First order operators in projective contact geometry - a classification

Svatopluk Krýsl

Charles University, Praha

Cartan geometries, Wien

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Structure of the lecture:

1. Projective contact geometry
2. Projective contact manifolds
3. Solution to the equivalence problem
4. Strongly invariant operators
5. Higher symplectic spinor modules
6. Classification results

Homogeneous model

Definition: (Klein's) homogeneous model of projective contact geometry $= \mathbb{RP}^{2n-1}$ seen as G/P where $G = Sp(2n, \mathbb{R})$ and P is the isotropy subgroup of the action of $Sp(2n, \mathbb{R})$ on the projective space $\mathbb{RP}^{2n-1}$ given by prescription $(g, [v]) \mapsto [gv]$, $g \in Sp(2n, \mathbb{R})$, $0 \neq v \in \mathbb{R}^{2n}$.

Facts:

1. The action $(g, [v]) \mapsto [gv]$ above is transitive.
2. P is a parabolic subgroup of G, i.e., $G^{\mathbb{C}}/P^{\mathbb{C}}$ is projective variety.
3. One can check $\mathfrak{p} \simeq (\mathfrak{sp}(2n-2, \mathbb{R}) \oplus \mathbb{R}) \oplus \mathbb{R}^{2n-2} \oplus \mathbb{R}$. Terminology - first summand $=: \mathfrak{g}_0$, second summand $=: \mathfrak{g}_1 \simeq \mathbb{R}^{2n-2}$ and third summand $=: \mathfrak{g}_2$. Here, $\mathfrak{g} = \mathfrak{sp}(2m, \mathbb{R})$.

Homogeneous model

Definition: Projective contact geometry = Cartan geometry $(\mathcal{G}, \omega)$ of type G/P where G and P are groups from the definition of the homogeneous model above.

To solve the equivalence problem, one should find some induced tangential structures or some partial affine connections.

Difficult because the structures should be "encoding" and "decoding" at once and also somehow independent on each other.

Appropriate structures induced on the manifold $\mathbb{RP}^{2n-1}$ from G/P

1. A contact subbundle of $T\mathbb{RP}^{2n-1}$
2. A class of projectively equivalent partial connections

Homogeneous model

Contact subbundle

$(\mathbb{R}^{2n}, \omega)$ symplectic vector space considered as a symplectic manifold (using the canonical parallelism), $\mathcal{C} := \mathbb{R}^{2n} \setminus \{0\}$, $p : \mathcal{C} \to \mathbb{RP}^{2n-1}$ is a principal $\mathbb{R}^{\times}$-bundle (p realizes the classical equivalence classes definition of the projective space), $\sigma : T\mathbb{RP}^{2n-1} \to T\mathcal{C}$ a bundle morphism such that $\sigma \circ p_* = \mathrm{Id}_{T\mathcal{C}}$, $q^{\sigma} : T\mathbb{RP}^{2n-1} \to \mathbb{R}$, $q^{\sigma}(\xi) := \omega(\sigma(\xi), E)$, $E := \sum_{i=1}^{2n} x^i \frac{\partial}{\partial x^i}$ Euler vector field,
$H := \mathrm{Ker}(q^{\sigma})$ independent of σ.

Statement: H **is a contact subbundle of** TM.

Homogeneous model

Projective class of connections

Take the Levi-Civita ∇ for the flat metric of $\mathbb{R}^{2n}$ or $\mathcal{C}$. Push it forward via a section σ of the bundle $p : \mathcal{C} \to \mathbb{RP}^{2n-1}$ and than by the map tangent map of p to get a connection ∇^σ. Set $N = \{h^\sigma \circ \nabla^\sigma, \nabla^\sigma \text{above}\}$. Here $h^\sigma : T\mathbb{RP}^{2n-1} \to H$ is the projection constructed with help of the Euler vector field and the section σ.

Statement: If γ is up to a parametrization a geodesics of a connection from N, it is a geodesic up to a parametrization for any of them.

Remark: The geodesics are supposed to go in the H-directions only.

Aim: Formulate the above statements independently from the model (sections, projections, Euler vector field etc.).

Projective contact manifolds

Definition: A manifold M is called contact, if it permits a contact subbundle $H =^{def}$ corank one subbundle of TM for which the Lewy form $L : H \times H \to TM/H$ defined by

$$L(X, Y) := [X, Y] \bmod H \quad \text{is nondegenerate.}$$

Because for $Q = TM/H$, $\mathrm{rank}(Q) = 1$, the term 'nondegenerate' makes sense.

Alternative definition: H is maximal nonintegrable subbundle of TM in the Frobenius sense.

$\Rightarrow$ M is of odd dimension.

Projective contact manifolds

Examples: Contact manifolds are arenas of time-dependent Hamiltonian mechanics. Recall from physics, the nonholonomic differential 1-form $dH = dt - \sum_{i=1}^{2n} p_i dq^i$ in time-dependent Hamiltonian mechanics. $Ker(dH)$ is a contact subbundle for the manifold $\mathbb{R}^{2n+1}[t, q^1, \ldots, q^n, p_1, \ldots, p_n]$.

Darboux type theorem holds (Moser, Weinstein).

Definition: The nondegeneracy of L implies for each $\Upsilon \in \Gamma(M, H^*)$ the existence of $\Upsilon^\sharp \in \Gamma(M, Hom(Q, H))$ given by the formula

$$L(\Upsilon^\sharp(X), Y) = \Upsilon(Y)(X),$$

where $X \in \Gamma(M, Q)$, $Y \in \Gamma(M, H)$.

Projective contact manifolds

∇ contact connection $=^{def}$

1. Partial affine connection $\nabla : \Gamma(H) \times \Gamma(H) \to \Gamma(H)$
2. $\nabla_\xi(\bigwedge_0^2 H) \subseteq \bigwedge_0^2 H$ for each $\xi \in \Gamma(H)$, where $\bigwedge_0^2 H := \mathrm{Ker}(\mathcal{L})$

Projective class of connections:
∇' and ∇ are called *projectively equivalent*

$$\nabla' \simeq \nabla \text{ iff } \nabla'_X Y - \nabla_X Y = \Upsilon(X)Y + \Upsilon(Y)X + \Upsilon^\sharp(\mathcal{L}(X,Y))$$

for all $X, Y \in \Gamma(M, H)$ and an $\Upsilon \in \Gamma(M, H^*)$.

Solution to the equivalence problem

Definition: The triple $(M, H, [\nabla])$ is called projective contact manifold. They are object in the category of projective contact manifolds. Morphisms in this category are defined to be local diffeomorphisms preserving the contact bundle and the projective class of contact connections.

Theorem: There is a bijective correspondence between the category of projective contact manifolds and the category of regular normal projective contact geometries.

Remark: Proof based on prolongation procedure (K. Yamaguchi; A. Čap, G. Schmalz).

Regularity $= \pm$ torsion-freeness; normality $= \partial^* \kappa = 0$, where κ is the curvature of $(\mathcal{G}, \omega)$ and $\partial^* =$ Killings adjoint of the Kostant/Chevalley-Eilenberg differential.

Strongly invariant operators

Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P)
For P-modules E, F, set $\mathcal{E} := \mathcal{G} \times_P E$, $\mathcal{F} := \mathcal{G} \times_P F$
J^1E inherits a canonical P-module structure (Slovák, Souček [1]) from E
$(\nabla^\omega s)(X)(u) = \mathcal{L}_{\omega_u^{-1}(X)} s$, $s \in \mathcal{C}^\infty(\mathcal{G}, E)^P = \Gamma(M, E)$, $u \in \mathcal{G}$, $X \in \Gamma(M, TM)$ - so called absolute covariant derivative

Definition: For each P-module homomorphism $\Phi : J^1E \to F$, $(D_{(\mathcal{G},\omega)} s)(u) = \Phi(s(u), \nabla^\omega s(u))$, is called first order strongly invariant operator.

Strongly invariant operators

Statement: {First order strongly invariant operators} $\leftrightarrow$ $\{P$-homomorphisms $\Phi : J^1 E \to F\}$

Denote the $\mathbb{C}$-vector space of first order strongly invariant operators for $(\mathcal{G}, \omega)$, E and F by $\mathrm{Diff}^1_{(\mathcal{G},\omega)}(\mathcal{E}, \mathcal{F})$.

Examples:

1. Dirac operator
2. Rarita-Schwinger operator
3. twistor operator

Strongly invariant operators

Problem: Classify all P-modules homomorphisms $\Phi : J^1E \to F$ for suitable P-modules E and F.

Method: Schur lemma + explicit formula for the P-module structure of J^1E + Casimir operators.

Suitable = Levi part G_0 of P acts in a reductive (e.g. irreducible) way and the unipotent part P^+ of P acts by identity.

Setting of the next theorem:
Let G be a complex simple Lie group and P a parabolic subgroup of G, G_0 the Levi part of P and G_0^{ss} its semisimple part. Suppose G_0 is connected and the subalgebra $\mathfrak{g}_0$ has an one dimensional center $\mathbb{C}Gr$. (The last condition is settled only for convenience.)

Strongly invariant operators

Theorem (Slovák, Souček [1]): Let $(\mathcal{G}, \omega)$ be a parabolic geometry of type G/P with G and P specified above. Let E be a finite dimensional irreducible P-module with the highest weight λ when considered as a $\mathfrak{g}_0^{ss}$-module. Let the grading element Gr acts by the complex number w. Further, let $\mathfrak{g}_1$ be an irreducible $\mathfrak{g}_0^{ss}$-module with the highest weight α. Assume that the tensor product $\mathfrak{g}_1 \otimes E$ is multiplicity free as $\mathfrak{g}_0^{ss}$-module. Then there exists a nonzero strongly invariant operator $D : \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{F})$ iff

1. $\mathcal{F} = \mathcal{G} \times_P F$ and F is an $\mathfrak{g}_0^{ss}$-irreducible summand in $\mathfrak{g}_1 \otimes E$
2. $w = c^{\mu}_{\lambda\alpha} := \frac{1}{2}[(\lambda, \lambda + 2\delta) + (\alpha, \alpha + 2\delta) - (\mu, \mu + 2\delta)]$, where μ is the highest weight of F and $(,)$ is the Killing form of $\mathfrak{g}_0^{ss}$.

Strongly invariant operators

Belehrung: "For each parabolic geometry and fixed irreducible P-modules, there is at most one strongly invariant operator."

Comments on the theorem:

1. The condition for $\mathfrak{g}_1$ to be an irreducible $\mathfrak{g}_0^{ss}$-module is settled for convenience only. Actually, $\mathfrak{g}_1$ is always a direct sum of irreducible $\mathfrak{g}_0^{ss}$-modules. For this more general situation, see Slovák, Souček [1].
2. Also, one can consider real Lie algebras (and their representations on complex vector spaces).

Strongly invariant operators

Context of strongly invariant operators: See Čap, Slovák, Souček [2].

1. There are *natural operators*
2. There are *invariant operators* (BGG theory, regular/singular, standard/nonstandard)
3. There are *strongly invariant operators* (first order strongly invariant operators = first order invariant operators in the broader sense above; not true in general - some invariant operators are not generated by the absolute covariant derivative)

M. Eastwood, J. Rice (Comm. Math. Phys., 1987), H. D. Fegan (Q. J. Math., 1976) - both conformal case.
Motivation: Try to find a invariant conformal field theory; particularly popular in the '60 and '70.

Strongly invariant operators

Summary and receipt:

1. Decompose the tensor product $\mathfrak{g}_1 \otimes E$ into irreducible $\mathfrak{g}_0^{ss}$-submodules explicitly.
2. Compute the Killing products of appropriate weights.
3. You get the "generalized conformal weight" w. This conformal weight fixes the action of the center of $\mathfrak{g}_0$. The operator exists according to the theorem of Slovák and Souček.
4. Integrate the Lie algebra information above to get a Lie group situation. (Topology, e.g., connectedness and simple connectedness of G_0, may be important here.) How to do it in a specific case, see e.g. Krýsl [6].

Higher symplectic spinor modules

We focus to modules from a specific class of infinite dimensional $\mathfrak{sp}(2m,\mathbb{C})$-modules and specific Cartan geometry. Namely, higher symplectic spinor modules and contact projective geometry.

Definition: A weighted $\mathfrak{sp}(2m,\mathbb{C})$-module L is called higher symplectic spinor module, if it is a module with bounded multiplicities, i.e., there is a $k \in \mathbb{N}_0$ such that for each $\nu \in \mathfrak{h}^*$, $\dim_{\mathbb{C}} L_\nu \leq k$.

Here $\mathfrak{h}$ Cartan subalgebra of $\mathfrak{sp}(2m,\mathbb{C})$ and L_ν is the weight space of L of weight $\nu \in \mathfrak{h}^*$.

If one can choose $k = 1$ in the previous definition and L is moreover supposed to be irreducible then L is either the defining module or the "odd" or the "even" part of the Fock module (see below).

Examples:

1. Each finite dimensional $\mathfrak{sp}(2m, \mathbb{C})$-module
2. Fock module = Harish-Chandra underlying module of the Segal-Shale-Weil representation (sometimes called oscillatory, metaplectic or, in Russian and Physics literature, Berezin representation)
3. Tensor products of the Fock-module representation and finite dimensional modules

Actually, the third class is exhausting all the higher symplectic spinor modules (Britten, Hooper, Lemire, Canad. Journ. Phys.) Fock module is the 'precise symplectic analogue' of the spinor modules for orthogonal or spin groups.

Higher symplectic spinor modules

Theorem: (Parametrization of higher symplectic spinor modules) $L(\mu)$ is an irreducible symplectic spinor module iff its highest weight $\mu \in \mathbb{A} :=$

$$\{\sum_{i=1}^{m} \lambda_i \varpi_i | \lambda_i \in \mathbb{N}_0, i = 0, \dots, m-1; \lambda_m \in \mathbb{Z} + \frac{1}{2}; \lambda_{m-1} + 2\lambda_m + 3 > 0\}.$$

Proof. Britten, Hooper, Lemire, Canad. Journ. Phys. □

Remark: Here $\{\varpi_i\}_{i=1}^m$ are the so called fundamental weights of $\mathfrak{sp}(2m, \mathbb{C})$ (for a choice of $\mathfrak{h}$ and a set of positive roots).

Higher symplectic spinor modules

Theorem: (Decomposition result) For $\mu \in \mathbb{A}$, we have

$$\mathfrak{g}_1 \otimes L(\mu) = \mathbb{R}^{2n-2} \otimes L(\mu) = \bigoplus_{\mu' \in \mathbb{A}_\mu} L(\mu'),$$

where $\mathbb{A}_\mu := \{\mu + \nu | \nu \in \Pi(\varpi_1)\} \cap \mathbb{A}$.
Proof. Krýsl [6]. $\square$

Here $\Pi(\varpi_1)$ the set of all weight of the defining representation $L(\varpi_1)$. It consists of $2m = 2n - 2$ elements, $\pm\epsilon_i$ where $\epsilon_i := \varpi_i - \varpi_{i-1}$, $i = 1, \ldots, m = n - 1$ with convention $\varpi_0 := 0$.

Remark: Method - certain character formula (due to C. Jantzen).

Classification result

Theorem: Let $(\mathcal{G}, \omega)$ be a contact projective geometry and $(\lambda, c, \gamma), (\mu, d, \gamma') \in \mathbb{A} \times \mathbb{C} \times \mathbb{Z}_2$. Let E, F be an admissible irreducible P-modules with highest weights λ, μ respectively and let Gr acts by c on E and by d on F. Suppose $-1 \in \mathfrak{Z}(G_0) \simeq \mathbb{R}^\times$ acts by γ on E and by γ' on F.
Then the vector space of first order differential operators

$$\dim \mathrm{Diff}^1_{(\mathcal{G},\omega)}(\mathcal{E}, \mathcal{F}) = \begin{cases} 1 \text{ if } c = c^{\mu}_{\lambda\varpi_1} = d - 1, \gamma = \gamma' \text{ and } \mu \in \mathbb{A}_\lambda, \\ 0 \text{ if otherwise.} \end{cases}$$

Proof. Krýsl [5]. $\square$
Remark: Similar methods to that of in Slovák, Souček [1], similar result, techniques a bit more difficult because of the infinite dimension of the modules in question.

Classification result

Examples:

1. Contact projective Dirac: $\lambda = -\frac{1}{2}\omega_m$, $\mu = \varpi_{m-1} - \frac{3}{2}\varpi_m$, $c = \frac{1+2m}{2}$.
2. Contact projective twistor: $\lambda = -\frac{1}{2}\omega_m$, $\mu = \varpi_1 - \frac{3}{2}\varpi_{m-1}$, $c = \frac{1}{2}$.
3. Contact projective Rarita-Schwinger: $\lambda = \varpi_1 - \frac{1}{2}\varpi_m$, $\mu = \varpi_1 + \varpi_{m-1} - \frac{3}{2}\varpi_m$, $c = \frac{1+2m}{2}$
   Reference: Dissertation thesis of L. Kadlčáková, Praha, 2002 and Krýsl [5].

Literature

[1] J. Slovák, V. Souček, Invariant operators of the first order on manifolds with a given parabolic structure, Sémin. Congr., Vol. 4, SMF, Paris 2001.
[2] A. Čap, J. Slovák, V. Souček, Bernstein-Gelfand-Gelfand Sequences, Ann. of Math. 154 (broad context on not only strongly invariant operators).
Specific to the contact projective geometry and higher symplectic spinor modules:
[3] A. Weil, Sur certains groups d'opérateurs unitaires, Acta Math. 111, 1964, (+/- pioneering literature on SSW).
[4] M. Vergne, M. Kashiwara, On the Segal-Shale-Weil Representations..., Invent. Math. (more info on SSW).
[5] S. Krýsl, Classification of first order invariant operators..., Diff. Geom. Appl., Elsevier. (classification)
[6] S. Krýsl, Decomposition of tensor product..., J. Lie theory, Darmstadt.