Some (locally) transitive actions on the Möbius sphere related to conformal holonomy

Jesse Alt
(joint w/ A. Di Scala, T. Leistner, F. Leitner)

University of the Witwatersrand

July 2011

Conformal holonomy

For a conformal semi-Riemannian manifold $(M, [g])$ of signature (p, q), we have the canonical Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P), where $G = O(p+1, q+1)$ and $P \cong CO(p, q) \ltimes (\mathbb{R}^{p,q})^*$ is the stabilizer in G of an isotropic ray $\mathbb{R}_+ v \subset \mathbb{R}^{p+1,q+1}$.

To $(\mathcal{G}, \omega)$ one can associate $(\widehat{\mathcal{G}}, \widehat{\omega})$, a $O(p+1, q+1)$-bundle with principal connection. The *conformal holonomy* $\mathrm{Hol}(M, [g])$ (either up to conjugation or with respect to base-points) is usually defined as the holonomy of $(\widehat{\mathcal{G}}, \widehat{\omega})$.

In particular, we have $\mathrm{Hol}(M, [g]) \subset O(p+1, q+1)$.

Decomposable conformal holonomy

Theorem (Leitner, Armstrong) If $(M, [g])$ has *decomposable holonomy*, i.e. if there is a non-trivial decomposition $\mathbb{R}^{p+1,q+1} = V \oplus W$ into $\mathrm{Hol}(M, [g])$-invariant, non-degenerate subspaces (of dimensions $r+1$, resp. $s+1$), then, on an open dense subset of M, there exists a metric $g_0 \in [g]$ which is locally isometric to a product of Einstein metrics of dimensions r, resp. s.

Remark The "singular set" in the above result (where g_0 is not defined) is a basic feature of holonomy reduction for Cartan connections, as explained in recent work of Čap/Gover/Hammerl.

Remark A global classification of decomposable conformal holonomy in Riemannian signature (classifying the possible singular sets) has also been obtained recently by Leitner/Armstrong.

Restrictions on irreducible conformal holonomy

Theorem (Di Scala/Olmos) The only irreducible connected subgroup of $O(n+1,1)$ is $SO_0(n+1,1)$. (Note: "irreducible subgroup" := irreducibly acting under the standard rep'n.)

Corollary: In Riemannian signature the only irreducible conformal holonomy is generic.

Theorem (Di Scala/Leistner) The only connected irreducible subgroups of $O(n,2)$ are:
$SO_0(n,2)$ for all n;
$S^1 \cdot SO(m,1)$, $U(m,1)$ and $SU(m,1)$ for even $n = 2m$;
$SO_0(2,1)_i \subset SO(3,2)$ for $n = 3$.

Corollary A: The only connected, irreducible conformal holonomy groups possible in Lorentzian signature are: $SO_0(n,2), SU(m,1)$ and $SO_0(2,1)_i$.

Restrictions on irreducible conformal holonomy

Theorem (J.A.) If $H \subset O(p+1, q+1)$ is a semi-simple, connected, irreducible subgroup which acts transitively on the Möbius sphere $S^{p,q}$ (or its double cover $S^p \times S^q$), then H is one of the following:
(i) $SO_0(p+1, q+1)$ for all p, q;
(ii) $SU(r+1, s+1)$ for $p = 2r+1, q = 2s+1$;
(iii) $Sp(r+1, s+1)$ for $p = 4r+3, q = 4s+3$;
(iv) $Sp(1) \cdot Sp(r+1, s+1)$ for $p = 4r+3, q = 4s+3$;
(v) $Spin_0(1,8) \subset SO(8,8)$ for $p = q = 7$;
(vi) $Spin_0(3,4) \subset SO(4,4)$ for $p = q = 3$;
(vii) $G_{2,2} \subset SO(4,3)$ for $p = 3, q = 2$.

Corollary B: These are the only possible connected, irreducible conformal holonomy groups which act transitively on $S^{p,q}$.

Remarks on these results

Corollaries A and B follow by using:

Lemma: If a connected conformal holonomy group $\mathrm{Hol}(M,[g]) \subset O(p+1,q+1)$ acts irreducibly on $\mathbb{R}^{p+1,q+1}$, then it is semi-simple.

This Lemma is a consequence of the fact (proven by Leitner) that $\mathrm{Hol}(M,[g]) \subset U((p+1)/2,(q+1)/2)$ implies $\mathrm{Hol}_0(M,[g]) \subset SU((p+1)/2,(q+1)/2)$.

Ignoring the group $Sp(1) \cdot Sp(r+1,s+1)$, they give two new candidates for irreducible conformal holonomy groups which haven't been studied yet:

(A) $SO_0(2,1)_i$ for conformal Lorentz 3-manifolds;

(B) $Spin_0(1,8)$ for conformal manifolds of signature $(7,7)$.

(Note that in case **B** the group acts transitively on $S^{7,7}$; in case **A** the action on $S^{2,1}$ isn't transitive, only *locally* transitive.)

Statement of the (negative) results

Theorem A (J. A., Di Scala, Leistner) If a conformal Lorentz 3-manifold has $\mathrm{Hol}(M,[g]) \subseteq SO_0(2,1)_i$, then it is conformally flat. In particular, $\mathrm{Hol}(M,[g])$ is discrete, so $SO_0(2,1)_i$ can't occur as a conformal holonomy group.

Theorem B (J. A., Leitner) The inclusion $Spin_0(1,8) \hookrightarrow SO_0(8,8)$ induces a Fefferman-type construction mapping conformal Riemannian spin manifolds of dimension 7 to conformal pseudo-Riemannian manifolds of signature $(7,7)$ (on an S^7 fiber bundle). The Cartan connection given by this construction is never normal unless the Riemannian spin 7-manifold is conformally flat. In particular, the irreducible conformal holonomy $Spin_0(1,8) \subset SO_0(8,8)$ can't be realized by such a construction.

The inclusion $SO(2,1)_i \hookrightarrow SO(3,2)$

Let $\mathbb{R}^{2,1}$ be $\mathbb{R}^3$ with the quadratic form $x_1^2 + x_2^2 - x_3^2$.
Let $\mathcal{S}_0$ denote the trace-free, self-adjoint endomorphisms of $\mathbb{R}^{2,1}$.
$SO(2,1)$ acts faithfully and irreducibly on $\mathcal{S}_0$ by conjugation, and the action preserves the coefficients of characteristic polynomials.
In particular, the coefficient q of the linear term (up to a constant the quadratic trace form), which gives a metric of signature $(3,2)$ on $\mathcal{S}_0$, and we identify $\mathbb{R}^{3,2} = (\mathcal{S}_0, q)$.

This gives the inclusion $SO(2,1)_i \subset SO(3,2)$ of $SO(2,1)$ as an irreducibly acting subgroup.

Orbits in $S^{2,1}$

Let $\mathcal{N} := \{s \in \mathcal{S}_0 : q(s) = 0\}$ be the null-cone and $\pi : \mathbb{R}^5 \to \mathbb{PR}^5$ the standard projection. The Möbius sphere is $S^{2,1} = \pi(\mathcal{N})$ and $SO(2,1)$ acts on points $[s] = \pi(s) \in S^{2,1}$ by conjugation:

$$A([s]) = [AsA^{-1}].$$

This action is not transitive:
Since $SO(2,1)_i$ and $S^{2,1}$ are both 3-dimensional, the stabilizer in $SO(2,1)_i$ of all $[s] \in S^{2,1}$ would have to be 0-dimensional. But one can easily write down $s \in \mathcal{N}$ for which this isn't true.
The orbit structure and stabilizers, under the action of $SO_0(2,1)_i$, are described by the following Proposition.

Orbits in $S^{2,1}$

Proposition: Let $0 \neq s \in \mathcal{N}$ and $[s] \in S^{2,1}$.
1. If $\det(s) \neq 0$, then the orbit of $[s]$ is open in $S^{2,1}$ and its stabilizer is the identity.
2. If $\det(s) = 0$, then: (a) If the 0-eigenspace of s is one-dimensional (i.e. s is 2-step nilpotent), the orbit of $[s]$ in $S^{2,1}$ is 2-dimensional and the stabilizer is given by a one-parameter subgroup of "Lorentz boosts"; (b) If the 0-eigenspace of s is two-dimensional (s is 1-step nilpotent), the orbit of $[s]$ in $S^{2,1}$ is 1-dimensional and the stabilizer is given by the stabilizer of a null line in $\mathbb{R}^{2,1}$.
Moreover, $S^{2,1}$ has precisely one $SO_0(2,1)$-orbit of each of these types. The 2- and 1-dimensional orbits are closed, and the open orbit is dense.
Proof: Uses Jordan normal form of s to analyze stabilizer and conjugacy class.

The tensor Υ

An alternative description of the orbits is given by identifying $SO(2,1)_i$ as the stabilizer in $O(3,2)$ of a (trace-free) symmetric trilinear form $\Upsilon \in \bigodot_0^3(\mathcal{S}_0^*)$: $\Upsilon(s_1, s_2, s_3) := \mathrm{Tr}(s_1 s_2 s_3)$.
The stabilizer of Υ is $SO(2,1)_i$ (compare similar result for $SO(3)_i \subset O(5)$ by Bobienski/Nurowski).
Now $[s] \mapsto \Upsilon(s,s,s)$ defines a conformal 3-density σ on $S^{2,1}$ (i.e. with respect to a metric g in the conformal class of $S^{2,1}$ we have a smooth function $[\sigma]_g$ which rescales by a factor of 3 under conformal change of g). On the zero-set of σ, we can similarly define a one-form τ on $S^{2,1}$ (up to conformal scaling) by $u + \mathbb{R}s \mapsto \Upsilon(s,s,u)$ for $u \in s^\perp$, since $T_s S^{2,1} \cong s^\perp/\mathbb{R}s$.
The vanishing of σ distinguishes between cases (1) and (2) in the Proposition. In case (2), vanishing of τ distinguishes between (a) and (b).

Application to $\mathrm{Hol}(M,[g]) \subset SO_0(2,1)_i$

The main tool needed now to prove Theorem A is the notion of holonomy reduction for Cartan geometries via "curved orbit decomposition" introduced recently by Čap/Gover/Hammerl:
Let $\mathbb{W}$ be a (finite-dimensional) G-module and $H \subset G$ the stabilizer of some $\alpha \in \mathbb{W}$, $H = \mathrm{Stab}_G(\alpha)$. Then we have $\mathcal{O} := G(\alpha) \cong G/H$ as homogeneous G-spaces.
For any other closed subgroup $P \subset G$ we have a decomposition of $\mathcal{O}$ into P-orbits:

$$\mathcal{O} = \bigsqcup_{\overline{\alpha} \in P \backslash \mathcal{O}} P(\overline{\alpha});$$

Each $\overline{\alpha} \in P \backslash \mathcal{O}$ corresponds to an H-orbit in G/P.

Application to $\mathrm{Hol}(M,[g]) \subset SO_0(2,1)_i$

For $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G,P) we have the associated (tractor) bundle $\mathcal{W} := \mathcal{G} \times_P \mathbb{W}$ with covariant derivative $\nabla^{\mathcal{W}}$ induced by ω. There is a $\nabla^{\mathcal{W}}$-parallel section $s \in \Gamma(\mathcal{W})$ of type α if and only if $\mathrm{Hol}(\omega) \subseteq H$.

Theorem (Čap/Gover/Hammerl) There is an induced decomposition

$$M = \bigsqcup_{\overline{\alpha} \in P \backslash \mathcal{O}} M_{\overline{\alpha}}$$

into initial submanifolds $M_{\overline{\alpha}}$ whose local structure is determined by the H-orbit in G/P corresponding to $\overline{\alpha}$. Moreover, each non-empty $M_{\overline{\alpha}} \subset M$ carries a natural Cartan geometry of type $(H, H \cap P)$ which reduces $(\mathcal{G}, \omega)$. In particular, the curvature is given by restricting the curvature of ω to a sub-bundle and it is torsion-free whenever ω is.

Application to $\mathrm{Hol}(M,[g]) \subset SO_0(2,1)_i$

Proof of Theorem A: The Theorem of Čap/Gover/Hammerl, together with the Proposition describing the orbits in $S^{2,1}$, imply that if $\mathrm{Hol}(M,[g]) \subseteq SO_0(2,1)_i$ for a conformal Lorentzian 3-manifold $(M,[g])$ we have:
On an open dense subset $M_0 \subset M$ there is a Cartan geometry $(\mathcal{H} \to M_0, \eta)$ of type $(SO_0(2,1), \mathbf{1})$ which reduces the canonical conformal Cartan geometry $(\mathcal{G}, \omega)$ of $(M,[g])$. In particular, since ω is torsion-free, so is η. But since η is of type $(SO_0(2,1), \mathbf{1})$, this means it has no curvature. It follows, using equivariance, that the curvature of ω vanishes over M_0, and hence everywhere by continuity, so $(M,[g])$ is conformally flat. By the Ambrose-Singer Theorem, $\mathrm{Hol}(M,[g])$ is discrete.

The inclusion $Spin_0(1,8) \hookrightarrow SO_0(8,8)$

Follows Bryant's "Remarks on Spinors in Low Dimensions".
Let $\mathbb{O}$ be the octonions, $<,>$ and $|.|$ the standard inner-product, resp. norm on $\mathbb{O}$. For $(r, \mathbf{x}) \in \mathbb{R} \oplus \mathbb{O}$, define

$$m_{(r,\mathbf{x})} = i \begin{bmatrix} rI_8 & CR_{\mathbf{x}} \\ -CL_{\mathbf{x}} & -rI_8 \end{bmatrix} \in \mathrm{End}_{\mathbb{C}}(\mathbb{C} \otimes \mathbb{O}^2).$$

Then $(m_{(r,\mathbf{x})})^2 = -(r^2 - |\mathbf{x}|^2)I_{16}$, and it follows that this induces a realization of $Cliff_{1,8}$. The connected component $Spin_0(1,8)$ is generated by products $m_{(r,\mathbf{x})}m_{(s,\mathbf{y})}$ with $r^2 - |\mathbf{x}|^2 = s^2 - |\mathbf{y}|^2 = \pm 1$. It follows that $Spin_0(1,8) \subset \mathrm{End}_{\mathbb{R}}(\mathbb{O}^2)$ and a computation shows that the generators preserve the quadratic form Q on $\mathbb{O}^2$:

$$Q(\mathbf{z}, \mathbf{w}) := |\mathbf{z}|^2 - |\mathbf{w}|^2.$$

This realizes $Spin_0(1,8) \subset SO_0(8,8)$ as an irreducibly acting subgroup.

Transitive action on $S^{7,7}$

The induced action of $Spin_0(1,8)$ on $S^7 \times S^7$ (and on its quotient $S^{7,7}$) is transitive:
Calculation shows that $Spin_0(1,8)$ contains the elements

$$\{\begin{bmatrix} L_{\bar{\mathbf{x}}}L_{\mathbf{y}} & 0 \\ 0 & R_{\bar{\mathbf{x}}}R_{\mathbf{y}} \end{bmatrix} : |\mathbf{x}|^2 = |\mathbf{y}|^2 = 1\}.$$

These generate the (maximal compact) subgroup $Spin(8) \subset Spin_0(1,8)$, cf. Bryant's "Remarks ...". From Bryant's discussion of $Spin(8)$, we see that this maximal compact subgroup acts transitively on

$$S^7 \times S^7 = \{(\mathbf{z}, \mathbf{w}) \in \mathbb{O}^2 : |\mathbf{z}|^2 = |\mathbf{w}|^2 = 1\}.$$

(Uses triality for $Spin(8)$ and the transitive action of $Spin(7) \subset Spin(8)$ on S^7.) Thus we have transitivity of $Spin_0(1,8) \subset SO(8,8)$ on $S^7 \times S^7$ and $S^{7,7}$.

The homogeneous Fefferman construction

Denote $G := Spin_0(1,8) \subset \widetilde{G} := O(8,8)$ and $\widetilde{P} \subset \widetilde{G}$ the parabolic subgroup stabilizing a null ray $\mathbb{R}_+.(\mathbf{z},\mathbf{w}) \subset \mathbb{O}^2$ under the standard action. By transitivity, we have

$$G/(G \cap \widetilde{P}) = \widetilde{G}/\widetilde{P} \approx S^7 \times S^7.$$

Let $P \subset G$ be the stabilizer of a null ray $\mathbb{R}_+.v \subset \mathbb{R}^{1,8}$ under the action given by $\lambda_{1,8} : G \to SO_0(1,8)$.
Then we verify that $G \cap \widetilde{P} \subset P$. In fact, we have

$$G \cap \widetilde{P} \cong (\mathbb{R}_+ \times G_2) \ltimes (\mathbb{R}^{1,8})^*; P \cong (\mathbb{R}_+ \times Spin(7)) \ltimes (\mathbb{R}^{1,8})^*.$$

Thus we have projections $G \to G/(G \cap \widetilde{P}) \cong S^7 \times S^7$ and $G/(G \cap \widetilde{P}) \to G/P \cong S^7$.

Normality: I. Definition

For $P \subset G$ a parabolic subgroup of a semi-simple Lie group, canonical Cartan geometries of type (G, P) are distinguished by a *normality* condition defined via Lie algebra cohomology:
At Lie algebra level $\mathfrak{p} \subset \mathfrak{g}$ gives a $|k|$-grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{+k}$ with $\mathfrak{p} = \mathfrak{g}_0 \oplus \ldots \oplus \mathfrak{g}_{+k}$, $\mathfrak{p}_+ := \mathfrak{g}_{+1} \oplus \ldots \oplus \mathfrak{g}_{+k}$ and $(\mathfrak{g}_-)^* \cong (\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$.
The curvature of a parabolic geometry $(\mathcal{G}, \omega)$ of type (G, P) can be considered as a P-equivariant C^∞ function $\kappa : \mathcal{G} \to \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g}$.
Def.: $(\mathcal{G}, \omega)$ is *normal* iff $\mathrm{Im}(\kappa) \subset \mathrm{Ker}(\partial^*)$, where the *Kostant codifferential* $\partial^* : \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g} \to \mathfrak{p}_+ \otimes \mathfrak{g}$ is given by

$$\begin{aligned}
&\partial^*(Z_1 \wedge Z_2 \otimes W) \\
&\qquad = (Z_1 \otimes [Z_2, W] - Z_2 \otimes [Z_1, W]) \quad - [Z_1, Z_2] \otimes W \\
&\qquad =: \partial_1^*(Z_1 \wedge Z_2 \otimes W) \qquad\qquad\qquad - \partial_2^*(Z_1 \wedge Z_2 \otimes W)
\end{aligned}$$

Normality: II. Some relevant properties

Remarks:
$\partial^* : \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g} \to \mathfrak{p}_+ \otimes \mathfrak{g}$ is P-equivariant.

If the grading of $\mathfrak{g}$ is a $|1|$-grading (like for conformal geometry), then $\partial^* = \partial_1^*$.

By a result of A. Čap, if a normal parabolic geometry which is torsion-free (i.e. $\mathrm{Im}(\kappa) \subset \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{p}$), then $\partial_1^* \circ \kappa \equiv \partial_2^* \circ \kappa \equiv 0$.

Conformal geometry is one parabolic geometry where the normal Cartan connection is always torsion-free.

Normality: III. Technical Lemma for Fefferman constructions

Let $\varphi : \mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ be an inclusion between semi-simple Lie algebras, and $\mathfrak{p} \subset \mathfrak{g}$, $\tilde{\mathfrak{p}} \subset \tilde{\mathfrak{g}}$ two parabolic subalgebras. At the Lie algebra level, the conditions for a Fefferman construction are: $\tilde{\mathfrak{g}} = \varphi(\mathfrak{g}) + \tilde{\mathfrak{p}}$ and $\varphi(\mathfrak{g}) \cap \tilde{\mathfrak{p}} \subset \varphi(\mathfrak{p})$. Then an element $\kappa \in \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$ defines $\tilde{\kappa} \in \Lambda^2\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{g}}$ by letting $\tilde{\kappa}(\varphi(X), \varphi(Y)) = \varphi(\kappa(X, Y))$ for any $X, Y \in \mathfrak{g}$.

Lemma: Suppose, in addition, that we have: $\varphi(\mathfrak{p}_+) \subset \tilde{\mathfrak{p}}$ and $\varphi(\mathfrak{g}_-) \subset \tilde{\mathfrak{g}}_- \oplus \tilde{\mathfrak{g}}_0$; $B(X, Y) = \tilde{B}(\varphi(X), \varphi(Y))$; and $B(X, \varepsilon) = c\tilde{B}(\varphi(X), \tilde{\varepsilon})$ for $B, \tilde{B}$ (multiples of) the Killing forms, $0 \neq c \in \mathbb{R}$ and $\varepsilon, \tilde{\varepsilon}$ the grading elements. Then:

$$\mathrm{proj}^{\perp}_{\varphi(\mathfrak{g})} \circ \tilde{\partial}^*_1(\tilde{\kappa}) \circ \varphi = \varphi \circ \partial^*_1(\kappa) \ (: \mathfrak{g}/\mathfrak{p} \to \varphi(\mathfrak{g})).$$

Application to $Spin_0(1,8) \subset O(8,8)$

Now consider again $G := Spin_0(1,8) \subset \widetilde{G} := O(8,8)$, $P \subset G$ and $\widetilde{P} \subset \widetilde{G}$ as before.

Corollary: If $(\widetilde{\mathcal{G}}, \widetilde{\omega})$ is a Cartan geometry of type $(\widetilde{G}, \widetilde{P})$ induced by a Fefferman construction from a Cartan geometry $(\mathcal{G}, \omega)$ of type (G, P), then $\widetilde{\omega}$ is normal only if ω is.

Proof: Basic computations verify that all the conditions of the previous Lemma are satisfied. With a Fefferman construction, the curvature $\tilde{\kappa}$ of the induced connection $\tilde{\omega}$ is related to the curvature κ of ω by $\tilde{\kappa} = \kappa$ on the sub-bundle $\mathcal{G} \subset \widetilde{\mathcal{G}}$, and this determines $\tilde{\kappa}$ by $\widetilde{P}$-equivariance. Applying the Lemma, we get

$$\mathrm{proj}_{\mathfrak{g}}^{\perp} \circ \tilde{\partial}^* \circ \tilde{\kappa} = \partial^* \circ \kappa$$

since $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are both $|1|$-graded. The LHS vanishes by normality of $\tilde{\omega}$, which shows $\partial^* \circ \kappa \equiv 0$, i.e. ω is normal.

Proof of Theorem B

It remains to show that, for $(\mathcal{G}, \omega)$ of type (G, P) normal, the Fefferman space $(\widetilde{\mathcal{G}}, \widetilde{\omega})$ of type $(\widetilde{G}, \widetilde{P})$ is only normal when $(\mathcal{G}, \omega)$ is flat (i.e. $\kappa \equiv 0$).
As noted above, because $(\widetilde{\mathcal{G}}, \widetilde{\omega})$ is of conformal type, normal implies torsion-free. So a necessary condition is that $\mathrm{Im}(\tilde{\kappa}) \subset \Lambda^2 \tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{p}}$. Since $\tilde{\kappa} = \kappa$ on $\mathcal{G} \subset \widetilde{\mathcal{G}}$, this means we must have $\kappa(u)(\mathfrak{g}, \mathfrak{g}) \subset (\mathfrak{g} \cap \tilde{\mathfrak{p}}) \subsetneq \mathfrak{p}$ for all $u \in \mathcal{G}$. Looking at the lowest non-vanishing homogeneity component κ_0 (which is given by the Weyl tensor), we have $\kappa_0(u)(\mathfrak{g}, \mathfrak{g}) \subset \mathfrak{so}(7) \subset \mathfrak{g}_0 \cong \mathbb{R} \oplus \mathfrak{so}(7)$. Since $\mathfrak{so}(7)$ is irreducible under the action of $Spin(7) \subset P$, this means unless κ_0 vanishes identically, we have $\kappa_0(\mathcal{G})(\mathfrak{g}, \mathfrak{g}) = \mathfrak{so}(7)$. Since $\mathfrak{so}(7) \not\subseteq (\mathfrak{g}_0 \cap \tilde{\mathfrak{p}}) \cong \mathbb{R} \oplus G_2$, this means that $\tilde{\omega}$ is not normal unless $\kappa_0 \equiv 0$, which in turn implies $\kappa \equiv 0$.

Conclusion: We get an exclusion of the conformal holonomy $SO_0(2,1)_i \subset SO_0(3,2)$ and a partial exclusion of the conformal holonomy $Spin_0(1,8) \subset SO_0(8,8)$ (it can't occur via a parabolic Fefferman-type construction).

Question: Can we completely exclude $Spin_0(1,8) \subset SO(8,8)$ as a conformal holonomy? Is there some other construction for non-flat $(M,[g])$ with $\mathrm{Hol}(M,[g]) \subseteq Spin_0(1,8)$?

Question: In case A, the reduced Cartan geometries corresponding to the 3 orbit types of $SO(2,1)_i$ acting on $S^{2,1}$ give distinguished geometric structures on the open dense subset $M_0 \subset M$ and on the submanifolds of dimensions 1 and 2. These structures are all flat when $\mathrm{Hol}(\omega) \subseteq SO_0(2,1)_i$ for ω torsion-free (in particular, for ω the normal conformal Cartan connection). Is there a class of $(M,[g])$ non-flat, with distinguished (non-normal) conformal Cartan connection $\widetilde{\omega}$ and $\mathrm{Hol}(\widetilde{\omega}) \subseteq SO_0(2,1)_i$?

Question: In case B, the Cartan connection induced by the Fefferman-type construction is not normal, but what is the relation between it and the normal conformal Cartan connection of the conformal structure of signature $(7,7)$ which it induces? What geometric properties do these conformal structures of signature $(7,7)$ have? Can they be characterized a la Sparling's criteria for (classical) Fefferman metrics?

Question: Suitable conformal analogue of "nearly-integrable special geometries" (a.k.a. weak holonomy, etc.)?

Question: What other irreducible subgroups $H \subset SO(p+1, q+1)$ exhibit similar phenomena as $SO_0(2,1)_i \subset SO(3,2)$?

Question: If an irreducible subgroup $H \subset SO(p+1, q+1)$ has *no* open orbits in $S^{p,q}$ can it be excluded as a conformal holonomy group (or its geometry otherwise classified)?

References

Leitner: *Normal conformal Killing forms*, preprint: arXiv.0406316 (2004).
Armstrong: *Definite signature conformal holonomy: a complete classifation*, J. Geom. Phys. **57** (2007).
Čap/Gover/Hammerl: *Holonomy reductions of Cartan geometries and curved orbit decompositions*, preprint: arXiv.1103.4497 (2011).
Leitner/Armstrong: *Decomposable conformal holonomy in Riemannian signature*, to appear in Math. Nachr.
Di Scala/Olmos: *The geometry of homogeneous submanifolds of hyperbolic space*, Math. Z., **237** (2001).
Di Scala/Leistner: *Connected subgroups of SO(2,n) acting irreducibly on* $\mathbb{R}^{2,n}$, to appear in Israel J. Math.
J.A.: *Transitive conformal holonomy groups*, preprint: arXiv.1107.0617 (2011).
Bobienski/Nurowski: *Irreducible* $SO(3)$ *geometry in dimension five*, J. reine angew. Math. **605** (2007).