TWO-JETS OF CONFORMAL FIELDS ALONG THEIR ZERO SETS IN ANY METRIC SIGNATURE

Andrzej Derdzinski (The Ohio State University)

July 11, 2011

Cartan Connections, Geometry of Homogeneous Spaces, and Dynamics

week one: CONFORMAL GEOMETRY AND GENERALIZATIONS

The Erwin Schrödinger International Institute for Mathematical Physics

Vienna, July 11 – 15, 2011

these notes are posted at
`http://www.math.ohio-state.edu/~andrzej/esi.pdf`

CONFORMAL VECTOR FIELDS

(M, g) always denotes a pseudo-Riemannian manifold of dimension $n \geq 3$.

A vector field v on M is called *conformal* if its local flow consists of conformal diffeomorphisms. Equivalently, for some $\phi : M \to \mathrm{I\!R}$,

$$2\nabla v \;=\; A \;+\; \phi\,\mathrm{Id}, \qquad \text{with } A^* = -A. \tag{1}$$

Here ∇v is treated as a bundle morphism $TM \to TM$ (which sends each vector field w to $\nabla_w v$), and $A = \nabla v - [\nabla v]^*$ is twice the skew-adjoint part of ∇v.

Note that $\operatorname{div} v = n\phi/2$.

Example: *Killing fields* v, characterized by $\phi = 0$.

THE SIMULTANEOUS KERNEL

Manifolds *need not be connected*. A submanifold is always endowed with the subset topology.

Z denotes the *zero set* of a given conformal field v.

If $x \in Z$, we use the symbol

$$\mathcal{H}_x = \mathrm{Ker}\,\nabla v_x \cap \mathrm{Ker}\,d\phi_x$$

for the *simultaneous kernel*, at x, of the differential $d\phi$ and the bundle morphism $\nabla v : TM \to TM$.

When x is fixed, we also write H instead of $\mathcal{H}_x$.

(NON)ESSENTIAL AND (NON)SINGULAR ZEROS

$x \in Z$ is an *essential* zero of v if no conformal change of g on any neighborhood U of x turns v into a Killing field for the new metric on U.

Otherwise, $x \in Z$ is a *nonessential* zero of v.

A *nonsingular zero* of v is any $x \in Z$ such that, for some neighborhood U of x in M, the intersection $Z \cap U$ is a submanifold of M.

Zeros of v not having a neighborhood with this property are from now on called *singular*.

BEIG'S THEOREM (1992)

$x \in Z$ is nonessential if and only if

$$\phi(x) = 0 \text{ and } \nabla\phi_x \in \nabla v_x(T_xM). \qquad (2)$$

In other words: *$x \in Z$ is essential if and only if*

$$\text{either } \phi(x) \neq 0, \quad \text{or } \phi(x) = 0 \text{ and } \nabla\phi_x \notin \nabla v_x(T_xM). \qquad (3)$$

For a proof, see a 1999 paper by Capocci.

ESSENTIAL/NONESSENTIAL COMPONENTS OF Z

Z is always locally pathwise connected. Thus, the connected components of Z are pathwise connected, closed subsets of M.

From now on they are simply called the *components* of Z.

A component of Z is referred to as *essential* if all of its points are essential zeros of v.

Otherwise, the component is said to be *nonessential*.

This definition allows a nonessential component N to contain some essential zeros of v. We'll see, however, that essential zeros in N then form a closed subset of N without relatively-interior points.

GEOMETRY OF AN ESSENTIAL COMPONENT Σ

Let Σ be an essential component. Then

(i) Σ is a null totally geodesic submanifold of (M, g), closed as a subset of M.

In addition, for any $x \in \Sigma$, with $\mathcal{H}_x = \operatorname{Ker} \nabla v_x \cap \operatorname{Ker} d\phi_x$,

(ii) $T_x\Sigma = \mathcal{H}_x \cap \mathcal{H}_x^\perp$,

(iii) the metric g_x restricted to $\mathcal{H}_x$ is semidefinite.

GEOMETRY OF A NONESSENTIAL COMPONENT N

Assume N to be nonessential, and let Σ denote the set of all essential zeros of Z lying in N. Then

(a) Σ, if nonempty, is a null totally geodesic submanifold of (M, g), closed as a subset of M,

(b) $N \smallsetminus \Sigma$ is a totally umbilical submanifold of M, with $\dim(N \smallsetminus \Sigma) > \dim \Sigma$, and g restricted to $N \smallsetminus \Sigma$ has the same sign pattern (including rank) at all points,

(c) Σ consists of singular, $N \smallsetminus \Sigma$ of nonsingular zeros of v.

For any $x \in \Sigma$ and $y \in N \smallsetminus \Sigma$, with $\mathcal{H}_x = \operatorname{Ker} \nabla v_x \cap \operatorname{Ker} d\phi_x$,

(d) $T_y(N \smallsetminus \Sigma) = \operatorname{Ker} \nabla v_y$ and $T_x\Sigma = \mathcal{H}_x \cap \mathcal{H}_x^\perp$,

(e) $\operatorname{rank} \nabla v_y = 2 + \operatorname{rank} \nabla v_x$,

(f) the metric g_x restricted to $\mathcal{H}_x$ is not semidefinite.

MORE ON NONESSENTIAL COMPONENTS N

Again, N is nonessential, Σ is the set of essential zeros of Z lying in N, and $x \in \Sigma$.

Let $C = \{u \in T_xM : g_x(u,u) = 0\}$ be the null cone, and $H = \mathcal{H}_x$ the simultaneous kernel at x, that is, $H = \mathrm{Ker}\,\nabla v_x \cap \mathrm{Ker}\,d\phi_x$.

For any sufficiently small neighborhoods U of 0 in T_xM and U' of x in M such that $\exp_x$ is a diffeomorphism $U \to U'$,

(g) $Z \cap U'$ corresponds under $\exp_x$ to $C \cap H \cap U$,

(h) $\Sigma \cap U'$ corresponds under $\exp_x$ to $H \cap H^\perp \cap U$.

INDUCED STRUCTURES ON Σ AND $N \smallsetminus \Sigma$

Let Σ be either an essential component, or the set of essential points (assumed nonempty) in a nonessential component N.

$N \smallsetminus \Sigma$ is endowed with a *possibly-degenerate conformal structure*, or, in other words, a symmetric 2-tensor field, defined only up to multiplications by functions without zeros, and having the same sign pattern at all points (see (b) on p. 7).

Σ carries a natural *projective structure* – a class of torsion-free connections having the same family of nonparametrized geodesics (see (i) on p. 6 and (a) on p. 7), as well as a distinguished *codimension-zero-or-one distribution*, which means: a 1-form ξ defined only up to multiplications by functions without zeros.

HOW g DETERMINES THE ONE-FORM ξ on Σ

Σ is again either an essential component, or the singular subset, assumed nonempty, of a nonessential component N.

$\phi = (2/n)\,\mathrm{div}\,v$ is constant along every component of Z (more on this later).

If $\phi = 0$ on Σ, then $\Sigma \ni x \mapsto \mathcal{H}_x = \mathrm{Ker}\,\nabla v_x \cap \mathrm{Ker}\,d\phi_x$ is, in both cases, a parallel subbundle of $T_\Sigma M$ contained in $\mathrm{Ker}\,\nabla v$ as a codimension-one subbundle, and we set $\xi = g(w,\,\cdot\,)$, on Σ, for any section w of $\mathrm{Ker}\,\nabla v$ over Σ with $d_w\phi = 1$.

If $\phi \neq 0$ on Σ, we set $\xi = 0$ (consistent with the above).

MORE ON THE ONE-FORM ξ ON Σ

In both cases, Σ the natural projective structure, and the codimension-zero-or-one distribution corresponding to ξ is *geodesic* (although not necessarily integrable): if $\Gamma \subseteq \Sigma$ is a geodesic segment and $T_x\Gamma \subseteq \mathrm{Ker}\,\xi_x$ for some $x \in \Gamma$, then the same is true for every $x \in \Gamma$.

Equivalently: for any (torsion-free) connection D within the projective structure,

$$\mathrm{sym}\,\nabla\xi \;=\; \mu \odot \xi \quad \text{for some 1-form } \mu \text{ on } \Sigma. \tag{4}$$

In coordinates: $\xi_{j,k} + \xi_{k,j} = \mu_j \xi_k + \mu_k \xi_j$.
Note the invariance under changing the connection within the projective structure, and multiplications by functions without zeros.

A UNIQUE CONTINUATION PROPERTY OF ξ

Due to the “geodesic” property, if ξ vanishes on a nonempty open subset of a connected component of Σ, then it must vanish on the whole connected component.

This remains true also if one replaces the words ‘nonempty open subset’ by ’codimension-one submanifold’.

EXAMPLE: RIEMANN EXTENSIONS

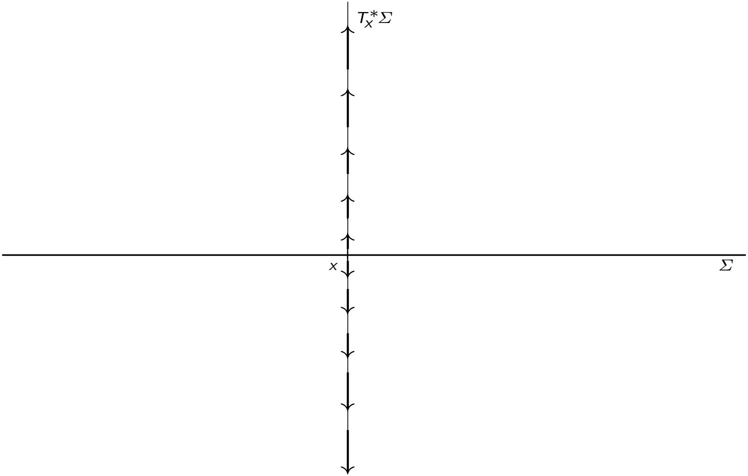
Let D be a connection on a manifold Σ (of any dimension).

We denote by $\pi : T^*\Sigma \to \Sigma$ the bundle projection of the cotangent bundle of Σ.

The Patterson-Walker *Riemann extension metric* on $M = T^*\Sigma$ is the neutral-signature metric g^{D} defined by requiring that

- all vertical and all D-horizontal vectors be g^{D}-null, while
- $g^{\mathrm{D}}_y(\zeta, w) = \zeta(d\pi_y w)$ for any $y \in M$, any vertical vector $\zeta \in \operatorname{Ker} d\pi_y = T^*_x\Sigma$, with $x = \pi(y)$, and any $w \in T_yM$.

THE RADIAL VECTOR FIELD v ON $T^*\Sigma$



The radial field v is conformal for any Riemann extension metric.

THE EXAMPLE, CONTINUED

If the original manifold Σ is connected, the zero section $\Sigma \subseteq M$ is an essential component with $\phi \neq 0$.

CONFORMAL EQUIVALENCE OF ONE-JETS OF v

For $x \in Z$, the endomorphism ∇v_x of T_xM, independent of the choice of ∇, is also known as the *linear part*, or *Jacobian*, or *derivative*, or *differential* of v at the zero x. It coincides with the infinitesimal generator of the local flow of v acting in T_xM.

Given $x, y \in Z$, we say that the 1-jets of v at x and y are *conformally equivalent* if, for some vertical-arrow conformal isomorphism $T_xM \to T_yM$, the following diagram commutes:

$$\begin{tikzcd} T_xM \arrow[r, "\nabla v_x"] \arrow[d] & T_xM \arrow[d] \\ T_yM \arrow[r, "\nabla v_y"] & T_yM \end{tikzcd}$$

ONE-JETS ALONG A NONESSENTIAL COMPONENT

For a nonessential component N, with $\Sigma \subseteq N$ denoting its set of essential points:

The 1-jets of v at all points of any connected component of $N \smallsetminus \Sigma$ are conformally equivalent to one another, but not conformally equivalent to the 1-jet of v at any $x \in \Sigma$.

In fact, ∇v is parallel along $N \smallsetminus \Sigma$ with respect to a connection D in $T_{N \smallsetminus \Sigma} M$ which also preserves the conformal structure.
The claim about $x \in \Sigma$ follows from (e) on p. 7.

Such D arises by gluing together, via a partition of unity on $N \smallsetminus \Sigma$, the Levi-Civita connections of locally-defined metrics conformal to g, for which v is a Killing field.

CONSTANCY OF THE CHARACTERISTIC POLYNOMIAL

Denote by $\mathcal{P}_n$ the space of all polynomials in one real variable with degrees not exceeding $n = \dim M$.

Let $\chi : M \to \mathcal{P}_n$ be the function assigning to each $x \in M$ the characteristic polynomial of $\nabla v_x : T_xM \to T_xM$.

Then χ is constant along every component of Z.

As a consequence, $\phi = (2/n)\,\mathrm{div}\, v$ is also constant along every component.

ONE-JETS ALONG Σ (THE GENERIC CASE)

Again, Σ is either an essential component, or the singular subset (assumed nonempty) of a nonessential component N.

Suppose that ξ is not identically zero on a given connected component of Σ.

Then the 1-jets of v at all points of this connected component of Σ are conformally equivalent to one another. (See p. 24.)

THE GENERAL CASE

Once more, Σ denotes either an essential component, or the singular set (assumed nonempty) in a nonessential component N, but, this time, no assumptions are made about ξ.

Then, if $\Gamma \subseteq \Sigma$ is any geodesic segment, ∇v restricted to Γ descends to a parallel section of the vector bundle $\mathfrak{conf}\,[(T\Gamma)^{\perp}/(T\Gamma)]$.

Equivalently: using the parallel transport to trivialize $T_\Gamma M$, we obtain, for any $x, y \in \Gamma$,

$$\nabla v_y - \nabla v_x = w \wedge u$$

where w, u are (variable) vectors along Γ, and u is tangent to Γ.

THE CONFORMAL-EQUIVALENCE TYPE MAY VARY

It may change not only when one moves from Σ to $N \smallsetminus \Sigma$, but also within a connected component of Σ (on which ξ is identically zero):

For a pseudo-Euclidean space $(V, \langle\,,\rangle)$ of dimension n, vectors $w, u \in V$, a skew-adjoint endomorphism B, and $c \in \mathrm{I\!R}$, setting

$$v_x = w + Bx + cx + 2\langle u, x\rangle x - \langle x, x\rangle u \tag{5}$$

we define a conformal field v. Choose n even, $\langle\,,\rangle$ neutral, B with null n-dimensional eigenspaces for eigenvalues $c, -c$, and u not lying in the $-c$ eigenspace, along with $w = 0$. Then $\mathrm{Ker}\,\nabla v_x$ decreases when one moves from $x = 0$ to nearby x in the $-c$ eigenspace, orthogonal to u.

CONFORMAL EQUIVALENCE OF TWO-JETS OF v

We say that the 2-jets of v at $x \in Z$ and $y \in Z$ are *conformally equivalent* if the restrictions of $d\phi$ to $\mathrm{Ker}\,\nabla v$ at x and y correspond to each other under some conformal isomorphism $T_xM \to T_yM$ that, at the same time, realizes the conformal equivalence of the 1-jets of v at x and y.

(As usual, $\phi = (2/n)\,\mathrm{div}\, v$.)

This happens if and only if some diffeomorphism F between neighborhoods of x and y, with $F(x) = y$, sends the one 2-jet to the other, while, at the same time, for some function $\tau : U \to \mathrm{I\!R}$, the metrics F^*h and $e^\tau g$ have the same 1-jet at x.

TWO-JETS ALONG A NONESSENTIAL COMPONENT

For a nonessential component N and its essential set $\Sigma \subseteq N$:

The 2-jets of v at all points of any connected component of $N \smallsetminus \Sigma$ are conformally equivalent to one another, but not conformally equivalent to the 2-jet of v at any $x \in \Sigma$.

The reason is precisely the same as for 1-jets, since $d\phi = 0$ at every essential zero of v.

TWO-JETS ALONG Σ, THE GENERIC CASE

For Σ as before:

Let $\xi \neq 0$ somewhere in a given connected component of Σ.

Then the 2-jets of v at all points of this connected component of Σ are conformally equivalent to one another.

In fact, for any geodesic segment $\Gamma \subseteq \Sigma$ with a parametrization $t \mapsto x(t)$, if $\dot{x}$ is not in the image of ∇v, we may choose $w = w(t) \in T_{x(t)}M$ so that $\nabla_w v$ equals $\nabla\phi$ plus a function times $\dot{x}$ and $d_w\phi = 0$. Then both ∇v and the restriction of $d\phi$ to $\mathrm{Ker}\,\nabla v$ are D-parallel for the metric connection D in $T_\Gamma M$ given by $2\mathrm{D}_{\dot{x}} = 2\nabla_{\dot{x}} + w \wedge \dot{x}$.

PROOFS: NONESSENTIAL ZEROS

If $x \in Z$ is a nonessential zero of v, we may assume that v is a Killing field (by changing the metric conformally near x).

Thus (Kobayashi, 1958): $x \in Z$ has a neighborhood U' in M such that, for some star-shaped neighborhood U of 0 in T_xM, the exponential mapping $\exp_x$ is a diffeomorphism $U \to U'$ and

$$Z \cap U' = \exp_x[H \cap U].$$

Here $H = \mathcal{H}_x = \operatorname{Ker} \nabla v_x$, since $\phi = 0$.

PROOFS: ESSENTIAL ZEROS

THEOREM 1 (D., Class. Quantum Gravity **28**, 2011, 075011): *Let Z be the zero set of a conformal vector field v on a pseudo-Riemannian manifold (M,g) of dimension $n \geq 3$.*
If x is an essential zero of v and $H = Ker\nabla v_x \cap Ker\, d\phi_x$, then

$$Z \cap U' \;=\; \exp_x[C \cap H \cap U],$$

for any sufficiently small star-shaped neighborhood U of 0 in T_xM mapped by $\exp_x$ diffeomorphically onto a neighborhood U' of x in M, where $C = \{u \in T_xM : g_x(u,u) = 0\}$ is the null cone.

In other words:

The zero set Z is, near any essential zero x, the $\exp_x$-image of a neighborhood of 0 in the null cone in the simultaneous kernel H.

THE COMPONENTS OF Z

In addition, ϕ is constant along each connected component of Z.

Away from singularities, the components of Z are totally umbilical submanifolds of (M, g), and their codimensions are even unless the component is a null totally geodesic submanifold.

BACKGROUND

- Kobayashi (1958): for a Killing field v on a Riemannian manifold (M, g), the connected components of the zero set of v are mutually isolated totally geodesic submanifolds of even codimensions.

- Blair (1974): if M is compact, this remains true for conformal vector fields, as long as one replaces the word 'geodesic' by 'umbilical' and the codimension clause is relaxed in the case of one-point connected components.

- Belgun, Moroianu and Ornea (J. Geom. Phys. **61**, no. 3, 2011, pp. 589–593): Blair's conclusion holds without the compactness hypothesis.

LINEARIZABILITY

- The last result is also a direct consequence of the following theorem of Frances (2009, arXiv:0909:0044v2): at any zero z, a conformal field is linearizable unless z has a conformally flat neighborhood.

- Frances and Melnick (2010, arXiv:1008.3781): the above statement is true in real-analytic Lorentzian manifolds as well.

- Leitner (1999): in Lorentzian manifolds, zeros of a conformal field with certain additional properties lie, locally, in a null geodesic.

SINGULARITIES OF THE ZERO SET Z

Consequently:

The singular subset of $Z \cap U'$ equals $\exp_z[H \cap H^\perp \cap U]$, if the metric restricted to H is not semidefinite, and is empty otherwise.

WHY TOTALLY UMBILICAL

Let b be the second fundamental form of a submanifold K in a manifold M endowed with a torsionfree connection ∇.

If $x \in M$, a neighborhood U of 0 in T_xM is mapped by $\exp_x$ diffeomorphically onto a neighborhood of x in M, and $K = \exp_x[V \cap U]$ for a vector subspace V of T_xM, then $b_x = 0$.

THE CONFORMAL-FIELD CONDITION, REWRITTEN

We always denote by $t \mapsto x(t)$ a geodesic of (M, g), by $\dot{x} = \dot{x}(t)$ its velocity, and write $\dot{f} = d[f(x(t))]/dt$, $\ddot{f} = d^2[f(x(t))]/dt^2$ for vector-valued functions f on M.

The equality $2\nabla v = A + \phi\,\mathrm{Id}$ with $A^* = -A$, rewritten as $\nabla v + [\nabla v]^* = \phi\,\mathrm{Id}$, or

$$v_{j,k} \;+\; v_{k,j} \;=\; \phi g_{jk}\,,$$

is obviously equivalent to the requirement that, along every geodesic,

$$\langle v, \dot{x}\rangle^{\boldsymbol{\cdot}} = \phi\langle \dot{x}, \dot{x}\rangle\,, \qquad (6)$$

IDENTITIES RELATED TO THE CARTAN CONNECTION

If $t \mapsto u(t) \in T_{x(t)}M$ and $\nabla_{\dot{x}} u = 0$, one has

$$2\nabla_{\dot{x}}\nabla_u v = 2R(v \wedge \dot{x})u + [(d\phi)(u)]\dot{x} + \dot{\phi}u - \langle \dot{x}, u \rangle \nabla\phi,$$
$$(1 - n/2)[(d\phi)(u)]^{\cdot} = \sigma(u, \nabla_{\dot{x}} v) + \sigma(\dot{x}, \nabla_u v) + [\nabla_v \sigma](u, \dot{x}),$$

$\sigma = \mathrm{Ric} - (2n-2)^{-1}\,\mathrm{Scal}\, g$ being the Schouten tensor. Thus,

$$\nabla_{\dot{x}}\nabla_{\dot{x}} v = R(v \wedge \dot{x})\dot{x} + \dot{\phi}\dot{x} - \langle \dot{x}, \dot{x} \rangle \nabla\phi/2,$$

$$(1 - n/2)\ddot{\phi} = 2\sigma(\dot{x}, \nabla_{\dot{x}} v) + [\nabla_v \sigma](\dot{x}, \dot{x}),$$

Hence: *if the geodesic is null and* v, $\nabla_{\dot{x}} v$, $\dot{\phi}$ *vanish for some* t, *then they vanish for every* t.

ONE INCLUSION – FOR FREE

Once again: *if the geodesic is null and* v, $\nabla_{\dot{x}} v$, $\dot{\phi}$ *vanish for some* t, *then they vanish for every* t.

Therefore, for any zero x of v, essential or not,

$$\exp_x[C \cap H \cap U] \ \subseteq \ Z \cap U',$$

where $H = \operatorname{Ker} \nabla v_x \cap \operatorname{Ker} d\phi_x$. In other words:

The $\exp_x$*-image of the null cone in the simultaneous kernel* H *always consists of zeros of* v.

The clause about constancy of ϕ will now follow immediately, once the above inclusion is shown to be an equality.

INTERMEDIATE SUBMANIFOLDS

Given a zero x of a section ψ of a vector bundle $\mathcal{E}$ over a manifold M, we denote by $\partial\psi_x$ the linear operator $T_xM \to \mathcal{E}_x$ with the components $\partial_j\psi^a$. (Thus, $\partial\psi_x = \nabla\psi_x$ if ∇ is a connection in $\mathcal{E}$.)

A trivial consequence of the rank theorem:
All zeros of ψ near x then lie in a submanifold $\mathit{\Pi} \subseteq M$ such that $T_x\mathit{\Pi} = \mathrm{Ker}\,\partial\psi_x$ and $\mathrm{Ker}\,\partial\psi_y \subseteq T_y\mathit{\Pi}$ for all $y \in \mathit{\Pi}$ with $\psi_y = 0$.

Note that the zero set Z of ψ can, in general, be any closed subset of M. An *intermediate submanifold* $\mathit{\Pi}$ chosen as above provides some measure of control over Z.

CONNECTING LIMITS

Whenever M is a manifold, $x \in M$, and $L \subseteq T_x M$ is a line through 0, while $y_j, z_j \in M$, $j = 1, 2, \ldots$, are sequences converging to x with $y_j \neq z_j$ whenever j is sufficiently large, let us call L a *connecting limit* for this pair of sequences if some norm $|\,|$ in $T_x M$ and some diffeomorphism Ψ of a neighborhood of 0 in $T_x M$ onto a neighborhood of x in M have the property that $\Psi(0) = x$ and $d\Psi_0 = \mathrm{Id}$, while the limit of the sequence $(w_j - u_j)/|w_j - u_j|$ exists and spans L, the vectors u_j, w_j being characterized by $\Psi(u_j) = y_j$, $\Psi(w_j) = z_j$ for large j.

RADIAL LIMIT DIRECTIONS

For M, x and y_j, z_j as above, neither L itself nor the fact of its existence depends on the choice of $|\,|$ and Ψ.

In the case where $\Pi \subseteq M$ is a submanifold, both sequences y_j, z_j lie in Π, and L is their connecting limit, one has $L \subseteq T_x\Pi$.

By a *radial limit direction* of a subset $Z \subseteq M$ at a point $x \in M$ we mean a connecting limit of for a pair of sequences as above, of which one is constant and equal to x, and the other lies in Z.

CASE I: $\phi(x) \neq 0$

Choose U, U' so that $\phi \neq 0$ everywhere in U'. For $y \in (Z \cap U') \smallsetminus \{x\}$, let $L_y = T_x \varGamma_y$ be the initial tangent direction of the geodesic segment $\varGamma_y$ joining x to y in U'.

Recall that

$$\langle v, \dot{x} \rangle^{\cdot} = \phi \langle \dot{x}, \dot{x} \rangle \, ,$$

and so $\varGamma_y$ is null.

CASE I: $\phi(x) \neq 0$ (CONTINUED)

Next, for U, U' small enough, $L_y \subseteq \operatorname{Ker} \nabla v_x$.

In fact, $\mathit{\Gamma}_y$ is rigid. Hence v is tangent to $\mathit{\Gamma}_y$, and $L_y \subseteq \operatorname{Ker}(\nabla v_x - \lambda_y \mathrm{Id})$ for some eigenvalue λ_y.

Now, if we had $\lambda_y \neq 0$ for some sequence $y \in (Z \cap U') \smallsetminus \{x\}$ converging to x, passing to a suitable subsequence such that $L_y \to L$ for some L we would get $\lambda_y = \lambda$ (independent of y), and a contradiction would ensue: $L \subseteq T_x \mathit{\Pi} = \operatorname{Ker} \partial \psi_x = \operatorname{Ker} \nabla v_x$, where $\mathit{\Pi}$ is an intermediate submanifold for $\psi = v$ and x.

CASE I: $\phi(x) \neq 0$ (STILL)

Furthermore, as $2\nabla v = A + \phi\,\mathrm{Id}$ with $A^* = -A$, it follows that

both $\mathrm{Ker}\,\nabla v_x$ *and* $H \subseteq \mathrm{Ker}\,\nabla v_x$ *are null subspaces of* $T_x M$.

If $\mathrm{Ker}\,\nabla v_x \subseteq \mathrm{Ker}\,d\phi_x$, so that $H = \mathrm{Ker}\,\nabla v_x$, the one inclusion we already have completes the proof.

CASE I: $\phi(x) \neq 0$ (FINAL STEP)

Therefore, assume that $\mathrm{Ker}\,\nabla v_x$ is *not* contained in $\mathrm{Ker}\, d\phi_x$. Thus, $K = \exp_x[H \cap U]$ is a codimension-one submanifold of $\varPi = \exp_x[\mathrm{Ker}\,\nabla v_x \cap U]$, while the restriction of ϕ to $\varPi$ has a nonzero differential at x, and $\phi = \phi(x)$ on K. Making U, U' smaller, we ensure that $\phi \neq \phi(x)$ everywhere in $\varPi \smallsetminus K$. This shows that no zero y of v lies in $\varPi \smallsetminus K$, for the existence of one would result in a contradiction: we have

$\nabla_{\dot x}\nabla_{\dot x}(v \wedge \dot x) = [R(v \wedge \dot x)\dot x] \wedge \dot x$ (for null geodesics) and

$\nabla_{\dot x}\nabla_{\dot x} v = \dot\phi\,\dot x$ (for null geodesics to which v is tangent);

integrating the latter, one obtains $\nabla_{\dot x} v = [\phi - \phi(x)]\dot x$.

CASE II: $\phi(x) = 0$ **AND** $\nabla\phi_x \notin \nabla v_x(T_xM)$

SUBCASE II-a: in addition, $\operatorname{Ker} \nabla v_x$ is not null.

For $K = \exp_x[H \cap H^\perp \cap U]$ and any $y \in K$:

the parallel transport from x to y sends the simultaneous kernel $H = \operatorname{Ker} \nabla v_x \cap \operatorname{Ker} d\phi_x$ onto $\mathcal{H}_y = \operatorname{Ker} \nabla v_y \cap \operatorname{Ker} d\phi_y$,

while

$\dim \mathcal{H}_y$ *is independent of $y \in K$*, and

if $\phi(x) = 0$, both $\operatorname{rank} \nabla v_y$ *and* $\dim \operatorname{Ker} \nabla v_y$ *are constant as functions of $y \in K$.*

SUBCASE II-a, PROOF OF THE ABOVE CLAIMS

From the "inclusion for free" and the second order identities related to the Cartan connection:

$$2\nabla_{\dot{x}}\nabla_u v = [(d\phi)(u)]\dot{x}, \qquad (1-n/2)[(d\phi)(u)]^{\boldsymbol{\cdot}} = \sigma(\dot{x}, \nabla_u v).$$

Uniqueness of solutions: the parallel transport sends $H = \mathcal{H}_x$ INTO $\mathcal{H}_y$. Now 'ONTO' follows as $\dim \mathcal{H}_y \le \dim \mathcal{H}_x$ (semicontinuity). Thus, for $y \in K$ and $p_y = \dim \operatorname{Ker} \nabla v_y$,

$$p_x - 1 \le p_y \le p_x.$$

As $\phi(y) = 0$, the codimension $n - p_y$ is even (note that $2\nabla v = A + \phi\,\mathrm{Id}$ with $A^* = -A$). Hence $p_y = p_x$.

SETS OF CONNECTING LIMITS

Suppose that M is a manifold, $Y, Z \subseteq M$, and $x \in M$.

We denote by $\mathbb{L}_x(Y, Z)$ the set of all connecting limits for pairs y_j, z_j of sequences in Y and, respectively, Z, converging to x, with $y_j \neq z_j$ for all j.

For instance:

$\mathbb{L}_x(\{x\}, Z)$ is the set of all radial limit directions of a subset $Z \subseteq M$ at a point $x \in M$.

INTERMEDIATE SUBMANIFOLDS REVISITED

As before: we are given a zero x of a section ψ of a vector bundle $\mathcal{E}$ over a manifold M.

For $r = \operatorname{rank} \partial\psi_x$, we choose an r-dimensional real vector space W and a base-preserving bundle morphism $G : \mathcal{E} \to M \times W$ such that $G_x : \partial\psi_x(T_xM) \to W$ is an isomorphism. Now we may set $\mathit{\Pi} = U \cap F^{-1}(0)$ for a suitable neighborhood U of x in M and $F : M \to W$ defined by $F(y) = G_y\psi_y$.

If ξ is a section of $\mathcal{E}^*$ and $\partial\psi_x(T_xM) \subseteq \operatorname{Ker}\xi_x$, then $Q = \xi(\psi) : \mathit{\Pi} \to \mathrm{I\!R}$ has a critical point at x with the Hessian of Q characterized by $\partial dQ_x(u,u) = \xi\,([\nabla_u(\nabla\psi)]u)$.

SUBCASE II-a CONTINUED

Recall: this means that

$$\phi(x) = 0, \quad \nabla\phi_x \notin \nabla v_x(T_xM), \quad \mathrm{Ker}\,\nabla v_x \text{ not null.}$$

Fix a section w of the bundle $\mathrm{Ker}\,\nabla v$ over $K = \exp_x[H \cap H^\perp \cap U]$ lying outside the subbundle $\mathrm{Ker}\,\nabla v \cap \mathrm{Ker}\,d\phi$, and apply the intermediate submanifold construction to $\psi = v$, $\mathcal{E} = TM$ and $\xi = 2g(w,\cdot\,)$.

Then $Q = 2g(w,v) : \varPi \to \mathrm{I\!R}$ has, at x, the Hessian

$$\partial dQ \,=\, d\phi \otimes g(w,\cdot\,) + g(w,\cdot\,) \otimes d\phi - [d\phi(w)]g\,.$$

THE MORSE-BOTT LEMMA

Given a manifold Π*, a submanifold* $K \subseteq \Pi$*, a function* $Q : \Pi \to \mathrm{I\!R}$*, and a point* $x \in K \cap Q^{-1}(0)$*, let* $dQ = 0$ *on* K*, and let* $\operatorname{rank} \partial dQ_x \geq \dim \Pi - \dim K$.

Then, for some diffeomorphism Ψ *between neighborhoods* U *of* 0 *in* $T_x\Pi$ *and* U' *of* x *in* Π*, such that* $\Psi(0) = x$ *and* $d\Psi_0 = \mathrm{Id}$*, the composition* $Q \circ \Psi$ *equals the restriction to* U *of the quadratic function of* ∂dQ_x*.*

Consequently, $U' \cap Q^{-1}(0) = \Psi(C \cap U)$ *and* $K \cap U' = \Psi(V \cap U)$*, where* $C, V \subseteq T_xM$ *are the null cone and nullspace of* ∂dQ_x*.*

QUADRICS

Given a subset Z of a manifold $\mathit{\Pi}$, and a point $x \in Z$, and a symmetric bilinear form $(\,,)$ in T_xM, we say that Z is a *quadric at* x *in* $\mathit{\Pi}$ *modelled on* $(\,,)$ if some diffeomorphism Ψ between neighborhoods of 0 in $T_x\mathit{\Pi}$ and of x in $\mathit{\Pi}$, with $\Psi(0) = x$ and $d\Psi_0 = \mathrm{Id}$, makes Z, (near x) correspond to the null cone of $(\,,)$ (near 0). For instance:

- the conclusion of the Morse-Bott lemma states, in particular, that $Q^{-1}(0)$ is a quadric at x in $\mathit{\Pi}$, modelled on ∂dQ_x,

- our Theorem 1 implies that the zero set Z is a quadric at x in $\exp_x[H \cap U]$, modelled on the restriction of g_x to H.

CONSEQUENCES OF THE MORSE-BOTT LEMMA

In Subcase II-a, one has the equality

$$Z \cap \phi^{-1}(0) \cap U' \;=\; \exp_x[C \cap H \cap U].$$

Secondly, lying in H but not in $H \cap H^\perp$ is forbidden for connecting limit between $Z \smallsetminus \phi^{-1}(0)$ and K:

$$\mathbb{L}_x(Z \smallsetminus \phi^{-1}(0), K) \cap \mathbb{P}(H) \;\subseteq\; \mathbb{P}(H \cap H^\perp),$$

where $\mathbb{P}(\;)$ is the projective-space functor. Note:

$H \cap H^\perp = T_x K$ and $T_x(\varPi \cap \phi^{-1}(0)) = H$ is a codimension-one subspace of $\operatorname{Ker} \nabla v_x = T_x \varPi$.

PROOF OF THE FIRST RELATION

For suitably chosen w, both $Q = 2g(w,v) : \Pi \to \mathrm{I\!R}$ and the restriction of Q to $\Pi \cap \phi^{-1}(0)$ satisfy, along with our x and $K = \exp_x[H \cap H^\perp \cap U]$, the hypotheses of the Morse-Bott lemma.

(FINALLY, the assumption "$\mathrm{Ker}\,\nabla v_x$ not null" is used!)

So:

$$Z \cap \phi^{-1}(0) \cap U' \;=\; \exp_x[C \cap H \cap U],$$

since two quadrics modelled on the same symmetric bilinear form, such that one contains the other, must, essentially, coincide.

OUTLINE OF PROOF OF THE SECOND RELATION

The Morse-Bott lemma for Q, Π and K allows us to identify Q with the quadratic function of a direct-sum symmetric bilinear form on $W \oplus V$, where the summand form on W is nondegenerate and that on V is zero.
If $L \in \mathbb{L}_x(Z \smallsetminus \phi^{-1}(0), K) \cap \mathbb{P}(H)$, we have the convergence

$$\frac{s_j u_j + y_j - z_j}{|s_j u_j + y_j - z_j|} \;\to\; cu + x \in L \;\text{ as } j \to \infty,$$

for a fixed Euclidean sphere $S \subseteq W$, a neighborhood K of 0 in V, some $u_j, u \in \Sigma$, $s_j \in \mathbb{R}$ and $y_j, z_j \in K$ with $u_j \to u$ and $|s_j| + |y_j| + |z_j| \to 0$. From the Hessian formula at the bottom of p. 46, $d\phi_x(u) \neq 0$. Hence $c = 0$, which proves that $L \in \mathbb{P}(H \cap H^\perp)$.

THE CRUCIAL IMPLICATION

In Subcase II-a, the inclusion

$$\mathbb{L}_x(Z \smallsetminus \phi^{-1}(0), K) \cap \mathbb{P}(H) \;\subseteq\; \mathbb{P}(H \cap H^\perp)$$

implies, BY ITSELF, that

$$Z \cap U' \;\subseteq\; \phi^{-1}(0).$$

Here is why.

NONVANISHING OF ϕ

First, for a fixed positive-definite metric h, any $y \in U' \smallsetminus K$ is joined by a "rigid" g-geodesic segment Γ_y to a point $p_y \in K$ is such a way that that Γ_y is h-normal to K at p_y. Now:

if $y \in (Z \cap U') \smallsetminus \phi^{-1}(0)$, *then* $\phi \neq 0$ *everywhere in* $\Gamma_y \smallsetminus \{y, p_y\}$.

For, otherwise, a subsequence of a sequence of points y falsifying this claim and converging to x would produce, as the limit of their $T_y\Gamma_y$, an element L of $\mathbb{L}_x(Z \smallsetminus \phi^{-1}(0), K) \cap \mathbb{P}(H)$, and hence of $\mathbb{P}(H \cap H^\perp)$, which cannot happen as L would also be h-orthogonal to $H \cap H^\perp = T_x K$.

PROOF OF THE CRUCIAL IMPLICATION, CONTINUED

Next, whenever a conformal vector field v is tangent to a null geodesic segment $\varGamma$, so that $x(0) = y$ and $\nabla_{\dot{x}} v = \lambda \dot{x}$ at $t = 0$ for some $y \in M$ and $\lambda \in \mathrm{I\!R}$, we have

- $\nabla_{\dot{x}} v = [\lambda + \phi - \phi(y)]\dot{x}$ *along* $\varGamma$,

- ∇v *restricted to* $\varGamma$ *descends to a parallel section of* $\mathfrak{conf}\,[(T\varGamma)^\perp/(T\varGamma)]$ *and has the same characteristic polynomial at all points of* $\varGamma$, *if, in addition,* ϕ *is constant along* $\varGamma$.

To see this, it suffices to integrate the equality $\nabla_{\dot{x}}\nabla_{\dot{x}} v = \dot{\phi}\dot{x}$ (see the final step of Case I), and, respectively, use the first one of the second-order identities related to the Cartan connection.

PROOF OF THE CRUCIAL IMPLICATION, FINAL STEP

We prove that $Z \cap U' \subseteq \phi^{-1}(0)$ by contradiction. Suppose that some points $y \in Z \cap U'$ with $\phi(y) \neq 0$ form a sequence converging to x. Our Γ_y are tangent to v, so (see p. 54) $\nabla_{\dot{x}} v = [\lambda + \phi - \phi(y)]\dot{x}$, $x(0) = y$, $x(1) = p_y$, where λ may depend on y, but not on the curve parameter t. Thus, $\dot{x}(1)$ is an eigenvector of ∇v at p_y for the eigenvalue $\lambda_y = \lambda - \phi(y)$. Constancy of the spectrum of ∇v along Γ (see p. 54) implies that λ_y is an eigenvalue of ∇v_x and, as the limit L of any convergent subsequence of the directions $T_y\Gamma_y$ must lie in $T_x\Pi = \operatorname{Ker} \nabla v_x$, we eventually have $\lambda_y = 0$, that is, $\lambda = \phi(y)$. The equality $\nabla_{\dot{x}} v = [\lambda + \phi - \phi(y)]\dot{x}$ now becomes $\nabla_{\dot{x}} v = \phi\dot{x}$, and Rolle's theorem contradicts the conclusion about nonvanishing of ϕ on p. 53.

SUBCASE II-a WRAPPED UP

The inclusion on p. 49 combined with the crucial implication (p. 52) shows that $\phi(y) = 0$ for every $y \in Z$, near x.

The equality on p. 49 now proves the assertion of Theorem 1 in Subcase II-a.

CASE II: $\phi(x) = 0$ **AND** $\nabla\phi_x \notin \nabla v_x(T_xM)$

SUBCASE II-b: in addition, $\mathrm{Ker}\,\nabla v_x$ is null.

Since $\mathrm{Ker}\,\nabla v_x$ is null, so is $H \subseteq \mathrm{Ker}\,\nabla v_x$. Hence $H = H \cap H^\perp$ and the inclusion on p. 31 is satisfied trivially. The crucial implication (p. 52) now gives $Z \cap U' \subseteq \phi^{-1}(0)$.

We choose an intermediate submanifold N containing $K = \exp_x[H \cap H^\perp \cap U]$ (that is, $K = \exp_x[H \cap U]$) as a codimension-one submanifold.

Since $T_x\varPi \cap \mathrm{Ker}\, d\phi_x = T_xK$, it follows that $U' \cap \phi^{-1}(0) \subseteq K$, and so $X \cap U' \subseteq K$.
This completes the proof of Theorem 1.