Kobayashi pseudodistances for parabolic geometries

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Outline

Brief history of the subject

- 1967 Kobayashi : holomorphic pseudodistance (quickly extended to complex spaces and more recently almost complex manifolds)
- 1894 Hilbert : properly convex domains in projective space (1957 Birkhoff - extension to cones in Banach space)
- 1977 Kobayashi : (normal) projective geometries (1978 Kobayashi-Sasaki, 1981 Wu, Podesta, Goldman, ...)
- 1979 M. : holomorphic projective geometries
- 1981 M. : conformal Lorentz geometries

c. 1980 M. – unsolved puzzle: how does one extend this to general parabolic geometries?

The missing puzzle pieces

A. Čap, J. Slovák, V. Žádník, *On Distinguished Curves in Parabolic Geometries*, Transform. Groups **9** no. 2 (2004), 143–166.

B. Doubrov, *Projective reparametrization of homogeneous curves*, Arch. Math. (Brno) **41** (2005), 129–133.

B. Doubrov, V. Žádník, *Equations and symmetries of generalized geodesics*, in: *Differential Geometry and Its Applications*, Elsevier, Amsterdam (2004), 203–216.

V. Žádník, *Generalized Geodesics*, Ph.D. Thesis, Masaryk University (Brno), 2003.

V. Žádník, *Remarks on Development of Curves*, in: *The Proceedings of 24th winter school Geometry and Physics (Srni 2004), Suppl. Rend. Circ. Mat. Palermo, Series II*.

Puzzle solved

Primary reference

[CS] A. Čap, J. Slovák, *Parabolic Geometries I: Background and General Theory*, Math. Surveys and Monographs **154**, Amer. Math. Soc., Providence, 2009
ISBN: 978-0-8218-2681-2

Cartan connections

Notation

G	:	a Lie group (with Lie algebra $\mathfrak{g}$)
P	:	a closed subgroup (with Lie algebra $\mathfrak{p}$)
M	:	a manifold (usually connected)
$\pi : \mathcal{G} \to M$	:	a principal P-bundle (with $\dim \mathcal{G} = \dim G$)

A *Cartan connection* on $\mathcal{G}$ is a 1–form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ satisfying:

- $(r^h)^* \omega = \mathrm{Ad}(h^{-1})\omega$ for all $h \in P$,
- $\omega(\zeta_X(u)) = X$ for *fundamental vector fields* ζ_X with $X \in \mathfrak{p}$,
- $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism at each point $u \in \mathcal{G}$.

Parabolic geometries

A *Cartan geometry of type (G,P)* on M is a principal P-bundle $\pi : \mathcal{G} \to M$ together with a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$.

A *morphism* between two Cartan geometries $(\mathcal{G} \to M, \omega)$ and $(\mathcal{G}' \to M', \omega')$ of the same type is a bundle map $\varphi : \mathcal{G} \to \mathcal{G}'$ that preserves the connections: $\varphi^* \omega' = \omega$.

A *parabolic geometry of type (G,P)* is a Cartan geometry $(\pi : \mathcal{G} \to M, \omega)$ with G semisimple and P parabolic.

$C^{(G,P)}$: category of Cartan geometries modeled on G/P

Semisimple GLAs

A *$|k|$-grading* on $\mathfrak{g}$ is a vector space decomposition:

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-k} + \cdots + \mathfrak{g}_{-1}}_{\mathfrak{g}_-} + \underbrace{\mathfrak{g}_0 + \underbrace{\mathfrak{g}_1 + \cdots + \mathfrak{g}_k}_{\mathfrak{p}_+ = \mathfrak{g}^1}}_{\mathfrak{p} = \mathfrak{g}^0}$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and $\mathfrak{g}_{-1}$ generates the subalgebra $\mathfrak{g}_-$.

The associated *filtration* is given by:

$$\mathfrak{g}^i = \mathfrak{g}_i + \cdots + \mathfrak{g}_k$$
$$[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$$

Curvature

The *curvature* of a parabolic geometry $(\mathcal{G} \to M, \omega)$ is defined to be the horizontal two–form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ given by the structure equation

$$K(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)].$$

It is often convenient to work instead with the *curvature function* $\kappa : \mathcal{G} \to \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ defined by

$$\kappa(u)(X, Y) = K(\omega^{-1}(X)(u), \omega^{-1}(Y)(u)).$$

ω is said to be:

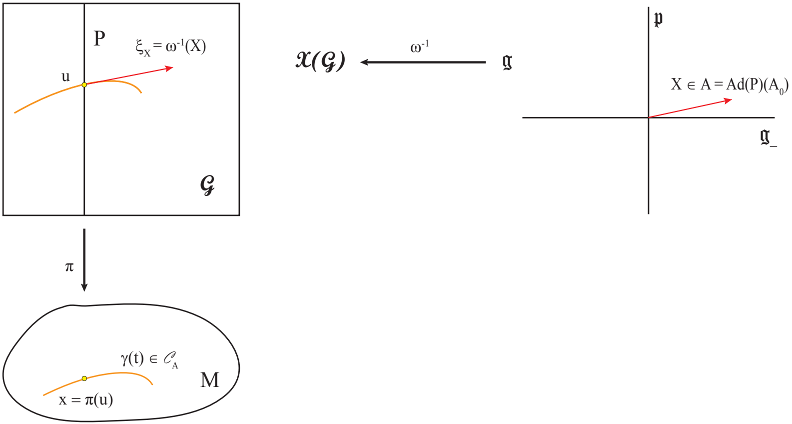
regular	if $\kappa(\mathfrak{g}_i, \mathfrak{g}_j) \subset \mathfrak{g}^{i+j+1} \quad \forall i, j < 0$
torsionfree	if $\kappa(\mathcal{G}) \subset \Lambda^2\mathfrak{g}^* \otimes \mathfrak{p}$
flat	if $\kappa \equiv 0$

Canonical curves: definition

$G_0 \subset P$	:	Levi subgroup of grading–preserving elements
$A_0 \subseteq \mathfrak{g}_-$	:	a G_0-invariant subset
A	:	$\mathrm{Ad}(P)(A_0)$
$(\pi : \mathcal{G} \to M, \omega)$	:	an object in the category $C^{(G,P)}$
J	:	open subinterval of $\mathbb{R}$

A smooth curve $\gamma(t) : J \to M$ is a *canonical curve of type A* on M if γ locally coincides up to a constant shift of parameter with the projection to M of the flow Fl_t^ξ of a *constant vector field* $\xi = \omega^{-1}(X) \in \mathfrak{X}(\mathcal{G})$ for some $X \in A$.

Canonical curves: diagram



Connectivity and completeness

We say that a parabolic geometry on a manifold M is:

A-connected if there exists a piecewise smooth canonical curve of type A joining any two points of M.

A-complete if Fl_t^{ξ} is defined for all $t \in \mathbb{R}$ regardless of the choice of $\xi = \omega^{-1}(X)$ with $X \in A$. In this case, each canonical curve of type A on M is infinitely extendible to a map $\gamma : \mathbb{R} \to M$.

complete if Fl_t^{ξ} is defined for all $t \in \mathbb{R}$ and all $\xi = \omega^{-1}(X)$ with $X \in \mathfrak{g}$. (Of course, the flat model space for any parabolic geometry is complete and therefore A-complete for any choice of A.)

Admissable parametrizations

Every canonical curve $\gamma(t)$ admits *affine reparametrizations*:

$$t \mapsto at + b \quad \text{for } a \neq 0, b \in \mathbb{R}.$$

Some $\gamma(t)$ even admit *projective reparametrizations*:

$$t \mapsto (at + b)/(ct + d) \quad \text{for } ad - bc \neq 0.$$

Theorem (CS,Thm. 5.3.5)

Either a canonical curve admits projective reparametrizations or it admits only affine reparametrizations.

Let $\mathcal{C}_A$ denote the class of canonical curves of type A on M. We say that $\mathcal{C}_A$ *admits projective reparametrizations* if all curves in $\mathcal{C}_A$ have this property.

On the abundance of suitable $\mathcal{C}_A$

Theorem (5.3.3ff; Čap, Slovák, Zádník)

Suppose that $A_0 \subset \mathfrak{g}_-$ is a G_0-invariant subset contained in a single grading component ($A_0 \subseteq \mathfrak{g}_j$ for some $j < 0$) and set $A = \mathrm{Ad}(P)(A_0)$. Then $\mathcal{C}_A$ admits projective reparametrizations and any curve in $\mathcal{C}_A$ defined on a connected interval is uniquely determined by its r-jet at a single point provided that $rj > k$.

In particular, if $A_0 \subseteq \mathfrak{g}_{-k}$, each curve in $\mathcal{C}_A$ defined on a connected interval is uniquely determined by its 2-jet at a single point. (These curves are called "chains.")

Pseudodistances

$d : M \times M \to [0, \infty]$ is a *pseudodistance* (or *'pseudometric'*) if, for all $x, y, z \in M$,

$$\begin{aligned} d(x,x) &= 0 \\ d(x,y) &= d(y,x), \quad \text{and} \\ d(x,y) &\leq d(x,z) + d(z,y). \end{aligned}$$

d is *finite* if

$$d(x,y) < \infty \quad \text{for all } x, y \in M.$$

d is *nondegenerate* (or a *'true distance'*) if

$$d(x,y) = 0 \quad \Longrightarrow \quad x = y$$

Nonexpansive maps

A map $f\colon M \to N$ between pseudometric spaces (M, d_M) and (N, d_N) is *nonexpansive* (or *'distance non-increasing'*) if

$$d_M(x, y) \quad \geq \quad d_N(f(x), f(y))$$

for all $x, y \in M$.

Notation

Given

$$\begin{array}{rcl} G & : & \text{a (real or complex) semisimple Lie group} \\ P & : & \text{a parabolic subgroup} \end{array}$$

we consider the following categories

$$\begin{array}{rcl} \mathcal{C}^{(G,P)} & : & \text{parabolic geometries modeled on } G/P \\ \mathcal{D} & : & \text{pseudometric spaces and nonexpansive maps} \\ \mathcal{S} & : & \text{sets and set maps} \end{array}$$

Intrinsic pseudodistance

Our goal is to define one or more functors $d_\bullet$ making the following diagram commutative:

$$\begin{tikzcd} C^{(G,P)} \arrow[rr, "d_\bullet"] \arrow[dr, "\mathcal{F}"'] & & D \arrow[dl, "\mathcal{F}'"] \\ & S & \end{tikzcd}$$

where $\mathcal{F}$ and $\mathcal{F}'$ are the forgetful functors.

$d_\bullet$ will depend upon the choice of a canonical curve class $\mathcal{C}_A$. For suitable $\mathcal{C}_A$, one can replace S with the topological category T by restricting to the full subcategory of $C^{(G,P)}$ comprised of regular parabolic geometries for which $d_\bullet$ is nondegenerate.

Poincaré metric

To measure distances where some projective invariance is available, it is natural to use the following projectively invariant metrics.

<u>real case</u>

$$I = \{u \in \mathbb{R} \mid |u| < 1\}$$

$$ds_I^2 = \frac{4du^2}{(1-u^2)^2}$$

<u>(almost-) complex case</u>

$$\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$$

$$ds_\Delta^2 = \frac{4\,dz\,d\bar{z}}{(1-|z|^2)^2}$$

Poincaré distance

The distance function on I corresponding to ds_I^2 is given by

$$\rho_I(u_1, u_2) = \left| log \frac{(1+u_1)(1-u_2)}{(1-u_1)(1+u_2)} \right|$$

Schwarz lemma: General linear fractional transformations of I (resp., Δ) are nonexpansive with respect to ρ_I (resp., ρ_Δ), while those which are isometries at a single point must be automorphism

(I, ρ_I) and (Δ, ρ_Δ) are used as "measuring rods."

Kobayashi construction: setup

$\mathfrak{g} = \mathfrak{g}_- + \mathfrak{g}_0 + \mathfrak{g}_+$	:	a $\lvert k \rvert$-graded semisimple Lie algebra
$\mathfrak{p} = \mathfrak{g}_0 + \mathfrak{g}_+$	:	the parabolic subalgebra
G	:	a Lie group with Lie algebra $\mathfrak{g}$
P	:	a parabolic subgroup of G
$G_0 \subset P$	:	the Levi subgroup
$A_0 \subseteq \mathfrak{g}_-$	:	a G_0-invariant subset
A	:	$\mathrm{Ad}(P)(A_0)$
assumption	:	*$\mathcal{C}_A$ admits projective reparametrizations*
$(\pi : \mathcal{G} \to M, \omega)$	:	a parabolic geometry of type (G, P)

Kobayashi construction: path length

Given $x, y \in M$, we define a *(Kobayashi) path of type A* from x to y to be a collection of points $x = x_0, x_1, \ldots, x_k = y \in M$, pairs of points $a_1, b_1, \ldots, a_k, b_k \in I$, and projectively parametrized canonical curves of type A $\gamma_1, \ldots, \gamma_k : I \to M$ such that

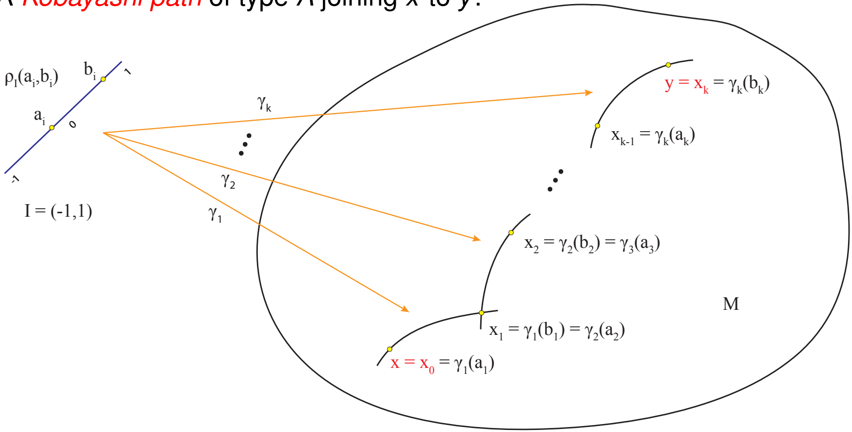
$$\gamma_i(a_i) = x_{i-1} \quad \text{and} \quad \gamma_i(b_i) = x_i \qquad \text{for} \quad i = 1, \ldots, k.$$

Denoting the above path by $\alpha = \{x_i, a_i, b_i, \gamma_i\}$, we define its *(Kobayashi) length* to be

$$L(\alpha) = \sum_{i=1}^{k} \rho_I(a_i, b_i).$$

Kobayashi construction: diagram

A *Kobayashi path* of type *A* joining *x* to *y*:



Kobayashi construction: definition

The *Kobayashi pseudodistance* on M associated with $\mathcal{C}_A$ is then given by

$$d_M^A(x,y) = \inf_\alpha L(\alpha),$$

where the infimum is taken over all Kobayashi paths α of type A joining x to y in M.

If no such path exists, we set $d_M^A(x,y) = \infty$.

Obviously $d_M^A(x,x) = 0$, $d_M^A(x,y) = d_M^A(y,x)$, and $d_M^A(x,y) \geq 0$ for all $x, y \in M$, so d_M^A is a symmetric extended real-valued function $d_M^A \colon M \times M \to [0,\infty]$.

Basic properties

Theorem

(a) d_M^A is a pseudodistance that depends only on the parabolic geometry $(\mathcal{G} \to M, \omega)$ and choice of G_0-invariant subset $A_0 \subset \mathfrak{g}_-$.

(b) If $f : I \to M$ is in $\mathcal{C}_A$, then $d_M^A(f(p), f(q)) \leq \rho(p, q)$ for all $p, q \in I$.

(c) if δ_M is any pseudodistance on M such that $\delta_M(f(p), f(q)) \leq \rho(p, q)$ for all $p, q \in I$ and all $f : I \to M$ in $\mathcal{C}_A$, then $\delta_M(x, y) \leq d_M^A(x, y)$ for all $x, y \in M$.

Basic properties (cont.)

Theorem

(d) Any morphism $\Phi : (\mathcal{G} \to M, \omega) \to (\mathcal{G}' \to M', \omega')$ *between two geometries of type* (G, P) *induces a local diffeomorphism* $\varphi : M \to M'$ *which is nonexpansive:*

$$d^A_{M'}(\varphi(x), \varphi(y)) \le d^A_M(x, y) \qquad \text{for all } x, y \in M.$$

(e) Each automorphism of $(\varphi : \mathcal{G} \to M, \omega)$ *is an isometry:*

$$d^A_M(\varphi(x), \varphi(y)) = d^A_M(x, y) \qquad \text{for all } x, y \in M.$$

(f) If M *is A-connected and A-complete,* $d^A_M \equiv 0$.

Basic properties: coverings

If $\pi : \tilde{M} \to M$ is a covering map and $(\mathcal{G} \to M, \omega)$ is a parabolic geometry of type (G, P) on M, then $(\pi^*\mathcal{G} \to \tilde{M}, \tilde{\omega} = \pi^*\omega)$ is a parabolic geometry of that type on $\tilde{M}$.

Theorem

The covering map $\pi : \tilde{M} \to M$ is nonexpansive with respect to the pseudodistance on M and that induced by π on $\tilde{M}$. In fact,

$$d^A_M(x, y) = \inf_{\tilde{y}} d^A_{\tilde{M}}(\tilde{x}, \tilde{y}) \quad \text{for all } x, y \in M,$$

where $\tilde{x}$ is any point of $\pi^{-1}(x)$ and the infimum is taken over all points $\tilde{y} \in \pi^{-1}(y)$. Consequently, $d^A_{\tilde{M}}$ is a (complete) distance if and only if d^A_M is.

A-connectivity and finiteness

For $A_0 \subseteq \mathfrak{g}_-$, G_0-invariant or not, we define $L(A_0) \subseteq \mathfrak{g}$ to be the smallest Lie subalgebra containing $\bar{A} = \operatorname{span} A_0 + \mathfrak{p}$.

We say that A_0 is *bracket-generating* if $L(A_0) = \mathfrak{g}$, *i.e.*, if every element of $\mathfrak{g}$ can be written as a linear combination of iterated brackets of elements of A_0 and $\mathfrak{p}$: set $\bar{A}_0 = \bar{A}$ and iteratively define $\bar{A}_k = \bar{A}_{k-1} + [\bar{A}, \bar{A}_{k-1}]$ for $k \geq 1$; then A_0 is bracket-generating if $\bar{A}_k = \mathfrak{g}$ for some k.

Proposition

If $A_0 \subseteq \mathfrak{g}_-$ is bracket-generating, then $\omega^{-1}(\bar{A}) \subseteq T\mathcal{G}$ defines a bracket-generating distribution on the bundle space of every regular parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P).

Finiteness (cont.)

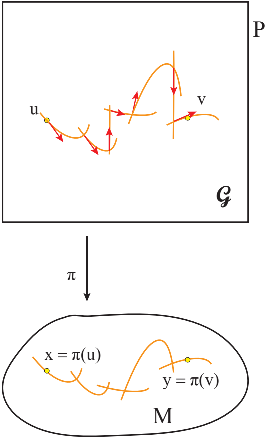
Theorem

Let $A_0 \subseteq \mathfrak{g}_-$ be a G_0-invariant bracket-generating subset, $A = \mathrm{Ad}(P)(A_0)$, and $(\pi : \mathcal{G} \to M, \omega)$ a regular parabolic geometry. If M is connected, then M is A-connected and d_M^A is finite.

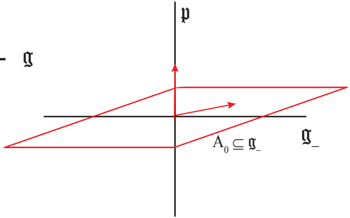
Examples: A_0 any G_0-invariant spanning subset of $\mathfrak{g}_{-1}$. (This works since $\mathfrak{g}_{-1}$ generates $\mathfrak{g}_-$.)

Proof: Choose basis $\bar{B} = \{X_1, \ldots, X_k, X_{k+1}, \ldots, X_l\}$ of $\bar{A}$ with $\{X_1, \ldots, X_k\} \subset A_0$ and $\{X_{k+1}, \ldots, X_l\} \subset \mathfrak{p}$, then apply the Chow-Rashevskii Theorem on accessibility to $\omega^{-1}(\bar{A})$.

A_0 bracket-generating implies d_M^A finite: diagram



$$\mathfrak{X}(\mathcal{G}) \xleftarrow[\xi_X = \omega^{-1}(X)]{\omega^{-1}} \mathfrak{g}$$



$$\bar{B} = \{\overbrace{X_1, \ldots, X_k}^{A_0}, \overbrace{X_{k+1}, \ldots, X_l}^{\mathfrak{p}}\} \text{ basis of } \bar{A}$$

Chow-Rashevskii $\Longrightarrow$

$$\exists\, \xi_{X_{i_1}}, \ldots, \xi_{X_{i_r}} \in \Gamma(\omega^{-1}(\bar{A})),\ t_1, \ldots, t_r \in \mathbb{R} \ni$$

$$v = \mathrm{Fl}_{t_1}^{\xi_{X_{i_1}}} \cdots \mathrm{Fl}_{t_r}^{\xi_{X_{i_r}}}(u)$$

Summary

In summary:

Theorem

Choose a G_0-invariant $A_0 \subseteq \mathfrak{g}_-$ so that with $A = Ad(P)A_0$, $\mathcal{C}_A$ admits projective reparametrizations. If $(\mathcal{G} \to M, \omega)$ is regular and A_0 is bracket-generating, d_M^A is finite. If furthermore d_M^A is nondegenerate, then d_M^A is finitely arcwise connected, hence inner.

Projective geometries: setup

$$
\begin{aligned}
\mathfrak{g} &= \mathfrak{sl}(n+1,\mathbb{R}) = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 \\
&= \left\{ \begin{pmatrix} a & y^t \\ x & A \end{pmatrix} \mid A \in \mathfrak{gl}(n,\mathbb{R}), a = -tr(A), x, y \in \mathbb{R}^n \right\} \\
\mathfrak{p} &= \mathfrak{g}_0 + \mathfrak{g}_1 \\
G &= SL(n+1,\mathbb{R}) \\
P &= \text{isotropy group of line through } e_1 \\
G_0 &= GL(n,\mathbb{R}) \\
G/P &= SL(n+1,\mathbb{R})/GL(n,\mathbb{R}) \ltimes \mathbb{R}^n = \mathbb{P}^n \\
A_0 &= \mathfrak{g}_{-1} \quad \text{(our only choice!)} \\
\mathcal{C}_A &= \text{the class of projectively parametrized geodesics}
\end{aligned}
$$

Projective geometries: results 1

Theorem (Kobayashi 1978-79)

Let (M, g) be a Riemannian manifold with distance function δ_M and $Ric \leq -c^2 g$. Then

$$d_M(x, y) \geq \frac{2c}{\sqrt{n-1}} \delta_M(x, y) \quad \forall x, y \in M.$$

If M is complete Einstein with $Ric = -c^2 g$, then we have equality above, so in this case the projective automorphism group of M coincides with its isometry group.

Theorem (Kobayashi-Sasaki 1978)

Let (M, ω) be a complete torsionfree affine connection with positive semidefinite Ricci tensor. Then $d_M \equiv 0$.

Projective geometries: results 2

Wu (1981)

- d_M is the integrated form of an infinitesimal “Royden pseudometric”
- slightly stronger versions of the nondegeneracy and triviality conditions (based on weaker assumptions on the Ricci tensor)
- an analog of Brody’s theorem: d_M is nondegenerate if and only if there is no complete, projectively parametrized geodesic

Conformal Lorentz geometries: setup

$$
\begin{aligned}
\mathfrak{g} &= \mathfrak{so}(r+1,2), \quad r+1 = m \geq 3, r \geq 0 \\
&= \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 = \mathbb{R}^m + \mathfrak{co}(r,1) + \mathbb{R}^{m*} \\
\mathfrak{p} &= \mathfrak{g}_0 + \mathfrak{g}_1 \\
G &= PO(r+1,2) = O(r+1,2)/\{\pm I\} \\
P &= \text{isotropy group of line through } e_1 \text{ (“Poincaré group”)} \\
G_0 &\cong CO(\mathfrak{g}_{-1}) \cong CO(r,1) \\
G/P &= PO(r+1,2)/CO(r,1) \ltimes \mathbb{R}^m \\
&= \text{Möbius space } S^{(r,1)} \text{ of null lines in } \mathbb{R}^{m+2} \\
A_0 &= \text{null cone in } \mathfrak{g}_{-1} \\
\mathcal{C}_A &= \text{the class of projectively parametrized null geodesics}
\end{aligned}
$$

Conformal Lorentz geometries: results 1

Theorem (M. 1981)

Let (M, g) be a null geodesically complete Lorentzian manifold with $Ric(X, X) \leq 0$ for all null tangent vectors X. Then $d_M \equiv 0$.

null convergence condition (NCC): $Ric(X, X) \geq 0$ $\forall$ null X.

null generic condition (NGC): $\exists$ a point along every inextendible null geodesic at which $Ric(\dot\gamma, \dot\gamma) \neq 0$.

Conformal Lorentz geometries: results 2

Theorem (M. 1981)

Let (M, g) be a Lorentzian manifold satisfying the NCC and the NGC. Then d_M is nondegenerate.

Corollary (M. 2011 (a variant of the Hawking-Penrose singularity theorems)**)**

Let (M, h) be an Einstein Lorentz manifold. Suppose that there is a metric in the conformal class of h satisfying the NCC and the NGC. Then d_M is nondegenerate and every affinely parametrized null geodesic of (M, h) is incomplete.

d_M nondegenerate seems to describe BIG BANG cosmologies. For black hole models, d_M can degenerate along null geodesics which avoid the singularity.

Conformal Lorentz geometries: results 3

M. (1982)

- studied d_M for Lorentzian warped products, obtaining sufficient conditions on the warping function for d_M to be nondegenerate (and for $d_M \equiv 0$).
- explicitly computed d_M for Einstein-deSitter space (called the "Poincaré-Lorentz upper half-plane" by Nomizu); $d_M(x, y)$ for null-separated events is essentially redshift.

Dobarro-Ünal (2009)

- studied various energy conditions on static spacetimes, obtaining more explicit conditions on the warping function for d_M to be nondegenerate (and for $d_M \equiv 0$).

Contact projective geometry: setup

$$\begin{aligned}
\mathfrak{g} &= \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 = \mathfrak{sp}(2n+2,\mathbb{R}), n \geq 1 \\
\mathfrak{p} &= \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 \\
G &= Sp(2n+2,\mathbb{R}) \\
P &= \text{stabilizer of an oriented line in } \mathbb{R}^{2n+2} \\
G_0 &\cong CSp(\mathfrak{g}_{-1}) \\
G/P &= Sp(2n+2,\mathbb{R})/CSp(2n) \ltimes \mathbb{R}^{2n+1} = S^{2n+1} \\
A_0 &= \mathfrak{g}_{-1}, \qquad B_0 = \mathfrak{g}_{-2} \\
\mathcal{C}_A &= \text{projectively parametrized contact geodesics} \\
\mathcal{C}_B &= \text{projectively parametrized } \textit{chains}
\end{aligned}$$

Note: $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = \mathfrak{g}_{-2}$ and $[\mathfrak{g}_{-2}, \mathfrak{g}_1] = \mathfrak{g}_{-1}$

Contact projective geometry: results

Fox (2005), Čap-Žádník (2008)

- The inclusion $G = SP(2n+2,\mathbb{R}) \hookrightarrow SL(2n+2,\mathbb{R}) = \tilde{G}$ induces a "Fefferman construction" from a contact projective geometry $(\mathcal{G} \to M, \omega)$ to a projective geometry $(\tilde{\mathcal{G}} \to M, \tilde{\omega})$ on the same manifold M.
- The paths of $\tilde{\omega}$ are the contact projective geodesics plus the chains:
$$\tilde{\mathcal{C}}_{\mathfrak{g}_{-1}} = \mathcal{C}_{A} \cup \mathcal{C}_{B}$$

Clearly $d_M^A \geq \tilde{d}_M$ and $d_M^B \geq \tilde{d}_M$, so the Kobayashi-Wu criteria for nondegeneracy of the projective pseudodistance $\tilde{d}_M$ can be applied to both contact projective pseudodistances.

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Parabolic contact geometries

$$\begin{aligned}\mathfrak{g} &= \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 \\ A_0 &= \mathfrak{g}_{-2} \\ \mathcal{C}_A &= \text{projectively parametrized } \textit{chains}\end{aligned}$$

Note: nondegeneracy of the bracket on $\mathfrak{g}_{-1}$ implies that $[\mathfrak{g}_{-2}, \mathfrak{g}_1] = \mathfrak{g}_{-1}$, so that d_M^A defined with chains is always finite. (See [C-S, lemma 4.2.2])

Example: C-R geometries

H. Jacobowitz (1985) considers a slightly different problem: he shows that the 'endpoint map' $\mathfrak{g}_{-2} \to M$ is onto in some neighborhood of every point in the strictly pseudoconvex case. He uses a 'formal solution to the CR embedding problem', a 'weak version' of Moser normal form, and rather ugly calculations in local coordinates. He also gives a counterexample in the indefinite signature case. L. Koch (1988) proved the same result using the Fefferman construction. N. Kruzhilin (1986)?

Holomorphic projective geometries

Theorem (McKay 2006)

Complete complex parabolic geometries are flat.

Theorem (Kobayashi-Ochiai 1980)

A compact Einstein-Kähler manifold M admits a normal holomorphic projective geometry if and only if it is of constant holomorphic sectional curvature. The possibilities are:

- $M = \mathbb{P}^n$,
- M *is covered by a complex torus* $\mathbf{T}^n \quad (c_1 = 0)$*, or*
- M *is covered by the unit ball* $\mathbf{B}^n \subset \mathbb{C}^n \quad (c_1 < 0)$.

Holomorphic projective geometries (cont.)

Consider flat holomorphic projective geometries on a complex torus $\mathbf{T}^n$ (or flat holomorphic affine connections on a compact Kähler manifold M).

Then $d_M^K \equiv 0$ for the Kobayashi holomorphic pseudodistance.

What about the holomorphic projective pseudodistance, d_M^{HP}? In other words, what happens if we restrict our measurements to chains of (projectively parametrized) complex geodesics?

Holomorphic projective geometries (cont.)

Theorem (Y. Matsushima 1968)

The holomorphic affine structures on $\mathbf{T}^n$ are in one-to-one correspondence with the commutative associative algebra structures over $\mathbb{C}$ on $\mathbb{C}^n$.

Theorem (M. 1979)

$d^{HP}_{\mathbf{T}^n}$ is nondegenerate for the flat projective geometry underlying a holomorphic affine structure on $\mathbf{T}^n$ if and only if the algebra corresponding to that affine structure is semisimple.

So semisimplicity characterizes the 'maximally incomplete' situation and, moreover, is a projective invariant.

Summary

- The Kobayashi intrinsic pseudodistance construction extends to general parabolic geometries.
- Connection-preserving (and in certain situations much more general morphisms) are nonexpansive.
- Each pseudodistance is a coarse, global measure of incompleteness for distinguished curves of some fixed type.

Addendum, p. 1

The following result can now be regarded as a template.

Theorem ('principle of the little Picard theorem')

Let X and Y be regular parabolic geometries of the same type with $d_X = 0$ and d_Y nondegenerate. Then every morphism $f\colon X \to Y$ is a constant map. More generally, if Y is nondegenerate modulo a subset Δ, then every morphism $f\colon X \to Y$ is either constant or $f(X) \subset \Delta$.

Referring to the holomorphic case, Kobayashi says that "This is a trivial consequence of the fact that *f* is distance-decreasing."

S. Kobayashi, *Intrinsic distances, measures, and geometric function theory*, Bull. Amer. Math. Soc. **82** (1976), 357–416.

Addendum, p. 2

The following result seems to have the same flavor as the question treated in Charles' talk on Monday.

Theorem

Let X be a complex manifold and A a complex subspace of codimension at least 2. Let Y be a complete hyperbolic space. Then every holomorphic map $f\colon X - A \to Y$ extends to a holomorphic map $f\colon X \to Y$.

M. H. Kwack, Generalization of the big Picard theorem, Ann. of Math. (2) 90 (1969), 9-22. (See Kobayashi's 1976 Bulletin survey for a discussion.)