor $\mathcal{D} \times (A_1 \times \dots \times B_m) \to \underline{Con}$ which is convenient in the second variable. This bifunctor in turn can be viewed as a convenient functor $(A_1 \times \dots \times B_m) \to \underline{Con}^{\mathcal{D}}$, so φ is in a natural way a module in $\underline{Con}^{\mathcal{D}}$. Thus for any diagram in the category $\mathrm{Mod}^{B_1-\dots-B_m}_{A_1-\dots-A_n}$ we get a canonical module structure on the limit and the colimit of the underlying convenient vector spaces by 4.25. To show that these modules are in fact the limit and the colimit in the categorical sense we only have to show that the maps induced by a family of module homomorphisms via the universal property are module homomorphisms. But this is clear from the definition of the module structures.

By 4.22 the morphism sets of $\mathrm{Mod}^{B_1-\dots-B_m}_{A_1-\dots-A_n}$ are convenient vector spaces and thus in particular abelian groups and the composition is bilinear. Moreover the zero object of $\underline{Con}$ clearly induces a zero object in $\mathrm{Mod}^{B_1-\dots-B_m}_{A_1-\dots-A_n}$ and since the existence of products and direct sums has been shown above the additivity of the category follows. $\square$

4.29. Free modules. Let E be a convenient vector space, A a convenient algebra. Then $\mathcal{F}(E) := A\tilde{\otimes}E$ has a canonical left and right A–module structure since it is the value on A of the smooth functor $.\tilde{\otimes}E$. This defines a functor $\mathcal{F} : \underline{Con} \to \mathrm{Mod}_A$ (respectively Mod^A). Moreover $e \mapsto e \otimes 1$ where 1 denotes the unit of A defines a bounded linear map $i : E \to \mathcal{F}(E)$.

We call $\mathcal{F}(E)$ the *free A–module* over E.

Proposition. *The A–module $\mathcal{F}(E)$ has the following universal property: If M is a left or right A–module and $f : E \to M$ is a bounded linear map then there is a unique A–module homomorphism $\tilde{f} : \mathcal{F}(E) \to M$ such that $\tilde{f} \circ i = f$. This property gives rise to an isomorphism of convenient vector spaces $L(E, M) \cong \operatorname{Hom}_A(\mathcal{F}(E), M)$ (respectively $L(E, M) \cong \operatorname{Hom}^A(\mathcal{F}(E), M)$). Thus the functor $\mathcal{F}$ is left adjoint to the forgetful functor.*

Proof. We prove the proposition only for left modules, the proof for right modules is similar. So let M be a left A–module with module action $\lambda : A \to L(M, M)$. For a bounded linear map $f : E \to M$ consider $F : A \times E \to M$ given by $F(a, e) := \lambda_M(a)(f(e))$. Obviously this is a bounded bilinear map and thus by the universal property of the tensor product it induces a bounded linear map $\tilde{f} : \mathcal{F}(E) \to M$. That this map is in fact a module homomorphism follows immediately from the definition of the A–module structure on $\mathcal{F}(E)$.

The map $f \mapsto \tilde{f}$ can be written as the composition of the bounded linear isomorphism $L(A, E; M) \cong L(A \tilde{\otimes} E, M)$ with the map $f \mapsto ev \circ (\lambda \times f)$ and thus is a bounded linear map $L(E, M) \to \operatorname{Hom}_A(\mathcal{F}(E), M)$ which is easily seen to be inverse to the bounded linear map i^*. $\square$

4.30. The free modules over finite dimensional vector spaces can be described very easily: $\mathcal{F}(\mathbb{R}^n) \cong A^n$ the direct sum (or equivalently direct product) of n copies of A.

Another interesting special case is the case of the free module over a module. If M is a left (right) A–module then the identity Id_M defines a canonical homomorphism of left (right) A–modules $\pi_M : \mathcal{F}(M) \to M$.

Definition. (1): An A–module M is called finitely generated iff there is a surjective homomorphism of A–modules $p : A^n \to M$ for some $n \in \mathbb{N}$ which has a bounded linear section, i.e. there is a bounded linear map $s : M \to A^n$ such that $p \circ s = Id_M$.
(2): An A–module M is called projective iff there is an A–module homomorphism $j : M \to \mathcal{F}(M)$ such that $\pi_M \circ j = Id$.
We write $\mathcal{P}(A)$ for the category of finitely generated projective right A–modules.

4.31. Proposition (Characterization of projective modules).
Let P be a left or right A–module. Then the following conditions are equivalent:
(1): P is projective.
(2): There is an A–module Q such that $P \oplus Q$ (direct sum as A–modules) is isomorphic to a free module.
(3): There is a free module F and a bounded module homomorphism $p : F \to F$ with $p \circ p = p$ such that P is isomorphic to the image of p.
(4): If $f : M \to N$ is a bounded homomorphism between A–modules and $g : P \to N$ is a bounded module homomorphism such that there is a bounded linear map $\psi : P \to M$ such that $f \circ \psi = g$ then there is also a bounded module homomorphism $h : P \to M$ with $f \circ h = g$.

If P is finitely generated then the free modules in (2) and (3) can be chosen to be A^n for some $n \in \mathbb{N}$.

Proof. (1)⇒(2): Put $Q := \operatorname{Ker}(\pi_P) \subset \mathcal{F}(P)$. As π_P is a bounded module homomorphism Q is a c^∞–closed submodule of $\mathcal{F}(P)$ and thus a convenient A–module. Moreover the map $j \oplus i : P \oplus Q \to \mathcal{F}(P)$, where j is a section of π_P and i is the inclusion of Q into $\mathcal{F}(P)$ is an isomorphism of A–modules.
(2)⇒(3): If $P \oplus Q \cong F$ for a free module F then the map $Id_P \oplus 0$ has the desired properties.
(3)⇒(4): Let us first show that free modules satisfy (4): Let $i : E \to \mathcal{F}(E)$ be the canonical map. The bounded linear map $\psi \circ i : E \to M$ corresponds to a unique bimodule homomorphism $h : \mathcal{F}(E) \to M$ and $h \circ i = \psi \circ i$ and thus $f \circ h \circ i = f \circ \psi \circ i = g \circ i$. Hence $f \circ h$ and g are module homomorphisms corresponding to the same linear map $E \to N$ and thus are equal.

In the general case assuming (3) the map $g \circ p : F \to N$ is a bounded bimodule homomorphism and $\psi \circ p$ is a bounded linear lift. Thus we get a bounded module homomorphism $\tilde{h} : F \to M$ such that $f \circ \tilde{h} = g \circ p$. Let h be the restriction of $\tilde{h}$ to the submodule $\mathrm{Im}(p) = \mathrm{Ker}(Id - p) \cong P$. As $f \circ \tilde{h} \circ p = g \circ p \circ p = g \circ p$ we see that $f \circ h = g$.
(4)$\Rightarrow$(1): Apply (4) to the case $f = \pi_P : \mathcal{F}(P) \to P$, $g = Id_P$ and $\psi = i$.

Let us now assume that P is finitely generated. Applying (4) to the case $M = A^n$, $N = P$, $f = p$, $g = Id_P$ and $\psi = s$ (with notation as in 4.30) we conclude that there is a bounded module homomorphism $h : P \to A^n$ such that $p \circ h = Id_P$. As above one concludes that conditions (2) and (3) are satisfied for $F = A^n$. $\square$

4.32. Corollary. *For any convenient algebra the category $\mathcal{P}(A)$ of finitely generated projective right A–modules is a convenient pseudo–abelian category. (c.f.* [Ka, I.6.7]*)*

Proof. $\mathcal{P}(A)$ is a full subcategory of the convenient category Mod^A and from 4.31(2) one immediately concludes that a finite sum of finitely generated projective modules is again finitely generated and projective, so $\mathcal{P}(A)$ is an additive category. To complete the proof we only have to show that for an object P of $\mathcal{P}(A)$ and a morphism $p : P \to P$ which satisfies $p \circ p = p$ the kernel of p is again projective. But this is obvious from 4.31(3). $\square$

4.33. Lemma. *Let A be a convenient algebra, $(M, \rho) \in \mathrm{Mod}^A$ and $(N, \lambda) \in \mathrm{Mod}_A$.*
(1): There is a convenient vector space $M \tilde{\otimes}_A N$ and a bounded bilinear map $b : M \times N \to M \tilde{\otimes}_A N$, $(m, n) \mapsto m \otimes_A n$ such that $b(\rho(a)(m), n) = b(m, \lambda(a)(n))$ for all $a \in A$, $m \in M$ and $n \in N$ which has the following universal property: If E is a convenient vector space and $f : M \times N \to E$ is a bounded bilinear map such that $f(\rho(a)(m), n) = f(m, \lambda(a)(n))$ then there is a unique bounded linear map $\tilde{f} : M \tilde{\otimes}_A N \to E$ with $\tilde{f} \circ b = f$. (2): For any convenient vector space E there is a natural isomorphism of convenient vector spaces $\mathrm{Hom}^A(M, L(N, E)) \cong L(M \tilde{\otimes}_A N, E)$.
(3): $.\tilde{\otimes}_A P : \mathrm{Mod}^A \to \underline{Con}$ and $M \tilde{\otimes}_A . : \mathrm{Mod}_A \to \underline{Con}$ are convenient functors.
(4): For $M \in \mathrm{Mod}^A$, $N \in \mathrm{Mod}_A^B$ and $P \in \mathrm{Mod}_B$ there is a canonical isomorphism of convenient vector spaces $M \tilde{\otimes}_A (N \tilde{\otimes}_B P) \cong (M \tilde{\otimes}_A N) \tilde{\otimes}_B P$.
(5): For $M \in \mathrm{Mod}_A^B$, $N \in \mathrm{Mod}_B^C$ and $P \in \mathrm{Mod}_A^C$ there is a natural isomorphism of convenient vector spaces
$\mathrm{Hom}_A^C(M \tilde{\otimes}_B N, P) \cong \mathrm{Hom}_A^B(M, \mathrm{Hom}^C(N, P))$.

Proof. We construct $M \tilde{\otimes}_A N$ as follows: Let $M \otimes N$ be the non completed tensor product of M and N constructed in 4.16 and let V be the closure in the associated locally convex topology of the subspace generated by all elements of the form $\rho(a)(m) \otimes n - m \otimes \lambda(a)(n)$ and define $M \tilde{\otimes}_A N$ to be the completion of $M \otimes_A N := (M \otimes N)/V$. As $M \otimes N$ has the universal property that bounded bilinear maps from $M \times N$ into arbitrary locally convex spaces induce bounded and hence continuous (in the locally convex topology) linear maps on $M \otimes N$, (1) is clear.
(2): The space $L(N, E)$ has a canonical right A–module structure by 4.25 and 4.26. Moreover one immediately checks that by definition of this structure a bounded linear map $f : M \to L(N, E)$ is a module homomorphism if and only if the associated map $\hat{f} : M \times N \to E$ satisfies the condition of (1). Thus by (1) we have a bijection between the two spaces. The map $L(M \tilde{\otimes}_A N, E) \to \mathrm{Hom}^A(M, L(N, E))$ which establishes this bijection is associated to b^* and thus it is bounded and linear. So we only have to show that the inverse is bounded.

From 4.16 we get a bounded linear map $\varphi : L(M, L(N, E)) \to L(M \otimes N, E)$ which is inverse to the map induced by the canonical bilinear map. Now let $L^{\mathrm{ann}\ V}(M \otimes N, E)$ be the closed linear subspace of $L(M \otimes N, E)$ consisting of all maps which annihilate V. Restricting we get a bounded linear map $\varphi : \mathrm{Hom}^A(M, L(N, E)) \to L^{\mathrm{ann}\ V}(M \otimes N, E)$.

Let $\psi : M \otimes N \to M \otimes_A N \to M \tilde{\otimes}_A N$ be the composition of the canonical projection with the inclusion into the completion. Then ψ induces a well defined linear map ψ_* :

$L^{\mathrm{ann}\,V}(M \otimes N, E) \to L(M\tilde{\otimes}_A N, E)$ and $\psi_* \circ \varphi$ is inverse to b^*. So it suffices to show that ψ_* is bounded.

This is the case if and only if the associated map $L^{\mathrm{ann}\,V}(M \otimes N, E) \times (M\tilde{\otimes}_A N) \to E$ is bounded. This in turn is equivalent to boundedness of the associated map $M\tilde{\otimes}_A N \to L(L^{\mathrm{ann}\,V}(M \otimes N, E), E)$. But this is just the prolongation to the completion of the bounded linear map $M\tilde{\otimes}_A N \to L(L^{\mathrm{ann}\,V}(M \otimes N, E), E)$ which sends x to the evaluation at x and thus it is bounded.
(3): We only consider the functor $.\tilde{\otimes} P$, the proof for the other one is similar. Thus we have to show that for two right A–modules M and N the map $\mathrm{Hom}^A(M, N) \to L(M\tilde{\otimes}_A P, N\tilde{\otimes}_A P)$ induced by functoriality of the tensor product is bounded and linear. First observe that the spaces $M \times P$ and $N \times P$ are naturally objects of Mod_A^A by using the right action on M respectively N and the left action on P. Forming the product with the identity map on P now induces a bounded linear map $\mathrm{Hom}^A(M, N) \to \mathrm{Hom}_A^A(M \times P, N \times P)$. Flipping the first P part we get a linear map to $\mathrm{Hom}^A(M, \mathrm{Hom}_A(P, N \times P))$ which is bounded by cartesian closedness. Next a short computation shows that $b_* : \mathrm{Hom}_A(P, N \times P) \to L(P, N\tilde{\otimes}_A P)$ is a homomorphism of right A–modules and thus it induces a bounded linear map $\mathrm{Hom}^A(M, \mathrm{Hom}_A(P, N \times P)) \to \mathrm{Hom}^A(M, L(P, N\tilde{\otimes}_A P))$. Finally we have the isomorphism $\mathrm{Hom}^A(M, L(P, N\tilde{\otimes}_A P)) \cong L(M\tilde{\otimes}_A P, N\tilde{\otimes}_A P)$ constructed in (2). A short computation shows that the composition of all these maps is exactly the map induced by functoriality of the tensor product.
(4): Straightforward computations show that both spaces have the following universal property: For any convenient vector space E and any bounded module homomorphism $f \in \mathrm{Hom}^A(M, \mathrm{Hom}^B(N, L(P, E)))$ there is a unique linear map prolonging f.
(5): One easily checks that the isomorphism constructed in (2) restricts to the claimed isomorphism. $\square$

4.34. Let $\varphi : A \to B$ be a bounded homomorphism between convenient algebras. Then B becomes in a natural way a convenient A–bimodule as follows: Let $\check{\mu} : B \to C^\infty(B, B)$ be the map associated via cartesian closedness to the multiplication $\mu : B \times B \to B$. Clearly $\check{\mu}$ is in fact a bounded algebra homomorphism $B \to L(B, B)$ and composing it with φ we get a left A–module structure on B. The right module structure on B is constructed in the same way using that φ is also an algebra homomorphism $A^{op} \to B^{op}$.

Using the left A–module structure on B we can now construct a functor $\varphi_* : \mathrm{Mod}^A \to \mathrm{Mod}^B$ between the categories of right modules as follows: For a right A–module M we define $\varphi_*(M) := M\tilde{\otimes}_A B$. By 4.33(3) this is a right B–module and φ_* is a convenient functor.

If $f, g : M \to N$ are two A–module homomorphisms then $\varphi_*(f+g)$ is the map induced by $(m, b) \mapsto (f(m)+g(m))\otimes_A b = f(m)\otimes_A b + g(m)\otimes_A b$ and as by 4.33(1) bounded linear maps from $M\tilde{\otimes}_A B$ to any convenient vector space are uniquely determined by their compositions with the canonical map $M \times B \to M\tilde{\otimes}_A B$ this implies that $\varphi_*(f+g) = \varphi_*(f) + \varphi_*(g)$ and thus φ_* is an additive functor. In particular this implies that φ_* commutes with finite direct sums (In fact the functor $\varphi_* = .\tilde{\otimes}_A B$ commutes with all colimits as it has a right adjoint by 4.33(2)).

4.35. Proposition. *φ_* restricts to a convenient additive functor $\mathcal{P}(\varphi) : \mathcal{P}(A) \to \mathcal{P}(B)$.*

Proof. We only have to show that if P is a finitely generated projective right A–module then $P\tilde{\otimes}_A B$ is finitely generated and projective as a B–module. First of all $A\tilde{\otimes}_A B$ is immediately seen to be isomorphic to B. As φ_* commutes with direct sums this implies that $A^n\tilde{\otimes}_A B \cong B^n$. Now if P is an arbitrary object of $\mathcal{P}(A)$. Then there is a right A–module Q such that $P \oplus Q \cong A^n$ for some n. Then we have $\varphi_*(P) \oplus \varphi_*(Q) \cong \varphi_*(P \oplus Q) \cong \varphi_*(A^n) \cong B^n$ and thus $\varphi_*(P)$ is finitely generated and projective. $\square$

5. Bundles of projective modules over base spaces

5.1. Definition. Let A be a convenient algebra and let X be a base space. By an *A–bundle over X* we mean a locally trivial fiber bundle $p : E \to X$ over X in which every fiber is a finitely generated projective right A–module and the transition functions are isomorphisms of A–modules. So any point $x \in X$ has an open neighborhood U_x such that there is a diffeomorphism $\varphi_x : p^{-1}(U_x) \to U_x \times P$ with $pr_1 \circ \varphi_x = p$, where P is a finitely generated projective right A–module and if x and y are such that $U_{xy} := U_x \cap U_y \neq \emptyset$ then the function $\varphi_x \circ \varphi_y^{-1} : U_{xy} \times P \to U_{xy} \times P$ is of the form $(z, u) \mapsto (z, \varphi_{xy}(z, u))$ where $\varphi_{xy} : U_{xy} \times P \to P$ is a smooth function which has the property that for any $z \in X$ the map $u \mapsto \varphi_{xy}(z, u)$ is a homomorphism of right A–modules. Using cartesian closedness one immediately concludes that the last condition is equivalent to the fact that the map $\check{\varphi}_{xy} : U_{x,y} \to \operatorname{Aut}(P)$ is smooth, where $\operatorname{Aut}(P)$ denotes the smooth group of all A–module isomorphisms of P. ($\operatorname{Aut}(P)$ is the group of invertible elements of the convenient algebra $\operatorname{Hom}^A(P, P)$ and thus is a smooth group by 4.20.)

Note that the isomorphism type of the module P may be different over different connected components of the space X.

If $p : E \to X$ is an A–bundle over X then we define a right action r of A on E by $\varphi_x(r(\varphi_x^{-1}(z, u), a)) := (z, \rho_P(a)(u))$ where $\rho_P : A \to L(P, P)$ defines the right module structure of P. This is well defined since the transition functions of the bundle are module homomorphisms.

If $p : E \to X$ and $p' : E' \to X$ are A–bundles over X then a morphism from E to E' is a smooth map $f : E \to E'$ which is fiber respecting and covers the identity, i.e. $p' \circ f = p$, and is equivariant for the actions of A constructed above. This is equivalent to the fact that the restriction of f to any fiber is a homomorphism of right A–modules.

By $\mathcal{E}_A(X)$ we denote the category of A–bundles over X with morphisms as described above.

5.2. Let $p : E \to X$ be an A–bundle over a connected base space X. Then in the terminology of 2.19 E is a smooth fiber bundle over X with fiber P and structure group $\operatorname{Aut}(P)$, the smooth group of all A–module automorphisms of P, where P is a finitely generated projective right A–module. Thus by 2.27 the set of isomorphism classes of such bundles (with P fixed) is in bijective correspondence with the set $[X, B\operatorname{Aut}(P)]$, where $B\operatorname{Aut}(P)$ is the classifying space of the smooth group $\operatorname{Aut}(P)$. If X is not connected then this applies to each connected component of X.

5.3. Let $p : E \to X$ be an A–bundle over a base space X. If u and v are in the same fiber, i.e. $p(u) = p(v)$ then we can add the two elements by adding them in a chart. This is well defined (independent of the chart) since the transition functions are module homomorphisms and thus in particular linear.

Lemma. *The fiberwise addition defines a smooth map $E \times_X E \to E$.*

Proof. Clearly addition is well defined as a map $E \times_X E \to E$. Let $q : E \times_X E \to X$ denote the natural map. Take a point $x \in X$ and let U_x be an open neighborhood such that there is a chart $\varphi_x : p^{-1}(U_x) \to U_x \times P$. Then $q^{-1}(U_x)$ is the set of all $(u, v) \in E \times E$ such that $p(u) = p(v) \in U_x$. On this open subset the map is given by $(u, v) \mapsto \varphi_x^{-1}(p(u), (pr_2 \circ \varphi_x)(u) + (pr_2 \circ \varphi_x)(v))$ and this is obviously smooth since the canonical maps $E \times_X E \to E$ are by definition smooth. $\square$

5.4. Direct sums. Let $p : E \to X$ and $p' : E' \to X$ be A–bundles over a base space X and define $E \oplus E' := E \times_X E'$. Then we have a canonical smooth map $q : E \oplus E' \to X$. Let x be a point in X and let U_x be an open neighborhood of x such that there are diffeomorphisms $\varphi_x : p^{-1}(U_x) \to U_x \times P$ and $\varphi'_x : p'^{-1}(U_x) \to U_x \times P'$. Then the open subset $q^{-1}(U_x)$ of $E \oplus E'$ consists of all pairs (u, v) such that $p(u) = p'(v) \in U_x$. Define $\psi_x : q^{-1}(U_x) \to U_x \times (P \oplus P')$ by $\psi_x(u, v) := (p(u), (i_P \circ pr_2 \circ \varphi_x)(u) + (i_{P'} \circ pr_2 \circ \varphi'_x(v)))$, where $i_P : P \to P \oplus P'$ and $i_{P'} : P' \to P \oplus P'$ are the canonical maps. Obviously this defines a smooth map. To see that it even is a diffeomorphism we proceed as follows: By 4.28 $P \oplus P'$ is isomorphic to $P \times P'$ and thus we have natural projections $\pi_P : P \oplus P' \to P$ and $\pi_{P'} : P \oplus P' \to P'$. The smooth maps $\varphi_x^{-1} \circ (Id \times \pi_P) : U_x \times (P \oplus P') \to E$ and $(\varphi'_x)^{-1} \circ (Id \times \pi'_P) : U_x \times (P \oplus P') \to E'$ induce by the universal property of the fibered product a smooth map $U_x \times (P \oplus P') \to q^{-1}(U_x) \subset E \oplus E'$ which is immediately seen to be inverse to ψ_x and thus ψ_x is a diffeomorphism.

To show that $q : E \oplus E' \to X$ is indeed an A–bundle over X we have to verify that the transition functions have the right form. So let x and y be such that $U_{xy} \neq \emptyset$. Then a short computation shows that $(\psi_x \circ \psi_y^{-1})(z, w) = (z, i_P(\varphi_{xy}(z, \pi_P(w))) + i_{P'}(\varphi'_{xy}(z, \pi_{P'}(w))))$, where φ_{xy} and φ'_{xy} are the transition functions of E and E' and this map is obviously A–linear in w since the maps i_P, $i_{P'}$, π_P and $\pi_{P'}$ are A–module homomorphisms and the maps φ_{xy} and φ'_{xy} are A–linear in w.

One easily checks that the fiberwise addition defined above is in fact a morphism of A–bundles $E \oplus E \to E$.

5.5. Lemma. *For any base space X and any convenient algebra A the category $\mathcal{E}_A(X)$ of A–bundles over X is an additive category.*

Proof. Let $f, g : E \to E'$ are two morphisms between A–bundles over X. Then these maps define a smooth map $f \times_X g : E \to E' \times_X E'$ by the universal property of the pullback and we define the sum of the two morphisms to be the composition of this map with the fiberwise sum $E' \times_X E' \to E'$. One easily checks that this sum is again a morphism and that with this definition the set of all morphisms between two A–bundles over X becomes an abelian group. It is also clear that the composition of morphisms is bilinear for this addition.

The zero object is the trivial A–bundle $Id_X : X \to X$. So we only have to show that $E \oplus E'$ is indeed the categorical coproduct of E and E'. First consider the map $0_{E'} : X \to E'$ defined by $x \mapsto (\varphi')^{-1}(x, 0)$ for some chart map φ' of E'. This is again independent of the choice of the chart by the form of the transition functions and it is obviously smooth. Together with the identity on E the composition of $0_{E'}$ with the projection p of E induces a smooth map $i_E : E \to E \oplus E'$ and in the same way we define $i_{E'} : E' \to E \oplus E'$. Now let F be another A–bundle over X and let $f : E \to F$ and $f' : E' \to F'$ be morphisms. We have to show that there is a unique morphism $g : E \oplus E' \to F$ such that $g \circ i_E = f$ and $g \circ i_{E'} = f'$. The composition of f and f' with the canonical maps $E \oplus E' \to E$ and $E \oplus E' \to E'$ defines by the universal property of the pullback a smooth map $\tilde{g} : E \oplus E' \to F \times_X F$ and we define $g : E \oplus E' \to F$ to be the composition of the fiberwise addition and $\tilde{g}$.

The smooth map g is immediately seen to be a morphism and uniqueness of g follows easily from uniqueness of $\tilde{g}$. $\square$

5.6. Lemma. *Let A be a convenient algebra, X a base space, $\pi : E \to X$ an A–bundle over X. Then there is a natural number n and an A–bundle F over X such that $E \oplus F \cong X \times A^n$ as an A–bundle.*

Proof. Let us first show that without loss of generality we may assume that the fiber of E is A^m for some m. Let $X_1, \dots, X_k$ be the finitely many connected components of X. Then the bundles $\pi : \pi^{-1}(X_i) \to X_i$ are locally trivial fiber bundles with fiber P_i and structure group $\mathrm{Aut}(P_i)$ for finitely generated projective right A–modules P_i. Now for any i there is an m_i such that P_i is a direct summand in A^{m_i}. Thus for any i we can find a finitely generated projective right A–module Q_i such that $P_i \oplus Q_i \cong A^m$ where m is the maximum of all m_i. Let $\tilde{E}_i$ be the trivial bundle $X_i \times Q_i \to X_i$. Then these bundles together form an A–bundle $\tilde{E}$ over X and $E \oplus \tilde{E}$ is an A–bundle with fiber A^m.

So we assume that E has fiber A^m. By compactness of X there is a finite atlas $(U_i,\varphi_i)_{i=1}^{\ell}$ for the bundle E, which has a subordinate partition of unity $\{\tilde f_i : i=1,\dots,\ell\}$. Then $x\mapsto \sum_{i=1}^{\ell}\tilde f_i^2(x)$ is a strictly positive smooth function and so also $f:X\to\mathbb{R}$ defined by $f(x):=\sqrt{\sum_{i=1}^{\ell}\tilde f_i^2(x)}$ is smooth and we define $f_i:X\to\mathbb{R}$ by $f_i(x):=\frac{\tilde f_i(x)}{f(x)}$. Then the family $\{f_i\}$ of smooth functions is subordinate to the cover $\{U_i\}$ and satisfies $\sum f_i^2=1$. Now we put $n:=\ell\cdot m$ and define $\Phi:E\to X\times A^n\cong X\times(\prod_1^{\ell}A^m)$ by

$$\Phi(u):=(p(u),(f_i(p(u))\cdot pr_2(\varphi_i(u)))_{i=1}^{\ell}).$$

This is well defined since the function f_i is zero locally around points where φ_i is not defined and it is smooth for the same reason. Moreover by definition it is a morphism of A–bundles over X.

Consider the map $\Psi: X\times(\prod_{i=1}^{\ell}A^m)\to E$ defined by

$$\Psi(x,(u_1,\dots,u_m)):=\sum_{i=1}^{m}f_i(x)\cdot\varphi_i^{-1}(x,u_i),$$

where the sum is the fiberwise addition. The map Ψ is then easily seen to be a smooth morphism of A–bundles. Moreover using that $\sum f_i^2=1$ one immediately shows that $\Psi\circ\Phi=Id_E$. Now consider $p:=Id-\Phi\circ\Psi: X\times A^n\to X\times A^n$. As $\Psi\circ\Phi=Id$ one immediately concludes that $p\circ p=p$. Moreover p is obviously a smooth morphism of A–bundles over X and clearly Φ induces an isomorphism of A–bundles $E\cong\operatorname{Ker}(p)$, where $\operatorname{Ker}(p)$ is the set of all (x,u) such that $p(x,u)=(x,0)$.

Next we define $\bar p:=\Phi\circ\Psi: X\times A^n\to X\times A^n$. Then obviously this is a morphism of A–bundles and if $\operatorname{Ker}(\bar p)$ is an A–bundle then $X\times A^n\cong\operatorname{Ker}(p)\oplus\operatorname{Ker}(\bar p)$ as an A–bundle over X. So to finish the proof it suffices to show that $\operatorname{Ker}(\bar p)$ is an A–bundle over X, i.e. that it is locally trivial.

For $i=1,\dots,\ell$ put $V_i:=f_i^{-1}((0,1])$. Then clearly (V_i) is an open covering of X. Now define $\psi_i: V_i\times A^n\to V_i\times A^{m(\ell-1)}$ by

$$\psi_i(x,(u_1,\dots,u_\ell)):=(x,(u_1,\dots,u_{i-1},u_{i+1},\dots,u_\ell))$$

and $\omega_i: V_i\times A^{m(\ell-1)}\to V_i\times A^n$ by

$$\begin{aligned}\omega_i(x,&(u_1,\dots,u_{i-1},u_{i+1},\dots,u_\ell)):=\\ &=(x,(u_1,\dots,u_{i-1},-\sum_{j\neq i}\tfrac{f_j(x)}{f_i(x)}\varphi_{ij}(x)(u_j),u_{i+1},\dots,u_\ell)).\end{aligned}$$

Here the φ_{ij} denote the transition functions of the bundle E, i.e. we have $(\varphi_i\circ\varphi_j^{-1})(x,u)=(x,\varphi_{ij}(x)(u))$ for all $x\in U_i\cap U_j$ and all $u\in A^m$. The map ω_i is well defined and smooth since f_i is positive on V_i and thus $1/f_i$ is smooth and since f_j is zero locally around points where φ_{ij} is not defined. Moreover both ψ_i and ω_i are obviously morphisms of A–bundles and clearly $\psi_i\circ\omega_i=Id$. We want to use the restriction of ψ_i to $\operatorname{Ker}(\bar p)\cap(V_i\times A^n)$ as a chart. Let us first show that ω_i has values in this subspace. A short computation shows that for $(x,(u_1,\dots,u_\ell))\in X\times A^n$ the k–th component of $(pr_2\circ\Phi\circ\Psi)(x,(u_1,\dots,u_\ell))$ is given by $\sum_{j=1}^{\ell}f_k(x)f_j(x)\varphi_{kj}(x)(u_j)$ and thus we get for the k–th component of $(pr_2\circ\Phi\circ\Psi\circ\omega_i)(x,(u_1,\dots,u_{i-1},u_{i+1},\dots,u_\ell))$:

$$\sum_{j\neq i}f_k(x)f_j(x)\varphi_{kj}(x)(u_j)-f_k(x)f_i(x)\sum_{j\neq i}\tfrac{f_j(x)}{f_i(x)}\varphi_{ki}(x)(\varphi_{ij}(x)(u_j))$$

which is zero since $\varphi_{ki}(x) \circ \varphi_{ij}(x) = \varphi_{kj}(x)$.

Next if $(x, (u_1, \dots, u_\ell)) \in \operatorname{Ker}(\bar{p}) \cap (V_i \times A^n)$ the i–th component of $(pr_2 \circ \Phi \circ \Psi)(x, (u_1, \dots, u_\ell))$ must be zero and thus we get $0 = \sum_{j=1}^{\ell} f_i(x) f_j(x) \varphi_{ij}(x)(u_j)$ and since $f_i(x) > 0$ and $\varphi_{ii}(x) = Id$ this is equivalent to $u_i = -\sum_{j \neq i} \frac{f_j(x)}{f_i(x)} \varphi_{ij}(x)(u_j)$ which shows that on this subset we have $\omega_i \circ \psi_i = Id$. Finally since ψ_i and ω_i are morphisms of A–bundles the transition functions are also morphisms of A–bundles and thus the proof is complete. $\square$

5.7. Corollary. *If $\pi : E \to X$ is an A–bundle over a base space X then there is a natural number n and a morphism of A–bundles $p : X \times A^n \to X \times A^n$ such that $p \circ p = p$ and $E \cong \operatorname{Ker}(p)$ as an A–bundle over X.*

5.8. Sections of vector bundles. By a *convenient vector bundle* we mean a locally trivial fiber bundle $\pi : E \to X$ with fiber a convenient vector space V and structure group $GL(V)$, the group of all bounded linear maps from V to V which have a bounded linear inverse. $GL(V)$ is a smooth group by 4.20. A *section* of the bundle $\pi : E \to X$ is a smooth map $s : X \to E$ such that $\pi \circ s = Id_X$.

A convenient vector bundle over a smoothly paracompact space (and thus in particular over a base space) has many sections. First we can construct the zero section 0_E as in the proof of 5.5. Then let $(U_\alpha, \varphi_\alpha)$ be an atlas for the bundle E, i.e. each $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times V$ is a diffeomorphism. As X is smoothly paracompact we may assume that there is a partition of unity $\{f_\alpha\}$ subordinate to the cover $\{U_\alpha\}$. For any α choose a smooth function $g_\alpha : U_\alpha \to V$. (There are many such smooth functions as there are many real valued smooth functions.) Now the smooth section $x \mapsto \varphi_\alpha^{-1}(x, f_\alpha(x) \cdot g_\alpha(x))$ defined on U_α can be extended by the zero section to the whole of X and thus defines a global smooth section which we denote by $f_\alpha g_\alpha$. Then one easily checks that $x \mapsto \sum_\alpha (f_\alpha g_\alpha)(x)$ where the sum is defined as in 5.3 also defines a smooth section.

5.9. Lemma. *Let $\pi : E \to X$ be a convenient vector bundle over a smooth space X, $\Gamma(E)$ the space of all smooth sections of the bundle. Then $\Gamma(E)$ has a natural structure of a convenient vector space.*

Proof. Let us first consider the case of a trivial bundle $pr_1 : X \times V \to X$. In this case the sections of the bundle correspond bijectively to the smooth maps from X to V, so $\Gamma(E) = C^\infty(X, V)$ which is a convenient vector space by 4.12. If we have a bundle which is isomorphic to a bundle of the form $X \times V$ then we clearly get an isomorphism (of vector spaces) $\Gamma(E) \cong C^\infty(X, V)$ and we define the smooth structure on $\Gamma(E)$ via this isomorphism.

In the general case let $(U_\alpha, \varphi_\alpha)$ be an atlas for the bundle E, and denote by E_α the convenient vector bundle $\pi : \pi^{-1}(U_\alpha) \to U_\alpha$. For any α the restriction defines a linear map $\Gamma(E) \to \Gamma(E_\alpha)$ and together these maps define an injective linear map $i : \Gamma(E) \to \prod_\alpha \Gamma(E_\alpha)$. We claim that the linear subspace $i(\Gamma(E))$ is c^∞–closed in $\prod_\alpha \Gamma(E_\alpha)$. For any α and any $x \in U_\alpha$ the evaluation at x defines a bounded linear map $\mathrm{ev}_x : C^\infty(U_\alpha, V) \to V$ and thus also $\mathrm{ev}_x : \Gamma(E_\alpha) \to V$. Define $\mathrm{ev}_{\alpha,x} : \prod_\alpha \Gamma(E_\alpha) \to V$ as $\mathrm{ev}_{\alpha,x} := \mathrm{ev}_x \circ pr_\alpha$. Now for any pair α, β of indices and any point $x \in U_\alpha \cap U_\beta$ we get a bounded linear map $\mathrm{ev}_{\alpha\beta,x} : \prod_\alpha \Gamma(E_\alpha) \to V$ defined by $\mathrm{ev}_{\alpha\beta,x} := \mathrm{ev}_{\alpha,x} - \mathrm{ev}_{\beta,x}$ and obviously $i(\Gamma(E))$ is exactly the intersection of all kernels of the maps $\mathrm{ev}_{\alpha\beta,x}$ and thus c^∞–closed and a convenient vector space and we put on $\Gamma(E)$ the smooth structure obtained in this way.

To see that this construction does not depend on the choice of the atlas one proceeds as follows: First one easily shows that one gets the same smooth structure on $\Gamma(E)$ if one replaces an atlas (U_i, φ_i) by an atlas (V_j, φ_j) where the covering $\{V_j\}$ is a refinement of $\{U_i\}$ and the maps φ_j are appropriate restrictions of the maps φ_i. Then if one has two different atlases one may without loss of generality assume that they are defined with respect to the same covering and the using the 'transition' functions between two charts defined on the same open set one easily shows that one gets the same structure on $\Gamma(E)$. $\square$

5.10. Proposition. *Let A be a convenient algebra, X a base space and $\pi : E \to X$ an A–bundle over X. Then the space of sections $\Gamma(E)$ is a convenient right module over the convenient algebra $C^\infty(X, A)$.*

Proof. Let us first define the module structure in the case of a trivial bundle $E = X \times P$, where P is a finitely generated projective right A–module. Let $\rho_P : A^{op} \to L(P, P)$ be the algebra homomorphism which defines the module structure on P and let $\hat{\rho}_P : A^{op} \times P \to P$ be the map associated to ρ_P via cartesian closedness. From 4.19 it is evident that $C^\infty(X, A)^{op} = C^\infty(X, A^{op})$. Consider the map $C^\infty(X, A^{op}) \times C^\infty(X, P) \times X \to P$ defined by $(f, s, x) \mapsto \hat{\rho}_P(f(x), s(x))$. This is easily seen to be smooth as it can be written as a composition of $\hat{\rho}_P$ with evaluation maps which are smooth by cartesian closedness. Thus the map $\hat{\rho} : C^\infty(X, A^{op}) \times C^\infty(X, P) \to C^\infty(X, P)$, given by $(\hat{\rho}(f, s))(x) = \hat{\rho}_P(f(x), s(x))$ is smooth and it is obviously linear in s and so again by cartesian closedness the associated map

$$\rho : C^\infty(X, A^{op}) \to C^\infty(C^\infty(X, P), C^\infty(X, P))$$

is smooth and has values in $L(C^\infty(X, P), C^\infty(X, P))$ and one easily checks that it is an algebra homomorphism. Thus this defines a right module structure on $C^\infty(X, P) \cong \Gamma(E)$.

In the general case let (U_i, φ_i) be an atlas for the bundle E and let P_i be the fiber over U_i for any i. Then from 5.9 we know that there is a bounded linear map $j : \Gamma(E) \to \prod_{i=1}^n C^\infty(U_i, P_i)$ which induces an isomorphism from $\Gamma(E)$ to a closed linear subspace of the second space. For any i let $j_i : U_i \hookrightarrow X$ be the inclusion. This induces bounded linear restriction mappings $j_i^* : C^\infty(X, A^{op}) \to C^\infty(U_i, A^{op})$ which are even homomorphisms of convenient algebras for the pointwise structures. Putting these together we obtain a homomorphism of convenient algebras $\varphi : C^\infty(X, A^{op}) \to \prod_{i=1}^n C^\infty(U_i, A^{op})$. Now we define $\tilde{\rho} : \prod_{i=1}^n C^\infty(U_i, A^{op}) \to L(\prod_{i=1}^n C^\infty(U_i, P_i), \prod_{i=1}^n C^\infty(U_i, P_i))$ by

$$\tilde{\rho}(f_1, \dots, f_n)(s_1, \dots, s_n) := (\rho_1(f_1)(s_1), \dots, \rho_n(f_n)(s_n))$$

, where ρ_i denotes the module action of $C^\infty(U_i, A^{op})$ on $C^\infty(U_i, P_i)$ constructed as above. Then this is obviously bounded and linear and thus also the composition $j^* \circ \tilde{\rho} \circ \varphi : C^\infty(X, A^{op}) \to L(\Gamma(E), \prod_{i=1}^n C^\infty(U_i, P_i))$ is a bounded linear map. Moreover one immediately checks that for any $f \in C^\infty(X, A^{op})$ and any $s \in \Gamma(E)$ the element $(i^* \circ \tilde{\rho} \circ \varphi)(f)(s)$ lies in the subspace $j(\Gamma(E))$. Using cartesian closedness twice one concludes that the induced mapping $\rho := j_* \circ j^* \circ \tilde{\rho} \circ \varphi : C^\infty(X, A^{op}) \to L(\Gamma(E), \Gamma(E))$ bounded and linear and one easily checks that it is an algebra homomorphism and thus $\Gamma(E)$ is a right module over $C^\infty(X, A)$. □

5.11. Theorem. *The pseudo–abelian category associated to the additive category $\mathcal{E}_A(X)$ of A–bundles over the base space X (c.f.* [Ka, I.6.10]*) is equivalent to the category $\mathcal{P}(C^\infty(X, A))$ of finitely generated projective right modules over the convenient algebra $C^\infty(X, A)$.*

Proof. By 4.32 $\mathcal{P}(C^\infty(X, A))$ is a pseudo–abelian category and thus by [Ka, I.6.12] it suffices to construct an additive functor

$$\Gamma : \mathcal{E}_A(X) \to \mathcal{P}(C^\infty(X, A))$$

which is fully faithful and such that any object of $\mathcal{P}(C^\infty(X, A))$ is a direct factor of an object in the image of Γ. In 5.10 we saw that for any A–bundle $\pi : E \to X$ the space $\Gamma(E)$ is a convenient right module over $C^\infty(X, A)$. By construction the module structure can be described as follows: Take $f \in C^\infty(X, A)$ and $s \in \Gamma(E)$, let $x \in X$ be a point and let $\varphi_x : \pi^{-1}(U_x) \to U_x \times P$ be a chart defined on a neighborhood U_x of x. Then we have $(\rho(f)(s))(y) = \varphi_x^{-1}(y, \rho_P(f(y))((pr_2 \circ \varphi_x \circ s)(y)))$ for $y \in U_x$, where ρ_P denotes the A–module structure of P. If $\pi' : E' \to X$ is another A–bundle and $\varphi : E \to E'$ is a morphism then we define $\Gamma(\varphi) : \Gamma(E) \to \Gamma(E')$ by $\Gamma(\varphi)(s) := \varphi_*(s) = \varphi \circ s$. As the restriction of φ to any fiber is an A–module homomorphism one immediately concludes from the description of the action of $C^\infty(X, A)$ above that $\Gamma(\varphi)$ is a homomorphism of $C^\infty(X, A)$–modules. Moreover it is clear that if φ_1 and φ_2 are two such morphisms then $\Gamma(\varphi_1 + \varphi_2) = \Gamma(\varphi_1) + \Gamma(\varphi_2)$. So it remains to show that $\Gamma(\varphi)$ is bounded. By definition of the smooth structure of $\Gamma(E)$ it suffices to do this in the case where both bundles are trivial, but there $\Gamma(\varphi)$ is just the map $\varphi_* : C^\infty(X, P) \to C^\infty(X, P')$ which is smooth by 1.8.

In the case of a trivial bundle $pr_1 : X \times A^n \to X$ we have $\Gamma(X \times A^n) = C^\infty(X, A^n) \cong C^\infty(X, A)^n$ (by 4.17). By 5.7 for an arbitrary A–bundle E over X there is a morphism $p : X \times A^n \to X \times A^n$ with $p \circ p = p$ such that $E \cong Ker(p)$. Then $\Gamma(p)$ is a module homomorphism and a projection and one easily sees that the isomorphism $E \cong Ker(p)$ induces an isomorphism $\Gamma(E) \cong Ker(\Gamma(p))$ and thus by 4.31 $\Gamma(E)$ is a finitely generated projective $C^\infty(X, A)$–module. Thus $\Gamma : \mathcal{E}_A(X) \to \mathcal{P}(C^\infty(X, A))$ is a functor and every object of the second category is a direct factor of an object in the image by 4.31. Moreover the functor Γ is immediately seen to be faithful, since for every point u in an A–bundle there is a section s such that $u = s(\pi(u))$. (c.f. 5.8)

Let us next show that any module homomorphism $g : C^\infty(X, A)^n \to C^\infty(X, A)^m$ is induced by a morphism of A–bundles $\varphi : X \times A^n \to X \times A^m$. Define $\varphi(x, u) := g(s_u)(x)$ where $s_u : X \to X \times A^n$ is the constant section $y \mapsto (y, u)$. φ is smooth as g and the evaluation are smooth and $u \mapsto s_u$ is the map associated via cartesian closedness to $pr_1 : A^n \times X \to A^n$, and thus φ is obviously a morphism of A–bundles. To show that $g = \Gamma(\varphi)$ it suffices to show that if $s(x) = u$ then $g(s)(x) = g(s_u)(x)$ and for this in turn it suffices to show that if $s(x) = 0$ then $g(s)(x) = 0$. Now let $u_1, \dots, u_n$ be the 'unit vectors' in A^n, i.e. the elements for which $pr_i(u_j)$ is 1_A for $i = j$ and 0 for $i \neq j$. Then since the projections $A^n \to A$ are smooth one easily concludes that any section s of $X \times A^n$ can be written in the form $s(y) = \sum s_{u_i}(y) \cdot f_i(y)$ for unique functions $f_i \in C^\infty(X, A)$. But then we have $g(s)(y) = \sum g(s_{u_i})(y) \cdot f_i(y)$ and if $s(x) = 0$ then $f_i(x) = 0$ for all i and thus $g(s)(x) = 0$.

Finally if E and E' are A–bundles over X and $g : \Gamma(E) \to \Gamma(E')$ is a module homomorphism then let n and m be such that $\Gamma(E) \cong Ker(\Gamma(p)) \subset C^\infty(X, A)^n$ and $\Gamma(E') \cong Ker(\Gamma(p')) \subset C^\infty(X, A)^m$ for projection morphisms p and p'. Then $i \circ g \circ \Gamma(Id - p)$ is a module homomorphism $C^\infty(X, A)^n \to C^\infty(X, A)^m$, where $i : Ker(\Gamma(p')) \to C^\infty(X, A)^m$ denotes the inclusion. By the argument above this homomorphism is induced by a morphism of A–bundles $X \times A^n \to X \times A^m$. Now one easily sees that this morphism has values in $Ker(p')$ and the restriction to $Ker(p)$ defines a morphism of A–bundles $\varphi : E \to E'$ which is easily seen to induce g. Thus the functor Γ is full and the proof is complete. □

5.12. Corollary. *For any convenient algebra A and any base space X the category $\mathcal{E}_A(X)$ is a convenient category.*

Proof. Via Γ we can identify $\mathcal{E}_A(X)$ with a full subcategory of the category $\mathcal{P}(C^\infty(X, A))$ which is a convenient category by 4.32. □

5.13. Corollary. *Let A be a convenient algebra such that for any finitely generated projective convenient module P over A the set of invertibles* $\mathrm{Aut}(P)$ *is c^∞–open in* $\mathrm{Hom}^A(P, P)$ *and the inversion* $\mathrm{Aut}(P) \to \mathrm{Aut}(P)$ *is smooth for the initial smooth structure with respect to the inclusion into* $\mathrm{Hom}^A(P, P)$*. (In particular these conditions are satisfied if A is a Banach algebra).*
Then for any base space X the categories $\mathcal{E}_A(X)$ and $\mathcal{P}(C^\infty(X, A))$ are equivalent.

Proof. It suffices to show that for any base space X the category $\mathcal{E}_A(X)$ is pseudo–abelian since then the result immediately follows from 5.11. So we have to show that if $\pi : E \to X$ is an A–bundle over X and $p : E \to E$ is a morphism such that $p \circ p = p$ then the kernel of p exists and to do this it suffices to prove that the bundle $\mathrm{Ker}(p)$ is locally trivial. Since this is a local question we may assume that the bundle E is trivial, i.e. $E = X \times P$ for some finitely generated projective A–module P.

Now write $p : E \to E$ as $(x, z) \mapsto (x, q(x, z))$. Then $q : X \times P \to P$ is smooth and thus the associated map $\check{q} : X \to C^\infty(P, P)$ is smooth and has by assumption values in the closed linear subspace $\operatorname{Hom}^A(P, P)$ and thus is smooth as a map to this space. Choose a point $x_0 \in X$ and consider the map $f : X \to \operatorname{Hom}^A(P, P)$ defined by $f(x) := Id - \check{q}(x) - \check{q}(x_0) + 2\check{q}(x_0) \circ \check{q}(x)$. Then f is obviously smooth and thus continuous for the $\tau_{\mathcal{C}}$–topologies and since $f(x_0) = Id$ and $\operatorname{Aut}(P)$ is c^∞–open in $\operatorname{Hom}^A(P, P)$ there is an open neighborhood U_{x_0} of x_0 such that $f(x)$ is an isomorphism for all $x \in U_{x_0}$. Now consider the map $\varphi : U_{x_0} \times P \to U_{x_0} \times P$ defined by $(x, z) \mapsto (x, f(x)(z))$. By construction this is an isomorphism of smooth A–bundles over U_{x_0} since the map $x \mapsto (f(x))^{-1}$ is smooth by smoothness of the inversion. One immediately verifies that $f(x) \circ \check{q}(x) = \check{q}(x_0) \circ f(x)$ and thus φ induces an isomorphism between the A–bundles $\operatorname{Ker}(p) \restriction U_{x_0}$ and $U_{x_0} \times \operatorname{Ker}(\check{q}(x_0))$ and clearly $\operatorname{Ker}(\check{q}(x_0))$ is a finitely generated projective A–module.

Let us finally show that the assumptions of the theorem are satisfied if A is a Banach algebra. In this case for any n the space A^n is a Banach space and since any finitely generated projective A–module is the kernel of a bounded and thus continuous projection defined on some A^n all these modules are Banach spaces. Thus the spaces $\operatorname{Hom}^A(P, P)$ are Banach algebras and so the invertibles form an open subset and the inversion is smooth by the implicit function theorem. $\square$

5.14. Proposition. *Let A and B be convenient algebras and let X and Y be base spaces.*
(1): Any smooth map $f : X \to Y$ induces an additive functor $f^* = \mathcal{E}_A(f) : \mathcal{E}_A(Y) \to \mathcal{E}_A(X)$.
(2): Any bounded algebra homomorphism $\varphi : A \to B$ induces an additive functor $\varphi_* = \mathcal{E}_\varphi(X) : \mathcal{E}_A(X) \to \mathcal{E}_B(X)$. In the case where X is a single point and thus $\mathcal{E}_A(X)$ is equivalent with $\mathcal{P}(A)$ this functor coincides with the one constructed in 4.35.

Proof. (1): If $p : E \to Y$ is an A–bundle over Y then we define f^*E to be the pullback in the category of smooth spaces. A short argument similar to the proof of 2.6 shows that this is an A–bundle over X. Using the universal property of the pullback one immediately concludes that a morphism $\alpha : E \to E'$ between A–bundles over Y induces a morphism of A–bundles $f^*\alpha : f^*E \to f^*E'$ and that f^* is indeed an additive functor.
(2): Let $p : E \to X$ be an A–bundle. Then for any $x \in X$ the fiber $E_x := p^{-1}(x)$ is a finitely generated projective right A–module. Thus we can apply the functor $\mathcal{P}(\varphi)$ constructed in 4.35 to any fiber to get a finitely generated projective right B–module $\mathcal{P}(\varphi)(E_x)$, and we define $\varphi_*(E)$ as a set to be the disjoint union of all the spaces $\mathcal{P}(\varphi)(E_x)$. Then there is an obvious projection $\tilde{p} : \varphi_*(E) \to X$ which sends $\mathcal{P}(\varphi)(E_x)$ to x. Next we have to define a smooth structure on $\varphi_*(E)$.

Let (U_i, u_i) be an A–bundle atlas on E, so any U_i is an open subset of X and $u_i : p^{-1}(U_i) \to U_i \times P_i$ is a fiber respecting diffeomorphism, where P_i is a finitely generated projective right A module. For any $x \in U_i$ the map u_i induces a smooth and thus bounded A–module homomorphism $u_{i,x} : E_x \to P_i$. Now we define a bijective map $\tilde{u}_i : \tilde{p}^{-1}(U_i) \to U_i \times \mathcal{P}(\varphi)(P_i)$ by $z \mapsto (\tilde{p}(z), \mathcal{P}(\varphi)(u_{i,\tilde{p}(z)})(z))$ and we define a smooth structure on $\varphi_*(E)$ by requiering that all maps $\tilde{u}_i$ are diffeomorphisms. To show that this is a correct definition it suffices to show that for any i, j such that $U_{ij} := U_i \cap U_j \neq \emptyset$ the map $\tilde{u}_i \circ \tilde{u}_j^{-1} : U_{ij} \times \mathcal{P}(\varphi)(P_j) \to U_{ij} \times \mathcal{P}(\varphi)(P_j)$ is a diffeomorphism. But by definition of an A–bundle we have $(u_i \circ u_j^{-1})(x, p) = (x, u_{ij}(x)(p))$ for some smooth map $u_{ij} : U_{ij} \to \operatorname{Hom}^A(P_i, P_j)$. By 4.35 $\mathcal{P}(\varphi)$ is a convenient functor and thus it induces a smooth linear map $\mathcal{P}(\varphi) : \operatorname{Hom}^A(P_i, P_j) \to \operatorname{Hom}^B(\mathcal{P}(\varphi)(P_i), \mathcal{P}(\varphi)(P_j))$. Now a short computation shows that $(\tilde{u}_i \circ \tilde{u}_j^{-1})(x, z) = (x, (\mathcal{P}(\varphi) \circ u_{ij})(x)(z))$, and the result follows. Moreover the same argument shows that the smooth structure on $\varphi_*(E)$ does not depend on the choice of the atlas and that $\varphi_*(E)$ is a B–bundle.

Next let $q : F \to X$ be another A–bundle and let $f : E \to F$ be a morphism of A–bundles. Then the restriction $f_x : E_x \to F_x$ of f to any fiber is a bounded A–module homomorphism, so applying $\mathcal{P}(\varphi)$ we get induced B–module homomorphisms $\mathcal{P}(\varphi)(E_x) \to \mathcal{P}(\varphi)(F_x)$ and thus an induced fiber respecting map $\varphi_*(f) : \varphi_*(E) \to \varphi_*(F)$. Using the description of atlasses given above and the fact that the functor $\mathcal{P}(\varphi)$ is convenient one immediately deduces that $\varphi_*(f)$ is smooth and thus a morphism of B–bundles. Moreover from this description it is clear that the functor φ_* is addittive. □

6. Elementary K–Theory

6.1. Proposition. *Let M be a commutative monoid. Then there is an up to isomorphism unique abelian group $K(M)$, called the Grothendieck group of M, with a homomorphism of monoids $s : M \to K(M)$ which has the following universal property: If G is an abelian group and $f : M \to G$ is a homomorphism of monoids then there is a unique group homomorphism $K(f) : K(M) \to G$ such that $K(f) \circ s = f$. This construction defines a functor from the category of commutative monoids to the category of abelian groups which is left adjoint to the forgetful functor.*

Proof. Let us write the algebraic structures as $+$. On $M \times M$ define an equivalence relation by declaring (m, n) to be equivalent to (m', n') if and only if there is a $p \in M$ such that $m+n'+p = m'+n+p$ and let $K(M)$ be the set of all equivalence classes. Let us write $[m, n]$ for the equivalence class of (m, n). Now define $[m_1, n_1]+[m_2, n_2] := [m_1+m_2, n_1+n_2]$. This is easily seen to be well defined and it is clearly associative and commutative. Moreover $[0, 0]$ is a neutral element and for $[m, n]$ the inverse is given by $[n, m]$. Thus $K(M)$ is an abelian group. Now define $s : M \to K(M)$ by $s(m) := [m, 0]$. Then this is clearly a monoid homomorphism. If G is an abelian group and $f : M \to G$ is a monoid homomorphism then we define $K(f) : K(M) \to G$ by $K(f)([m, n]) := f(m) - f(n)$. Then this is a well defined group homomorphism and $K(f) \circ s = f$. Uniqueness of $K(f)$ follows immediately from the fact that $[m, n] = s(m) - s(n)$. The functoriality of the construction is obvious. □

6.2. Lemma. *(1): Any element of $K(M)$ can be written as $s(m)-s(n)$ for some elements $m, n \in M$.*
(2): The map $s : M \to K(M)$ is injective if and only if the monoid M has cancellation, i.e. iff for any $m, n, p \in M$ the identity $m + p = n + p$ implies $m = n$.
(3): If the monoid M is equipped with an associative multiplication which is distributive with respect to addition and such that $0 \cdot 0 = 0$ then there is a natural ring structure on $K(M)$ such that the map s is compatible with the multiplications.

Proof. (1) was already observed in the proof of 6.1.
(2): $s(m) = s(n)$ if and only if there is an element $p \in M$ such that $m + p = n + p$.
(3): Define $[m, n] \cdot [m', n'] := [mm' + nn', mn' + nm']$. All properties are then easily verified. □

6.3. Definition. First we define the Grothendieck group of an additive category $\mathcal{C}$ as follows: Let $\Phi(\mathcal{C})$ denote the set of isomorphism classes of objects of $\mathcal{C}$. Then this is a commutative monoid with addition given by the coproduct and we define $K(\mathcal{C}) := K(\Phi(\mathcal{C}))$. If $\mathcal{D}$ is another additive category and $\varphi : \mathcal{C} \to \mathcal{D}$ is an additive functor then φ commutes with finite direct sums and thus induces a monoid homomorphism $\Phi(\mathcal{C}) \to \Phi(\mathcal{D})$ and hence a group homomorphism $K(\mathcal{C}) \to K(\mathcal{D})$.
For a convenient algebra A define $K(A) = K_0(A) := K(\mathcal{P}(A))$.
If A is a convenient algebra and X is a base space then we put $K_A(X) = K^0_A(X) := K(\mathcal{E}_A(X))$.

6.4. Proposition. *(1): $A \mapsto K(A)$ defines a covariant functor from the category of convenient algebras and bounded homomorphisms to the category of abelian groups.*
(2): $(A, X) \mapsto K_A(X)$ defines a functor which is covariant in A and contravariant in X.

Proof. (1): From 4.35 we know that a bounded algebra homomorphism $\varphi : A \to B$ induces an additive functor $\mathcal{P}(\varphi) : \mathcal{P}(A) \to \mathcal{P}(B)$ and this induces by 6.3 a group homomorphism $K(\varphi) : K(A) \to K(B)$ and using 4.33(4) one easily shows that this indeed defines a functor.
(2): By 5.14(1) a smooth function $f : X \to Y$ between base spaces induces an additive functor $f^* : \mathcal{E}_A(Y) \to \mathcal{E}_A(X)$ and thus a group homomorphism $K_A(f) : K_A(Y) \to K_A(X)$ and one easily shows that this defines a functor.

By 5.14(2) a bounded homomorphism $\varphi : A \to B$ induces an additive functor $\mathcal{E}_\varphi(X) : \mathcal{E}_A(X) \to \mathcal{E}_B(X)$ for any smooth space X and thus a group homomorphism $K_\varphi(X) : K_A(X) \to K_B(X)$ and again one easily checks that this defines a functor. $\square$

6.5. Lemma. *Let A be a convenient algebra.*
(1): Any element of $K_0(A)$ can be written as $[P] - [A^n]$ for some finitely genrated projective right A–module P and some $n \in \mathbb{N}$, where $[P]$ denotes the class of P in $K_0(A)$.
(2): Two finitely generated projective right A–modules P and Q represent the same class in $K_0(A)$ if and only if $P \oplus A^n \cong Q \oplus A^n$ for some n.

Proof. (1): By 6.2(1) any element of $K_0(A)$ can be written as $[P]-[Q]$ for finitely generated projective right A–modules P and Q. By 4.31 there is a finitely generated projective right A–module R such that $Q \oplus R \cong A^n$ for some n. Thus $(P \oplus R) \oplus Q \cong P \oplus A^n$ and hence $[P \oplus R] - [A^n] = [P] - [Q]$ in $K_0(A)$.
(2): P and Q represent the same class in $K_0(A)$ if and only if there is a finitely generated projective right A–module R such that $P \oplus R \cong Q \oplus R$. By 4.31 there is a finitely generated projective right A–module S such that $R \oplus S \cong A^n$ for some n and thus the result follows. $\square$

6.6. Lemma. *Let A be a convenient algebra, X a base space. Let θ_n be the trivial bundle over X with fiber A^n, i.e. $\theta_n = X \times A^n$. Then we have:*
(1): Any element of $K_A(X)$ can be represented as $[E] - [\theta_n]$ for some A–bundle E over X and some $n \in \mathbb{N}$. (Here $[E]$ denotes the class of E in $K_A(X)$.)
(2): Two A–bundles E and F represent the same class in $K_A(X)$ if and only if $E \oplus \theta_n \cong F \oplus \theta_n$ for some n.

Proof. This is proved as 6.5 above using 5.6 instead of 4.31. $\square$

6.7. Reduced K–Theory. Let pt be the smooth space consisting of a single point. Then for any smooth space X there is a unique smooth map $p : X \to pt$. For any convenient algebra A this smooth map induces a group homomorphism $K_A(p) : K_A(pt) = K(A) \to K_A(X)$ and we define the reduced K–theory $\tilde{K}_A(X)$ of X to be the cokernel of this homomorphism. Thus we have an exact sequence

$$0 \to K(A) \to K_A(X) \to \tilde{K}_A(X) \to 0$$

If we choose a point $x_0 \in X$ then the inclusion of x_0 induces a group homomorphism $K_A(X) \to K(A)$ which is left inverse to $K_A(p)$ and thus we get a splitting $K_A(X) \cong \tilde{K}_A(X) \oplus K(A)$.

6.8. Let A be a convenient algebra, $\pi : E \to X$ a smooth A–bundle over a base space X. Then for two points of X which are in the same connected component the fibers over the two points are isomorphic and thus the bundle E determines a locally constant function $X \to \Phi(\mathcal{P}(A))$ and composing this with the map to the Grothendieck group we get a locally constant function $r_E : X \to K(A)$, i.e. an element of the abelian group $H^0(X, K(A))$, the Čech cohomology of X with values in $K(A)$. Note that if we put on $K(A)$ the discrete smooth structure then $H^0(X, K(A)) = [X, K(A)]$, the set of smooth homotopy classes of smooth maps from X to $K(A)$.

Clearly r_E depends only on the isomorphism class of E and $E \mapsto r_E$ defines a homomorphism of monoids $\Phi(\mathcal{E}_A(X)) \to H^0(X, K(A))$. By the universal property of the Grothendieck group we thus get a group homomorphism $K_A(X) \to H^0(X, K(A))$ and we denote by $K'_A(X)$ the kernel of this homomorphism. Thus we have an exact sequence

$$0 \to K'_A(X) \to K_A(X) \to H^0(X, K(A)) \to 0$$

Now let $f : X \to \Phi(\mathcal{P}(A))$ be a locally constant function. Then f is constant on any of the connected components $X_1, \dots, X_n$ of X. For any i choose a finitely generated projective A–module P_i which is in the isomorphism class to which f maps X_i. Then consider the A–bundle over X given over X_i by $X_i \times P_i$. Forming the isomorphism class and composing with the map to the Grothendieck group we get a monoid homomorphism $H^0(X, \Phi(\mathcal{P}(A))) \to K_A(X)$. Clearly the Grothendieck group of $H^0(X, \Phi(\mathcal{P}(A)))$ is just $H^0(X, K(A))$ and thus by the universal property we get a group homomorphism $H^0(X, K(A)) \to K_A(X)$. Obviously this homomorphism splits the above exact sequence and thus we get a natural isomorphism $K_A(X) \cong K'_A(X) \oplus H^0(X, K(A))$.

If X is connected then there is a natural isomorphism $H^0(X, K(A)) \cong K(A)$ and the split map $K(A) \to K_A(X)$ constructed above is the same as the map induced by the projection of X to a point and thus in this case the groups $\tilde{K}_A(X)$ and $K'_A(X)$ are isomorphic.

6.9. Our next task is to give a homotopy theoretic interpretation of $K_A(X)$. This requires an intermediate step. For any $n \in \mathbb{N}$ let $\Phi^A_n(X)$ denote the set of all isomorphism classes of smooth A–bundles over X with fiber A^n. Let $GL(n, A)$ be the smooth group of all isomorphisms of right A–modules $A^n \to A^n$, i.e. $GL(n, A) = \operatorname{Aut}(A^n)$. Then by 2.28 there is a bijection $\Phi^A_n(X) \cong [X, BGL(n, A)]$, the set of homotopy classes of smooth maps from X to the classifying space of the smooth group $GL(n, A)$. Adding trivial bundles with fiber A we get maps $\Phi^A_n \to \Phi^A_{n+1}$ we denote by $\Phi^A(X)$ the direct limit of the so obtained inductive system of sets. The direct sum of A–bundles defines maps $\Phi^A_n(X) \times \Phi^A_m(X) \to \Phi^A_{n+m}(X)$. As the category of sets is cartesian closed $\Phi^A_n(X) \times \Phi^A(X)$ is the direct limit of the sets $\Phi^A_n(X) \times \Phi^A_m(X)$ and thus these maps induce a map $\Phi^A_n(X) \times \Phi^A(X) \to \Phi^A(X)$ for any n which in turn induce a map $\Phi^A(X) \times \Phi^A(X) \to \Phi^A(X)$ which defines the structure of a commutative monoid on $\Phi^A(X)$.

Now let θ_n be the bundle $X \times A^n$ for any n. Then we define a map $\Phi^A_n(X) \to K'_A(X)$ by $E \mapsto [E] - [\theta_n]$. Obviously these maps define a homomorphism of monoids $\sigma : \Phi^A(X) \to K'_A(X)$.

6.10. Lemma. *The homomorphism $\sigma : \Phi^A(X) \to K'_A(X)$ defined above is an isomorphism. Thus $\Phi^A(X)$ is an abelian group.*

Proof. If $[E]-[\theta_n] = [F]-[\theta_m]$ in $K_A(X)$ then $E \oplus \theta_m$ and $F \oplus \theta_n$ represent the same class in $K_A(X)$ and thus by 6.6(2) $E \oplus \theta_m \oplus \theta_p \cong F \oplus \theta_n \oplus \theta_p$ for some p, and since $\theta_m \oplus \theta_p = \theta_{m+p}$ the bundles E and F represent the same element of $\Phi^A(X)$ and the homomorphism is injective.

On the other hand by 6.6(1) any element of $K_A(X)$ can be written as $[E] - [\theta_n]$ for some A–bundle E and some n. Let X_i, $i = 1, \dots, k$ be the connected components of X and let P_i be the fiber of E over X_i. If $[E]-[\theta_n]$ lies in the subgroup $K'_A(X)$ then for any i the modules P_i and A^n represent the same class in $K_0(A)$. Thus for any i there is a finitely generated projective right A–module Q_i such that $P_i \oplus Q_i \cong A^n \oplus Q_i$. Now for any i we can choose a module R_i such that $Q_i \oplus R_i \cong A^m$ for some fixed m. Thus we get $P_i \oplus A^m \cong A^{n+m}$ for each i. By definition $[E \oplus \theta_m] - [\theta_{n+m}] = [E] - [\theta_n]$ in $K_A(X)$ and we just saw that $E \oplus \theta_m$ is an A bundle with fiber A^{n+m}, so $[E \oplus \theta_m] - [\theta_{n+m}]$ is in the image of σ and thus σ is surjective. $\square$

6.11. Let us now express the last result in terms of sets of homotopy classes. From 6.9 we know that $\Phi_n^A(X) \cong [X, BGL(n, A)]$. Now $f \mapsto f \oplus Id_A$ induces a smooth homomorphism $GL(n, A) \to GL(n+1, A)$ and thus a smooth map between the classifying spaces which in turn induces a map $[X, BGL(n, A)] \to [X, BGL(n+1, A)]$ that clearly corresponds to the map $\Phi_n^A(X) \to \Phi_{n+1}^A(X)$ constructed in 6.9. Let $[X, BGL(A)]$ denote the direct limit of the so obtained inductive system. Then there is a bijection $\Phi^A(X) \cong [X, BGL(A)]$. Thus we have an isomorphism $K_A'(X) \cong [X, BGL(A)]$ where the group structure on $[X, BGL(A)]$ is induced by the direct sum of homomorphisms.
Define $[X, K_0(A) \times BGL(A)]$ to be the direct limit of the sets $[X, K_0(A) \times BGL(n, A)]$ where the connecting maps are constructed as above and $K_0(A)$ is equipped with the discrete smooth structure. Now one easily verifies that for arbitrary smooth spaces we have $[X, Y \times Z] \cong [X, Y] \times [X, Z]$ and by cartesian closedness of the category of sets the product with a fixed set commutes with direct limits and so we have:

$$[X, K_0(A) \times BGL(A)] \cong [X, K_0(A)] \times [X, BGL(A)] \cong K_A(X).$$

Let us finally compare the functorial properties of $K_A(X)$ and $[X, K_0(A) \times BGL(A)]$. So let Y be another base space and $f : Y \to X$ a smooth function and let $[E] - [\theta_n]$ be an element of $K_A(X)$. By construction this element is mapped by $K_A(f)$ to the element $[f^*E] - [\theta_n]$ of $K_A(Y)$. Clearly the element of $[Y, K_0(A)]$ corresponding to this is given by f^* of the element of $[X, K_0(A)]$ corresponding to $[E] - [\theta_n]$.On the other hand f induces maps $f^* : [X, BGL_n(A)] \to [Y, BGL_n(A)]$ which in turn induce a map $f^* : [X, BGL(A)] \to [Y, BGL(A)]$. If $[E] - [\theta_n]$ lies in $K_A'(X)$ then the classifying homotopy class of f^*E is given by $f^*([g])$ where $[g]$ is the homotopy class corresponding to the bundle E. Thus we have an isomorphism of functors between $X \mapsto K_A(X)$ and $X \mapsto [X, K_0(A) \times BGL(A)]$
Similarly from the definiton of the functor $\mathcal{E}_\varphi(X)$ induced by a bounded algebra homomorphism $\varphi : A \to B$ one easily concludes that there is an isomorphism of functors between $A \mapsto K_A(X)$ and $A \mapsto [X, K_0(A) \times BGL(A)]$. Putting these results together we get:

6.12. Theorem. *There is a natural isomorphism of bifunctors*

$$K_A(X) \cong [X, K_0(A) \times BGL(A)].$$

6.13. Higher K–groups. For any smooth space X we define X^+ to be the disjoint union (coproduct) of X and a single point x^+. Clearly this constrution defines a functor $^+ : \underline{C^\infty} \to \underline{C_0^\infty}$ from the category of smooth spaces to the category of pointed smooth spaces. Moreover one immediately checks that this functor is left adjoint to the forgetful functor. Finally it is easy to see that for any smooth space X the space X^+ is well pointed, i.e. the inclusion $x^+ \hookrightarrow X^+$ is a smooth cofibration.

Now in analogy to classical topological K–theory we define for any convenient algebra A, any base space X and $n > 0$ the groups $K_A^{-n}(X) := \tilde{K}_A(S^n(X^+))$, where S^n denotes the n–fold unreduced suspension (c.f. 3.43). (Note that clearly X^+ is a base space and thus by 3.12 $S^n(X^+)$ is a base space.)

Next for any convenient algebra A and $n > 0$ we define $K_n(A) := K_A^{-n}(pt)$, where pt denotes the smooth space consisting of a single point.

Finally note that the higher K–groups have obvious functorial properties. For a smooth map $f : X \to Y$ we define

$$K_A^{-n}(f) := \tilde{K}_A(S^n(f^+)) : K_A^{-n}(X) \to K_A^{-n}(Y)$$

and for a bounded algebra homomorphism $\varphi : A \to B$ we put

$$K_\varphi^{-n}(X) := \tilde{K}_\varphi(S^n(X^+)) : K_A^{-n}(X) \to K_B^{-n}(X)$$
$$K_n(\varphi) := \tilde{K}_\varphi^{-n}(pt) : K_n(A) \to K_n(B)$$

Our next task is now to find several different expressions of K–groups in terms of homotopy theory.

6.14. Proposition. *Let (X, x_0) be a well pointed smooth space. Then the inclusion $x_0 \times I \to SX$ is a smooth cofibration. Thus the natural projection $SX \to S'X$ from the unreduced to the reduced suspension is a homotopy equivalence and the reduced suspension is a well pointed smooth space.*

Proof. By 3.27 it suffices to show that $(SX, x_0 \times I)$ is a smooth NDR–pair in order to proove the first statement. Again by 3.27 (X, x_0) is a smooth NDR–pair and thus there are smooth maps $u : X \to I$ and $h : X \times I \to X$ such that $u(x_0) = 0$, $h|_{X\times\{0\}} = Id_X$, $h(x_0, t) = x_0$ for all t and $h(x, 1) = x_0$ for all x such that $u(x) < 1$.

Now let $\varphi \in C^\infty(\mathbb{R}, I)$ be a smooth map such that $\varphi(t) = 0$ if $t \le \varepsilon$ or $t \ge 1 - \varepsilon$ where ε is some small positive number and $\varphi(t) = 1$ for $1/4 \le t \le 3/4$ and define $\tilde{U} : X \times I \to I$ by $\tilde{U}(x, t) := u(x) \cdot \varphi(t)$. Since $\tilde{U}(x, 0) = \tilde{U}(x, 1) = 0$ this factors to a smooth map $U : SX \to I$ which by construction vanishes on $x_0 \times I$. Next let $\alpha : X \times I \to CX \to SX$ be the natural map and define $\tilde{H} : X \times I \times I \to SX$ by

$$\tilde{H}(x, s, t) := \begin{cases} \alpha(h(x,t), s(1 - u(x)\varphi(1/2 - s)t)) & s \le 1/2 \\ \alpha(h(x,t), s + (1-s)u(x)\varphi(3/2 - s)t) & s \ge 1/2 \end{cases}$$

Then this is well defined and smooth since by construction of φ we have $\tilde{H}(x, s, t) = \alpha(h(x,t), s)$ for $1/2 - \varepsilon < s < 1/2 + \varepsilon$. Next since $\tilde{H}(x, 0, t) = \alpha(h(x,t), 0) = \alpha(x_0, 0)$ and $\tilde{H}(x, 1, t) = \alpha(h(x,t), 1) = \alpha(x_0, 1)$ the map $\tilde{H}$ factors to a smooth map $H : SX \times I \to SX$. Then since $\tilde{H}(x, s, 0) = \alpha(x, s)$ we get $H|_{SX\times\{0\}} = Id_{SX}$ and since $u(x_0) = 0$ we have $\tilde{H}(x_0, s, t) = \alpha(x_0, s)$, so H is a homotopy relative to $x_0 \times I$. Finally suppose that $\alpha(x, s) \in SX$ is a point such that $U(\alpha(x, s)) < 1$. Then either $u(x) < 1$ or $u(x) = 1$ and $\varphi(s) < 1$. If $u(x) < 1$ then $h(x, 1) = x_0$ and thus $H(\alpha(x, s), 1) \in x_0 \times I$. On the other hand if $\varphi(s) < 1$ then $s < 1/4$ or $s > 3/4$. For $s < 1/4$ we have $1/2 - s > 1/4$ and thus $\varphi(1/2 - s) = 1$, so if $u(x) = 1$ we get $H(\alpha(x, s), 1) = \alpha(h(x, 1), 0) = \alpha(x_0, 0)$ and in the same way one shows that for $s > 3/4$ and $u(x) = 1$ we get $H(\alpha(x, s), 1) = \alpha(h(x, 1), 1) = \alpha(x_0, 1)$. Thus the pair (U, H) statisfies all conditions of 3.26 and so the first part of the proof is complete.

Now the natural projection $SX \to S'X$ is given by contracting the contractible subset $x_0 \times I$ to a single point and thus it is a homotopy equivalence by 3.13. Finally the inclusion of the base point (the point to which $x_0 \times I$ is contracted) is a smooth cofibration by 3.5. $\square$

6.15. Corollary. *For any well pointed smooth space X and any $n > 0$ there is a canonical smooth homotopy equivalence between $S^n X$ and ${S'}^n X$.*

Proof. From the proof above we get a smooth homotopy equivalence $p : SX \to S'X$. Now one immediately checks that then $S(p) : SSX \to SS'X$ is a smooth homotopy equivalence, too. Since $S'X$ is again well pointed we get a smooth homotopy equivalence $SS'X \to S'S'X$, so its composition with $S(p)$ gives a smooth homotopy equivalence $S^2X \to {S'}^2X$. Now the result follows by induction. $\square$

6.16. Next we want to show that the K–groups can also be expressed via sets of homotopy classes of base point preserving smooth maps. This needs some preparation. First let (X, A) be a smooth NDR–pair, Y an arbitrary smooth space, $f, g : X \to Y$ and $\varphi : A \times I \to Y$ smooth maps. Then we say that f is homotopic to g along φ and write $f \sim_\varphi g$ iff there is a smooth homotopy $H : X \times I \to Y$ between f and g such that $H|_{A\times I} = \varphi$.

Lemma. *In a setting as above assume that $\psi : A \times I \to Y$ is a smooth map which is smoothly homotopic to φ relative to $A \times \{0,1\}$. If $f \sim_\varphi g$ then there is a smooth function $\gamma \in C^\infty(I, I)$ with $\gamma(i) = i$ for $i = 0, 1$ such that $f \sim_{\psi\circ(id_A\times\gamma)} g$.*

Proof. By assumption there is a smooth map $h : A\times I\times I \to Y$ such that $h(a, t, 0) = \varphi(a, t)$, $h(a, t, 1) = \psi(a, t)$ and $h(a, i, s) = \varphi(a, i) = \psi(a, i)$ for $i = 0, 1$ and all s and t and a smooth map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ and $H(a, t) = \varphi(a, t)$ for all $a \in A$. Now since (X, A) is a smooth NDR–pair it follows from 3.4(2) that the inclusion of $A \times I$ into $X \times I$ is a smooth cofibration. Applying the cofibration property to the maps h and H we get a smooth map $\tilde H : X \times I \times I \to Y$ such that $\tilde H(a, t, s) = h(a, t, s)$ and such that $\tilde H|_{X\times I\times\{0\}}$ is smoothly homotopic to H relative to $A \times I$. Thus there is a smooth map $\mathcal{H} : X \times I \times I \to Y$ such that $\mathcal{H}(x, t, 0) = H(x, t)$, $\mathcal{H}(x, t, 1) = \tilde H(x, t, 0)$ and $\mathcal{H}(a, t, s) = H(a, t) = \varphi(a, t)$.

Now we consider the following homotopies: $\mathcal{H}|_{X\times\{0\}\times I}$ is a homotopy between $f = H|_{X\times\{0\}}$ and $\tilde H|_{X\times\{0\}\times\{0\}}$ and by construction we have $\mathcal{H}(a, 0, s) = \varphi(a, 0) = \psi(a, 0)$. Next $\tilde H|_{X\times\{0\}\times I}$ connects $\tilde H|_{X\times\{0\}\times\{0\}}$ to $\tilde H|_{X\times\{0\}\times\{1\}}$ and $\tilde H(a, 0, s) = h(a, 0, s) = \psi(a, 0)$. Then $\tilde H|_{X\times I\times\{1\}}$ connects $\tilde H|_{X\times\{0\}\times\{1\}}$ to $\tilde H|_{X\times\{1\}\times\{1\}}$ and $\tilde H(a, t, 1) = h(a, t, 1) = \psi(a, t)$. Next $\tilde H|_{X\times\{1\}\times I}$ backwards connects $\tilde H|_{X\times\{1\}\times\{1\}}$ to $\tilde H|_{X\times\{1\}\times\{0\}}$ and $\tilde H(a, 1, s) = h(a, 1, s) = \psi(a, 1)$ and finally $\mathcal{H}|_{X\times\{1\}\times I}$ backwards connects $\tilde H|_{X\times\{1\}\times\{0\}}$ to $H|_{X\times\{1\}} = g$ and $\mathcal{H}(a, 1, s) = H(a, 1) = \varphi(a, 1) = \psi(a, 1)$. Thus piecing these five homotopies together smoothly we get a smooth homotopy between f and g along a reparametrization of ψ. □

6.17. Corollary. *Assume that (X, x_0) is a well pointed smooth space, (Y, y_0) a pointed smooth space and that $u : I \to Y$ is a smooth path with $u(0) = u(1) = y_0$ and let $f, g : X \to Y$ be base point preserving smooth maps. If $f \sim_u g$ and u is homotopic to the constant path y_0 relative to $\{0, 1\}$, then f and g are smoothly homotopic as base point preserving maps.*

6.18. Now for any smooth group G we define the base point of the classifying space BG to be the orbit of the point $(1, e, 0, e, 0, e, \dots) \in EG$, where e denotes the unit element of G. (In fact the choice of the base point in BG is not important since BG is the smooth image of the contractible smooth space EG and thus is smoothly path connected.)

Now let (X, x_0) be a pointed smooth space and let A be a convenient algebra. Then we have the smooth homomorphisms $i_n : GL(n, A) \to GL(n + 1, A)$ considered in 6.11. The induced maps $B(i_n) : BGL(n, A) \to BGL(n + 1, A)$ are clearly base point preserving and thus they induce maps $[X, BGL(n, A)]_0 \to [X, BGL(n + 1, A)]_0$ between the sets of homotopy classes of base point preserving smooth maps and we define $[X, BGL(A)]_0$ to be the direct limit of the so obtained inductive system. For any n we have a forgetful map $v_n : [X, BGL(n, A)]_0 \to [X, BGL(n, A)]$ and clearly these maps induce a map $v : [X, BGL(A)]_0 \to [X, BGL(A)]$.

6.19. Theorem. *If X is well pointed then $v : [X, BGL(A)]_0 \to [X, BGL(A)]$ is bijective.*

Proof. First let $f : X \to BGL(n, A)$ be an arbitrary map. Since $BGL(n, A)$ is smoothly path connected there is a smooth path from $f(x_0)$ to the base point of $BGL(n, A)$. Applying the cofibration property of (X, x_0) to this path and f we get a smooth map $H : X \times I \to BGL(n, A)$ such that $H(x_0, 1)$ is the base point of $BGL(n, A)$ and such that $H|_{X\times\{0\}}$ is smoothly homotopic to f (even relative to x_0). Thus f is homotopic to a base point preserving map, so for any n the map v_n is surjective and thus v is surjective.

To complete the proof it now suffices to show that if $f_0, f_1 : X \to BGL(n,A)$ are two base point preserving maps which are freely homotopic then $B(\varphi) \circ f_1$ and $B(\varphi) \circ f_2$ are homotopic as base point preserving maps, where $\varphi : GL(n,A) \to GL(2n,A)$ is the map $g \mapsto g \oplus Id_{A^n}$.

First we define a smooth map $EGL(n,A) \times EGL(n,A) \to EGL(2n,A)$ as follows: Let $k \mapsto (i(k), j(k))$ be a bijection between $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$ such that $i(0) = j(0) = 0$ and we map the element $((t_i, g_i), (s_j, h_j)) \in EGL(n,A)\times EGL(n,A)$ to the element of $EGL(2n,A)$ which has as k–th t–coordinate $t_{i(k)} \cdot s_{j(k)}$ and as k–th g–coordinate $g_{i(k)} \oplus h_{j(k)}$. Then one immediately checks that this map is well defined and smooth. Next we have the canonical projections $EGL(n,A) \times EGL(n,A) \to BGL(n,A) \times BGL(n,A)$ and $EGL(2n,A) \to BGL(2n,A)$ and one immediately checks that the map defined above factors to a map $\bullet : BGL(n,A) \times BGL(n,A) \to BGL(2n,A)$ which is smooth since the map $EGL(n,A) \times EGL(n,A) \to BGL(n,A) \times BGL(n,A)$ is the projection of a locally trivial fiber bundle and thus a final morphism.

Now let $*$ be the base point of $BGL(n,A)$ and consider the map $z \mapsto z \bullet *$. On the level of $EGL(n,A)$ this map is given by sending a sequence (t_i, g_i) to the sequence (s_k, h_k) where $s_k = 0$ if $j(k) \neq 0$ and $s_k = t_{i(k)}$ if $j(k) = 0$ and $h_k = g_{i(k)} \oplus id_{A^n}$. In particular we have $s_0 = t_0$ and $h_0 = g_0 \oplus id_{A^n}$. We can construct a homotpy between this map and the map $E(\varphi)$ as follows: Using the homotopy A constructed in the proof of 2.20 but starting moving not in the 0– but in the 1–coordinate we deform $E(\varphi)$ to a map having zeros in all odd t–coordinates and our map to one having zeros in all even t–coordinates but t_0 and then connect the points affinely (c.f. the proof of 2.23). Projecting this homotopy to the BGL level we get a smooth homotopy between $z \mapsto z \bullet *$ and $B(\varphi)$ as base point preserving maps.

In the same way we can construct a homotopy of base point preserving maps between $z \mapsto * \bullet z$ and $B(\psi)$ where $\psi : GL(n,A) \to GL(2n,A)$ is the map $g \mapsto Id_{A^n} \oplus g$. Finally we want to show that $B(\varphi)$ and $B(\psi)$ are smoothly homotopic as base point preserving maps. Consider the smooth map $\alpha : GL(n,A) \times I \to L_A(A^{2n}, A^{2n})$ given in matrix notation by

$$g \mapsto \begin{pmatrix} (1-t)g & t \cdot id_{A^n} \\ -tg & (1-t) \cdot id_{A^n} \end{pmatrix}.$$

Then a short computation shows that this matrix is invertible for all t with inverse given by

$$\begin{pmatrix} (1-t)g^{-1} & -tg^{-1} \\ t \cdot id_{A^n} & (1-t) \cdot id_{A^n} \end{pmatrix}.$$

Thus α is smooth as a map $GL(n,A) \times I \to GL(2n,A)$. Thus it induces a smooth map $EGL(n,A) \times I \to EGL(2n,A)$ and since for any t we have $\alpha(gh,t) = \alpha(g,t)\varphi(h)$ this map factors to a map $BGL(n,A)\times I \to BGL(2n,A)$ which is smooth since the map $EGL(n,A)\times I \to BGL(n,A) \times I$ is the projection of a locally trivial fiber bundle and thus a final morphism. So we get a smooth homotopy of base point preserving maps between $B(\varphi)$ and $B(\omega)$, where $\omega : GL(n,A) \to GL(2n,A)$ is the map $g \mapsto \begin{pmatrix} 0 & id_{A^n} \\ -g & 0 \end{pmatrix}$. But now for any element $g \in GL(n,A)$ we have $\varphi(g) = \omega(g) \cdot \begin{pmatrix} 0 & -id_{A^n} \\ id_{A^n} & 0 \end{pmatrix}$ and thus $B(\varphi) = B(\omega)$.

Together we have now shown that the maps $z \mapsto z\bullet*$ and $z \mapsto *\bullet z$ are smoothly homotopic to $B(\varphi)$ as base point preserving maps. So let us now assume that $f_0, f_1 : X \to BGL(n,A)$ are two base point preserving maps which are freely homotopic, so we have a smooth map $H : X \times I \to BGL(n,A)$ such that $H|_{X\times\{i\}} = f_i$. Then $c : I \to BGL(n,A)$ defined by $c(t) := H(x_0, t)$ is a smooth curve such that $c(0) = c(1) = *$. Now define $h : X \times I \to BGL(2n,A)$ by $h(x,t) := c(1-t) \bullet f_1(x)$. Then by constrution this is a homotopy from $* \bullet f_1$ to itself along the path $t \mapsto c(1-t) \bullet *$. From the argument above we see that

$t \mapsto c(1-t) \bullet *$ is smoothly homotopic to the path $t \mapsto B(\varphi)(c(1-t))$, and so by lemma 6.16 there is a smooth fuction $\gamma \in C^\infty(I,I)$ and a smooth homotopy $\tilde{h}$ from $* \bullet f_1$ to itself along the path $t \mapsto B(\varphi)(c(\gamma(1-t)))$.
Now consider the following homotopies: First take $(x,t) \mapsto B(\varphi)(H(x,\gamma(t)))$ which is a homotopy from $B(\varphi) \circ f_0$ to $B(\varphi) \circ f_1$ along the path $c \circ \gamma$. Then take a homotopy between $B(\varphi) \circ f_1$ and $* \bullet f_1$ as base point preserving maps. Next take the homotopy $\tilde{h}$ from $* \bullet f_1$ to itself along the path $t \mapsto B(\varphi)(c(\gamma(1-t)))$ and finally take a homotpy between $* \bullet f_1$ and $B(\varphi) \circ f_1$ as base point preserving maps. Piecing these four homotopies together smoothly we get a smooth homotopy between $B(\varphi) \circ f_0$ and $B(\varphi) \circ f_1$ in which the base point runs first along some path and then runs back the same path. Clearly this path of the base point is homotopic to the constant path $*$ relative to $\{0,1\}$ and thus by corollary 6.17 $B(\varphi) \circ f_0$ and $B(\varphi) \circ f_1$ are homotopic as base point preserving smooth maps. □

6.20. Corollary. *Let X be a well pointed base space and let $[X, K_0(A) \times BGL(A)]_0$ be the direct limit of the system of sets $[X, K_0(A) \times BGL(n,A)]_0$. Then there is a canonical isomorphism $\tilde{K}_A(X) \cong [X, K_0(A) \times BGL(A)]_0$.*

Proof. As in 6.11 one shows that

$$[X, K_0(A) \times BGL(A)]_0 \cong [X, K_0(A)]_0 \times [X, BGL(A)]_0.$$

Using 6.19 we conclude that

$$[X, K_0(A) \times BGL(A)]_0 \cong \operatorname{Ker}(K_A(i) : K_A(X) \to K_0(A)),$$

where $i : pt \to X$ denotes the inclusion of the base point. By 6.7 there is a short exact sequence

$$0 \to K_0(A) \to K_A(X) \to \tilde{K}_A(X) \to 0$$

and $K_A(i)$ is a canonical split of this sequence, so the claimed isomorphism follows. □

6.21. Corollary. *For any base space X and any convenient algebra A there is a canonical isomorphism $K_A(X) \cong \tilde{K}_A(X^+)$.*

Proof. Let Y be an arbitrary pointed smooth space and let $i : X \to X^+$ be the natural inclusion. Then one immediately checks that the composition $[X^+, Y]_0 \to [X^+, Y] \to [X, Y]$ of i^* and the forgetful map is a bijection. Applying this to $BGL(n,A)$ and passing to limits one immediately concludes that there is an induced isomorphism $[X^+, K_0(A) \times BGL(A)]_0 \cong [X, K_0(A) \times BGL(A)]$ and consequently the claimed isomorphism follows from 6.20 and 6.12. □

6.22. Lemma. *Let E^{n+1} and S^n be the closed $n+1$–cell and the n–sphere as in 1.19. Then the spaces $E^{n+1} \cup_{S^n} E^{n+1}$ and $CS^n \cup_{S^n} CS^n$ are smoothly homotopy equivalent.*

Proof. Consider S^n and E^{n+1} as being embedded into $\mathbb{R}^{n+1}$ in the usual way. Define $\tilde{f} : S^n \times I \to E^n$ by $\tilde{f}(x,t) := tx$. This is oviously smooth and sends $S^n \times \{0\}$ to 0 and thus factors to a smooth bijective map $f : CS^n \to E^{n+1}$. The inverse of f (which is not smooth) is given by $f^{-1}(x) = p(\frac{1}{\|x\|}x, \|x\|)$ for $x \neq 0$, where $p : S^n \times I \to CS^n$ denotes the canonical map and $f^{-1}(0)$ is the base point of CS^n. Now let $\varphi \in C^\infty(I,I)$ be a smooth increasing map such that φ is zero locally around zero and $\varphi(t) = 1$ for all $t \geq 1/3$ and define $g : E^{n+1} \to CS^n$ by $g(x) := f^{-1}(\frac{\varphi(\|x\|)}{\|x\|}x)$. Obviously $x \mapsto \frac{\varphi(\|x\|)}{\|x\|}x$ is a smooth map $E^{n+1} \to E^{n+1}$. To show that g is smooth it suffices to show that for any smooth function $\psi \in C^\infty(CS^n, \mathbb{R})$ the function $\psi \circ g$ is smooth on E^{n+1}. By definition of the smooth structure on CS^n smoothness of ψ is equivalent to smoothness of $\psi \circ p$, so we assume that ψ is given as a smooth function on $S^n \times I$ which is constant on $S^n \times \{0\}$. Without loss we generality we may assume that ψ is zero on $S^n \times \{0\}$.

Let $c : \mathbb{R} \to E^{n+1}$ be a smooth curve and let $r \in I$ be the maximum value such that $\varphi(r) = 0$. If $\|c(t)\| < r$ then $\psi \circ g \circ c$ is identically zero and thus smooth locally around t. Thus we may without loss of generality assume that $c(t) \neq 0$ for all t. But then c can be written uniquely as $c(t) = r(t) \cdot x(t)$, where $r : \mathbb{R} \to I$ and $x : \mathbb{R} \to S^n$ are smooth curves. (Recall that the smooth curves into E^{n+1} are exactly those which are smooth as curves into $\mathbb{R}^{n+1}$.)

As $\psi : S^n \times I \to \mathbb{R}$ is smooth we conclude from cartesian closedness that the associated map $\check{\psi} : S^n \to C^\infty(I, \mathbb{R})$ is smooth, too. In [Kriegl, 1990] it is proved that for a convex subset K of $\mathbb{R}^n$ with nonempty interior a function $K \to \mathbb{R}$ is smooth for the initial smooth structure if and only if it is smooth (in the usual sense) in the interior of K and all derivatives on the interior extend to continuous maps on K. In the special case of I this implies that the smooth real valued functions are exactly the restrictions of smooth real valued functions defined on $\mathbb{R}$. Thus for any $\alpha \in C^\infty(I, \mathbb{R})$ the integral $\int_0^1 \alpha'(s)ds$ is well defined and we have $\alpha(t) - \alpha(0) = t\int_0^1 \alpha'(ts)ds$. Applying this to $\check{\psi}(x)$ one gets $\psi(x,t) = t\int_0^1 \frac{\partial}{\partial s}\psi(x,ts)ds$ and one easily shows that the map $h : S^n \times I \to \mathbb{R}$ defined by $h(x,t) := \int_0^1 \frac{\partial}{\partial s}\psi(x,ts)ds$ is smooth. Thus we have

$$(\psi \circ g \circ c)(t) = \psi(x(t), \varphi(r(t))) = \varphi(r(t))h(x(t), \varphi(r(t)))$$

for $r(t) > r$. But this equation also holds for $r(t) \leq r$ since there both sides vanish. So $\psi \circ g \circ c$ is smooth and thus g is smooth.

The map $f \circ g : E^{n+1} \to E^{n+1}$ is given by $x \mapsto \frac{\varphi(\|x\|)}{\|x\|}x$ and thus $(x,t) \mapsto (t + (1-t)\frac{\varphi(\|x\|)}{\|x\|})x$ is a smooth homotopy from $f \circ g$ to the identity which equals the identity for any t on the subspace S^n. On the other hand $g \circ f$ is induced by the map $S^n \times I \to S^n \times I$, $(x,t) \mapsto (x, \varphi(t))$ and thus a homotopy to the identity is induced by $(x,t,s) \mapsto (x, st + (1-s)\varphi(t))$. Again this map equals the identity for all s on the subspace $S^n \times \{1\}$. Thus all maps and homotopies induce maps on $E^{n+1} \cup_{S^n} E^{n+1}$ and $CS^n \cup_{S^n} CS^n$, respectively and the result follows. $\square$

6.23. Lemma. *The smooth space $X := E^n \cup_{S^{n-1}} E^n$ is smoothly homotopy equivalent to the smooth manifold S^n.*

Proof. Let S^n be embedded into $\mathbb{R}^{n+1}$ in the usual way and consider $\mathbb{R}^n$ as the subspace of all vectors for which the last coordinate is zero. Now define $f_\pm : E^n \to S^n$ by

$$f_\pm(x) = f_\pm(x_1, \dots, x_n) = \tfrac{1}{1+\|x\|^2}(2x_1, \dots, 2x_n, \pm(1 - \|x\|^2)).$$

Then $f_\pm$ is obviously a smooth map from E^n to the upper (respectively lower) half of S^n which is even a diffeomorphism on the open interiors since we have $f_\pm^{-1}(x_1, \dots, x_{n+1}) = \frac{1}{1 \pm x_{n+1}}(x_1, \dots, x_n)$. Moreover the restrictions of f_+ and f_- to S^{n-1} obviously both coincide with the inclusion of S^{n-1} into S^n and thus we get an induced smooth bijective map $f : X \to S^n$.

The map f^{-1} is not smooth since X has an 'edge' in S^{n-1} so we have to deform it to a smooth map. Let $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ be an increasing smooth map such that $\varphi(t) = t$ for all $t \leq 2/3$ and $\varphi(t) = 1$ for all $t \geq 1 - \varepsilon$ where ε is some small positive number. Then clearly $x \mapsto \frac{\varphi(\|x\|)}{\|x\|}$ defines a smooth function on $\mathbb{R}^n$ and we define $\psi : S^n \to S^n$ on the upper half sphere by $\psi(x) := f_+(\frac{\varphi(\|f_+^{-1}(x)\|)}{\|f_+^{-1}(x)\|}f_+^{-1}(x))$ and on the lower half sphere in the same way using f_- instead of f_+. Then ψ is smooth as it is obviously smooth on the open half spheres and by construction of φ it is given on an open neighborhood of S^{n-1} as a smooth retraction of this neighborhood to S^{n-1}. Then define $g : S^n \to X$ by $g := f^{-1} \circ \psi$.

To see that g is smooth let $c : \mathbb{R} \to S^n$ be a smooth curve and let $h : X \to \mathbb{R}$ be a smooth function and consider $h \circ g \circ c$. By definition of the smooth structure of X the restriction $\tilde{h}$ of h to S^{n-1} is a smooth map. Now assume that $t_0 \in \mathbb{R}$ is such that $c(t_0) \in S^{n-1} \subset S^n$. Then $\psi \circ c$ stays in $S^{n-1} \subset S^n$ locally around t_0 and since the restriction of f^{-1} to S^{n-1} is the identity we see that $h \circ g \circ c$ is smooth locally around t_0. So we may without loss of generality assume that the curve c stays in one open half sphere. But the restrictions of g to the half spheres are obviously smooth as maps into E^n and thus $g \circ c$ is smooth as a curve into one of the n–cells, so $h \circ g \circ c$ is smooth by definition of the smooth structure of X.

The composition $f \circ g : S^n \to S^n$ is just ψ. A smooth homotopy from ψ to the identity can be constructed as $p \circ H$ where $H : S^n \times I \to \mathbb{R}^{n+1} \setminus \{0\}$ is defined by $H(x,t) := (1-t)\psi(x) + tx$ and $p : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ is the smooth map $x \mapsto \frac{x}{\|x\|}$. On the other hand $g \circ f : X \to X$ is induced by the map $\alpha : E^n \to E^n$ on both copies of E^n which is given by $\alpha(x) = \frac{\varphi(\|x\|)}{\|x\|}x$. By cartesian closedness we have $X \times I = (E^n \times I) \cup_{S^{n-1} \times I} (E^n \times I)$. Now define $H : E^n \times I \to E^n$ by $H(x,t) := (1-t)\alpha(x) + tx$. This map on both copies of E^n induces then a smooth homotopy between $g \circ f$ and the identity. $\square$

6.24. Proposition. *Define $\mathfrak{S}^n := S^n(pt^+)$, where pt denotes the smooth space consisting of a single point. Then for any n the space $\mathfrak{S}^n$ is smoothly homotopy equivalent to the smooth manifold S^n.*

Proof. By definition pt^+ is consists of two points with the discrete smooth structure, so $pt^+ = S^0$. So let us inductively assume that we have found a homotopy equivalence $f : S^n \to \mathfrak{S}^n$. Then $S(f) : S(S^n) \to S\mathfrak{S}^n = \mathfrak{S}^{n+1}$ is a homotopy equivalence, too. On the other hand by 6.22 and 6.23 the manifold S^{n+1} is smoothly homotopy equivalent to $CS^n \cup_{S^n} CS^n$, which in turn is homotopy equivalent to $S(S^n)$ by 3.14 since the spaces are the homotpy cofiber and the cofiber of the cofibration $S^n \to CS^n$. $\square$

6.25. Corollary. *For any convenient algebra A and any natural number n there is an isomorphism $K_n(A) \cong [S^n, BGL(A)]_0$.*

Proof. By definition we have $K_n(A) = K^n_A(pt) = K^0_A(\mathfrak{S}^n)$ and by 6.12 the this group is isomorphic to $[\mathfrak{S}^n, BGL(A)]$ which clearly by 6.24 is isomorphic to $[S^n, BGL(A)]$ and finally by 6.19 this group is isomorphic to $[S^n, BGL(A)]_0$. $\square$

7. Relative K–groups and long exact sequences

The main aim of this section is to establish the two fundamental long exact sequences of K–goups, the one induced by a smooth map and the one induced by an algebra homomorphism. To formulate thse results we need the definitions of the relative K–group corresponding to a smooth map or an algebra homomorphism. To define these groups we start from the general definition of the K–group of a functor due to Karoubi (c.f. [Ka]). Then we show that for the groups we are interested in there are nice interpretations in terms of homotopy theory and derive the long exact sequences from the Puppe–sequences constructed in chapter 3.

7.1. Definition. Let $\mathcal{C}$ and $\mathcal{C}'$ be additive convenient categories (c.f.4.21), $\varphi : \mathcal{C} \to \mathcal{C}'$ a convenient additive functor. Let $\Gamma(\varphi)$ denote the set of all triples (E, F, α) where E and F are objects of $\mathcal{C}$ and α is an isomorphism between $\varphi(E)$ and $\varphi(F)$. Two triples (E, F, α) and (E', F', α') are called isomorphic iff there are isomorphisms $f : E \to E'$ and $g : F \to F'$

such that the following diagram commutes.

$$\begin{array}{ccc} \varphi(E) & \xrightarrow{\alpha} & \varphi(F) \\ \Big\downarrow{\scriptstyle \varphi(f)} & & \Big\downarrow{\scriptstyle \varphi(g)} \\ \varphi(E') & \xrightarrow{\alpha'} & \varphi(F') \end{array}$$

A triple (E, F, α) is called *elementary* iff $E = F$ and α is homotopic to $id_{\varphi(E)}$ as an automorphism of $\varphi(E)$. (This makes sense since $\mathcal{C}'(\varphi(E), \varphi(E))$ is a convenient algebra and thus the automorphisms form a smooth group.) Finally we define the sum of two triples by

$$(E, F, \alpha) + (E', F', \alpha') := (E \oplus E', F \oplus F', \alpha \oplus \alpha').$$

Now we define the K–group $K(\varphi)$ of the functor φ to be the quotient of $\Gamma(\varphi)$ with respect to the equivalence relation defined by declaring two elements σ and σ' to be equivalent if and only if there are elementary triples τ and τ' such that $\sigma + \tau$ and $\sigma' + \tau'$ are isomorphic. We write $d(E, F, \alpha)$ for the class of the tripel in $K(\varphi)$.

Obviously the addition defined above factors to an addition on $K(\varphi)$ which defines the structure of a commutative monoid on $K(\varphi)$.

7.2. Proposition. *In $K(\varphi)$ we have the relation $d(E, F, \alpha) + d(F, E, \alpha^{-1}) = 0$. Thus $K(\varphi)$ is an abelian group.*

Proof. (c.f. [Ka, II.2.14]) Let (E, F, α) be an arbitrary element of $\Gamma(\varphi)$. Then by definition we have $d(E, F, \alpha) + d(F, E, \alpha^{-1}) = d(E \oplus F, F \oplus E, \alpha \oplus \alpha^{-1})$. Clearly the tripel $(E \oplus F, F \oplus E, \alpha \oplus \alpha^{-1})$ is isomorphic to the triple $(E \oplus F, E \oplus F, \beta)$, where β is defined by the matrix $\begin{pmatrix} 0 & -\alpha^{-1} \\ \alpha & 0 \end{pmatrix}$. In the group of automorphisms of $\varphi(E) \oplus \varphi(F)$ we have:

$$\begin{pmatrix} 0 & -\alpha^{-1} \\ \alpha & 0 \end{pmatrix} = \begin{pmatrix} id & -\alpha^{-1} \\ 0 & id \end{pmatrix} \begin{pmatrix} id & 0 \\ \alpha & id \end{pmatrix} \begin{pmatrix} id & -\alpha^{-1} \\ 0 & id \end{pmatrix}.$$

Now let $f \in \mathcal{C}'(\varphi(E), \varphi(F))$ be any morphism. Then the matrix $\begin{pmatrix} id & f \\ 0 & id \end{pmatrix}$ defines an automorphism of $\varphi(E) \oplus \varphi(F)$ with inverse given by $\begin{pmatrix} id & -f \\ 0 & id \end{pmatrix}$ and thus $t \mapsto \begin{pmatrix} id & tf \\ 0 & id \end{pmatrix}$ defines a smooth curve in the smooth group of automorphisms. Similarly for any f as above $t \mapsto \begin{pmatrix} id & 0 \\ tf & id \end{pmatrix}$ defines a smooth curve. Thus we can define a smooth curve c in the group of automorphisms by

$$c(t) := \begin{pmatrix} id & -t\alpha^{-1} \\ 0 & id \end{pmatrix} \begin{pmatrix} id & 0 \\ t\alpha & id \end{pmatrix} \begin{pmatrix} id & -t\alpha^{-1} \\ 0 & id \end{pmatrix}.$$

Then $c(0) = id_{\varphi(E) \oplus \varphi(F)}$ and $c(1) = \beta$ and thus the triple $(E \oplus F, E \oplus F, \beta)$ is elementary and the result follows. □

7.3. Let X and Y be base spaces, $f : X \to Y$ a smooth map and let A be a convenient algebra. Then f induces the additive functor $f^* = \mathcal{E}_A(f) : \mathcal{E}_A(X) \to \mathcal{E}_A(Y)$ (c.f. 5.14) and we want to describe the K–group of this functor. So let (E, F, α) be an element of $\Gamma(\mathcal{E}_A(f))$. Then E and F are A–bundles over Y and $\alpha : f^*E \to f^*F$ is an isomorphism. Thus we can assign to this triple a locally constant fuction $Y \to K_0(A)$ by assigning to any point the difference of the classes in $K_0(A)$ of the fiber of E and the fiber of F over the point.

Clearly this gives the constant function zero on elementary triples and is compatible with the addition and thus it induces a group homomorphism $r : K(\mathcal{E}_A(f)) \to H^0(Y, K_0(A))$. Now the function f induces a group homomorphism $f^* : H^0(Y, K_0(A)) \to H^0(X, K_0(A))$ and as the bundles f^*E and f^*F are isomorphic the homomorphism r has values in the subgroup $\mathrm{Ker}(f^*) \subset H^0(Y, K_0(A))$. Moreover one immediately checks that r is surjective onto this subgroup and we define $K'(\mathcal{E}_A(f))$ to be the kernel of r. So we have a short exact sequence

$$0 \to K'(\mathcal{E}_A(f)) \to K(\mathcal{E}_A(f)) \to \mathrm{Ker}(f^*) \to 0.$$

Now let g be an element of $\mathrm{Ker}(f^*)$. Then $g : Y \to K_0(A)$ is a locally constant function and $g \circ f$ is identically zero. We assign to g an element of $K(\mathcal{E}_A(f))$ as follows: Let $Y_1, \ldots, Y_n$ be the connected components of Y which are not contained in the image of f. Then by 6.2 for any i we can write $g(Y_i) = [P_i] - [Q_i] \in K_0(A)$, where the P_i and Q_i are finitely generated projective right A–modules. Now let E be the A–bundle over Y which is over Y_i given by $Y_i \times P_i$ and F the one given by $Y_i \times Q_i$ while over the components which are hit by f both bundles are zero, i.e. the identitiy map. Then clearly this defines a splitting of r and thus we have an isomorphism $K(\mathcal{E}_A(f)) \cong K'(\mathcal{E}_A(f)) \oplus \mathrm{Ker}(f^*)$. Finally one immediately checks that $\mathrm{Ker}(f^*) = [C_f, K_0(A)]_0$, the group of pointed homotopy classes of smooth maps from the mapping cone C_f of f to the discrete smooth group $K_0(A)$, so there is an isomorphism $K(\mathcal{E}_A(f)) \cong K'(\mathcal{E}_A(f)) \oplus [C_f, K_0(A)]_0$.

7.4. Our next task is to give an interpretation of the group $K'(\mathcal{E}_A(f))$ via homotopy theory. This requires an intermediate step. Consider the set of all pairs (E, α), where E is an A–bundle over Y with fiber A^n and α is an isomorphism between f^*E and the trivial bundle $X \times A^n$. Two such pairs (E, α) and (E', α') are said to be equivalent if and only if there is an isomorphisms $\varphi : E \to E'$ such that $\alpha' \circ f^*\varphi$ is homotopic to α as an isomorphism from f^*E to $X \times A^n$. Let $\Phi_n(\mathcal{E}_A(f))$ be the set of all equivalence classes.

Now $(E, \alpha) \mapsto (E \oplus \theta_1, \alpha \oplus id)$, where θ_1 denotes the trivial 'line' bundle $Y \times A$ over Y defines a map $\Phi_n(\mathcal{E}_A(f)) \to \Phi_{n+1}(\mathcal{E}_A(f))$ and we define $\Phi(\mathcal{E}_A(f))$ to be the direct limit of the so obtained inductive system. Next $((E, \alpha), (F, \beta)) \mapsto (E \oplus F, \alpha \oplus \beta)$ defines a map $\Phi_n(\mathcal{E}_A(f)) \times \Phi_m(\mathcal{E}_A(f)) \to \Phi_{n+m}(\mathcal{E}_A(f))$ and as in 6.9 one shows that this induces the structure of a commutative monoid on $\Phi(\mathcal{E}_A(f))$.

Now we define a map $\sigma_n : \Phi_n(\mathcal{E}_A(f)) \to K'(\mathcal{E}_A(f))$ by $\sigma_n(E, \alpha) := d(E, Y \times A^n, \alpha)$. This is easily seen to be well defined and clearly it induces a monoid homomorphism $\sigma : \Phi(\mathcal{E}_A(f)) \to K'(\mathcal{E}_A(f))$.

7.5. Proposition. *The homomorphism σ defined above is bijective, so $\Phi(\mathcal{E}_A(f))$ is an abelian group.*

Proof. Let (E, F, α) be an element of $\Gamma(\mathcal{E}_A(f))$ and let us write θ_k for the trivial A–bundle $Y \times A^k$ over Y. By 5.6 there is an A–bundle G over Y, a natural number n and an isomorphism $\psi : F \oplus G \to \theta_n$ of A–bundles and clearly the tripel (E, F, α) is equivalent to $(E \oplus G, F \oplus G, \alpha \oplus id_{f^*G})$ which in turn is equivalent to $(E \oplus G, \theta_n, f^*\psi \circ (\alpha \oplus id))$. Now let $Y_1, \ldots, Y_k$ be the connected components of Y and let P_i be the fiber of $E \oplus G$ over Y_i for any i. Then if the class of the above triple in $K(\mathcal{E}_A(f))$ lies in the subgroup $K'(\mathcal{E}_A(f))$ then for any i the modules P_i and A^n represent the same class in $K_0(A)$. Thus for any i there is a finitely generated projective right A–module Q_i such that $P_i \oplus Q_i \cong A^n \oplus Q_i$. Now for any i we can choose a module R_i such that $Q_i \oplus R_i \cong A^m$ for some fixed m. Thus we get $P_i \oplus A^m \cong A^{n+m}$ for each i. Now the above triple is equivalent to $(E \oplus G \oplus \theta_m, \theta_{n+m}, (f^*\psi \circ (\alpha \oplus id)) \oplus id)$ which by construction is in the image of σ since $E \oplus G \oplus \theta_m$ has fiber A^{n+m}, and thus σ is surjective.

To prove that σ is injective assume that E and F are A–bundles over Y with fiber A^n and A^m, respectively such that (E, θ_n, α) and (F, θ_m, β) are equivalent. Then by 7.2 we have

$d(E \oplus \theta_m, \theta_n \oplus F, \alpha \oplus \beta^{-1}) = 0$, so there are elementary triples (G, G, γ) and (L, L, λ) such that the triples $(E \oplus \theta_m \oplus G, \theta_n \oplus F \oplus G, \alpha \oplus \beta^{-1} \oplus \gamma)$ and (L, L, λ) are isomorphic. By 5.6 we can find an A–bundle G' over Y, a natural number p and an isomorphism $g : G \oplus G' \to \theta_p$ of A–bundles. Then clearly the triple $(E \oplus \theta_m \oplus G \oplus G', \theta_n \oplus F \oplus G \oplus G', \alpha \oplus \beta^{-1} \oplus \gamma \oplus id_{f^*G'})$ is isomorphic to $(L \oplus G', L \oplus G', \lambda \oplus id_{f^*G'})$. Thus by definiton there are isomorphisms $\varphi : E \oplus \theta_m \oplus G \oplus G' \to L \oplus G'$ and $\psi : \theta_n \oplus F \oplus G \oplus G' \to L \oplus G'$ such that $f^*\psi \circ (\alpha \oplus \beta^{-1} \oplus \gamma \oplus id_{f^*G'}) = (\lambda \oplus id_{f^*G'}) \circ f^*\varphi$. Since by construction γ and λ are homotopic to identity mappings as isomorphisms we may conclude from this that $f^*(\psi^{-1}) \circ f^*\varphi$ and $(\alpha \oplus \beta^{-1} \oplus id_{G \oplus G'})$ are homotopic as isomorphisms between $f^*(E \oplus \theta_m \oplus G \oplus G')$ and $f^*(\theta_n \oplus F \oplus G \oplus G')$.

Now let $\Phi : E \oplus \theta_m \oplus \theta_p \to F \oplus \theta_n \oplus \theta_p$ be the isomorphism defined by $\Phi := (\tau \oplus g) \circ \psi^{-1} \circ \varphi \circ (id_{E \oplus \theta_m} \oplus g^{-1})$, where $\tau : \theta_n \oplus F \to F \oplus \theta_n$ is the canonical isomorphism and $g : G \oplus G' \to \theta_p$ is the isomorphism from above. Then we have:

$$\begin{aligned}&(\beta \oplus id \oplus id) \circ f^*\Phi = \\ &\quad =(id \oplus \beta \oplus f^*g) \circ f^*(\psi^{-1}) \circ f^*\varphi \circ (id \oplus id \oplus f^*(g^{-1})),\end{aligned}$$

and this is homotopic as an isomorphism to

$$\begin{aligned}&(id \oplus \beta \oplus f^*g) \circ (\alpha \oplus \beta^{-1} \oplus id) \circ (id \oplus id \oplus f^*(g^{-1})) = \\ &=(\alpha \oplus id \oplus id).\end{aligned}$$

Thus the pairs (E, α) and (F, β) represent the same element in $\Phi(\mathcal{E}_A(f))$ and hence σ is injective. □

7.6. Next we define a map $u_n : \Phi_n(\mathcal{E}_A(f)) \to [C_f, BGL(n, A)]_0$ for any n, where C_f denotes the mapping cone of f, as follows: Clearly any pair (E, α) is equivalent to a pair in which the bundle is of the form $g^*EGL(n, A)[A^n]$ (c.f.2.25 and 2.27) and since any isomorphism of A–bundles is an isomorphism of associated bundles and thus induced by a unique isomorphism of the corresponding frame bundles (c.f. 2.24 and 2.27) we may assume that we have given a smooth map $g : Y \to BGL(n, A)$ and an isomorphism $\alpha : f^*g^*EGL(n, A) \to X \times GL(n, A)$ of principal bundles.

Now the composition of the canonical maps $f^*g^*EGL(n, A) \to g^*EGL(n, A) \to EGL(n, A)$ with α^{-1} and the canonical section $x \mapsto (x, id)$ of the bundle $X \times GL(n, A)$ is a smooth map $s : X \to EG$ and $p \circ s = g \circ f$, where $p : EGL(n, A) \to BGL(n, A)$ is the projection of the universal bundle. Now the space $EGL(n, A)$ is contractible and thus s is homotopic to the constant map $(1, id, 0, id, 0, id, \ldots)$. The image of a nullhomotopy under p is a homotopy between $g \circ f$ and the constant map $*$, where $*$ denotes the base point of $BGL(n, A)$ so it factors to a smooth map $H : CX \to BGL(n, A)$ which maps the peak of the cone to the base point and coincides on the subspace X with $g \circ f$.

Recall that the mapping cone C_f of f is defined as the push out

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ CX & \longrightarrow & C_f \end{array}$$

Thus the maps g and H induce a smooth base point preserving map $C_f \to BGL(n, A)$ the homotopy class of which we assign to the pair $(g^*EGL(n, A), \alpha)$.

7.7. Lemma. *The map u_n defined in 7.6 above is well defined.*

Proof. We keep the notation of 7.6 and we write G for $GL(n,A)$. Let us first fix g and α. Then we have to show that the definition does not depend on the choice of the nullhomotopy. So let us assume that $h_1, h_2 : CX \to EG$ are smooth maps which both restrict to s on X and map the peak of CX to the point $(1, id, 0, id, 0, id, \dots)$. There is a natural inclusion $X^+ \to CX$ which restricts to the natural inclusion on X and maps the poin x^+ to the peak of CX. By assumption the two maps h_1 and h_2 define a smooth map $h_1 \cup h_2 : CX \cup_{X^+} CX \to EG$ which is again homotopic to the constant map $(1, id, 0, id, \dots)$. Thus there is a homotopy $\tilde{h} : (CX \cup_{X^+} CX) \times I \to EG$ such that $\tilde{h}(z,0) = (h_1 \cup h_2)(z)$ and $\tilde{h}(z,1) = (1, id, 0, id, \dots)$. By cartesian closedness we have $(CX \cup_{X^+} CX) \times I = (CX \times I) \cup_{X^+ \times I} (CX \times I)$ and thus from $\tilde{h}$ we get two homotopies $\tilde{h}_1, \tilde{h}_2 : CX \times I \to EG$ which coincide on the subspace $X^+ \times I$ and satisfy $\tilde{h}_i|_{CX \times \{0\}} = h_i$ and $\tilde{h}_i|_{CX \times \{1\}}$ is the constant map $(1, id, 0, id, \dots)$. Now let $\gamma \in C^\infty(\mathbb{R}, I)$ be a smooth increasing map such that $\gamma(t) = 0$ for $t \le \varepsilon$ and $\gamma(t) = 1$ for $t \ge 1 - \varepsilon$, where ε is some small positive number and define $\tilde{\mathcal{H}} : X \times I \times I \to EG$ by

$$\tilde{\mathcal{H}}(x,s,t) := \begin{cases} \tilde{h}_1(\pi(x,s), (1-s)\gamma(4t)) & t \le 1/4 \\ \tilde{h}_1(\pi(x, s + (1-s)\gamma(4t-1)), (1-s)) & 1/4 \le t \le 1/2 \\ \tilde{h}_2(\pi(x, 1 - (1-s)\gamma(4t-2)), (1-s)) & 1/2 \le t \le 3/4 \\ \tilde{h}_2(\pi(x,s), (1-s)(1-\gamma(4t-3))) & 3/4 \le t, \end{cases}$$

where $\pi : X \times I \to CX$ denotes the canonical projection. One immediately checks that this map is well defined and smooth. Moreover by construction $\tilde{\mathcal{H}}(x, 0, t)$ is independent of x and thus the map $\tilde{\mathcal{H}}$ factors to a smooth map $\mathcal{H} : CX \times I \to EG$ which is obviously a homotpy between h_1 and h_2. On the subspace $X^+ \times I$ this homotopy behaves as follows: On $X \times I$ it equals $s \circ pr_1$ while the point x^+ runs through a curve starting and ending at the point $(1, id, \dots)$ and then runs the same curve backwards. Clearly this map is smoothly homotopic relative to $X^+ \times \{0,1\}$ to the map which equals $s \circ pr_1$ on $X \times I$ and the constant map $(1, id, \dots)$ on $x^+ \times I$. Now one easily verifies that (CX, X^+) is a smooth NDR–pair, and thus we conclude from lemma 6.16 that there is a homotopy H of base point preserving maps between h_1 and h_2 which restricts to $s \circ pr_1$ on $X \times I$. Thus together with the map $g \circ pr_1 : Y \times I \to BG$ the map $p \circ H : CX \times I \to BG$ induces a smooth map $C_f \times I \to BG$ which is by construction a smooth base point preserving homotopy between the maps constructed using h_1 and h_2, respectively.

So let us now assume that (g, α) and $(\tilde{g}, \tilde{\alpha})$ give rise to equivalent pairs. Then there is an isomorphism $\varphi : g^*EG \to \tilde{g}^*EG$ such that $\tilde{\alpha} \circ f^*\varphi$ and α are homotopic as isomorphisms between f^*g^*EG and $X \times G$. Thus we get an isomorphism $\Psi : f^*g^*EG \times I \to X \times I \times G$ of principal bundles over $X \times I$ which restricts to α on $f^*g^*EG \times \{0\}$ and to $\tilde{\alpha} \circ f^*\varphi$ on $f^*g^*EG \times \{1\}$. On the other hand by 2.22 there is a homomorphism $\Phi : g^*EG \times I \to EG$ of principal bundles which restricts to p^*g on $g^*EG \times \{0\}$ and to $p^*\tilde{g} \circ \varphi$ on $g^*EG \times \{1\}$. Now we define a map $h : X \times I \to EG$ by $h(x,t) := (\Phi \circ (u \times id) \circ \Psi^{-1})(x, t, id)$, where $u : f^*g^*EG \to g^*EG$ is the canonical map. Then clearly $h|_{X \times \{0\}}$ is the map constructed from α in 7.6 and from the commutative diagram

$$\begin{array}{ccc} f^*g^*EG & \xrightarrow{u} & g^*EG \\ \downarrow{\scriptstyle f^*\varphi} & & \downarrow{\scriptstyle \varphi} \\ f^*\tilde{g}^*EG & \xrightarrow{\tilde{u}} & \tilde{g}^*EG \end{array}$$

we conclude that $h|_{X \times \{1\}}$ is the map constructed from $\tilde{\alpha}$ as in 7.6. Since EG is contractible the map h is homotopic to the constant map $(1, id, 0, id, 0, id, \dots)$, so there is a map $\tilde{H}$:

$X \times I \times I \to EG$ such that $\tilde{H}(x,0,t) = (1, id, 0, id, \dots)$ and $\tilde{H}(x,1,t) = h(x,t)$. Since $\tilde{H}(x,0,t)$ is independent of x this map factors to a smooth map $H : CX \times I \to EG$ and by construction the map $p \circ H$ together with the map $Y \times I \to BG$ induced by Φ defines a smooth map $C_f \times I \to BG$ which is a homotopy of base point preserving maps between the maps associated to the two pairs. $\square$

7.8. Theorem. *For any $n \in \mathbb{N}$ the map $u_n : \Phi_n(\mathcal{E}_A(f)) \to [C_f, BGL(n,A)]_0$ is bijective. Together these maps induce an isomorphism of abelian groups $\Phi(\mathcal{E}_A(f)) \cong [C_f, BGL(A)]_0$.*

Proof. **Step 1:** u_n is surjective.
Let $g : C_f \to BG$ be a base point preserving smooth map where as before we write G for $GL(n,A)$. Next let $j : Y \to C_f$, $k : CX \to C_f$ and $\pi : X \times I \to CX$ be the canonical mappings and let $\gamma \in C^\infty(\mathbb{R}, I)$ be a smooth increasing map such that $\gamma(t) = 0$ for $t \leq \varepsilon$ and $\gamma(t) = 1$ for $t \geq 1 - \varepsilon$, where ε is some small positive number. Define $\varphi : X \times I \to BG$ by $\varphi(x,t) := (g \circ k \circ \pi)(x, \gamma(t))$ and set $P := (g \circ j)^* EG$. Then by construction we have $\varphi(x,0) = *$, the base point of BG and $\varphi(x,1) = g \circ j \circ f$, so we can identify f^*P with a subbundle of $\varphi^* EG$. Moreover φ satisfies the conditions of 2.11, so there is an isomorphism $\Phi : f^*P \times I \to \varphi^* EG$ of principal bundles over $X \times I$ which restricts to the natural inclusion on $f^*P \times \{1\}$. By construction the map $(p^*\varphi \circ \Phi)|_{X \times \{0\}} : f^*P \to EG$ has values in the fiber over the base point of BG which is canonically diffeomorphic to G and thus together with the projection of f^*P it defines an isomorphism $\alpha : f^*P \to X \times G$ of principal bundles. Now consider the smooth map $H : X \times I \to EG$ defined by $H(x,t) := (p^*\varphi \circ \Phi)(\alpha^{-1}(x, id), t)$. Then by construction we have $H(x,0) = (1, id, 0, id, \dots)$, while $H|_{X \times \{1\}}$ is the map constructed from α as in 7.6. Thus the map associated to the pair $(P[A^n], \alpha)$ is induced by $g \circ j$ on Y and $\pi(x,s) \mapsto (g \circ k)(\pi(x, \gamma(s)))$, and one immediately verifies that it is homotopic to g as a base point preserving map.

Step 2: Assume that $h : C_f \to BG$ is the smooth map associated to a pair (g, α) and let $\tilde{\alpha} : f^*g^*EG \to X \times G$ be the isomorphism constructed from h as in Step 1. Then we show that (g, α) and $(g, \tilde{\alpha})$ are equivalent. First note that the isomorphism α can be reconstructed from the corresponding map $s : X \to EG$ as follows: Let $\tau : EG \times_{BG} EG \to G$ be the smooth map constructed in 2.3. Then a short computation shows that $\alpha(z) = \tau(s((g \circ f)^* p(z)), p^*(g \circ f)(z))$, where $(g \circ f)^* p$ is the projection of the bundle f^*g^*EG and $p^*(g \circ f) : f^*g^*EG \to EG$ is the canonical map. Conversely given a map s as above with $p \circ s = g \circ f$ the formula above defines an isomorphism $f^*g^*EG \to X \times G$.

Now let $s, \tilde{s} : X \to EG$ be the maps corresponding to α and $\tilde{\alpha}$, respectively. From the construction above we get a nullhomotopy $\tilde{H} : X \times I \to EG$ with $\tilde{H}(x,1) = \tilde{s}(x)$. On the other hand from 7.6 we get a nullhomotopy H with $H(x,1) = s(x)$ which clearly can be reparametrized in a way such that $p \circ H = p \circ \tilde{H} : X \times I \to BG$. Thus we can define $\sigma : X \times I \to G$ by $\sigma(x,t) := \tau(H(x,t), \tilde{H}(x,t))$. Since both H and $\tilde{H}$ are nullhomotopies we get $\sigma(x,0) = id$ and by construction we have $s(x) = \tilde{s}(x) \cdot \sigma(x,1)$ where the dot denotes the principal action. Now we define an isomorphism of principal bundles $\Phi : f^*g^*EG \times I \to X \times I \times G$ by $\Phi(z,t) := (x, t, (\sigma(x,t))^{-1} \cdot \tilde{\alpha}(z))$, where $x = (g \circ f)^* p(z)$. Then one immediately verifies that this is a homotopy between $\tilde{\alpha}$ and α and thus (g, α) and $(g, \tilde{\alpha})$ are indeed equivalent.

Step 3: To complete the proof it suffices to show that two pairs are equivalent if theire associated maps are homotopic as base point preserving maps. Recall that by cartesian closedness $C_f \times I$ is the push out

$$\begin{array}{ccc} X \times I & \xrightarrow{f \times id} & Y \times I \\ \downarrow & & \downarrow{\scriptstyle j \times id} \\ CX \times I & \xrightarrow{k \times id} & C_f \times I. \end{array}$$

So assume that we have given a smooth map $H : C_f \times I \to BG$ such that $H(C_f \times \{1\}) = H(k(\pi(x,0)),t) = *$, which is a homotopy between the maps associated to two pairs (g,α) and $(\tilde{g},\tilde{\alpha})$. Without loss of generality we may assume that H satisfies the conditions of 2.11 (i.e. can be constantly extended to $C_f \times \mathbb{R}$). Put $P := (H\circ(j\times id))^*EG$. By construction we have $H\circ(j\times id)|_{Y\times\{0\}} = g$, so by 2.11 there is an isomorphism of principal bundles $\Phi : g^*EG\times I \to P$ which restricts to the natural inclusion on $g^*EG\times\{0\}$ and we define an isomorphism $\varphi : g^*EG \to \tilde{g}^*EG$ as $\varphi := \Phi|_{g^*EG\times\{1\}}$.

Next let $\gamma \in C^\infty(\mathbb{R}, I)$ be the map used in step 1, and define $\psi : X\times I\times I \to BG$ by $\psi(x,s,t) := H(k(\pi(x,\gamma(s))),t)$. Then we have $\psi|_{X\times\{1\}\times I} = H\circ((j\circ f)\times I)$, so by 2.11 there is an isomorphism $\Psi : f^*P\times I \to \psi^*EG$ which restricts to the identity on $f^*P\times\{1\}$. On the other hand $\psi(x,0,t) = *$ and so $(p^*\psi\circ\Psi)|_{f^*P\times\{0\}}$ induces an isomorphism $\Omega : f^*P \to X\times I\times G$ and clearly Ω restricts to the isomorphism β constructed from $H|_{C_f\times\{0\}}$ as in step 1 and to $\tilde{\beta}$ constructed from $H|_{C_f\times\{1\}}$ as in step 1 on $f^*P|_{X\times\{0\}} = g^*EG$ and $f^*P|_{X\times\{1\}} = \tilde{g}^*EG$, respectively. Now Φ induces an isomorphism $(f\times id)^*\Phi : f^*g^*EG\times I \to f^*P$ and thus we get an isomorphism $\Omega\circ f^*\Phi : f^*g^*EG\times I \to X\times I\times G$, and by construction this is a homotopy of isomorphisms between β and $\tilde{\beta}\circ f^*\varphi$. In view of step 2 this proves that (g,α) and $(\tilde{g},\tilde{\alpha})$ are equivalent.

So it remains to discuss the compatibility of the maps u_n with the algebraic structures. One easily verifies that the map $[C_f, BGL(n,A)]_0 \times [C_f, BGL(m,A)]_0 \to [C_f, BGL(n+m,A)]_0$ which induces the addition on $[C_f, BGL(A)]$ is induced by the obvious analogs $\bullet : BGL(n,A)\times BGL(m,A) \to BGL(n+m,A)$ of the map constructed in the proof of 6.19. Let us also denote by $\bullet$ the corresponding map $EGL(n,A)\times EGL(m,A) \to EGL(n+m,A)$. Let (g,α) and $(\tilde{g},\tilde{\alpha})$ be two pairs in dimensions n and m, respectively and consider the map $G : C_f \to BGL(n+m,A)$ induced by their sum. Then the composition of G with the natural map $Y \to C_f$ clearly is $g\bullet\tilde{g}$. Let $s : X \to EGL(n,A)$ and $\tilde{s} : X \to EGL(m,A)$ be the maps constructed from α and $\tilde{\alpha}$ as in 7.6. Then clearly $s\bullet\tilde{s} : X \to EGL(n+m,A)$ is the map constructed from $\alpha\oplus\tilde{\alpha}$ as in 7.6. Consequently given nullhomotopies h and $\tilde{h}$ for s and $\tilde{s}$ the map $h\bullet\tilde{h}$ is a nullhomotopy for the map $s\bullet\tilde{s}$. Thus we see that we can construct the map associated to the sum of the pairs as the $\bullet$–product of the individual maps. This shows that the maps u_n induce a map $u : \Phi(\mathcal{E}_A(f)) \to [C_f, BGL(A)]$ and that u is a group homomorphism. $\square$

7.9. In order to define homomorphisms between absolute and relative K–groups it is more convenient to work with the map $f^+ : X^+ \to Y^+$ associated to f (c.f. 6.13). So let C_{f^+} be the mapping cone of f^+.

The inclusion $X \to X^+$ induces an inclusion $CX \to C(X^+)$ and together with the inclusion $Y \to Y^+$ this map defines a smooth map $i : C_f \to C_{f^+}$ which is base point preserving if we choose for both spaces the 'peaks' of the cones as base points.

Proposition. *The map $i : C_f \to C_{f^+}$ defined above is a smooth homotopy equivalence of pointed smooth spaces.*

Proof. First note that by cartesian closedness the space $X^+\times I$ is the coproduct of $X\times I$ and $\{x^+\}\times I$. Thus we can define a smooth map $X^+\times I \to CX$ by requiering that it is the natural map on $X\times I$ and maps $\{x^+\}\times I$ to the peak of CX. Together with the map $Y^+ \to C_f$ which is the natural map on Y and maps y^+ to the base point of C_f this map induces a base point preserving smooth map $j : C_{f^+} \to C_f$, and one immediately verifies that $j\circ i$ is the identity. So it remains to show that $i\circ j : C_{f^+} \to C_{f^+}$ is smoothly homotopic to the identity as a base point preserving map.

Consider the map $X^+\times I\times I \to X^+\times I$ defined by $(x,s,t)\mapsto(x,s)$ for $x\neq x^+$ and $(x^+,s,t)\mapsto(x^+,ts)$, which is smooth by cartesian closedness. Moreover since for all x we have $(x,0,t)\mapsto(x,0)$ this map induces a smooth map $C(X^+)\times I \to C(X^+)$ and composing the natural map $C(X^+) \to C_{f^+}$ with this one we get a smooth map $h : C(X^+)\times I \to C_{f^+}$.

On the other hand define a map $Y^+ \times I \to C_{f^+}$ by mapping (y,t) to the image of y under the canonical map $Y^+ \to C_{f^+}$ for $y \neq y^+$ and mapping (y^+,t) to $h(x^+,1,t)$. Then this map is smooth by cartesian closedness and since again by cartesian closedness the diagram

$$\begin{array}{ccc} X^+ \times I & \longrightarrow & Y^+ \times I \\ \downarrow & & \downarrow \\ C(X^+) \times I & \longrightarrow & C_{f^+} \times I \end{array}$$

is a push out it induces together with h a smooth map $H : C_{f^+} \times I \to C_{f^+}$ which is easily seen to be a homotopy of base point preserving maps between $i \circ j$ and the identity. □

7.10. Corollary. *For any convenient algebra A and any smooth map $f : X \to Y$ between base spaces there is a canonical isomorphism of abelian groups $K(\mathcal{E}_A(f)) \cong \tilde{K}^0_A(C_{f^+})$.*

Proof. From 7.8 we get an isomorphism $K'(\mathcal{E}_A(f)) \cong [C_f, BGL(A)]_0$. So from 7.3 we see that $K(\mathcal{E}_A(f)) \cong [C_f, K_0(A) \times BGL(A)]_0$. Next by 7.9 we get an isomorphism $[C_f, K_0(A) \times BGL(A)]_0 \cong [C_{f^+}, K_0(A) \times BGL(A)]_0$ and thus the result follows from 6.20. □

7.11 Definition. In anlogy to 6.13 we now define $K^{-n}(\mathcal{E}_A(f)) := \tilde{K}_A(S^n(C_{f^+}))$ for $n > 0$. (Note that in view of 6.15 we could also use the reduced suspension.)

From the above description one immediately sees that the relative K–groups have functorial properties as follows: Suppose that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \end{array}$$

is a commutative diagram of base spaces and smooth maps. Then $C(\varphi^+)$ and ψ^+ induce a base point preserving smooth map $C_{f^+} \to C_{\tilde{f}^+}$ and thus group homomorphisms $K^{-n}(\mathcal{E}_A(f)) \to K^{-n}(\mathcal{E}_A(\tilde{f}))$ for any $n \geq 0$.

On the other hand for a homomorphism $\varphi : A \to B$ between convenient algebras $\tilde{K}_\varphi(S^n(C_{f^+}))$ is a group homomorphism $K^{-n}(\mathcal{E}_A(f)) \to K^{-n}(\mathcal{E}_B(f))$.

Moreover we also get homomorphisms between absolute and relative K–groups as follows: The natural map $Y^+ \to C_{f^+}$ induces maps $S^n(Y^+) \to S^n(C_{f^+})$ and consequently homomorphisms $K^{-n}(\mathcal{E}_A(f)) \to K_A^{-n}(Y)$ for $n > 0$. For $n = 0$ we compose this map with the isomorphism $K_A(Y^+) \to \tilde{K}_A(Y)$ from 6.21 to get a homomorphism $K(\mathcal{E}_A(f)) \to K_A(Y)$.

On the other hand the map $q : C_{f^+} \to S(X^+)$ constructed in 3.41 (this is the map induced by contracting Y^+ to a point) induces smooth maps $S^n(C_{f^+}) \to S^{n+1}(X^+)$ and thus group homomorphisms $K_A^{-n-1}(X) \to K^{-n}(\mathcal{E}_A(f))$ for all $n \geq 0$.

7.12. With the aid of the relative K–groups we can now also give a new interpretation of the higher K–groups. Let X be a base space and let $i : X \to C(X)$ be the inclusion. Since this map is a smooth cofibration we conclude from 3.14 that the mapping cone C_i of i is smoothly homotopy equivalent to $S(X)$. Thus from 7.8 and 6.19 we conclude that $\tilde{K}_A(SX) \cong \Phi(\mathcal{E}_A(i))$.

On the other hand for a pointed base space let $[X, GL(n,A)]_0$ be the set of all pointed smooth homotopy classes of base point preserving smooth maps from X to the smooth group $GL(n,A)$, where as the base point of $GL(n,A)$ we take the identity. The smooth homomorphisms $GL(n,A) \to GL(n+1,A)$ induce maps $[X, GL(n,A)]_0 \to [X, GL(n+1,A)]_0$ and we denote by $[X, GL(A)]_0$ the direct limit of the so obtained inductive system. Clearly the pointwise multiplication of smooth maps induces a group structure on $[X, GL(A)]_0$. Similarly for an arbitrary base space we define $[X, GL(A)]$, which is also a group.

7.13. Theorem. *For any pointed smooth space* (X, x_0) *there is a natural isomorphism* $\tilde{K}_A(SX) \cong [X, GL(A)]_0$*. In particular* $[X, GL(A)]_0$ *is always an abelian group.*

Proof. For any $n > 0$ We define a map $v_n : [X, GL(n, A)] \to \Phi_n(\mathcal{E}_A(i))$ by assigning to a smooth map $f : X \to GL(n, A)$ the pair $(CX \times A^n, \alpha_f)$, where $\alpha_f : X \times A^n \to X \times A^n$ is the isomorphism defined by $\alpha_f(x, u) := (x, f(x)(u))$. Then one easily shows that for a map $g : X \to GL(n, A)$ which is smoothly homotopic to f the isomorphism α_g is smoothly homtopic to α_f in the space of isomorphisms and so v_n is well defined. Moreover from the defintions of the connecting maps one immediately concludes that the maps v_n induce a map $v : [X, GL(A)] \to \Phi(\mathcal{E}_A(i))$.

Next we show that each map v_n is surjective. So let (E, α) be a pair representing a class in $\Phi_n(\mathcal{E}_A(i))$. Since CX is smoothly contractible the bundle E must be isomorphic to $CX \times A^n$, so without loss of generality we may assume that $E = CX \times A^n$. Then clearly $i^*E = X \times A^n$, so α is an automorphism of $X \times A^n$. The second component of α is then a smooth map $X^+ \times A^n \to A^n$ and we denote by $\tilde{\alpha} : X \to C^\infty(A^n, A^n)$ the map associated to α via cartesian closedness. Then by definition $\tilde{\alpha}$ has values in subspace of invertible A–module homomorphisms and since α^{-1} is smooth we conclude that $\tilde{\alpha}$ is smooth as a map $X \to GL(n, A)$. Now let $f : X \to GL(n, A)$ be the smooth map defined by $f(x) := \tilde{\alpha}(x) \cdot (\tilde{\alpha}(x_0))^{-1}$. Then $\alpha_f = \alpha \circ i^*\varphi$, where $\varphi : CX \times A^n \to CX \times A^n$ is the isomorphism defined by $\varphi(z, u) = (z, (\tilde{\alpha}(x_0))^{-1}(u))$. Thus by definition the pairs (E, α) and $(CX \times A^n, \alpha_f)$ are equivalent and the map v_n is surjective.

On the other hand assume that $f, g : X \to GL(n, A)$ are smooth maps such that the pairs $(CX \times A^n, \alpha_f)$ and $(CX \times A^n, \alpha_g)$ are equivalent. Then there is an isomorphism $\varphi : CX \times A^n \to CX \times A^n$ such that α_f and $\alpha_g \circ i^*\varphi$ are homotopic as isomorphisms. The isomorphism φ can be viewed as a homotopy of isomorphisms between $i^*\varphi = \varphi|_X$ and the isomorphism $\bar{\varphi} : X \times A^n \to X \times A^n$ defined by $\bar{\varphi}(x, u) = (x, pr_2(\varphi(*, u)))$ where $*$ denotes the 'peak' of $C(X^+)$. So α_f is homotopic as an isomorphism to $\alpha_g \circ \bar{\varphi}$ and thus there is an isomorphism $H : X \times I \times A^n \to X \times I \times A^n$ which restricts to α_f over $X \times \{0\}$ and to $\alpha_g \circ \bar{\varphi}$ over $X \times \{1\}$. As above we construct from H a smooth map $\tilde{H} : X \times I \to GL(n, A)$ and by construction $\tilde{H}$ restricts f on $X \times \{0\}$ and to $g \cdot c$ on $X \times \{1\}$ where c is some constant map. But then the map $\mathcal{H} : X \times I \to GL(n, A)$ defined by $\mathcal{H}(x, t) := \tilde{H}(x, t) \cdot (\tilde{H}(x_0, t)^{-1})$ is easily seen to be a smooth base point preserving homotopy between f and g. Thus each map v_n is bijective and so v is bijective.

Let us now show that v is a group homomorphism. Let $f : X \to GL(n, A)$ and $g : X \to GL(m, A)$ be smooth maps. To compute the product we may without loss of generality assume that $n = m$ and then the product is represented by the map $X \to GL(2n, A)$ defined by $x \mapsto (f(x) \cdot g(x)) \oplus IdA^n$. On the other hand one easily computes that the map corresponding to the sum of the pairs $(CX \times A^n, \alpha_f)$ and $(CX \times A^n, \alpha_g)$ is given by $x \mapsto f(x) \oplus g(x)$. In matrix notation the maps correspond to $\begin{pmatrix} f \cdot g & 0 \\ 0 & Id \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & Id \end{pmatrix}$ and $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & 0 \\ 0 & g \end{pmatrix}$, respectively. Now for any $(x, t) \in X \times I$ the matrix $\begin{pmatrix} cos(t)g(x) & sin(t)g(x) \\ -sin(t)Id & cos(t)Id \end{pmatrix}$ is invertible with inverse $\begin{pmatrix} cos(t)g(x)^{-1} & -sin(t)Id \\ sin(t)g(x)^{-1} & cos(t)Id \end{pmatrix}$. Thus sutably reparametrized this formula defines a smooth homotopy between $\begin{pmatrix} g & 0 \\ 0 & Id \end{pmatrix}$ and $\begin{pmatrix} 0 & g \\ -Id & 0 \end{pmatrix}$. In the same way one constructs a smooth homotopy between $\begin{pmatrix} Id & 0 \\ 0 & g \end{pmatrix}$ and $\begin{pmatrix} 0 & g \\ -Id & 0 \end{pmatrix}$. Consequently the maps corresponding to the two products are freely homotopic and multiplying with the pointwise inverse of the path of the base point under such a homotopy we get a base point preserving homotopy, so v is a group isomorphism. □

7.14. Corollary. *There is a natural isomorphism of bifunctors $K_A^{-1}(X) \cong [X, GL(A)]$.*

Proof. By defintion $K_A^{-1}(X) = \tilde{K}_A(S(X^+))$, so by 7.13 we get $K_A^{-1}(X) \cong [X^+, GL(A)]_0$. Clearly the obvious isomorphisms $[X^+, GL(n,A)]_0 \cong [X, GL(n,A)]$ induce an isomorphism $[X^+, GL(A)]_0 \cong [X, GL(A)]$ and the result follows. □

7.15. Corollary. *For any convenient algebra A and any $n > 0$ there is an isomorphism $K_n(A) \cong [S^{n-1}, GL(A)]_0$.*

Proof. This follows immediately from 7.13 and 6.24. □

7.16. Theorem (The long exact sequence of a smooth map). *Let $f : X \to Y$ be a smooth map between base spaces and let A be a convenient algebra. Then there is a long exact sequence of abelian groups and group homomorphisms*

$$\begin{aligned} \dots \to K_A^{-n-1}(X) \to K^{-n}(\mathcal{E}_A(f)) \to & \\ \to K_A^{-n}(Y) \xrightarrow{K_A^{-n}(f)} K_A^{-n}(X) \to \dots & \\ \to K_A^{-1}(X) \to K(\mathcal{E}_A(f)) \to K_A^0(Y) \xrightarrow{K_A^0(f)} K_A^0(X) & \end{aligned}$$

which is natural in f and A.

Proof. For any k the space $BGL(k,A)$ is smoothly path connected since it is the smooth image of the contractible space $EGL(k,A)$. Thus we can apply the Puppe sequence 3.43 to the map $f^+ : X^+ \to Y^+$ and $BGL(k,A)$ to get an exact sequence of pointed sets

$$\begin{aligned} (1) \quad \dots \to [S^{n+1}(X^+), BGL(k,A)] \xrightarrow{S^n(q)^*} [S^n C_{f^+}, BGL(k,A)] \xrightarrow{S^n(g)^*} & \\ \to [S^n(Y^+), BGL(k,A)] \xrightarrow{S^n(f^+)^*} [S^n(X^+), BGL(k,A)] \to \dots & \\ \dots \to [S(X^+), BGL(k,A)] \xrightarrow{q^*} [C_{f^+}, BGL(k,A)] \xrightarrow{g^*} & \\ \to [Y, BGL(k,A)] \xrightarrow{(f^+)^*} [X, BGL(k,A)] & \end{aligned}$$

Clearly the Puppe sequence is natural for maps in the second variable, so using the maps $BGL(k,A) \to BGL(k+1,A)$ constructed in 6.11 we get a large commutative diagram with exact rows. Thus passing to the direct limit we get an induced sequence as (1) with $BGL(k,A)$ replaced by $BGL(A)$ and one easily checks that this sequence is again an exact sequence of pointed sets. By 6.11 for any base space Z there is a natural isomorphism $[Z, BGL(A)] \cong K'_A(Z)$ and by 6.8 for smoothly path connected base spaces we have $K'_A(Z) \cong \tilde{K}_A(Z)$. Since the suspension over any smooth space is clearly smoothly path connected we thus get the claimed sequence in all but the last three terms where we have:

$$\dots \to K'_A(C_f^+) \to K'_A(Y^+) \to K'_A(X^+).$$

Now consider the sequence

$$0 \to [C'_{f^+}, K(A)]_0 \to [Y^+, K(A)]_0 \xrightarrow{(f^+)^*} [X^+, K(A)]_0.$$

Since the image of Y^+ in C_{f^+} meets any path component this sequence is exact at $[C'_{f^+}, K(A)]_0$. On the other hand the map $X^+ \to C_{f^+}$ maps th whole space X^+ to the path component of the base point of C_{f^+}, while a function $Y^+ \to K(A)$ which composed with f^+ is the zero function can be extended by zero on $C(X^+)$ to a function on C_{f^+} and thus the whole sequence is exact. So we can add these three terms to the last three terms in the sequence above without destroying the exactness. Using 6.20 we conclude the the last three terms now look as

$$\cdots \to \tilde{K}_A(C_f^+) \to \tilde{K}_A(Y^+) \to \tilde{K}_A(X^+),$$

so by 6.21 we get the claimed sequence. Naturality of the sequence follows immediately from the obvious naturality properties of the Puppe sequence. □

7.17. Next we discuss the relative K–group associated to an algebra homomorphism. So let X be a base space and let $\varphi : A \to D$ be a bounded homomorphism between convenient algebras. Then φ induces an additive functor $\varphi_* = \mathcal{E}_\varphi(X) : \mathcal{E}_A(X) \to \mathcal{E}_D(X)$ (c.f. 5.14). Thus by definition the elements of $\Gamma(\mathcal{E}_\varphi(X))$ are triples (E, F, α) where E and F are A–bundles over X and α is an isomorphism of D–bundles between $\varphi_*(E)$ and $\varphi_*(F)$. To such a triple we assign a locally constant function $X \to K_0(A)$ by assining to any point the difference in $K_0(A)$ of the classes of the fiber of E and the fiber of F over this point. Obviously this induces a group homomorphism $r : K(\mathcal{E}_\varphi(X)) \to H^0(X, K_0(A))$. Now by 6.4 φ induces a group homomorphism $K_0(\varphi) : K_0(A) \to K_0(D)$ and thus also a group homomorphism $K_0(\varphi)_* : H^0(X, K_0(A)) \to H^0(X, K_0(D))$. Since the bundles $\varphi_*(E)$ and $\varphi_*(F)$ are by definition isomorphic the map r has in fact values in $\mathrm{Ker}(K_0(\varphi)_*)$ and we define $K'(\mathcal{E}_\varphi(X))$ to be the kernel of r.

Now let f be an element of $\mathrm{Ker}(K_0(\varphi)_*)$. Then we assign to f an element of $K(\mathcal{E}_\varphi(X))$ as follows: Let $X_1, \dots X_n$ be the path connected components of X on shich f is nonzero. Then by 6.2 for any i we can write $f(X_i) = [P_i] - [Q_i] \in K_0(A)$, where P_i and Q_i are finitely generated projective right A–modules. Now let E be the A–bundle over X which is over X_i given by $X_i \times P_i$ and F the one given by $X_i \times Q_i$ while over the components on which f is zero both bundles are zero, i.e. the identitiy map. Then one immediately sees that this defines a group homomorphism $\mathrm{Ker}(K_0(\varphi)_*) \to K(\mathcal{E}_\varphi(X))$, which is right inverse to r, so

$$0 \to K'(\mathcal{E}_\varphi(X)) \to K(\mathcal{E}_\varphi(X)) \to \mathrm{Ker}(K_0(\varphi)_*) \to 0$$

is a split short exact sequence of abelian groups and thus there is an isomorphism $K(\mathcal{E}_\varphi(X)) \cong K'(\mathcal{E}_\varphi(X)) \oplus \mathrm{Ker}(K_0(\varphi)_*)$.

7.18. As before we next want to interpret the group $K'(\mathcal{E}_\varphi(X))$ using homotopy theory and this needs an intermediate step. Consider the set of all pairs (E, α), where E is an A–bundle over X with fiber A^n and α is an isomorphism of D–bundles between $\varphi_*(E)$ and $X \times D^n$. Two such pairs (E, α) and (E', α') are said to be equivalent if and only if there is an isomorphism $f : E \to E'$ such that α and $\alpha' \circ \varphi_* f$ are homotopic as isomorphisms from $\varphi_*(E)$ to $X \times D^n$. By $\Phi_n(\mathcal{E}_\varphi(X))$ we denote the set of all equivalence classes.

Now $(E, \alpha) \mapsto (E \oplus \theta_1, \alpha \oplus id)$, where θ_1 denotes the trivial 'line' bundle $X \times A$ over X defines a map $\Phi_n(\mathcal{E}_\varphi(X)) \to \Phi_{n+1}(\mathcal{E}_\varphi(X))$ and we define $\Phi(\mathcal{E}_\varphi(X))$ to be the direct limit of the so obtained inductive system. Next $((E, \alpha), (F, \beta)) \mapsto (E \oplus F, \alpha \oplus \beta)$ defines a map $\Phi_n(\mathcal{E}_\varphi(X)) \times \Phi_m(\mathcal{E}_\varphi(X)) \to \Phi_{n+m}(\mathcal{E}_\varphi(X))$ and as in 6.9 one shows that this induces the structure of a commutative monoid on $\Phi(\mathcal{E}_\varphi(X))$.

Now we define a map $\sigma_n : \Phi_n(\mathcal{E}_\varphi(X)) \to K'(\mathcal{E}_\varphi(X))$ by $\sigma_n(E, \alpha) := d(E, X \times A^n, \alpha)$. This is easily seen to be well defined and clearly it induces a monoid homomorphism $\sigma : \Phi(\mathcal{E}_\varphi(X)) \to K'(\mathcal{E}_\varphi(X))$.

7.19. Proposition. *The homomorphism σ defined above is bijective. Thus $\Phi(\mathcal{E}_\varphi(X))$ is an abelian group.*

Proof. Let (E, F, α) be an element of $\Gamma(\mathcal{E}_\varphi(X))$ and let us write θ_k for the trivial A–bundle $X \times A^k$ over X. By 5.6 there is an A–bundle G over X, a natural number n and an isomorphism $f : F \oplus G \to \theta_n$ of A–bundles and clearly the tripel (E, F, α) is equivalent to $(E \oplus G, F \oplus G, \alpha \oplus id_{\varphi_* G})$ which in turn is equivalent to $(E \oplus G, \theta_n, \varphi_* f \circ (\alpha \oplus id))$. Now let $X_1, \dots, X_k$ be the connected components of X and let P_i be the fiber of $E \oplus G$ over X_i for any i. Then if the class of the above triple in $K(\mathcal{E}_\varphi(X))$ lies in the subgroup $K'(\mathcal{E}_\varphi(X))$ then for any i the modules P_i and A^n represent the same class in $K_0(A)$. Thus for any i there is a finitely generated projective right A–module Q_i such that $P_i \oplus Q_i \cong A^n \oplus Q_i$. Now for any i we can choose a module R_i such that $Q_i \oplus R_i \cong A^m$ for some fixed m. Thus we get $P_i \oplus A^m \cong A^{n+m}$ for each i. Now the above triple is equivalent to $(E \oplus G \oplus \theta_m, \theta_{n+m}, (\varphi_* f \circ (\alpha \oplus id)) \oplus id)$ which by construction is in the image of σ since $E \oplus G \oplus \theta_m$ has fiber A^{n+m}, and thus σ is surjective.

To prove that σ is injective assume that E and F are A–bundles over X with fiber A^n and A^m, respectively, such that (E, θ_n, α) and (F, θ_m, β) are equivalent. Then by 7.2 we have $d(E \oplus \theta_m, \theta_n \oplus F, \alpha \oplus \beta^{-1}) = 0$, so there are elementary triples (G, G, γ) and (L, L, λ) such that the triples $(E \oplus \theta_m \oplus G, \theta_n \oplus F \oplus G, \alpha \oplus \beta^{-1} \oplus \gamma)$ and (L, L, λ) are isomorphic. By 5.6 we can find an A–bundle G' over X, a natural number p and an isomorphism $\psi : G \oplus G' \to \theta_p$ of A–bundles. Then clearly the triple $(E \oplus \theta_m \oplus G \oplus G', \theta_n \oplus F \oplus G \oplus G', \alpha \oplus \beta^{-1} \oplus \gamma \oplus id_{\varphi_* G'})$ is isomorphic to $(L \oplus G', L \oplus G', \lambda \oplus id_{\varphi_* G'})$. Thus by definiton there are isomorphisms $f : E \oplus \theta_m \oplus G \oplus G' \to L \oplus G'$ and $g : \theta_n \oplus F \oplus G \oplus G' \to L \oplus G'$ such that $\varphi_* g \circ (\alpha \oplus \beta^{-1} \oplus \gamma \oplus id_{\varphi_* G'}) = (\lambda \oplus id_{\varphi_* G'}) \circ \varphi_* f$. Since by construction γ and λ are homotopic to identity mappings as isomorphisms we may conclude from this that $\varphi_*(g^{-1}) \circ \varphi_* f$ and $(\alpha \oplus \beta^{-1} \oplus id_{G \oplus G'})$ are homotopic as isomorphisms between $\varphi_*(E \oplus \theta_m \oplus G \oplus G')$ and $\varphi_*(\theta_n \oplus F \oplus G \oplus G')$.

Now let $h : E \oplus \theta_m \oplus \theta_p \to F \oplus \theta_n \oplus \theta_p$ be the isomorphism defined by $h := (\tau \oplus \psi) \circ g^{-1} \circ f \circ (id_{E \oplus \theta_m} \oplus \psi^{-1})$, where $\tau : \theta_n \oplus F \to F \oplus \theta_n$ is the canonical isomorphism and $\psi : G \oplus G' \to \theta_p$ is the isomorphism from above. Then we have:

$$\begin{aligned}
&(\beta \oplus id \oplus id) \circ \varphi_* h = \\
&=(id \oplus \beta \oplus \varphi_* \psi) \circ \varphi_*(g^{-1}) \circ \varphi_* f \circ (id \oplus id \oplus \varphi_*(\psi^{-1})),
\end{aligned}$$

and this is homotopic as an isomorphism to

$$\begin{aligned}
&(id \oplus \beta \oplus \varphi_* \psi) \circ (\alpha \oplus \beta^{-1} \oplus id) \circ (id \oplus id \oplus \varphi_*(\psi^{-1})) = \\
&=(\alpha \oplus id \oplus id).
\end{aligned}$$

Thus the pairs (E, α) and (F, β) represent the same element in $\Phi(\mathcal{E}_\varphi(X))$ and hence σ is injective. □

7.20. Before we can proceed we have to discuss the functor $\mathcal{E}_\varphi(X)$ in terms of principal bundles and classifying spaces. Let G and H be smooth groups, $\psi : G \to H$ a smooth homomorphism and $p : P \to X$ a smooth principal G–bundle. Via ψ we define a smooth left action of G on H by $g \cdot h := \psi(g)h$ and we form the associated bundle to P with fiber H with respect to this action (c.f. 2.25). So we have to consider the space of orbits of the action $(z, h) \cdot g = (z \cdot g, g^{-1} \cdot h)$ on $P \times H$, where on P we have the principal right action. From the proof of 2.26 we see that taking an atlas (U_i, u_i) of P with smooth transition functions $u_{ij} : U_{ij} \to G$ we get and atlas $(U_i, \tilde{u}_i)$ for $P[H]$ with the same transition functions, but by definition of the action this atlas defines the structure of a smooth principal H–bundle on $P[H]$ and we denote this bundle by $\psi_* P$.

Now let $P \to X$ be a smooth principal G–bundle, $Q \to Y$ a smooth principal H–bundle. We define a *ψ–homomorphism* $f : P \to Q$ to be a fiber respecting smooth map which is equivariant over ψ for the principal right actions, i.e. we have $f(z \cdot g) = f(z) \cdot \psi(g)$. Note that since f is fiber respecting it covers a smooth map $\underline{f} : X \to Y$

In the construction of the associated bundle $P[H]$ we have the natural map $q : P \times H \to P[H] = \psi_* P$ and we define a map $\psi_P : P \to \psi_* P$ by $\psi_P(z) = q(z, e)$, where e denotes the unit element of H. Clearly this map is smooth and fiber respecting and since $q(z \cdot g, e) = q(z, g \cdot e) = q(z, \psi(g))$ we conclude that ψ_P is a ψ–homomorphism covering the identity. This ψ–homomorphism is universal in the following sense:

7.21. Proposition. *Let $f : P \to Q$ be a ψ–homomorphism. Then there is a unique smooth homomorphism of principal H–bundles $\tilde{f} : \psi_* P \to Q$ such that $f = \tilde{f} \circ \psi_P$.*

Proof. Consider the map $P \times H \to Q$ defined by $(z, h) \mapsto f(z) \cdot h$ where the dot denotes the principal right action. Since f is a ψ–homomorphism this map is immediately seen to be invariant for the G–action on $P \times H$ and thus it factors to a smooth map $\tilde{f} : P[H] \to Q$ which is by construction a smooth homomorphism of principal bundles and clearly $\tilde{f} \circ \psi_P = f$. Uniqueness of f immediately follows from the fact that the image of P under ψ_P meets any fiber of $\psi_* P$. $\square$

7.22. Using this universal property we can now extend ψ_* to a functor defined on the category of principal G–bundles as follows: Let $f : P_1 \to P_2$ be a smooth homomorphism of principal G–bundles. Then clearly $\psi_{P_2} \circ f : P_1 \to \psi_* P_2$ is a ψ–homomorphism and thus it induces a unique smooth homomorphism of principal H–bundles $\psi_* f : \psi_* P_1 \to \psi_* P_2$ and obviously this defines a functor.

The smooth group homomorphism ψ induces a ψ–homomorphism $E\psi : EG \to EH$ which covers a smooth map $B\psi : BG \to BH$. Now let $f : X \to BG$ be a smooth map. Then the composition of $E\psi$ with the canonical map $f^* EG \to EG$ together with the projection $f^* EG \to X$ induce a ψ–homomorphism $f^* EG \to (B\psi \circ f)^* EH$. Thus by 7.21 there is a unique induced principal bundle homomorphism $\psi_*(f^* EG) \to (B\psi \circ f)^* BH$ which clearly is an isomorphism, and we will identify these two bundles from now on. In particular this shows that there is a natural isomorphism $\psi_* EG \to (B\psi)^* EH$.

Now let us return to the case of a bounded algebra homomorphism $\varphi : A \to D$ between convenient algebras. The convenient functor $\mathcal{P}(\varphi)$ constructed in 4.35 induces for any finitely generated projective right A–module P a smooth homomorphism between the automorphism groups of P and $\mathcal{P}(\varphi)(P)$. In particular we get smooth homomorphisms $\varphi_n : GL(n, A) \to GL(n, D)$ for any n. For an A–bundle of the form $E = f^* EGL(n, A)[A^n]$ over X one easily shows that there is a natural isomorphism $\mathcal{E}_\varphi(X)(E) \cong (\varphi_n)_*(f^* EGL(n, A))[D^n]$. Moreover let $F = g^* EGL(n, A)[A^n]$ be another bundle of this form and let $\psi : E \to F$ be an isomorphism. Then ψ is induced by a homomorphism of principal bundles $\tilde{\psi} : f^* EGL(n, A) \to g^* EGL(n, A)$ and one easily checks that $\mathcal{E}_\varphi(X)(\psi)$ is the isomorphism induced by $(\varphi_n)_* \tilde{\psi}$.

7.23. For any $n > 0$ the smooth group homomorphism $\varphi_n : GL(n, A) \to GL(n, D)$ induces a smooth map $B(\varphi_n) : BGL(n, A) \to BGL(n, D)$. Let $\mathcal{F}_n(\varphi)$ be the homotopy fiber of this smooth map. Recall from 3.21 that $\mathcal{F}_n(\varphi)$ is defined by the pullback

$$\begin{array}{ccc}
\mathcal{F}_n(\varphi) & \longrightarrow & P(BGL(n, D)) \\
\downarrow & & \downarrow \\
BGL(n, A) & \xrightarrow{B(\varphi_n)} & BGL(n, D)
\end{array}$$

where P denotes the path fibration. Now we define a map $u_n : \Phi_n(\mathcal{E}_\varphi(X)) \to [X, \mathcal{F}_n(\varphi)]$ as follows: As in 7.6 we may restrict to pairs determined by a smooth map $f : X \to BGL(n, A)$ and an isomorphism $\alpha : \varphi_*(f^* EGL(n, A)) \to X \times GL(n, D)$.

Recall from 7.22 the natural isomorphism

$$\varphi_*(f^* EGL(n, A)) = (B(\varphi_n) \circ f)^* EGL(n, D).$$

Thus there is a homomorphism of principal bundles $\psi : \varphi_*(f^*EGL(n,A)) \to EGL(n,D)$ which covers the map $B(\varphi_n) \circ f : X \to BGL(n,D)$. Now define $s : X \to EGL(n,D)$ as $s(x) := \psi(\alpha^{-1}(x,id))$. Since $EGL(n,D)$ is contractible the map s is smoothly homotopic to the constant map $(1,id,0,id,\dots)$, so there is a smooth map $H : X \times I \to EGL(n,D)$ such that $H(x,0) = (1,id,0,id,\dots)$ and $H(x,1) = s(x)$. Then the map $h : X \to C^\infty(I,BGL(n,D))$ which is associated via cartesian closedness to the map $p \circ H : X \times I \to BGL(n,D)$ has values in the subspace $P(BGL(n,D))$ and is smooth as a map to this space by definition of the smooth structure. Moreover by construction the maps f and h induce a smooth map $X \to \mathcal{F}_n(\varphi)$, the homotopy class of which we assign to the pair determined by f and α.

7.24. Lemma. *The map u_n defined above is well defined.*

Proof. Let us first fix f and α. Then we have to show that the definition is independent of the choice of the nullhomotopy H. So let us assume that $H_0, H_1 : X \times I \to EGL(n,D)$ are smooth maps such that $H_i(x,0) = (1,id,\dots)$ and $H_i(x,1) = s(x)$ for $i = 0,1$. By the first condition the maps factor to smooth mappings $\tilde{H}_i : CX \to EGL(n,D)$. Then as in the proof of lemma 7.7 we conclude that there is a smooth homotopy $\tilde{\mathcal{H}} : CX \times I \to EGL(n,D)$ between $\tilde{H}_0$ and $\tilde{H}_1$ which restricts to $s \circ pr_1$ on $X \times I$ and maps the fiber over the 'peak' of CX to the point $(1,id,\dots)$. Now we define $\mathcal{H} : X \times I \times I \to EGL(n,D)$ by $\mathcal{H}(x,s,t) := \tilde{\mathcal{H}}(\pi(x,t),s)$, where $\pi : X \times I \to CX$ is the canonical map. Then by construction we have $\mathcal{H}(x,s,0) = (1,id,\dots)$, $\mathcal{H}(x,s,1) = s(x)$ and $\mathcal{H}(x,i,t) = H_i(x,t)$ for $i = 0,1$. Consequently the map $X \times I \to C^\infty(I,BGL(n,D))$ associated via cartesian closedness to $p \circ \mathcal{H} : X \times I \times I \to EGL(n,D)$ has values in the subspace $\mathcal{P}(BGL(n,D))$ and together with the map $f \circ pr_1 : X \times I \to BGL(n,A)$ it induces a smooth homotopy between the maps constructed using H_0 and H_1 as in 7.24.

So let us assume that (f,α) and $(\tilde{f},\tilde{\alpha})$ give rise to equivalent pairs. Then there is an isomorphism $g : f^*EGL(n,A) \to \tilde{f}^*EGL(n,A)$ such that $\tilde{\alpha} \circ \varphi_* g$ and α are homotopic as isomorphisms between $\varphi_* f^*EGL(n,A)$ and $X \times GL(n,D)$. Thus there is an isomorphism $\Psi : \varphi_* f^*EGL(n,A) \times I \to X \times I \times GL(n,D)$ of principal bundles over $X \times I$ which restricts to α on $\varphi_* f^*EGL(n,A) \times \{0\}$ and to $\tilde{\alpha} \circ \varphi_* g$ on $\varphi_* f^*EGL(n,A) \times \{1\}$. On the other hand by 2.22 there is a smooth homomorphism of pricipal bundles $f^*EGL(n,A) \times I \to EGL(n,A)$ which restricts to the canonical map $f^*EGL(n,A) \to EGL(n,A)$ on $f^*EGL(n,A) \times \{0\}$ and to the composition of the canonical map $\tilde{f}^*EGL(n,A) \to EGL(n,A)$ with the isomorphism g on $f^*EGL(n,A) \times \{1\}$. By 7.22 there is an induced homomorphism of pricipal $GL(n,D)$–bundles $\varphi_*\Phi : \varphi_* f^*EGL(n,A) \times I \to \varphi_* EGL(n,A) \cong (B\varphi_n)^*EGL(n,D)$ and we denote by $h : X \times I \to BGL(n,A)$ the induced smooth map. Next let $u : (B\varphi_n)^*EGL(n,D) \to EGL(n,D)$ be the canonical map and define $\sigma : X \times I \to EGL(n,D)$ as $\sigma(x,t) := (u \circ \varphi_*\Phi \circ \Psi^{-1})(x,t,id)$. Then one easily shows that $\sigma|_{X\times\{0\}} : X \to EGL(n,D)$ and $\sigma|_{X\times\{1\}}$ are the maps constructed from (f,α) and $(\tilde{f},\tilde{\alpha})$ as in 7.23. Since $EGL(n,D)$ is contractible the map σ is smoothly homotopic to the constant map $(1,id,\dots)$, so there is a smooth map $H : X \times I \times I \to EGL(n,D)$ such that $H(x,t,0) = (1,id,\dots)$ and $H(x,t,1) = \sigma(x,t)$. Consequently the map $X \times I \to C^\infty(I,BGL(n,D))$ associated via cartesian closedness to $p \circ H$ has values in the subspace $\mathcal{P}(BGL(n,D))$ and by construction together with the map $h : X \times I \to BGL(n,A)$ it induces a smooth homotopy $X \times I \to \mathcal{F}_n(\varphi)$ between the map constructed from (f,α) and $(\tilde{f},\tilde{\alpha})$. $\square$

7.25. Proposition. *For any $n > 0$ the map $u_n : \Phi_n(\mathcal{E}_\varphi(X)) \to [X, \mathcal{F}_n(\varphi)]$ is bijective.*

Proof. **Step 1:** First we show that u_n is surjective. So let $g : X \to \mathcal{F}_n(\varphi)$ be a smooth map. Let us denote by $j : \mathcal{F}_n(\varphi) \to BGL(n,A)$ and $k : \mathcal{F}_n(\varphi) \to \mathcal{P}(BGL(n,D))$ the canonical mappings. Next let $\gamma \in C^\infty(\mathbb{R},I)$ be a smooth increasing map such that $\gamma(t) = 0$ for all $t \leq \varepsilon$ and $\gamma(t) = 1$ for all $t \geq 1-\varepsilon$, where ε is some small positive number. Clearly the map

$\psi : X \times I \to BGL(n, D)$ defined by $\psi(x, t) := k(g(x))(\gamma(t))$ is smooth. On the other hand let P be the smooth principal $GL(n, A)$–bundle $(j \circ g)^* EGL(n, A)$ over X. By construction $\psi(x, 1) = (B\varphi_n \circ j \circ g)(x)$ and thus we can identify $\varphi_* P \cong (B\varphi_n \circ j \circ g)^* EGL(n, D)$ with a subbundle of $\psi^* EGL(n, D)$. Moreover by construction ψ satisfies the conditions of 2.11 and thus there is an isomorphism $\Phi : \varphi_* P \times I \to \psi^* EGL(n, D)$ of principal bundles over $X \times I$ which restricts to the natural inclusion on $\varphi_* P \times \{1\}$. Next consider the natural map $p^*\psi : \psi^* EGL(n, D) \to EGL(n, D)$. Since by definition we have $\psi(x, 0) = *$, the base point in $BGL(n, D)$, for every $x \in X$ we see that $(p^*\psi \circ \Phi)|_{\varphi_* P \times \{0\}}$ has values in the fiber over $*$ which is canonically diffeomorphic to $GL(n, D)$. Thus together with the bundle projection $\varphi_* P \to X$ this map induces an isomorphism $\alpha : \varphi_* P \to X \times GL(n, D)$ of principal bundles. Now define $H : X \times I \to EGL(n, D)$ by $H(x, t) := (p^*\psi \circ \Phi)(\alpha^{-1}(x, id), t)$. Then by definition of α we have $H(x, 0) = (1, id, \dots)$ while by construction of Φ the map $H|_{X \times \{1\}} : X \to EGL(n, D)$ is the map associated to α as in 7.23. Thus H can be used for the construction of the map associated to the pair (P, α) which shows that the homotopy class associated to this pair contains the map $X \to \mathcal{F}_n(\varphi)$ induced by $j \circ g : X \to BGL(n, A)$ and the map $X \to \mathcal{P}(BGL(n, D))$ associated via cartesian closedness to $\psi : X \times I \to BGL(n, D)$. Clearly this map is homotopic to g and thus u_n is surjective.

Step 2: Let $f : X \to BGL(n, A)$ be a smooth map, $\alpha : \varphi_* f^* EGL(n, A) \to X \times GL(n, D)$ an isomorphism, $g : X \to \mathcal{F}_n(\varphi)$ the smooth map constructed from the pair (f, α) as in 7.23 and $\tilde{\alpha} : \varphi_* f^* EGL(n, A) \to X \times GL(n, D)$ the isomorphism constructed from g as in step 1. Then as in the proof of 7.8 one shows the the pairs corresponding to (f, α) and $(f, \tilde{\alpha})$ are equivalent.

Step 3: To complete the proof it remains to show that two pairs which give rise to homotopic maps are equivalent. So let $H : X \times I \to \mathcal{F}_n(\varphi)$ be a smooth map such that $H|_{X \times \{0\}}$ is a map associated to a pair (f, α) and $H|_{X \times \{1\}}$ is associated to a pair (g, β). Without loss of generality we may assume that H can be constantly extended to $X \times \mathbb{R}$. Then let P be the smooth principal $GL(n, A)$–bundle $(j \circ H)^* EGL(n, A)$ over $X \times I$. By 2.11 there is an isomorphism $\Phi : f^* EGL(n, A) \times I \to P$ which restricts to the natural inclusion on $f^* EGL(n, A) \times \{0\}$ and we can define an isomorphism $\omega : f^* EGL(n, A) \to g^* EGL(n, A)$ of principal bundles by $\omega := \Phi|_{f^* EGL(n, A) \times \{1\}}$. Next let $\gamma \in C^\infty(\mathbb{R}, I)$ be the map used in step 1 and define $\psi : X \times I \times I \to BGL(n, D)$ by $\psi(x, t, s) := k(H(x, t))(\gamma(s))$. Clearly this defines a smooth map. By construction we have $\psi(x, t, 1) = (B\varphi_n \circ j \circ H)(x, t)$ and thus we may identify $\varphi_* P \cong (B\varphi_n \circ j \circ H)^* EGL(n, D)$ with a subbundle of $\psi^* EGL(n, D)$. Moreover the map ψ satifies the conditions of 2.11 and thus there is an isomorphism $\Psi : \varphi_* P \times I \to \psi^* EGL(n, D)$ which restricts to the natural inclusion on $\varphi_* P \times \{1\}$.

Next by 7.22 the isomorphism Φ induces an isomorphism $\varphi_* \Phi : \varphi_* f^* EGL(n, A) \times I \to \varphi_* P$ and we consider the composition $p^*\psi \circ \Psi|_{\varphi_* P \times \{0\}} \circ \varphi_* \Phi : \varphi_* f^* EGL(n, A) \times I \to EGL(n, D)$. Since $\psi(x, t, 0)$ is the base point of $BGL(n, D)$ for all x and t this map has values in the fiber over the base point which is canonically diffeomorphic to $GL(n, D)$ and thus together with the bundle projection of $\varphi_* f^* EGL(n, A) \times I$ it induces an isomorphism of smooth principal bundles $\varphi_* f^* EGL(n, A) \times I \to X \times I \times GL(n, D)$. Now one easily sees that restricted to $\varphi_* f^* EGL(n, A) \times \{0\}$ this isomorphism equals $\tilde{\alpha}$ while restricted to $\varphi_* f^* EGL(n, A) \times \{1\}$ it equals $\tilde{\beta} \circ \varphi_* \omega$, where $\tilde{\alpha} : \varphi_* f^* EGL(n, A) \to X \times GL(n, D)$ and $\tilde{\beta} : \varphi_* g^* EGL(n, A) \to X \times GL(n, D)$ are the isomorphisms constructed from $H|_{X \times \{0\}}$ and $H|_{X \times \{1\}}$ as in step 1. Thus we see that the pairs $(f, \tilde{\alpha})$ and $(g, \tilde{\beta})$ are equivalent and in view of step 2 this completes the proof. $\square$

7.26. Recall from 6.11 that there are smooth maps $BGL(n, A) \to BGL(n+1, A)$ induced by the natural inclusion $GL(n, A) \to GL(n+1, A)$. For the algebra D the corresponding maps also induce smooth maps $\mathcal{P}(BGL(n, D)) \to \mathcal{P}(BGL(n+1, D))$. Composing with the natural maps from $\mathcal{F}_n(\varphi)$ we get smooth maps $\{_n(\varphi) \to BGL(n+1, A)$ and $\mathcal{F}_n(\varphi) \to \mathcal{P}(BGL(n+1, D))$ and one immediately verifies that these maps induce smooth maps $\mathcal{F}_n(\varphi) \to \mathcal{F}_{n+1}(\varphi)$

for any $n \in \mathbb{N}$. Consequently for any base space X there are induced (set) maps $[X, \mathcal{F}_n(\varphi)] \to [X, \mathcal{F}_{n+1}(\varphi)]$ and we define $[X, \mathcal{F}(\varphi)]$ to be the direct limit of the so obtained inductive system. Next one immediately verifies that for any $n \in \mathbb{N}$ the diagram

$$\begin{tikzcd} \Phi_n(\mathcal{E}_\varphi(X)) \arrow[r, "u_n"] \arrow[d] & {[X, \mathcal{F}_n(\varphi)]} \arrow[d] \\ \Phi_{n+1}(\mathcal{E}_\varphi(X)) \arrow[r, "u_{n+1}"] & {[X, \mathcal{F}_{n+1}(\varphi)]} \end{tikzcd}$$

is commutative and consequently the maps u_n induce a bijection $u : \Phi(\mathcal{E}_\varphi(X)) \to [X, \mathcal{F}(\varphi)]$ and we define the structure of an abelian group on $[X, \mathcal{F}(\varphi)]$ by requiering that u is a group homomorphism. Thus from 7.19 we conclude that $K'(\mathcal{E}_\varphi(X)) \cong [X, \mathcal{F}(\varphi)]$.

7.27. Let us next study the functorial properties of the relative K–group associated to a bounded algebra homomorphism. First let X and Y be base spaces and let $f : X \to Y$ be a smooth map. Then we get induced set maps $f^* : [Y, \mathcal{F}_n(\varphi)] \to [X, \mathcal{F}_n(\varphi)]$. From the proof of 7.25 one concludes that the map $u_n^{-1} \circ f^* \circ u_n : \Phi_n(\mathcal{E}_\varphi(Y)) \to \Phi_n(\mathcal{E}_\varphi)(X)$ is given by mapping the class of a pair (E, α) to the class of $(f^*E, f^*\alpha)$, and from this description it is clear that there is an induced map $[Y, \mathcal{F}(\varphi)] \to [X, \mathcal{F}(\varphi)]$ which is a group homomorphism. Next f induces a group homomorphism $f^* : H^0(Y, K_0(A)) \to H^0(X, K_0(A))$ and $H^0(X, K_0(\varphi)) \circ f^* = f^* \circ H^0(Y, K_0(\varphi))$, so f^* restricts to a homomorphism $\operatorname{Ker}(H^0(Y, K_0(\varphi))) \to \operatorname{Ker}(H^0(X, K_0(\varphi)))$. In view of 7.17 this shows that f induces a group homomorphism $K(\mathcal{E}_\varphi(Y)) \to K(\mathcal{E}_\varphi(X))$.

On the other hand assume that $A, D, \tilde{A}$ and $\tilde{D}$ are convenient algebras and that we have a commutative diagram of bounded algebra homomorphisms:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\psi"'] & D \arrow[d, "\bar{\psi}"] \\ \tilde{A} \arrow[r, "\tilde{\varphi}"] & \tilde{D} \end{tikzcd} \tag{1}$$

Passing to the groups $GL(n, \quad)$ and further to their classifying spaces we get a commutative diagram

$$\begin{tikzcd} BGL(n, A) \arrow[r, "B\varphi_n"] \arrow[d, "B\psi_n"'] & BGL(n, D) \arrow[d, "B\bar{\psi}_n"] \\ BGL(n, \tilde{A}) \arrow[r, "B\tilde{\varphi}_n"] & BGL(n, \tilde{D}) \end{tikzcd}$$

Moreover using the map $\mathcal{P}(B\bar{\psi}_n) : \mathcal{P}(BGL(n, D)) \to \mathcal{P}(BGL(n, \tilde{D}))$ we get an induced map $\mathcal{F}_n(\varphi) \to \mathcal{F}_n(\tilde{\varphi})$ and thus an induced map $(\psi, \bar{\psi})_* : [X, \mathcal{F}_n(\varphi)] \to [X, \mathcal{F}_n(\tilde{\varphi})]$. Again from the proof of 7.25 One concludes that the map $u_n^{-1} \circ (\psi, \bar{\psi})_* \circ u_n : \Phi_n(\mathcal{E}_\varphi(X)) \to \Phi_n(\mathcal{E}_{\tilde{\varphi}}(X))$ is induced by mapping the class of a pair (E, α) to the class of $(\psi_* E, \bar{\psi}_* \alpha)$ and from this description it is clear that there is an induced map $[X, \mathcal{F}(\varphi)] \to [X, \mathcal{F}(\tilde{\varphi})]$ which is a group homomorphism. Next the commutative diagram (1) leads to a commutative diagram

$$\begin{tikzcd} H^0(X, K_0(A)) \arrow[r, "H^0(X, K_0(\varphi))"] \arrow[d, "H^0(X, K_0(\psi))"'] & H^0(X, K_0(D)) \arrow[d, "H^0(X, K_0(\bar{\psi}))"] \\ H^0(X, K_0(\tilde{A})) \arrow[r, "H^0(X, K_0(\tilde{\varphi}))"] & H^0(X, K_0(\tilde{D})) \end{tikzcd}$$

and this implies that $H^0(X, K_0(\psi))$ induces a group homomorphism

$$\operatorname{Ker}(H^0(X, K_0(\varphi))) \to \operatorname{Ker}(H^0(X, K_0(\tilde{\varphi}))).$$

Using 7.17 we see that the diagram (1) induces group homomorphisms $K(\mathcal{E}_\varphi(X)) \to K(\mathcal{E}_{\tilde{\varphi}}(X))$ for any base space X.

7.28. Our next task is to construct homomorphisms between absolute and relative K–groups. The first part of this is very easy: For any n there is the natural map $\mathcal{F}_n(\varphi) \to BGL(n,A)$ and thus an induced map $[X,\mathcal{F}_n(\varphi)] \to [X,BGL(n,A)]$. On the level of bundles this map is given by mapping the class of a pair (E,α) to the isomorphism class of E and thus there is an induced map $[X,\mathcal{F}(\varphi)] \to [X,BGL(A)]$ which is a group homomorphism. Moreover the inclusion of $\mathrm{Ker}(H^0(X,K_0(\varphi)))$ into $H^0(X,K_0(A))$ is a group homomorphism and thus in view of 7.17, 6.11 and6.8 we get a group homomorphism

$$\begin{aligned}K(\mathcal{E}_\varphi(X)) \cong [X,\mathcal{F}(\varphi)] \oplus \mathrm{Ker}(H^0(X,K_0(\varphi))) \to \\ \to [X,BGL(A)] \oplus H^0(X,K_0(A)) \cong K^0_A(X)\end{aligned}$$

On the other hand we have to construct a homomorphism $K_D^{-1}(X) \to K(\mathcal{E}_\varphi(X))$. For this we proceed as follows: Consider a base point preserving smooth map $f : S'(X^+) \to BGL(k,D)$ for some $k \in \mathbb{N}$. Then there is the canonically associated smooth map $\check{f} : X^+ \to \Omega(BGL(k,D))$, the loop space of $BGL(k,D)$ which is also smooth and base point preserving (c.f. 3.50). The space $\Omega(BGL(k,D))$ is just the fiber over the base point of the path fibration $\mathcal{P}(BGL(k,D)) \to BGL(k,D)$, so we can view $\check{f}$ as a map to $\mathcal{P}(BGL(k,D))$ and together with the constant map to the base point of $BGL(k,A)$ it induces a smooth base point preserving map $X^+ \to \mathcal{F}_n(\varphi)$. Now one easily checks that this construction defines a map $[S'(X^+),BGL(k,D)]_0 \to [X^+,\mathcal{F}_n(\varphi)]_0$.

Next recall the constructions of the connecting maps $[S'(X^+),BGL(k,D)]_0 \to [S'(X^+),BGL(k+1,D)]_0$ and $[X^+,\mathcal{F}_n(\varphi)]_0 \to [X^+,\mathcal{F}_{n+1}(\varphi)]_0$. In the first case this map is given by mapping the homotopy class of f to the class of $Bi_k^D \circ f$, where $Bi_k^D : BGL(k,D) \to BGL(k+1,D)$ is the map induced by the natural inclusion $i_k^D : GL(k,D) \to GL(k+1,D)$, while the second map is induced by composition with the map $\mathcal{F}_n(\varphi) \to \mathcal{F}_{n+1}(\varphi)$ which is induced by $Bi_k^A : BGL(k,A) \to BGL(k+1,A)$ and $\mathcal{P}(Bi_k^D) : \mathcal{P}(BGL(k,D)) \to \mathcal{P}(BGL(k+1,D))$. Next one immediately verifies that $(Bi_k^D \circ f)^\vee = \Omega(Bi_k^D) \circ \check{f}$ and since $\Omega(Bi_k^D)$ is just the restriction to $\Omega(BGL(k,D))$ of the map $\mathcal{P}(Bi_k^D)$ this easily implies that the maps constructed above induce a map $[S'(X^+),BGL(D)]_0 \to [X^+,\mathcal{F}(\varphi)]_0$. Now from 6.11 and 6.19 we see that $[S'(X^+),BGL(D)]_0 \cong \tilde{K}_D(S'(X^+))$ and there is an obvious isomorphism $[X^+,\mathcal{F}(\varphi)]_0 = (\mathcal{E}_\varphi(X))$. On the other hand using 6.15 we see that there is a natural group isomorphism $K_D^{-1}(X) = \tilde{K}_D(S(X^+)) \to \tilde{K}_D(S'(X^+))$. Thus it remains to show that the map $\tilde{K}_D(S'(X^+)) \to K(\mathcal{E}_\varphi(X))$ constructed above is a group homomorphism. In order to do this we have to express the map in the language of bundles where we have a better description of the group structures.

By 6.10 and 6.19 every element of $\tilde{K}_D(S'(X^+))$ can be written as $[f^*EGL(n,D)] - [S'(X^+) \times D^n]$ where $f : S'(X^+) \to BGL(n,D)$ is a base point preserving smooth map. From the proof of 7.25 and from 7.18 we see that this element is mapped to the class of a tripel $(X \times A^n, X \times A^n, \psi)$, where $\psi : X \times D^n \to X \times D^n$ is an isomorphism which can be described as follows: Let $\gamma \in C^\infty(\mathbb{R},I)$ be an increasing function such that $\gamma(t) = 0$ for all $t \le \varepsilon$ and $\gamma(t) = 1$ for all $t \ge 1-\varepsilon$ where ε is some small positive number and define $\tilde{\pi} : X^+ \times I \to S'(X^+)$ by $\tilde{\pi}(x,t) := \pi(x,\gamma(t))$ where $\pi : X^+ \times I \to S'X^+$ is the canonical map. Now consider the bundle $(f\circ\tilde{\pi})^*EGL(n,D)$. Since f is base point preserving the map $f\circ\tilde{\pi}$ maps the subspaces $X^+ \times \{0\}$ and $X^+ \times \{1\}$ to the base point of $BGL(n,D)$ so there are canonical trivializations of the restrictions of $(f\circ\tilde{\pi})^*EGL(n,D)$ to these subspaces. By 2.11 we thus get an isomorphism $\Psi : X \times I \times D^n \to (f\circ\tilde{\pi})^*EGL(n,D)$ which restricts to the natural inclusion on $X \times \{0\} \times D^n$ and ψ is just the restriction of such an isomorphism to $X \times \{1\} \times D^n$.

Now having given two elements $[f^*EGL(n,D)] - [S'(X^+) \times D^n]$ and $[\tilde{f}^*EGL(m,D)] - [S'(X^+) \times D^m]$ and corresponding isomorphisms $\Psi : X \times I \times D^n \to (f\circ\tilde{\pi})^*EGL(n,D)$ and

$\tilde{\Psi} : X \times I \times D^m \to (\tilde{f} \circ \tilde{\pi})^* EGL(m, D)$ we see that

$$\Psi \oplus \tilde{\Phi} : X \times I \times D^{n+m} \to (f \circ \tilde{\pi})^* EGL(n, D) \oplus (\tilde{f} \circ \tilde{\pi})^* EGL(m, D)$$

restricts to the identity over $X \times \{0\} \times D^{n+m}$ and from this we conclude that the sum of the two elements is mapped to the class of the tripel $(X \times A^{n+m}, X \times A^{n+m}, \psi \oplus \tilde{\psi})$ and thus the map is indeed a group homomorphism.

7.29. Definition. Let $\varphi : A \to D$ be a bounded algebra homomorphism and let X be a base space. For any $n \in \mathbb{N}$ the unique smooth map $S^n(X^+) \to pt$ induces by 7.28 above a group homomorphism $K(\mathcal{E}_\varphi(pt)) \to K(\mathcal{E}_\varphi(S^n(X^+)))$ which is obviously injective and we define the group $K^{-n}(\mathcal{E}_\varphi(X))$ to be the cokernel of this homomorphism. Thus for any n there is a short exact sequence

$$0 \to K(\mathcal{E}_\varphi(pt)) \to K(\mathcal{E}_\varphi(S^n(X^+))) \to K^{-n}(\mathcal{E}_\varphi(X)) \to 0$$

This sequence even splits canonically via the homomorphism induced by the inclusion of the base point (of X^+) into $S^n(X^+)$.

7.30.. Obviously the higher relative K–groups have functorial properties: If $f : X \to Y$ is a smooth map between base spaces then there is the induced map $S^n(f^+) : S^n(X^+) \to S^n(Y^+)$ which by 7.28 induces a group homomorphism $K(\mathcal{E}_\varphi(S^n(Y^+))) \to K(\mathcal{E}_\varphi(S^n(X^+)))$ and clearly we have a commutative diagram

$$\begin{tikzcd} K(\mathcal{E}_\varphi(pt)) \arrow[r, equal] \arrow[d] & K(\mathcal{E}_\varphi(pt)) \arrow[d] \\ K(\mathcal{E}_\varphi(S^n(Y^+))) \arrow[r] & K(\mathcal{E}_\varphi(S^n(X^+))) \end{tikzcd}$$

and thus an induced homomorphism $K^{-n}(\mathcal{E}_\varphi(Y)) \to K^{-n}(\mathcal{E}_\varphi(X))$.

Similarly one shows that a commutative diagram of bounded algebra homomorphisms

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\psi"'] & D \arrow[d, "\bar{\psi}"] \\ \tilde{A} \arrow[r, "\tilde{\varphi}"] & \tilde{D} \end{tikzcd}$$

like in 7.26 induces group homomorphisms $K^{-n}(\mathcal{E}_\varphi(X)) \to K^{-n}(\mathcal{E}_{\tilde{\varphi}}(X))$ for any base space X and any $n \in \mathbb{N}$.

7.31. Next we construct homomorphisms between higher absolute and relative K–groups. First we need a homomorphism $K^{-n}(\mathcal{E}_\varphi(X)) \to K_A^{-n}(X)$. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & K(\mathcal{E}_\varphi(pt)) \arrow[r] \arrow[d] & K(\mathcal{E}_\varphi(S^n(X^+))) \arrow[r] \arrow[d] & K^{-n}(\mathcal{E}_\varphi(X)) \arrow[r] & 0 \\ 0 \arrow[r] & K_0(A) \arrow[r] & K_A(S(X^+)) \arrow[r] & K_A^{-n}(X) \arrow[r] & 0 \end{tikzcd}$$

The rows in this diagram are exact by definition and the two vertical homomorphisms were constructed in 7.28 and one immediately concludes that the diagram is commutative. Thus there is a unique groups homomorphism $K^{-n}(\mathcal{E}_\varphi(X)) \to K_A^{-n}(X)$ fitting into the diagram. On the other hand we have to construct homomorphisms $K_D^{-n-1}(X) \to K^{-n}(\mathcal{E}_\varphi(X))$. This needs one more result:

7.32. Lemma. *Let Y be a base space, A a convenient algebra. Then there is an exact sequence of abelian groups and group homomorphisms*

$$K_1(A) \to \tilde{K}_A(S(Y^+)) \to \tilde{K}_A(S(Y)) \to 0$$

which is natural in Y.

Proof. Let $i : Y \to Y^+$ be the natural mapping, which is obviously a smooth cofibration and apply the Puppe sequence 3.43 for this map to $BGL(n,A)$. Then the third up to the sixth term of this sequence read as:

$$[S(C_i), BGL(n,A)] \to [S(Y^+), BGL(n,A)] \to \\ \to [SY, BGL(n,A)] \to [C_i, BGL(n,A)].$$

Since i is a smooth cofibration it follows from 3.14 the the homotopy cofiber C_i of i is smoothly homotopy equivalent to the cofiber of i which is just pt^+, the two point space. Thus we get an exact sequence if in the sequence above we replace C_i by pt^+. Moreover since $BGL(n,A)$ is smoothly path connected it follows that $[pt^+, BGL(n,A)] = 0$, and so we get an exact sequence

$$[S(pt^+), BGL(n,A)] \to [S(Y^+), BGL(n,A)] \to [SY, BGL(n,A)] \to 0.$$

Clearly this sequence is natural for maps in the second variable, so we can pass to the direct limit to get the same sequence with $BGL(n,A)$ replaced by $BGL(A)$. Then in this sequence all sets are abelian groups, namely since suspensions are always smoothly path connected the $\tilde{K}_A(\quad)$ groups of the spaces, and all maps are group homomorphisms since they are induced by smooth maps between the spaces. So we only have to use the identification $\tilde{K}_A(S(pt^+)) = K_A^{-1}(pt) = K_1(A)$ to get the claimed exact sequence. The naturality of the sequence in Y is obvious from the construction. □

7.33. Now consider the diagram

$$\begin{tikzcd}
 & K_1(D) \arrow[r] \arrow[d] & \tilde{K}_A(S(S^n(X^+)^+)) \arrow[r] \arrow[d] & K_D^{-n-1}(X) \arrow[r] & 0 \\
0 \arrow[r] & K(\mathcal{E}_\varphi(pt)) \arrow[r] & K(\mathcal{E}_\varphi(S^n(X^+))) \arrow[r] & K^{-n}(\mathcal{E}_\varphi(X)) \arrow[r] & 0
\end{tikzcd}$$

The top row in this diagram is exact by 7.32 applied to D and $Y = S^n(X^+)$, while the bottom row is exact by definition. The two vertical homomorphisms were constructed in 7.28 and one immediately verifies that the diagram is commutative. Thus there is a unique homomorphism $K_D^{-n-1}(X) \to K^{-n}(\mathcal{E}_\varphi(X))$ fitting into the diagram.

7.34. Proposition. *Let $\varphi : A \to D$ be a bounded homomorphism between convenient algebras. Then for any base space X there is an exact sequence of abelian groups and group homomorphisms*

$$\tilde{K}_A(S(X^+)) \to \tilde{K}_D(S(X^+)) \to K(\mathcal{E}_\varphi(X)) \to K_A(X) \to K_D(X)$$

which is natural in X.

Proof. For $n \in \mathbb{N}$ consider the last five terms of the Puppe sequence 3.53 for X associated to the smooth map $B\varphi_n : BGL(n,A) \to BGL(n,D)$ induced by φ. These read as

$$[X, \Omega(BGL(n,A))] \to [X, \Omega(BGL(n,D))] \to [X, \mathcal{F}_n(\varphi)] \to \\ \to [X, BGL(n,A)] \to [X, BGL(n,D)].$$

Since all maps in this sequence are induced by base point preserving smooth maps this sequence remains exact if we replace X by X^+ and free homotopy classes by pointed homotopy classes in the first two terms. Using the canonical isomorphism between $[\ \ ,\Omega(\ \ )]_0$ and $[S'(\ \ ),\ \ ]$ (c.f. 3.50) we get an exact sequence of pointed sets

$$\begin{aligned}[S'(X^+), BGL(n,A)]_0 \to [S'(X^+), BGL(n,D)]_0 \to [X, \mathcal{F}_n(\varphi)] \to \\ \to [X, BGL(n,A)] \to [X, BGL(n,D)].\end{aligned}$$

Next consider the diagram

$$\begin{tikzcd}
{[S'(X^+), BGL(n,A)]_0} \arrow[r] \arrow[d] & {[S'(X^+), BGL(n+1,A)]_0} \arrow[d] \\
{[S'(X^+), BGL(n,D)]_0} \arrow[r] \arrow[d] & {[S'(X^+), BGL(n+1,D)]_0} \arrow[d] \\
{[X, \mathcal{F}_n(\varphi)]} \arrow[r] \arrow[d] & {[X, \mathcal{F}_{n+1}(\varphi)]} \arrow[d] \\
{[X, BGL(n,A)]} \arrow[r] \arrow[d] & {[X, BGL(n+1,A)]} \arrow[d] \\
{[X, BGL(n,D)]} \arrow[r] & {[X, BGL(n+1,D)]}
\end{tikzcd}$$

We claim that this is commutative. For the top and the bottom square this is quite obvious while for the second sqaure from below it follows immediately from the definition of the connecting map $\mathcal{F}_n(\varphi) \to \mathcal{F}_{n+1}(\varphi)$. So let us consider the second square from above. The way $[S'(X^+), BGL(n,D)]_0 \to [X, \mathcal{F}_n(\varphi)] \to [X, \mathcal{F}_{n+1}(\varphi)]$ can be described as follows: Given $f : S'(X^+) \to BGL(n,D)$ take the associated map $\check{f} : X^+ \to \Omega(BGL(n,D))$, restrict it to X and compose it with the natural maps $\Omega(BGL(n,D)) \to \mathcal{F}_n(\varphi) \to \mathcal{F}_{n+1}(\varphi)$. But from the construction of the map $\mathcal{F}_n(\varphi) \to \mathcal{F}_{n+1}(\varphi)$ one immediately sees that this last map is the same as the natural map $\Omega(BGL(n,D)) \to \Omega(BGL(n+1,D)) \to \mathcal{F}_{n+1}(\varphi)$. This and the fact that $(g \circ f)^\vee = \Omega(g) \circ \check{f}$ implies that the square uncer consideration is commutative even on the level of maps and not only on the level of homotopy classes.

Now we can pass to the direct limit to get an exact sequence

$$\begin{aligned}[S'(X^+), BGL(A)]_0 \to [S'(X^+), BGL(D)]_0 \to \\ \to [X, \mathcal{F}(\varphi)] \to [X, BGL(A)] \to [X, BGL(D)].\end{aligned}$$

But now all sets are abelian groups and all maps are group homomorphisms (c.f. 6.12 and 7.28). So finally we only have to add the obvious exact sequence

$$0 \to \mathrm{Ker}(H^0(X, K_0(\varphi))) \to H^0(X, K_0(A)) \xrightarrow{H^0(X,K_0(\varphi))} H^0(X, K_0(D))$$

to the last three terms and use the natural isomorphism $[S'(X^+), BGL(A)]_0 \cong \tilde{K}_A(S(X^+))$ discussed in 7.28 for A and D to get the claimed exact sequence. Finally the naturality in X is obvious from the construction. $\square$

7.35. Theorem (The long exact sequence of a bounded algebra homomorphism). *Let $\varphi : A \to D$ be a bounded homomorphism between convenient algebras. Then for any base space X there is a long exact sequence of abelian groups and group homomorphisms*

$$\begin{aligned} \ldots \to K_D^{-n-1}(X) \to K^{-n}(\mathcal{E}_\varphi(X)) \to & \\ \to K_A^{-n}(X) \xrightarrow{K_\varphi^{-n}(X)} K_D^{-n}(X) \to \ldots & \\ \to K_D^{-1}(X) \to K(\mathcal{E}_\varphi(X)) \to K_A^0(X) \xrightarrow{K_\varphi^{0}(X)} K_D^0(X) & \end{aligned}$$

which is natural in X.

Proof. We show that for any $n \geq 0$ there is an exact sequence

$$K_A^{-n-1}(X) \to K_D^{-n-1}(X) \to K^{-n}(\mathcal{E}_\varphi(X)) \to K_A^{-n}(X) \to K_D^{-n}(X).$$

For $n = 0$ this is just the sequence constructed in 7.34. For $n > 0$ consider the following diagram:

$$\begin{tikzcd}
 & K_1(A) \arrow[r] \arrow[d] & \tilde{K}_A(S(S^n(X^+)^+)) \arrow[r] \arrow[d] & K_A^{-n-1}(X) \arrow[r] \arrow[d] & 0 \\
 & K_1(D) \arrow[r] \arrow[d] & \tilde{K}_D(S(S^n(X^+)^+)) \arrow[r] \arrow[d] & K_D^{-n-1}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K(\mathcal{E}_\varphi(pt)) \arrow[r] \arrow[d] & K(\mathcal{E}_\varphi(S^n(X^+))) \arrow[r] \arrow[d] & K^{-n}(\mathcal{E}_\varphi(X)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_0(A) \arrow[r] \arrow[d] & K_A(S^n(X^+)) \arrow[r] \arrow[d] & K_A^{-n}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_0(D) \arrow[r] & K_D(S^n(X^+)) \arrow[r] & K_D^{-n}(X) \arrow[r] & 0
\end{tikzcd}$$

By 7.32 the two top rows are exact and the two top squares are commutative. All other rows are exact by definition and the rest of the diagram is commutative by definition. Next the leftmost colum is just the exact sequence 7.34 for a single point, while the middle colum is the exact sequence 7.34 for the space $S^n(X^+)$. Thus from a standard diagram chase one concludes that the rightmost colum is exact at $K_D^{-n-1}(X)$ and at $K^{-n}(\mathcal{E}_\varphi(X))$. Next note that the inclusion of the base point (of X^+) into $S^n(X^+)$ induces homomorphisms $K_A(S(X^+)) \to K_0(A)$ and $K_D(S(X^+)) \to K_0(D)$ which are right inverse to the maps ocurring in the diagram above and clearly the diagram

$$\begin{tikzcd}
K_A(S(X^+)) \arrow[r] \arrow[d] & K_0(A) \arrow[d] \\
K_D(S(X^+)) \arrow[r] & K_0(D)
\end{tikzcd}$$

commutes. Using this fact one shows again by a diagram chase that the rightmost colum in the above diagram is also exact at $K_A^{-n}(X)$. Finally the naturality in X is obvious from the construction. $\square$

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