Schubert Cell Decomposition and Homology of Generalized Flag Manifolds

Diplomarbeit zur Erlangung
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Contents

Preface

The aim of my diploma thesis is to describe how generalized flag manifolds can be decomposed into so-called Schubert cells and how to compute the homology of these spaces using this cellular decomposition.

Given a complex semisimple Lie group G and a parabolic subgroup P of G we define a generalized flag manifold to be the homogeneous space G/P. For $G = SL_n(\mathbb{C})$, flag manifolds can be identified with the sets of partial flags of a certain type $(l_1, l_2, ..., l_k)$, that is

$$G/P = \{0 \subset V_1 \subset V_2 \subset ... \subset V_k \subset \mathbb{C}^n \text{ such that } \dim V_i = l_i\}.$$

Hence, complex Grassmann manifolds and complex projective spaces are examples for generalized flag manifolds.

I will now try to sketch the construction of the cellular decomposition of spaces G/P. If W denotes the Weyl group and B a Borel subgroup of G then we have

$$G = \bigsqcup_{w \in W} BwB.$$

This result is due to Bruhat and therefore called Bruhat decomposition of a semisimple Lie group. (In the case $G = SL_n(\mathbb{C})$ this means for example that every matrix in $G = SL_n(\mathbb{C})$ can be written in the form $b_1 w b_2$, where b_1 and b_2 are upper triangular matrices and w is a matrix in $G = SL_n(\mathbb{C})$ with exactly one non-zero entry in each row and column.)

It follows immediately from the Bruhat decomposition of the Lie group G that

$$G/B = \bigsqcup_{w \in W} Bwo,$$

where o is the image of B in G/B. Moreover, we will see that each orbit Bwo is diffeomorphic to a $\mathbb{C}^n$, where n is equal to the length of the Weyl group element w. Thus, we obtain the required cellular decomposition of G/B.

In case we consider a generalized flag manifold G/P, where P is an arbitrary parabolic subgroup the homogeneous space is the union of the orbits Bwo, o being the image of P in G/P, but they are not necessarily disjoint. We can realize G/P as a disjoint union of cells if we consider only cosets Bwo with w in a certain subset of the Weyl group called the Hasse diagram W^p associated to the parabolic P. Moreover, it turns out that the closure of a cell Bwo is the union of Bwo and all cells corresponding to elements of the Hasse diagram which are smaller than w in a certain ordering, called Bruhat ordering. By definition of the Bruhat ordering this in turn implies that the closure of a cell Bwo only contains cells of lower dimension. The cells Bwo are called Schubert cells and its closures are called Schubert varieties.

Finally, I give a brief description of the structure of the text:

We start the first chapter with a short review of the structure theory of complex semisimple Lie algebras and their representations. We will also study parabolic subalgebras and generalized flag manifolds will be introduced. References for Lie groups and Lie algebras are [**5**], [**9**] and [**10**].

The second chapter is devoted to the Weyl group. We will investigate the Hasse diagram corresponding to a parabolic subalgebra and the Bruhat ordering of the Weyl group. References for this chapter are [**1**] and [**4**].

In the third chapter we finally construct the cellular decomposition of spaces G/P. Moreover, we prove that the closure of a Schubert cell is the union of the cell with cells of lower dimension. The proof of this statement is taken from a paper by Bernstein, Gelfand and Gelfand [**2**].

In the fourth and final chapter we will use the preceding results on the cellular decomposition of G/P to determine its singular homology groups. This is done with the help of Borel-Moore homology, as in Borel and Haefliger [**3**] and in Appendix B of the book *Young Tableaux* by William Fulton [**6**]. It turns out that $H_*(G/P) = \bigoplus_{i \in \mathbb{Z}} H_i(G/P)$ is free abelian with one generator for each Schubert cell.

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CHAPTER 1

Lie Algebras and Generalized Flag Manifolds

1.1. Structure Theory of Complex Semisimple Lie Algebras

In the first section we will summarize various facts about complex semisimple Lie algebras required to understand what follows. We will omit the proofs of most of the results. A detailed account on the topic can be found for example in Knapp's book [**10**].

Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra, i.e. a finite-dimensional complex Lie algebra, that has no nonzero solvable ideals.

Every such $\mathfrak{g}$ contains a maximal abelian subalgebra $\mathfrak{h}$, which acts diagonally on $\mathfrak{g}$ by its adjoint representation $\mathrm{ad} : \mathfrak{h} \to \mathrm{End}_{\mathbb{C}}(\mathfrak{g})$. We call such a subalgebra $\mathfrak{h}$ a **Cartan subalgebra**.

Any two of these are conjugate by an inner automorphism.

Having fixed a Cartan subalgebra, we get a decomposition of $\mathfrak{g}$ as a direct sum of simultanous eigenspaces under the adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$, called **root-space decomposition**. The zero eigenspace is $\mathfrak{h}$ itself, since otherwise we could find an element X, not contained in $\mathfrak{h}$, such that $\mathrm{ad}(\mathfrak{h})X = 0$. But this would imply that $\mathfrak{h} \oplus \mathbb{C}X$ is an abelian subalgebra properly containing $\mathfrak{h}$, in contradiction with $\mathfrak{h}$ being maximal abelian. Hence we have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where each α is a nonzero linear functional on $\mathfrak{h}$ called a **root**, Δ denotes the finite set of these roots and $\mathfrak{g}_\alpha$ is defined as

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{ad}(H)X = \alpha(H)X\}$$

and called **root space**. All root spaces are one-dimensional.

Every finite-dimensional Lie algebra $\mathfrak{g}$ admits a symmetric bilinear form, the **Killing form**, as follows: if $X,\ Y \in \mathfrak{g}$ we set

$$B(X,Y) = \mathrm{Tr}(\mathrm{ad}X\mathrm{ad}Y).$$

The Killing form is non-degenerate if and only if the Lie algebra is semisimple. It is invariant under the group of automorphisms of $\mathfrak{g}$ as the following computation shows: let a be in $\mathrm{Aut}(\mathfrak{g})$, then

$$\begin{aligned} B(aX, aY) = \mathrm{Tr}(\mathrm{ad}(aX)\mathrm{ad}(aY)) &= \mathrm{Tr}(a(\mathrm{ad}X)a^{-1}a(\mathrm{ad}Y)a^{-1}) \\ &= \mathrm{Tr}(\mathrm{ad}X\mathrm{ad}Y) = B(X,Y). \end{aligned}$$

The Jacobi identity yields

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$$

with $\mathfrak{g}_0 = \mathfrak{h}$ and $\mathfrak{g}_{\alpha+\beta}$ understood to be zero if $\alpha+\beta$ is not a root. Hence $\mathrm{ad}(\mathfrak{g}_\alpha)\mathrm{ad}(\mathfrak{g}_\beta)$ carries $\mathfrak{g}_\gamma$ into $\mathfrak{g}_{\gamma+\alpha+\beta}$. Consequently, if $\beta \neq -\alpha$ then $\mathrm{ad}(\mathfrak{g}_\alpha)\mathrm{ad}(\mathfrak{g}_\beta)$ when written as a matrix in terms of a basis compatible with the root-space decomposition, has zero in every diagonal entry and thus its trace is zero. This tells us immediately, that $\mathfrak{g}_\alpha$ is perpendicular to $\mathfrak{g}_\beta$ with respect to the Killing form for all β other than $-\alpha$, which in turn implies that the Killing form is non-degenerate on $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$ and in particular on $\mathfrak{h} \times \mathfrak{h}$.

It follows, that to each α in $\mathfrak{h}^*$ corresponds an element H_α in $\mathfrak{h}$ satisfying $\alpha(H) = B(H, H_\alpha)$ for all $H \in \mathfrak{h}$. It turns out that if $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_{-\alpha}$, then

$$[X, Y] = B(X, Y)H_\alpha.$$

An important tool in the study of semisimple complex Lie algebras $\mathfrak{g}$ is the fact that every such $\mathfrak{g}$ contains embedded copies of $\mathfrak{sl}_2(\mathbb{C})$ and that the representations of $\mathfrak{sl}_2(\mathbb{C})$ are well known (see Section 1.2). With the help of these $\mathfrak{sl}_2(\mathbb{C})$-copies we can prove that we indeed have

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$$

if $\alpha, \beta \in \Delta \cup \{0\}$ and $\alpha + \beta \in \Delta$.

Next note that we can transfer the restriction of the Killing form to $\mathfrak{h}$ to a bilinear form on the dual $\mathfrak{h}^*$, by the definition

$$\langle \alpha, \beta \rangle := B(H_\alpha, H_\beta) = \alpha(H_\beta)$$

for $\alpha, \beta \in \mathfrak{h}^*$.

The $\mathbb{R}-$linear span of Δ, we denote it by V, is a real form of the vector space $\mathfrak{h}^*$. Denoting the $\mathbb{R}-$linear span of the H_α's for $\alpha \in \Delta$ by $\mathfrak{h}_0$, the elements of V are precisely those linear functionals that are real on $\mathfrak{h}_0$ and restriction from $\mathfrak{h}$ to $\mathfrak{h}_0$ is an $\mathbb{R}-$linear isomorphism of V onto $\mathfrak{h}_0^*$.

It turns out that the bilinear form $\langle . , . \rangle$ restricts to a positive definite inner product on $\mathfrak{h}_0^*$. Relative to this inner product, we introduce for $\alpha \in \Delta$ the reflection s_α in the hyperplane perpendicular to α, i.e.

$$s_\alpha(\varphi) = \varphi - \frac{2\langle \varphi, \alpha \rangle}{|\alpha|^2}\alpha$$

for $\varphi \in \mathfrak{h}_0^*$. The group generated by these root reflections is called the **Weyl group** W. W carries roots into roots. (A proof of this assertion can be found at the end of Section 1.2).

Next, we introduce a notion of positivity on $\mathfrak{h}_0^*$. To do so, we choose an ordered basis $\{H_1, ..., H_n\}$ of $\mathfrak{h}_0$ and define an element φ of $\mathfrak{h}_0^*$ to be positive if there exists some index k such that $\varphi(H_k) > 0$ and $\varphi(H_i) = 0$ for $1 \leq i < k$. This induces a total ordering, that is preserved under addition and multiplication by positive scalars by defining $\varphi > \psi$ if $\varphi - \psi$ is positive. We denote the set of positive roots by Δ^+. A positive root is called **simple** if it cannot be decomposed as a sum of two positive roots. Let $\Pi = \{\alpha_1, ..., \alpha_n\}$ be the set of simple roots. It's a fact that Π is a basis for $\mathfrak{h}_0^*$ and every root β can be written as $\beta = x_1\alpha_1 + ... + x_n\alpha_n$, where all the x_j are integers and have the same sign.

Conversely, we can start from a simple system, that is a set of roots $\{\alpha_i\}_{i=1}^n$ such that every element of Δ is a linear combination of these α_i's with integral coefficients all having the same sign, and define Δ^+ to be the the set of those roots which have only non-negative coefficients. Consider the ordered basis $\{H_i\}_{i=1,..,n}$ of $\mathfrak{h}_0$ defined by $\alpha_j(H_i) = \delta_{i,j}$, then any root is positive in the ordering induced by this basis if and only if it is in Δ^+.

Thus the Π's that can arise as sets of simple roots are precisely the possible simple systems.

Example 1.1.1. Take $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, let $\mathfrak{h}$ be the subalgebra of all diagonal matrices in $\mathfrak{g}$ then $\mathfrak{h}_0$ is the subalgebra of real diagonal matrices. Define a matrix $E_{i,j}$ by putting 1 in the (i,j)-th place and 0 elsewhere and an element e_i of $\mathfrak{h}^*$ by

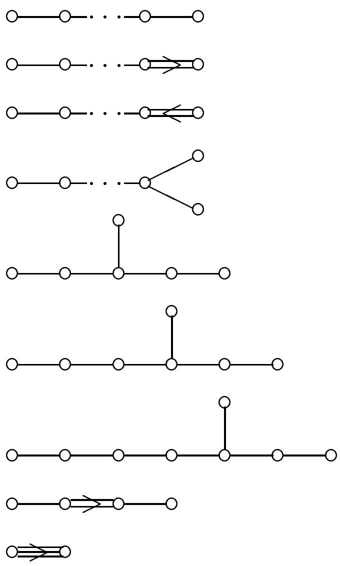
$$e_i \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{pmatrix} = h_i.$$

We have $\mathrm{ad}(H)E_{i,j} = [H, E_{i,j}] = (e_i(H) - e_j(H))E_{i,j}$. The subalgebra $\mathfrak{h}$ is maximal abelian and for each $H \in \mathfrak{h}$, $\mathrm{ad}(H)$ is diagonalized by the basis of $\mathfrak{g}$ consisting of elements of $\mathfrak{h}$ and the $E_{i,j}$ for $i \neq j$. Hence $\mathfrak{h}$ is a Cartan algebra and the linear functionals $e_i - e_j$ for $i \neq j$ are the roots of $\mathfrak{sl}_n(\mathbb{C})$. Choosing $\Delta^+ = \{e_i - e_j : i < j\}$, the associated set of simple roots is $\Pi = \{e_1 - e_2, e_2 - e_3, ..., e_{n-1} - e_n\}$.

It is a central fact in the classification theorem of complex semisimple Lie algebras, that we can associate to every complex semisimple Lie algebra $\mathfrak{g}$ a graph, called **Dynkin diagram**, that determines $\mathfrak{g}$ up to isomorphism.

We obtain the Dynkin diagram, by drawing one node for each simple root and joining two nodes, corresponding to distinct simple roots say α_i and α_j, by $\frac{4\langle\alpha_i,\alpha_j\rangle^2}{\langle\alpha_i,\alpha_i\rangle\langle\alpha_j,\alpha_j\rangle}$ edges. It turns out that the only possible values for $\frac{4\langle\alpha_i,\alpha_j\rangle^2}{\langle\alpha_i,\alpha_i\rangle\langle\alpha_j,\alpha_j\rangle}$ are $0, 1, 2$ and 3. Moreover, $\frac{2\langle\alpha_j,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}$ always is an integer. Hence, whenever a double or triple edge occurs, one of the roots is shorter than the other. In this case we add an arrow pointing towards the shorter of the two roots. The Dynkin diagrams of the complex simple Lie algebras are precisely:

Lie algebra	*Dynkin diagram*
$A_n = \mathfrak{sl}_{n+1}(\mathbb{C}), n \geq 1$	
$B_n = \mathfrak{so}_{2n+1}(\mathbb{C}), n \geq 2$	
$C_n = \mathfrak{sp}_n(\mathbb{C}), n \geq 3$	
$D_n = \mathfrak{so}_{2n}(\mathbb{C}), n \geq 4$	
E_6	
E_7	
E_8	
F_4	
G_2	



1.2. Some Representation Theory

In this section we shall be interested in representations of $\mathfrak{g}$ on a finite dimensional complex vector space V, i.e. complex-linear Lie algebra homomorphisms from $\mathfrak{g}$ into $\mathrm{End}_{\mathbb{C}}(V)$.

Let $\varphi : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$ be any finite dimensional representation. Whenever λ is in the dual $\mathfrak{h}^*$, we define V_λ to be the subspace

$$\{v \in V : \varphi(H)v = \lambda(H)v \,\text{for all}\, H \in \mathfrak{h}\}.$$

If $V_\lambda \neq 0$, we say that V_λ is a **weight space** and λ a **weight** of φ.

Next, we will prove that V is the direct sum of all weight spaces and that every weight is in $\mathfrak{h}_0^*$. To do so, we will make use of the fact, that every complex semisimple Lie algebra $\mathfrak{g}$ is spanned by embedded copies of $\mathfrak{sl}_2(\mathbb{C})$ and that the representations of $\mathfrak{sl}_2(\mathbb{C})$ are well known:

Every finite-dimensional complex-linear representation of $\mathfrak{sl}_2(\mathbb{C})$ is the direct sum of irreducible ones and for each integer $m \geq 1$, there is up to equivalence a unique irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ on a complex m-dimensional vectorspace. Consider the following basis of $\mathfrak{sl}_2(\mathbb{C})$:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These elements satisfy the bracket relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

It can be proved that if φ is an irreducible representation on a $(n+1)$-dimensional complex vector space V, then φ is unique up to equivalence. There is a basis $\{v_0, \varphi(f)v_0, \varphi(f)^2 v_0, ..., \varphi(f)^n v_0\}$ in V, such that $\varphi(e)v_0 = 0$, $\varphi(f)^{n+1}v_0 = 0$ and all of the vectors $\varphi(f)^i v_0$ are eigenvectors for $\varphi(h)$ with eigenvalues $n, n-2, ..., -n+2, -n$. Moreover we have $\varphi(e)\varphi(f)^m v_0 = m(n-m+1)\varphi(f)^{m-1}v_0$.

Next we will see that every complex semisimple Lie algebra contains subalgebras which are isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Choose any root α and take elements $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$ and H_α. We then have

$$[H_\alpha, X] = \alpha(H_\alpha)X,\ [H_\alpha, Y] = -\alpha(H_\alpha)Y \text{ and } [X, Y] = B(X, Y)H_\alpha.$$

Hence if we choose X and Y such that $B(X, Y) = \frac{2}{\alpha(H_\alpha)}$ and if we put $H'_\alpha = \frac{2}{\alpha(H_\alpha)}H_\alpha$ then the mapping

$$X \mapsto e, \quad Y \mapsto f, \quad H'_\alpha \mapsto h$$

extends linearly to an isomorphism

$$\mathfrak{sl}_\alpha := \operatorname{span}\{X, Y, H'_\alpha\} \cong \mathfrak{sl}_2(\mathbb{C}).$$

Now we restrict the representation of $\mathfrak{g}$ to $\mathfrak{sl}_\alpha$, and conclude from our discussion of $\mathfrak{sl}_\alpha$-representations, that $\varphi(H'_\alpha)$ is diagonable with integer eigenvalues. Recall that $\mathfrak{h}$ is abelian. Hence $\varphi(\mathfrak{h})$ is a commuting family of diagonable endomorphisms of V and therefore simultanously diagonable. It is now clear that V is indeed the direct sum of all weight spaces, and that weights are real-valued on $\mathfrak{h}_0$. Moreover, roots and weights are related by

$$\varphi(\mathfrak{g}_\alpha)V_\lambda \subseteq V_{\lambda+\alpha},$$

as can be seen by the following computation: If $X \in \mathfrak{g}_\alpha$, $v \in V_\lambda$ and $H \in \mathfrak{h}$, then

$$\begin{aligned}\varphi(H)\varphi(X)v = \varphi(X)\varphi(H)v + \varphi([H, X])v = \lambda(H)\varphi(X)v \\ +\alpha(H)\varphi(X)v = (\lambda+\alpha)(H)\varphi(X)v.\end{aligned}$$

Recall, that the form $\langle . , . \rangle$ restricts to an inner product on $\mathfrak{h}_0^*$. For each $\alpha \in \triangle$, W_α is the hyperplane in $\mathfrak{h}_0^*$ perpendicular to α. These hyperplanes partition $\mathfrak{h}_0^*$ into so-called **Weyl chambers** which are the connected components of $\mathfrak{h}_0^* \setminus \bigcup_{\alpha\in\Delta} W_\alpha$.

A linear functional is called **regular** if it is contained in the interior of some Weyl chamber. We define the fundamental chamber by requiring its elements λ to satisfy $\langle\lambda, \alpha\rangle > 0$ for all $\alpha \in \Pi$. We say an an element of the closure of the fundamental chamber is **dominant** for $\mathfrak{g}$. An element λ of $\mathfrak{h}^*$ is **algebraically integral** if $\frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ is in $\mathbb{Z}$ for all $\alpha \in \Pi$. Given $\Pi = \{\alpha_1, .., \alpha_n\}$, we define the set of **fundamental weights** $\{\lambda_1, ..., \lambda_n\}$ by requiring $\frac{2\langle\lambda_j,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle} = \delta_{i,j}$. The sum of the fundamental weights $\rho = \sum \lambda_i$ is contained in the fundamental chamber. Clearly, an element of $\mathfrak{h}^*$ is dominant if and only if it can be written as a linear combination of the fundamental weights with non-negative coefficients. It is algebraically integral if and only if all coefficients are integers. In Section 1.1 we have introduced an ordering on $\mathfrak{h}_0^*$. The largest weight in this ordering is called the **highest weight** of φ.

We can now formulate the classification theorem of finite-dimensional irreducible representations of $\mathfrak{g}$, the **Theorem of the Highest Weight**:

THEOREM 1.2.1. *Apart from equivalence the finite-dimensional irreducible representations φ of a complex semisimple Lie algebra $\mathfrak{g}$ are in one-to-one correspondence with the dominant algebraically integral linear functionals λ on $\mathfrak{h}$. The correspondence is given by the mapping that assigns to each finite-dimensional irreducible representation of $\mathfrak{g}$ its highest weight λ.*

We will not prove this theorem here but show some additional properties of the highest weight λ. (A proof can be found for example in [**10**, p.225]). But first we introduce the universal enveloping algebra of a Lie algebra.

DEFINITION 1.2.2. The **universal enveloping algebra** of a complex Lie algebra $\mathfrak{g}$ is a pair $(U(\mathfrak{g}),\ \iota)$ such that $U(\mathfrak{g})$ is an associative algebra with identity, $\iota : \mathfrak{g} \to U(\mathfrak{g})$ is a linear map satisfying $\iota([X,Y]) = \iota(X)\iota(Y) - \iota(Y)\iota(X)$ for $X, Y \in \mathfrak{g}$ and the following universal property holds: Whenever A is a complex associative algebra with identity, and $\pi : \mathfrak{g} \to A$ is a linear mapping such that $\pi(X)\pi(Y) - \pi(Y)\pi(X) = \pi([X,Y])$ for all X,Y $\in \mathfrak{g}$, then there exists a unique algebra homomorphism $\widetilde{\pi} : U(g) \to A$, $\widetilde{\pi}(1) = 1$, and the diagram

$$\begin{tikzcd} & U(\mathfrak{g}) \arrow[d, "\widetilde{\pi}"] \\ \mathfrak{g} \arrow[ur, "\iota"] \arrow[r, "\pi"'] & A \end{tikzcd}$$

commutes.

Applying the definition to $A = \mathrm{End}_{\mathbb{C}}(\mathfrak{g})$ we obtain to each representation φ an algebra homomorphism $\widetilde{\varphi}$. The mapping $\varphi \to \widetilde{\varphi}$ induces a one-to-one correspondence between representations of $\mathfrak{g}$ on complex vector spaces and left $U(\mathfrak{g})$-modules.

The theorem of Poincare-Birkhoff-Witt gives a basis of $U(\mathfrak{g})$ in terms of an ordered basis of $\mathfrak{g}$. More precisely we have: If $\{X_i\}_{i\in I}$ is a basis of $\mathfrak{g}$ and the index set I admits a total ordering, then the set of all monomials of the form $(\iota X_{i_1})^{j_1}...(\iota X_{i_n})^{j_n}$ with $i_1 < ... < i_n$ and all $j_k \geq 0$ is a basis of $U(\mathfrak{g})$.

We now return to the discussion of the properties of the highest weight λ of an irreducible finite-dimensional representation of $\mathfrak{g}$. Assume that $\alpha \in \Delta^+$, $X \in \mathfrak{g}_\alpha$ and $v \in V_\lambda$, then $\varphi(X)v \in V_{\lambda+\alpha}$ but $\lambda + \alpha$

exceeds λ and thus cannot be a weight, hence $V_{\lambda+\alpha} = 0$. It follows that *any root vector* $X \in \mathfrak{g}_\alpha$ *for* $\alpha \in \Delta^+$ *annihilates the members of* V_λ.

Next we show that *the weight space* V_λ *is one-dimensional.* This is where the universal enveloping algebra comes in. By the theorem of Poincare-Birkhoff-Witt a basis of $U(\mathfrak{g})$ is given by monomials of the form $X^{k_1}_{-\alpha_1}...X^{k_i}_{-\alpha_i}H^{l_1}_1...H^{l_n}_n X^{m_1}_{\alpha_1}...X^{m_i}_{\alpha_i}$, where $\alpha_j \in \Delta^+$, $1 \le j \le i$, $X_{\alpha_j} \in \mathfrak{g}_{\alpha_j}$ and $H_1, ..., H_n$ form a basis of $\mathfrak{h}$. Applying X_{α_j} to $v \in V_\lambda$ gives zero, as we have seen above. The H's multiply by constants and the $X_{-\alpha_j}$'s push the weight down. Since φ is irreducible, $\widetilde{\varphi}(U(\mathfrak{g}))v = V$. But the only elements in V_λ that can be obtained by applying $\widetilde{\varphi}(U(\mathfrak{g}))$ to v are multiples of v and we conclude that V_λ is one-dimensional.

Moreover, we see that *every weight of* φ *is of the form* $\lambda - \sum_{i=1}^n n_i\alpha_i$ *with* $n_i \ge 0$ *and* $\alpha_i \in \Pi$.

We can also prove that *the members of* V_λ*are the only vectors in* V*, that are annihilated by all root vectors* $X \in \mathfrak{g}_\alpha$ *for* $\alpha \in \Delta^+$. Assume that $v \notin V_\lambda$ satisfies $\varphi(X_\alpha)v = 0$ for all $\alpha \in \Delta^+$. If v has a part in V_λ we subtract it and hence we can assume that the largest weight μ such that v has a nonzero part in the corresponding weight space is smaller than λ. Let v' be that part, then $\varphi(X_\alpha)v' = 0$ for all $\alpha \in \Delta^+$. Consequently, any v in V is of the form $\mathbb{C}X^{k_1}_{-\alpha_1}...X^{k_i}_{-\alpha_i}v'$, but none of these is in V_λ which is in contradiction with λ being a weight.

Finally, we show that the *Weyl group maps weights into weights.* It clearly suffices to show that for any root α, and any weight μ, $s_\alpha(\mu) = \mu - \frac{2\langle\mu,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha$ is a weight. Consider the $\mathfrak{sl}_\alpha$-invariant subspace $\bigoplus_{n\in\mathbb{Z}} V_{\mu+n\alpha}$ of V. The endomorphism $\varphi(H'_\alpha)$ sends $v \in V_\mu$ to $\mu(H'_\alpha)v$ and since the eigenvalues of $\mathfrak{sl}_2(\mathbb{C})$ are symmetric about the origin also

$$-\mu(H'_\alpha) = \mu(H'_\alpha) - \frac{2\mu(H'_\alpha)}{\alpha(H'_\alpha)}\alpha(H'_\alpha) = s_\alpha\mu(H'_\alpha)$$

is an eigenvalue of $\varphi(H'_\alpha)$, hence $V_{s_\alpha(\mu)} \neq 0$ and the claim follows. Moreover we have $\dim V_\mu = \dim V_{s_\alpha(\mu)}$.

In particular, this argument shows that the roots of a semisimple Lie algebra are preserved under the Weyl group.

1.3. On Parabolic Subalgebras

In this section we will be concerned with a certain class of subalgebras of complex semisimple Lie algebras.

The simplest of these, the maximal solvable subalgebras of $\mathfrak{g}$, are called **Borel subalgebras**.

Recall, that a choice of a Cartan subalgebra $\mathfrak{h}$ determines a root-space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$. If in addition we choose a set of positive roots Δ^+, we can associate to $\mathfrak{g}$ and Δ^+ a subalgebra

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+, \text{ where } \mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha.$$

Using $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$, we see that $\mathfrak{n}^+$ is nilpotent and $[\mathfrak{b}, \mathfrak{b}] \subseteq \mathfrak{n}^+$, so $\mathfrak{b}$ is solvable.

Note, that if any element X is contained in a subalgebra of the Lie algebra $\mathfrak{g}$ containing $\mathfrak{h}$ and X decomposes as $X = H + \sum_{i=1}^k X_{\alpha_i}$ with $H \in \mathfrak{h}$ and $X_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$, then also H and the X_{α_i}'s are contained in this subalgebra. Since the root spaces are eigenspaces for $\mathrm{ad}(H)$, $H \in \mathfrak{h}$, this follows from the fact that projections onto eigenspaces of an operator are polynomials in the operator. Hence any subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ which contains $\mathfrak{h}$ is the direct sum of $\mathfrak{h}$ and some of the root spaces. Consequently, if $\mathfrak{s}$ strictly contains $\mathfrak{b}$, it has to contain at least one negative root space, say $\mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta^+$. But then $\mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha] \oplus \mathfrak{g}_\alpha$ is a subalgebra of $\mathfrak{s}$ isomorphic to the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Since any subalgebra of a solvable Lie algebra is solvable we conclude that $\mathfrak{s}$ cannot be solvable and thus $\mathfrak{b}$ is maximal solvable, hence Borel. We call $\mathfrak{b}$ the **standard Borel subalgebra** associated to $\mathfrak{h}$ and Δ^+.

It is a general fact that any two Borel subalgebras are conjugate by an inner automorphism, i.e. every Borel subalgebra is conjugate to a standard Borel subalgebra. A proof of this result can be found for example in [**9**].

A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called **parabolic**, if it contains a Borel subalgebra. In the following we fix a Cartan subalgebra $\mathfrak{h}$ and a set of positive roots Δ^+. Since any Borel subalgebra is conjugate to a standard Borel subalgebra, any parabolic subalgebra is conjugate to a **standard parabolic subalgebra**, i.e. a subalgebra that contains $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$.

We can now classify the parabolic subalgebras up to conjugacy: As we have observed above, any standard parabolic subalgebra must be a direct sum of $\mathfrak{b}$ and some negative root spaces, so there must be a subset $\Phi \subset \Delta^+$, such that

$$\mathfrak{p} = \bigoplus_{\alpha \in -\Phi} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

In order for $\mathfrak{p}$ to be a subalgebra, we must have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta} \subseteq \mathfrak{p}$ for $\mathfrak{g}_\alpha, \mathfrak{g}_\beta \subseteq \mathfrak{p}$. Consequently, for $\alpha, \beta \in -\Phi \cup \Delta^+$ such that $\alpha + \beta \in \Delta$ this implies $\alpha + \beta \in -\Phi \cup \Delta^+$. If we take $\alpha, \beta \in -\Phi$ the sum $\alpha + \beta$ is negative and thus $\alpha + \beta$ is contained in $-\Phi$. We denote the set of simple roots in Φ by Σ. Then the above argument shows that any linear combination of roots in Σ with positive coefficients which is a root has to be contained in Φ.

Conversely, if $\alpha \in \Phi \subseteq \Delta^+$ is not simple, there is by definition a simple root α_i such that $\alpha - \alpha_i \in \Delta^+$. But then $-\alpha_i = (\alpha - \alpha_i) - \alpha \in -\Phi \cup \Delta^+$and since $-\alpha_i$ is negative it follows that $-\alpha_i \in -\Phi$, hence $\alpha_i \in \Phi$. Inductively we conclude that all simple roots that occur in the decomposition of α with a nonzero coefficient are elements of Σ. Thus, Φ is precisely the set of all linear combinations of simple roots in Σ with positive coefficients, that are roots.

Putting the preceding results together, we conclude that up to conjugation parabolic subalgebras $\mathfrak{p}$ of $\mathfrak{g}$ are in one-to-one correspondence with subsets $\Sigma \subseteq \Pi$: We associate to a standard parabolic $\mathfrak{p}$ the subset $\Sigma = \{\alpha \in \Pi : \mathfrak{g}_{-\alpha} \in \mathfrak{p}\}$ and conversely the parabolic subalgebra associated to a set Σ of simple roots is the direct sum of $\mathfrak{b}$ and the root spaces $\mathfrak{g}_{-\alpha}$ where α is a sum of elements of Σ. The above discussion shows that the two mappings are inverse to each other.

Next we introduce a notation for standard parabolics $\mathfrak{p} \subseteq \mathfrak{g}$ in terms of Dynkin diagrams: We substitute all nodes in the Dynkin diagram of $\mathfrak{g}$ which correspond to simple roots in $\Pi \setminus \Sigma$ by crosses.

Example 1.3.1. In the case of $\mathfrak{sl}_n(\mathbb{C})$ the standard Borel subalgebras consist of upper triangular matrices and the standard parabolics are block-upper-triangular matrices: if $e_i - e_j$ is a root which is not contained in Σ, then for any $k \geq j$ and $l \leq i$ the (k,l)-th entry of every element in the corresponding parabolic subalgebra is 0.

Consider for example $\mathfrak{sl}_4(\mathbb{C})$. Here the parabolic subalgebra corresponding to the set $\Sigma = \{e_1 - e_2,\ e_3 - e_4\}$, in Dynkin diagramm notation $\circ\!-\!\!\!-\!\times\!-\!\!\!-\!\circ$, consists of matrices of the form

$$\mathfrak{p} = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \in \mathfrak{sl}_4(\mathbb{C}) \right\}$$

and the parabolic subalgebra corresponding to $\Sigma = \{e_2 - e_3,\ e_3 - e_4\}$, in Dynkin diagram notation $\times\!-\!\!\!-\!\circ\!-\!\!\!-\!\circ$ is

$$\mathfrak{p}' = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \in \mathfrak{sl}_4(\mathbb{C}) \right\}.$$

Defining the so-called **Levi factor** $\mathfrak{l} = \bigoplus_{\alpha \in \Phi \cup -\Phi} \mathfrak{g}_\alpha \oplus \mathfrak{h}$ and the **nilpotent radical** $\mathfrak{u} = \bigoplus_{\alpha \in \Delta^+ \setminus \Phi} \mathfrak{g}_\alpha$, we have $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$. We now prove the following Proposition:

Proposition 1.3.2. *Let $\mathfrak{p}$ be a standard parabolic subalgebra of a semisimple complex Lie algebra $\mathfrak{g}$ and let $\mathfrak{l}$ and $\mathfrak{u}$ be defined as above, then*

(a) $\mathfrak{l}$ and $\mathfrak{u}$ are subalgebras and $\mathfrak{u}$ even is an ideal in $\mathfrak{p}$

(b) $\mathfrak{l}$ is reductive with center $\mathfrak{h}' = \bigcap_{\alpha \in \Phi \cup -\Phi} \ker \alpha$ and with semisimple part $\mathfrak{l}_{ss} = \mathfrak{h}'' \oplus \bigoplus_{\alpha \in \Phi \cup -\Phi} \mathfrak{g}_\alpha$, where $\mathfrak{h}'' = \sum_{\alpha \in \Phi \cup -\Phi} \mathbb{C} H_\alpha$.

Proof. We first prove (a). To see that $\mathfrak{l}$ is a subalgebra of $\mathfrak{p}$, just note that $\Phi \cup -\Phi = (\mathrm{span}\Sigma) \cap \Delta$ and thus if $\alpha, \beta \in \Phi \cup -\Phi$ and $\alpha + \beta$ is a root we also have $\alpha + \beta \in \Phi \cup -\Phi$. The fact that $\mathfrak{u}$ is an ideal can be seen as follows: If α is any root in $\Delta^+ \setminus \Phi$, its decomposition into simple roots contains at least one simple root $\alpha_i \notin \Sigma$ with coefficient, say n_i. Therefore, to have $\alpha + \beta \in \Phi \cup -\Delta^+$ we must take a root $\beta = \sum m_j \alpha_j$ such that α_i occurs with coefficient $-n_i$. However, there is no such root β in $-\Phi \cup \Delta^+$.

(b) Since $\mathfrak{h}'$ is spanned by those elements in $\mathfrak{h}$ satisfying $\alpha(H) = 0$ for all $\alpha \in \Phi \cup -\Phi$ it is contained in the center of $\mathfrak{l}$. Next observe that $\mathfrak{h}' \cap \mathfrak{h}'' = \{0\}$, since we have $\alpha(H_\alpha) = B(H_\alpha, H_\alpha) \neq 0$ for any root α. The H_α, $\alpha \in \Delta$, form a basis for $\mathfrak{h}$, the H_α, $\alpha \in \Sigma$, form a basis for $\mathfrak{h}''$ and $\mathfrak{h}' = \bigcap_{\alpha\in\Sigma} \ker\, \alpha$. Hence $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$ by dimensional reasons. It follows that $\mathfrak{l} = \mathfrak{h}' \oplus \mathfrak{l}_{ss}$.

To prove that $\mathfrak{l}$ is reductive, i.e. that it is the Lie algebra direct sum of a semisimple Lie algebra and an abelian Lie algebra, it is enough to show that $\mathfrak{l}_{ss}$ has no solvable ideals. Since any ideal in $\mathfrak{l}_{ss}$ clearly is an ideal in $\mathfrak{l}$ it suffices to show that every solvable ideal in $\mathfrak{l}$ is contained in $\mathfrak{h}'$. From the fact that that projections onto eigenspaces of an operator are polynomials in the operator we conclude, as in the beginning of this section, that any ideal in $\mathfrak{l}$ is the direct sum of a subspace of $\mathfrak{h}$ and possibly some of the root spaces. It follows, that every ideal in $\mathfrak{l}$ not contained in $\mathfrak{h}'$ contains at least either a root space or an element of $\mathfrak{h} \setminus \mathfrak{h}'$. Let I be such an ideal. In order to see that I has to contain at least one of the root spaces, assume that it contains an element H of $\mathfrak{h} \setminus \mathfrak{h}'$. Then there is a root space, say $\mathfrak{g}_\alpha$, on which this element acts nontrivially, that is $[H, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha$ and it follows that $\mathfrak{g}_\alpha$ is contained in I. But then also $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha]$ and $\mathfrak{g}_{-\alpha} = [[\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha], \mathfrak{g}_{-\alpha}]$ are contained in I and consequently so is $\mathfrak{sl}_\alpha = \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha] \oplus \mathfrak{g}_\alpha$. Since $\mathfrak{sl}_\alpha$ is not solvable, I cannot be solvable either. This proves the claim.

Moreover, since the center of $\mathfrak{l}$ is a solvable ideal in $\mathfrak{l}$ it coincides with $\mathfrak{h}'$ as asserted. □

Remark 1.3.3. The number of crosses in the Dynkin diagram of a parabolic subalgebra $\mathfrak{p}$ equals the dimension of the center $\mathfrak{h}'$ of the Levi factor. We obtain the Dynkin diagram of the semisimple part $\mathfrak{l}_{ss}$ if we remove from the Dynkin diagram of $\mathfrak{p}$ all crossed nodes as well as all edges joining any node with a crossed one.

Example 1.3.4. Consider the subalgebra $\mathfrak{p}$ of $\mathfrak{sl}_4(\mathbb{C})$ as in Example 1.3.1 again. Then the subalgebra $\mathfrak{l}$ consists of the diagonal blocks

$$\mathfrak{l} = \left\{ \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \in \mathfrak{sl}_4(\mathbb{C}) \right\}$$

and nilpotent radical in this example is

$$\mathfrak{u} = \left\{ \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sl}_4(\mathbb{C}) \right\}.$$

The center $\mathfrak{h}'$ of $\mathfrak{l}$ is spanned by the element $\mathrm{diag}(1, 1, -1, -1)$ and the semisimple part $\mathfrak{l}_{ss}$ equals $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.

1.4. Generalized Flag Manifolds

Assume as before that $\mathfrak{g}$ is a complex semisimple Lie algebra, $\mathfrak{p} \subset \mathfrak{g}$ a parabolic subalgebra and let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Our next aim is to introduce and describe the homogeneous spaces associated to parabolic subalgebras.

To begin with, we introduce the notion of a **parabolic subgroup** of G. It turns out that the appropriate definition is the following

$$P := N_G(\mathfrak{p}) = \{g \in G \mid \mathrm{Ad}(\mathfrak{g})\mathfrak{p} \subseteq \mathfrak{p}\}.$$

Since P is closed, it is a Lie subgroup of G. The groups associated to Borel subalgebras are called **Borel subgroups**.

Let us observe that the Lie algebra of a parabolic subgroup P is $\mathfrak{p}$. To do so we have to verify that

$$\mathfrak{p} = N_{\mathfrak{g}}(\mathfrak{p}) = \{X \in \mathfrak{g} \mid \mathrm{ad}(X)\mathfrak{p} \subseteq \mathfrak{p}\}.$$

Clearly $N_{\mathfrak{g}}(\mathfrak{p})$ is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{p}$. As we have noted in Section 1.3, every subalgebra of $\mathfrak{g}$ is the direct sum of $\mathfrak{h}$ and some root spaces. This implies that $N_{\mathfrak{g}}(\mathfrak{p})$ is the direct sum of $\mathfrak{p}$ and, possibly, some negative root spaces not contained in $\mathfrak{p}$. Suppose that $X \in \mathfrak{g}_{-\alpha}$ is not in $\mathfrak{p}$, then also $[X, H] \notin \mathfrak{p}$ for all $H \in \mathfrak{h}$, hence $X \notin N_{\mathfrak{g}}(\mathfrak{p})$. Thus we see that $N_{\mathfrak{g}}(\mathfrak{p})$ and $\mathfrak{p}$ coincide.

Recall that every parabolic subalgebra $\mathfrak{p}'$ is conjugate by an inner automorphism to a standard parabolic $\mathfrak{p}$, that is there is a $g \in G$ such that $\mathrm{Ad}(g)\mathfrak{p}' = \mathfrak{p}$. Now let p' be any element in $P' = N_G(\mathfrak{p}')$. Then we have

$$\mathrm{Ad}(gg'g^{-1})\mathfrak{p} = \mathrm{Ad}(g)\mathrm{Ad}(g')\mathrm{Ad}(g^{-1})\mathfrak{p} \subseteq \mathfrak{p}.$$

Hence we conclude that $gP'g^{-1} = P$, i.e. every parabolic subgroup is conjugate to a standard parabolic subgroup.

We can now introduce the central object in this paper.

Definition 1.4.1. Let G be a complex semisimple Lie group and P a parabolic subgroup. Then homogeneous spaces of the form

$$X = G/P,$$

are called **generalized flag manifolds.**

As we have seen, every parabolic subgroup is conjugate to a standard parabolic subgroup. Since the homogeneous spaces G/P and G/gPg^{-1} are diffeomorphic this implies that in the above definition we can assume P to be standard parabolic.

Moreover, it is no restriction to assume that G is simply connected: Let $X = G/P$ and let $\pi : \widetilde{G} \to G$ be its universal covering, then $\widetilde{G}$ acts transitively on X by $\widetilde{g} * x = \pi(\widetilde{g})x$ with stabilizer $\widetilde{P} = \pi^{-1}(P)$. Since $T_e\pi \circ \mathrm{Ad}(\pi(\widetilde{g})) = \mathrm{Ad}(\widetilde{g}) \circ T_e\pi$ and $T_e\pi = \mathrm{id}$ we have $\pi^{-1}(N_G(\mathfrak{p})) = N_{\widetilde{G}}(\mathfrak{p})$, hence $\widetilde{P}$ is parabolic.

Thus, assigning to a homogenous space G/P the Dynkin diagram of its associated standard parabolic subalgebra, yields a classification of generalized flag manifolds in terms of Dynkin diagrams.

In the case $G = SL_n(\mathbb{C})$ we just speak of flag manifolds. The following example shows where this term comes from:

Example 1.4.2. Let $V = \mathbb{C}^n$ and $(l_1, ..., l_k)$ a strictly increasing sequence of natural numbers $0 < l_1 < ... < l_k < n = \dim V$. Then a flag of type $(l_1, ..., l_k)$ is a sequence of nested subspaces

$$V_1 \subset ... \subset V_i \subset ... \subset V_k \subset V$$

such that $\dim V_i = l_i$. If $\{v_i\}_{1 \leq i \leq n}$ is the standard basis of V, then the standard flag of type $(l_1, ..., l_k)$ is given by $V_i =< v_1, ..., v_{l_i} >$.

Consider the set $F_{l_1,...,l_k}$ of all flags of a certain type $(l_1, ..., l_k)$. Clearly $SL_n(\mathbb{C})$ acts transitively on $F_{l_1,...,l_k}$ and the stabilizer P of the standard flag of type $(l_1, ..., l_k)$ consists of matrices of the form

$$\begin{pmatrix} A_1 & * & * & * \\ & A_2 & * & * \\ & & \ddots & * \\ & & & A_{l+1} \end{pmatrix}$$

where the A_i are $(l_i - l_{i-1}) \times (l_i - l_{i-1})$ blocks, with $l_0 = 0$ and $l_{k+1} = n$. We will see in Example 1.5.1 that P is exactly the standard parabolic subgroup corresponding to the set $\Sigma = \Pi \setminus \{e_{l_1} - e_{l_1+1}, .., e_{l_k} - e_{l_k+1}\}$. Identifying $F_{l_1,...,l_k}$ with $SL_n(\mathbb{C})/P$ we turn this set into a flag manifold. Conversely, this shows that every flag manifold can be identified with the set of all flags of a certain type.

Note that F_l is the Grassmann manifold of all l-dimensional subspaces and F_1 is the $(n-1)$-dimensional projective space, hence these are particular examples of flag manifolds.

Example 1.4.3. Assume that V is a complex n-dimensional vector space equipped with a nondegenerate bilinear form $Q : V \times V \to \mathbb{C}$. We call a flag $V_1 \subset ... \subset V_k \subset V$ isotropic, if $Q \mid_{V_k \times V_k} = 0$. Now let $V_1 \subset ... \subset V_k \subset V$ and $V_1' \subset ... \subset V_k' \subset V$ be two isotropic flags of type $(l_1, ..., l_k)$ and choose bases $\{v_p\}_{1 \leq p \leq l_k}$ of V_k and $\{v_p'\}_{1 \leq p \leq l_k}$ of V_k', respectively. Then we find an $g \in GL_n(\mathbb{C})$ such that $gv_p = v'_p$. Since we consider isotropic flags, we have $Q(v_p, v_q) = 0 = Q(gv_p, gv_q)$ for all p and q and therefore we see that $g : V_k \to V'_k$ is an isometry. Applying Witt's theorem we conclude that there exists an isometry $\overline{g} : V \to V'$ satisfying $\overline{g}(V_i) = V'_i$, $1 \leq i \leq k$.

In case the bilinear form Q is skew-symmetric the above argument shows that $Sp_n(\mathbb{C})$ acts transitively on the set of all isotropic flags of a certain type. It turns out that the stabilizer of a fixed flag is a parabolic subgroup of $Sp_n(\mathbb{C})$. Moreover, the sets of isotropic flags of a certain type are exactly the generalized flag manifold for $G = Sp_n(\mathbb{C})$.

If Q is symmetric we can only conclude from Witt's theorem that $O_n(\mathbb{C})$ acts transitively on the set of all isotropic flags of a certain type. It turns out that the action of $SO_n(\mathbb{C})$, $n = 2m+1$, is transitive as well. However, if $n = 2m$ the group $SO_n(\mathbb{C})$ acts only transitively on the set of all isotropic flags of type $(l_1, ..., l_k)$ if $l_k < m$. If $l_k = m$, it has two orbits. In order to see this, choose a basis $\{v_1, ..., v_{2m}\}$ for $\mathbb{C}^{2m}$ in terms of which the symmetric bilinear form Q is given by

$$Q(v_i, v_{i+m}) = Q(v_{i+m}, v_i) = 1$$

and

$$Q(v_i, v_j) = 0 \text{ if } j \neq i \pm m.$$

Then the subspaces $V = \langle v_1, ..., v_m \rangle$ and $W = \langle v_1, ..., v_{m-1}, v_{2m} \rangle$ are both isotropic subspaces. However, there is no element in $SO_{2m}(\mathbb{C})$ that takes V to W. This follows from the fact that the representation of $\mathfrak{so}_{2m}$ on $\bigwedge^m \mathbb{C}^{2m}$ has two irreducible factors with highest weight vectors $v_1 \wedge ... \wedge v_m$ and $v_1 \wedge .. \wedge v_{m-1} \wedge v_{2m}$, respectively (see [**5**, p. 287]).

It can be shown that generalized flag manifolds are exactly the compact homogenous spaces associated to semisimple complex Lie groups, i.e. $P \subseteq G$ is parabolic if and only if G/P is compact. Here, we will only indicate how to see that generalized flag manifolds are compact:

Consider a complex semisimple Lie group G with Lie algebra $\mathfrak{g}$. It turns out that $\mathfrak{g}$ contains a compact real form, that is a compact real Lie algebra such that its complexification is just the given complex Lie algebra. (A Lie algebra $\mathfrak{k}$ is called compact if the group $\mathrm{Int}\mathfrak{k}$ is compact). A compact real form of $\mathfrak{g}$ can be constructed by putting

$$\mathfrak{k} = i\mathfrak{h}_0 + \sum_{\alpha\in\Delta} \mathbb{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha\in\Delta} i\mathbb{R}(X_\alpha + X_{-\alpha})$$

for certain vectors $X_\alpha \in \mathfrak{g}_\alpha$ and any other is conjugate to this one. See [**10**, Chapter 5] for all this.

In the following we view the complex Lie algbera $\mathfrak{g}$ as a real one. Observe that $\mathfrak{n}^- = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}$ is contained in $\mathfrak{k} + \mathfrak{n}^+$: Any $Z \in \mathfrak{n}^-$ can be written as $\sum c_i X_{-\alpha_i}$where the c_i's are complex numbers and the α_i's are some positive roots. Then the element $\overline{Z} = \sum \overline{c_i} X_{\alpha_i}$, $\overline{c_i}$ being the complex conjugate of c_i, is in $\mathfrak{n}^+$ and $Z = (Z - \overline{Z}) + \overline{Z} \in \mathfrak{k} + \mathfrak{n}^+$. Hence,

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ = \mathfrak{k} \oplus \mathfrak{h}_0 \oplus \mathfrak{n}^+.$$

This result is called the Iwasawa decomposition of the Lie algebra $\mathfrak{g}$.

There is a corresponding decomposition on Lie group level. Let K, A and N denote the connected real Lie subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{h}_0$ and $\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$, respectively. Then the Iwasawa-decomposition of a complex semisimple Lie group says that the multiplication map $K \times A \times N \to G$ is a diffeomorphism onto. Moreover, one can show that the connected real Lie subgroup K of G with Lie algebra $\mathfrak{k}$ is a maximal compact subgroup of G. A proof of the Iwasawa-decomposition can be found for example in [**10**, p.317]. Using this decomposition it is not difficult to see that G/P is indeed compact: Since $AN \subseteq B$ we have $G = KB$ and this yields that K acts transitively on G/P, if $P \subseteq B$. It follows that G/P is diffeomorphic to $K/(K \cap P)$ and therefore it is compact.

1.5. Projective Realisation of G/P

Let $X = G/P$ be a generalized flag manifold and take G to be simply connected. Let $\Sigma \subseteq \Pi$ be the set of simple roots assigned to the standard parabolic subalgebra $\mathfrak{p}$ and define a weight

$$\rho^p = \sum_{i \text{ such that } \alpha_i \in \Pi\setminus\Sigma} \lambda_i$$

where λ_i is th fundamental weight corresponding to the simple root α_i.

Then we have

$$\frac{2\langle \rho^p, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle} = \begin{cases} 0 & \text{if } \alpha_i \in \Sigma \\ 1 & \text{if } \alpha_i \in \Pi \setminus \Sigma \end{cases}.$$

Due to the Theorem of the Highest Weight there is a finite-dimensional irreducible representation $\varphi' : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$ with highest weight ρ^p. Since G is simply connected this representation exponentiates to a representation $\varphi : G \to \mathrm{Gl}_{\mathbb{C}}(V)$ of G.

Let f be a nonzero weight vector of weight ρ^p and $[f]$ the corresponding line in the projective space $\mathbb{P}(V)$. Our aim is to prove that φ factorizes to a diffeomorphism

$$G/P \to G[f] \subseteq \mathbb{P}(V).$$

Therefore, we need to show that P and the stabilizer $S = \{g \in G \mid \varphi(g)[f] = [f]\}$ coincide.

Let us observe first that $\mathfrak{p}$ is equal to the Lie algebra of S, that is to the stabilizer of $[f]$ in $\mathfrak{g}$: Clearly $\mathfrak{h}$ preserves f up to scalars. Being a highest weight vector, f is annihilated by the positive root spaces of

$\mathfrak{g}$. We will show that $\mathfrak{l}$ annihilates f as well: For any $\alpha \in \Sigma$ consider subrepresentation

$$\varphi' : \mathfrak{sl}_\alpha \to \mathrm{End}_{\mathbb{C}}(\bigoplus_{n\in\mathbb{Z}} V_{\rho^p+n\alpha})$$

of $\mathfrak{sl}_\alpha$ on V. Since $\rho^p(H_\alpha) = \frac{2\langle\rho^p,\alpha\rangle}{\langle\alpha,\alpha\rangle} = 0$ and since $\rho^p + \alpha$ is not a weight by maximality of ρ^p, we conclude from the symmetry of the eigenvalues of $\mathfrak{sl}_\alpha$ about the origin that $\rho^p - \alpha$ cannot be a weight. Hence $\mathfrak{g}_\alpha$ annihilates f for every α in the span of Σ and these are precisely the $\mathfrak{g}_\alpha \in \mathfrak{l}$, so the claim follows. Summarizing, we have seen that $\mathfrak{p}$ stabilizes $[f]$.

Conversely, take any element which is not contained in $\mathfrak{p}$, say $Y \in \mathfrak{g}_{-\beta} \subseteq \mathfrak{u}^- = \bigoplus_{\alpha\in-(\Delta^+\setminus\Phi)} \mathfrak{g}_\alpha$. Let $X \in \mathfrak{g}_\beta$, then the following computation shows that Y does not annihilate f:

$$\begin{aligned} XYf = XYf - YXf = [X,Y]f &= B(X,Y)H_\beta f \\ &= B(X,Y)\,\langle\rho^p,\beta\rangle\, f \neq 0. \end{aligned}$$

We conclude that $\mathfrak{p}$ is precisely the stabilizer of $[f]$ in $\mathfrak{g}$, as asserted.

To continue the proof of the fact that P and the stabilizer $S = \{g \in G : \varphi(g)[f] = [f]\}$ coincide, note that $\varphi(\exp(\mathfrak{p})) = \exp(\varphi'(\mathfrak{p}))$ implies that $\exp(\mathfrak{p})$ fixes $[f]$. It follows that we have

$$g\exp(X)g^{-1} \in S' = \{g' \in G : \varphi(g')[\varphi(g)f] = [\varphi(g)f]\}$$

for every X in $\mathfrak{p}$. Differentiating $g\exp(tX)g^{-1}$ at $t = 0$ we deduce that

$$\mathrm{Ad}(g)X \in T_eS' = \{Y \in \mathfrak{g} : \varphi'(Y)[\varphi(g)f] = [\varphi(g)f]\}$$

for all $X \in \mathfrak{p}$.

Choosing g in S, the above argument shows that $\mathrm{Ad}(g)$ preserves $\mathfrak{p}$, that is $g \in P$. Thus, we have seen that $S \subseteq P$.

For the converse inclusion, take any $g \in P$. Our plan is to verify that $\varphi(g)f$ is a highest weight vector, since we could then conclude from the fact that any highest weight vector is unique up to scalars that $\varphi(g)f$ is a multiple of f. To carry this out, observe that every $X \in \mathfrak{n}^+$ is of the form $X = \mathrm{Ad}(g)Y$ with $Y \in \mathfrak{p}$. By what we have seen above, it follows that X maps $[\varphi(g)f]$ into itself. On the other hand, recall that $\varphi'(\mathfrak{g}_\alpha)V_\lambda \subseteq V_{\lambda+\alpha}$ for any weight λ and any root α. We must therefore have $\varphi(X)\varphi(g)f = 0$ for all $X \in \mathfrak{n}^+$ and thus $\varphi(g)f$ is indeed a highest weight vector. This shows $P \subseteq S$.

All together we have obtained the desired result that G/P is diffeomorphic to the G-orbit of $[f]$ in $\mathbb{P}(V)$. Since $G[f] \cong G/P$ is compact it is closed in $\mathbb{P}(V)$ and thus it is a complex submanifold of $\mathbb{P}(V)$. Chow's theorem (see **[7]**) yields that it is algebraic and hence even a smooth algebraic sub-variety of $\mathbb{P}(V)$.

EXAMPLE 1.5.1. 1) Using the above description of G/P as an orbit in projective space, we will verify that the parabolic subgroups of

$SL_n(\mathbb{C})$ are block-upper-triangular matrices, as asserted in Example 1.4.2.

We start with describing some specific $\mathfrak{sl}_n(\mathbb{C})$-representations. The usual representation of $\mathfrak{sl}_n(\mathbb{C})$ on $\bigwedge^l \mathbb{C}^n$ is given by

$$\varphi_l(X)(v_{j_1} \wedge ... \wedge v_{j_l}) = \sum_{i=1}^{l}(v_{j_1} \wedge .. \wedge Xv_{j_i} \wedge .. \wedge v_{j_l}).$$

Here $\{v_i\}_{i=1}^n$ denotes the standard basis of $\mathbb{C}^n$.
Now take $H = \mathrm{diag}(h_1 h_2 ... h_n) \in \mathfrak{h}$. Then

$$\varphi_l(H)(v_{j_1} \wedge ... \wedge v_{j_l}) = \sum_{i=1}^{l} h_{j_i}(v_{j_1} \wedge ... \wedge v_{j_l}).$$

Hence $(v_{j_1} \wedge ... \wedge v_{j_l})$ is a weight vector of weight $\sum_{i=1}^l e_{j_i}$. As usual e_i denotes the linear functional that extracts the i-th entry of every matrix $H \in \mathfrak{h}$. The highest weight of this representation relative to the usual positive system is the fundamental weight $\lambda_l = \sum_{i=1}^l e_i$.

The tensor product $\varphi_{l_1} \otimes ... \otimes \varphi_{l_k}$ is the representation of $\mathfrak{sl}_n(\mathbb{C})$ on $\bigwedge^{l_1} \mathbb{C}^n \otimes ... \otimes \bigwedge^{l_k} \mathbb{C}^n$ given by

$$(\varphi_{l_1} \otimes ... \otimes \varphi_{l_k})(X)(w_1 \otimes ... \otimes w_k) = \sum_{i=1}^{k}(w_1 \otimes .. \otimes Xw_i \otimes .. \otimes w_k).$$

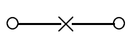
Its heighest highest weight is $\sum_{m=1}^k \lambda_{l_m}$ and a nonzero highest weight vector is $(v_1 \wedge ... \wedge v_{l_1}) \otimes ... \otimes (v_1 \wedge ... \wedge v_{l_k})$. In general the representation is not irreducible. However, the $\mathfrak{sl}_n(\mathbb{C})$-submodule generated by the vector $(v_1 \wedge ... \wedge v_{l_1}) \otimes ... \otimes (v_1 \wedge ... \wedge v_{l_k})$ is an irreducible representation with highest weight $\sum_{m=1}^k \lambda_{l_m}$.

It follows, that in the case $G = SL_n(\mathbb{C})$ and P corresponding to $\Sigma = \Pi \setminus \{\alpha_{l_1}, ..., \alpha_{l_k}\}$, we can take as an irreducible $\mathfrak{g}$-representation φ' with highest weight $\rho^p = \sum_{i \text{ such that } \alpha_i \in \Pi \setminus \Sigma} \lambda_i$ the representation $\varphi_{l_1} \otimes ... \otimes \varphi_{l_k}$. The corresponding heighest weight vector is $f = [(v_1 \wedge ... \wedge v_{l_1}) \otimes ... \otimes (v_1 \wedge ... \wedge v_{l_k})]$. Hence the parabolic subgroup P is precisely the stabilizer of the line through $(v_1 \wedge ... \wedge v_{l_1}) \otimes ... \otimes (v_1 \wedge ... \wedge v_{l_k})$. In other words this means that P is the stabilizer of the flag

$$\langle v_1, ..., v_{l_1} \rangle \subset \langle v_1, ..., v_{l_2} \rangle \subset ... \subset \langle v_1, ..., v_{l_k} \rangle .$$

Thus, it is indeed the subgroup of block-upper-triangular matrices with block sizes $l_1, l_2 - l_1, ..., n - l_k$.

2) Next we consider the Grassmann manifold $Gr_2(\mathbb{C}^4)$ of 2-dimensional subspaces in $\mathbb{C}^4$. We want to describe its realization as a subvariety in projective space in more detail.

The Dynkin diagram associated to $Gr_2(\mathbb{C}^4)$ is  . The weight ρ^p equals the fundamental weight $\lambda_2 = e_1 + e_2$. Hence $Gr_2(\mathbb{C}^4)$

can be embedded as follows

$$\begin{array}{rccc} \iota : & Gr_2(\mathbb{C}^4) & \to & SL_4(\mathbb{C})[f] \subseteq \mathbb{P}(\bigwedge^2 \mathbb{C}^4) \\ & \langle Av_1, Av_2 \rangle & \mapsto & [Av_1 \wedge Av_2] \end{array}$$

where A is in $SL_4(\mathbb{C})$ and $\langle Av_1, Av_2 \rangle$ denotes the plane spanned by the vectors Av_1 and Av_2.

That is, the image of $Gr_2(\mathbb{C}^4)$ in $\mathbb{P}(\bigwedge^2 \mathbb{C}^4)$ consists of those elements that can be written as $w_1 \wedge w_2$, $w_1, w_2 \in \mathbb{C}^4$. Let $w_1 = av_1 + bv_2 + cv_3 + dv_4$ and $w_2 = Av_1 + Bv_2 + Cv_3 + Dv_4$. Then we have

$$\begin{aligned} w_1 \wedge w_2 = (aB - bA)(v_1 \wedge v_2) &+ (aC - cA)(v_1 \wedge v_3) \\ +(aD - dA)(v_1 \wedge v_4) &+ (bC - cB)(v_2 \wedge v_3) \\ +(bD - dB)(v_2 \wedge v_4) &+ (cD - dC)(v_3 \wedge v_4). \end{aligned}$$

Denoting the coefficient of $v_i \wedge v_j$ by $x_{i,j}$ direct computation shows that

$$x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{1,4}x_{2,3} = 0.$$

Conversely any linear combination of the vectors $v_i \wedge v_j$ whose coefficients satisfy this equation is decomposable and hence the corresponding line in projective space lies in $\iota(Gr_2(\mathbb{C}^4))$. Therefore $\iota(Gr_2(\mathbb{C}^4))$ is a quadric hypersurface in $\mathbb{P}(\bigwedge^2 \mathbb{C}^4)$ defined by the above equation.

The embedding $\iota : Gr_2(\mathbb{C}^4) \hookrightarrow \mathbb{P}(\bigwedge^2 \mathbb{C}^4)$ is a special case of the classical Plücker embedding for Grassmann manifolds

$$\iota : Gr_k(\mathbb{C}^n) \hookrightarrow \mathbb{P}(\bigwedge^k \mathbb{C}^n)$$

sending the subspace generated by vectors $v_1, ..., v_k$ to $v_1 \wedge ... \wedge v_k$.

CHAPTER 2

On the Weyl Group

2.1. Basic Facts

We will collect some properties of the Weyl group that we shall need in the rest of this chapter and in Chapter 3. Further details and proofs of the statements can be found for example in [**10**] and [**9**].

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, α a root and let s_α be the reflection in the hyperplane W_α perpendicular to α, i.e. $s_\alpha(\varphi) = \varphi - \frac{2\langle\varphi,\alpha\rangle}{|\alpha|^2}\alpha$ for $\varphi \in \mathfrak{h}_0^*$. Then we defined the Weyl group W as the subgroup of the orthogonal group of $\mathfrak{h}_0^*$ generated by the reflections for $\alpha \in \Delta$. In fact it is sufficient to deal with the simple roots only: the s_α, $\alpha \in \Pi$, generate W.

We immediately see that the Weyl group is finite, since W maps the finite set Δ into itself and Δ spans $\mathfrak{h}_0^*$.

Recall that the connected components of $\mathfrak{h}_0^* \setminus \bigcup_{\alpha\in W_\alpha}$ are called Weyl chambers. They are in one-to-one correspondence with the sets of simple roots:

In every Weyl chamber lies a regular element $\gamma \in \mathfrak{h}_0^*$. We obtain a positive system defining $\Delta^+(\gamma) = \{\alpha \in \Delta : \langle\gamma,\alpha\rangle > 0\}$ and the positive system determines a set of simple roots $\Pi(\gamma)$. By definition γ' is in the same Weyl chamber as γ if and only if $\Delta^+(\gamma') = \Delta^+(\gamma)$, that is if and only if $\Pi(\gamma') =\Pi(\gamma)$. It follows that the mapping that assigns to every Weyl chamber the set $\Pi(\gamma)$ of simple roots is well defined and one-one. We will show that it is also onto: Let Π be any set of simple roots and define $\rho = \sum \lambda_i$, where the λ_i are the fundamental weights, then $\langle\rho,\alpha\rangle > 0$ for all $\alpha \in \Pi$. Clearly ρ lies in some Weyl chamber and the simple set associated to this Weyl chamber is just Π.

Furthermore, it can be proved that if Π and Π' are two simple systems for Δ, then there exists exactly one element $w \in W$ such that $w\Pi = \Pi'$.

It follows that the Weyl group acts simply transitively on the set of all Weyl chambers.

Next note that s_{α_i} for any $\alpha_i \in \Pi$ permutes the positive roots other than α_i: Write $\alpha \in \Delta^+$, $\alpha \neq \alpha_i$, as $\alpha = \sum c_j\alpha_j$, then at least one of the c_j in the decomposition for α is > 0 for $j \neq i$. It follows that

$$s_{\alpha_i}(\alpha) = \sum c_j\alpha_j - \frac{2\langle\alpha,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}\alpha_i$$

must be positive since all coefficients of $s_{\alpha_i}(\alpha)$ written as a linear combination of simple roots must have the same sign (zero allowed to be positive or negative) and the coefficient c_j is positive. Clearly s_{α_i} sends α_i to $-\alpha_i$.

This implies that for any $\alpha_i \in \Pi$ we have

$$s_{\alpha_i}(\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha) = (\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha) - \alpha_i.$$

Comparing this result with the definition of of s_{α_i} yields

$$\frac{2\left\langle \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha, \alpha_i\right\rangle}{\langle\alpha_i,\alpha_i\rangle} = 1.$$

Since Π is a basis for $\mathfrak{h}_0^*$ this in turn shows that

$$\rho = \sum_{i \text{ such that } \alpha_i\in\Pi}\lambda_i = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha.$$

Finally we will give an equivalent description of the Weyl group in terms of the connected Lie group G with Lie algebra $\mathfrak{g}$.
Let $N_G(\mathfrak{h})$ be the normalizer

$$N_G(\mathfrak{h}) = \{g \in G : \operatorname{Ad}(g)\mathfrak{h} = \mathfrak{h}\},$$

then Ad maps $N_G(\mathfrak{h})$ into $\operatorname{Aut}(\mathfrak{h})$ and dualizing gives an element of $\operatorname{Aut}(\mathfrak{h}^*)$. Moreover, if $g \in N_G(\mathfrak{h})$, α is a root, X in $\mathfrak{g}_\alpha$ and H in $\mathfrak{h}$, then

$$(*)\quad \begin{aligned}[H, \operatorname{Ad}(g)X] &= \operatorname{Ad}(g)[\operatorname{Ad}(g)^{-1}H, X] = \operatorname{Ad}(g)(\alpha(\operatorname{Ad}(g)^{-1}H)X)\\ &= (\alpha\circ\operatorname{Ad}(g)^{-1})(H)\operatorname{Ad}(g)X = \operatorname{Ad}^*(g)(\alpha)(H)\operatorname{Ad}(g)X.\end{aligned}$$

Hence $\operatorname{Ad}^*(g)$ maps roots into roots. Even more is true:

THEOREM 2.1.1. *Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$, let $Z_G(\mathfrak{h})$ be the centralizer $Z_G(\mathfrak{h}) = \{g \in G : Ad(g)H = H$ for every $H \in \mathfrak{h}\}$ and W the Weyl group. Then Ad^* induces an isomorphism*

$$N_G(\mathfrak{h})/Z_G(\mathfrak{h}) \cong W.$$

PROOF. To see that the action of any element w in W is given by $\operatorname{Ad}^*(\widetilde{w})$ for for some $\widetilde{w} \in N_G(\mathfrak{h})$ it sufficies to deal with the case of a root reflection s_α for some $\alpha \in \Delta$. We show that for every root α there is an element k_α in $N_G(\mathfrak{h})$ that acts as s_α on $\mathfrak{h}_0^*$. Choose elements $X_\alpha \in \mathfrak{g}_\alpha$ and $Y_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha, Y_{-\alpha}] = H'_\alpha = \frac{2}{\alpha(H_\alpha)}H_\alpha$. Then we define k_α by

$$k_\alpha := \exp(\frac{\pi}{2}(X_\alpha - Y_{-\alpha})).$$

We put $E_\alpha = \frac{\pi}{2}(X_\alpha - Y_{-\alpha})$. For any H in $\mathfrak{h}$ with $\alpha(H) = 0$ we have

$$\operatorname{Ad}(k_\alpha)H = \operatorname{Ad}(\exp(E_\alpha))H = \exp(\operatorname{ad}(E_\alpha))H = H.$$

Since the $H's$ with $\alpha(H) = 0$ together with H'_α span $\mathfrak{h}$, it remains to calculate the action of $\mathrm{Ad}(k_\alpha)$ on H'_α. First note that

$$(\mathrm{ad}(E_\alpha))H'_\alpha = \frac{\pi}{2}([X_\alpha, H'_\alpha] + [-Y_{-\alpha}, H'_\alpha]) = \pi(-X_\alpha - Y_{-\alpha})$$

and

$$(\mathrm{ad}(E_\alpha))^2 H'_\alpha = \frac{\pi^2}{2}[X_\alpha - Y_{-\alpha}, -X_\alpha - Y_{-\alpha}] = -\pi^2 H'_\alpha.$$

Using this we obtain

$$\begin{aligned} \mathrm{Ad}(k_\alpha)H'_\alpha &= \textstyle\sum_{n=0}^\infty \frac{1}{n!}(\mathrm{ad}(E_\alpha))^n H'_\alpha \\ &= \textstyle\sum_{n=0}^\infty \frac{1}{(2n)!}(\mathrm{ad}(E_\alpha))^{2n} H'_\alpha + \sum_{n=0}^\infty \frac{1}{(2n+1)!}(\mathrm{ad}(E_\alpha))^{2n+1} H'_\alpha \\ &= \textstyle\sum_{n=0}^\infty \frac{(-1)^n}{(2n)!}(\pi)^{2n} H'_\alpha + \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}(\pi)^{2n+1}(-X_\alpha - Y_{-\alpha}) \\ &= (\cos\pi)H'_\alpha + (\sin\pi)(-X_\alpha - Y_{-\alpha}) \\ &= -H'_\alpha \end{aligned}$$

Hence k_α is in $N_G(\mathfrak{h})$ and $\mathrm{Ad}^*(k_\alpha) = s_\alpha$.

Now take an arbitrary g in $N_G(\mathfrak{h})$. It follows from $(*)$ that $\mathrm{Ad}^*(g)$ permutes the roots. Let $\Pi = \{\alpha_1, ..., \alpha_n\}$ be the set of simple roots of $\mathfrak{g}$. For every root α we can write $\mathrm{Ad}^*(g^{-1})$ as a linear combination of the α_i with integral coefficients, all of the same sign. Applying $\mathrm{Ad}^*(g)$ we see that α may be written as an interal linear combination of the roots $\mathrm{Ad}^*(g)\alpha_i$ with coefficients of the same sign. Therefore, these roots form a simple system. The Weyl group acts simply transitively on the set of all simple systems, hence there is a unique element w in W such that $\mathrm{Ad}^*(g)\Pi = w\Pi$. Since simple roots form a basis for $\mathfrak{h}$, we conclude that $\mathrm{Ad}^*(g) = w$. This proves the theorem. $\square$

Now let $\varphi : \mathfrak{g} \to End_{\mathbb{C}}(V)$ be a finite-dimensional representation, λ a weight and $\widetilde{w} \in N_G(\mathfrak{h})$ a representative of the Weyl group element w. Then we have

$$\widetilde{\varphi}(\widetilde{w})V_\lambda = V_{w\lambda}.$$

Taking any H in $\mathfrak{h}$ and v in V_λ, this follows since

$$\widetilde{\varphi}(\exp(H))\widetilde{\varphi}(\widetilde{w})v = \widetilde{\varphi}(\widetilde{w}\widetilde{w}^{-1}\exp(H)\widetilde{w})v = \widetilde{\varphi}(\widetilde{w})\widetilde{\varphi}(\exp(\mathrm{Ad}(\widetilde{w}^{-1})H))v$$
$$= e^{\lambda(\mathrm{Ad}(\widetilde{w}^{-1})H)}\widetilde{\varphi}(\widetilde{w})v = e^{w\lambda(H)}\widetilde{\varphi}(\widetilde{w})v.$$

We conclude this section with an example:

EXAMPLE 2.1.2. Consider $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and the simple system $\Pi = \{e_1 - e_2, e_2 - e_3\}$ and denote the simple roots by α_1 and α_2. We have

$$s_{\alpha_2}(\alpha_1) = e_1 - e_2 - \frac{2\langle e_1 - e_2, e_2 - e_3\rangle}{\langle e_2 - e_3, e_2 - e_3\rangle} e_2 - e_3 = e_1 - e_3,$$

hence $s_{e_1 - e_3} = s_{s_{\alpha_2}(\alpha_1)} = s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}$ and we see that the reflections in the simple roots generate the Weyl group of $\mathfrak{sl}_3(\mathbb{C})$. Since $s_{\alpha_1}s_{\alpha_2}s_{\alpha_1} = s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}$ the Weyl group consists of the elements

$$W = \{\mathrm{id}, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_1}s_{\alpha_2}, s_{\alpha_2}s_{\alpha_1}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}\}.$$

It is easily verified that s_{α_1} and s_{α_2} correspond to the transpositions (1,2) and (2,3) respectively and that the Weyl group is just the symmetric group S_3.

In general, The Weyl group of $\mathfrak{sl}_n(\mathbb{C})$ consists of all permutations of $e_1, e_2, ..., e_n$, that is it is the symmetric group S_n. For $\mathfrak{so}_{2n+1}(\mathbb{C})$ and $\mathfrak{sp}_n(\mathbb{C})$ the Weyl group is generated by all permutations of $e_1, e_2, ..., e_n$ and all sign changes of the coefficients of $e_1, e_2, ..., e_n$. In the case of $\mathfrak{so}_{2n}(\mathbb{C})$ the Weyl group is generated by all permutations of $e_1, e_2, ..., e_n$ and all even sign changes.

2.2. The Bruhat Ordering of the Weyl Group

Given any Weyl group element w let the **length** $l(w)$ of w be the number of roots $\alpha > 0$ such that $w^{-1}\alpha < 0$. Defining the set

$$\Phi_w := \{\alpha \in \Delta^+ : w^{-1}\alpha \in -\Delta^+\} = w(-\Delta^+) \cap \Delta^+$$

this yields $l(w) = |\Phi_w|$.

Next observe that the set Φ_w is saturated, i.e. if $\alpha, \beta \in \Phi_w$ and $\alpha + \beta \in \Delta$ then $\alpha + \beta \in \Phi_w$: Suppose that $\alpha, \beta \in \Phi_w$ are given, such that $\alpha + \beta \in \Delta$ hence $\alpha + \beta \in \Delta^+$, then $w^{-1}(\alpha + \beta) = w^{-1}(\alpha) + w^{-1}(\beta)$ lies in Δ and since it is the sum of two negative roots, it must be negative, too.

Similarly we can show that the complement $\Delta^+ \setminus \Phi_w$ is saturated. Finally we define $< \Phi_w > := \sum_{\alpha \in \Phi_w} \alpha$.

Proposition 2.2.1. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, Δ the corresponding set of roots, Π a chosen set of simple roots and $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha = \sum \lambda_i$. Then we have:*

(1) $w(\rho) = \rho - < \Phi_w >$ for all $w \in W$.

(2) The map $w \mapsto \Phi_w$ defines a bijection between W and the set of all subsets $\Phi \subseteq \Delta^+$ such that both Φ and $\Delta^+ \setminus \Phi$ are saturated.

Proof. (1) Defining $s(\alpha) = \left\{ \begin{array}{rl} 1 & \text{if}\, \alpha \in \Delta^+ \\ -1 & \text{if}\, \alpha \in -\Delta^+ \end{array} \right.$,
we can rewrite

$$\rho = \tfrac{1}{2}\textstyle\sum_{\alpha \in \Delta^+} \alpha = \tfrac{1}{4}\sum_{\alpha \in \Delta^+} \alpha + \tfrac{1}{4}\sum_{\alpha \in -\Delta^+}(-\alpha) = \tfrac{1}{4}\sum_{\alpha \in \Delta} s(\alpha)\alpha.$$

Since any $w \in W$ permutes the roots this equals $\frac{1}{4}\sum_{\alpha \in \Delta} s(w\alpha)w\alpha$. On the other hand we get

$$w(\rho) = w(\tfrac{1}{4}\textstyle\sum_{\alpha \in \Delta} s(\alpha)\alpha) = \tfrac{1}{4}\sum_{\alpha \in \Delta} s(\alpha)w\alpha$$

and consequently

$$\rho - w(\rho) = \tfrac{1}{4}\textstyle\sum_{\alpha \in \Delta}(s(w\alpha) - s(\alpha))w\alpha.$$

We see that $(s(w\alpha) - s(\alpha))$ equals 2 if $\alpha \in -\Delta^+$ and $w\alpha \in \Delta^+$, i.e. if $w\alpha \in \Phi_w$. It equals -2 if $\alpha \in \Delta^+$ and $w\alpha \in -\Delta^+$, i.e. if $-w\alpha \in \Phi_w$

and it is zero otherwise. Hence we have

$$\begin{aligned} \rho - w(\rho) &= \tfrac{1}{4}\textstyle\sum_{\alpha\in-\Delta^+}(s(w\alpha)-s(\alpha))w\alpha + \tfrac{1}{4}\textstyle\sum_{\alpha\in\Delta^+}(s(w\alpha)-s(\alpha))w\alpha \\ &= \tfrac{1}{4}\textstyle\sum_{w\alpha\in\Phi_w} 2w\alpha + \tfrac{1}{4}\textstyle\sum_{-w\alpha\in\Phi_w}(-2)w\alpha = <\Phi_w> \end{aligned}$$

and the claim follows.

(2) Since both Φ_w and $\Delta^+ \setminus \Phi_w$ are saturated, the map $w \mapsto \Phi_w$ is well defined.

Assume that $\Phi_w = \Phi_{w'}$ for some elements $w, w' \in W$, then (1) implies that $w(\rho) = w'(\rho)$. Recall that ρ lies in the interior of the dominant Weyl chamber and the Weyl group acts simply transitively on the set of all Weyl chambers. This implies that the Weyl group orbit of ρ is in one-to-one correspondence with W, i.e. $w(\rho) = w'(\rho)$ if and only if $w = w'$ and the mapping is one-one.

Let us prove that it is onto: Given a subset $\Phi \subseteq \Delta^+$ such that both Φ and $\Delta^+ \setminus \Phi$ are saturated, we consider $\Gamma := -\Phi \cup (\Delta^+ \setminus \Phi)$. Then Δ is the disjoint union of Γ and $-\Gamma$. If $\alpha, \beta \in \Gamma$ such that $\alpha+\beta \in \Delta$, then $\alpha + \beta \in \Gamma$. The set of all elements of Γ which may not be decomposed as a sum of two elements of Γ is a simple system for Δ, hence there is an element $w \in W$ mapping Π to that subset of Γ. But this immediately implies $w(\Delta^+) = \Gamma$. Hence $w(-\Delta^+) \cap \Delta^+ = -\Gamma \cap \Delta^+ = \Phi$ and we conclude that $\Phi = \Phi_w$. $\square$

As an immediate consequence of this result we conclude that there is a unique element $w_0 \in W$ of maximal length $l(w_0) = |\Delta^+|$, namely the Weyl group element corresponding to $\Phi_{w_0} = \Delta^+$.

We will prove an equivalent characterization of the length of a Weyl group element in terms of reflections in simple roots. To do so, we first need the following lemma:

LEMMA 2.2.2. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, W the Weyl group and fix a set of simple roots $\Pi = \{\alpha_1, ..., \alpha_n\}$. If α_i is in Π and w in W, then*

$$l(s_{\alpha_i}w) = \begin{cases} l(w)+1 & if\ \alpha_i \notin \Phi_w \\ l(w)-1 & if\ \alpha_i \in \Phi_w \end{cases}$$

PROOF. Recall that for any $\alpha_i \in \Pi$, s_{α_i} permutes the positive roots other than α_i and sends α_i into its negative.
Thus we have

$$\begin{aligned} \left|\Phi_{s_{\alpha_i}w} \setminus \{\alpha_i\}\right| &= |\{\beta \in \Delta^+ : \beta \neq \alpha_i \,\text{and}\, (s_{\alpha_i}w)^{-1}\beta < 0\}| \\ &= |\{\gamma \in \Delta^+ : \gamma \neq \alpha_i \,\text{and}\, w^{-1}\gamma < 0\}| = |\Phi_w \setminus \{\alpha_i\}|\,. \end{aligned}$$

Now assume that $w^{-1}\alpha_i > 0$, then $\alpha_i \notin \Phi_w$ and the right side of the above equation is precisely $l(w)$. In this case $\alpha_i \in \Phi_{s_{\alpha_i}w}$ and we have to add 1 to the left side to obtain $l(s_{\alpha_i}w)$. If $w^{-1}\alpha_i < 0$ we need to add 1 to the right side to obtain $l(w)$ and the left side is just $l(s_{\alpha_i}w)$. $\square$

Using this we deduce the following proposition.

PROPOSITION 2.2.3. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, W the Weyl group and $\Pi = \{\alpha_1, ..., \alpha_n\}$ a set of simple roots. Then the length of any $w \in W$ is the smallest integer k such that w can be written as a product of k reflections in simple roots.*

PROOF. In the previous section we stated the fact that the s_{α_i} for simple roots α_i generate the Weyl group. It follows that any $w \in W$ has at least one expansion as a product of simple reflections. Let $w = s_{\alpha_{i_1}} ... s_{\alpha_{i_m}}$ be such an expansion. From the above lemma we conclude that the length of w is at most m.

To complete the proof we show by induction on $l(w)$ that if $l(w) = k$, there exists an expression of w as a product of k reflections s_{α_i} corresponding to simple roots α_i.

If $k = 0$, w^{-1} sends Δ^+ to Δ^+, hence $w^{-1}\Pi = \Pi$ and since the Weyl group acts simply on the sets of simple roots we conclude $w = \mathrm{id}$.

If $k > 0$, w^{-1} sends at least one simple root α_j into a negative root, i.e. $w^{-1}\alpha_j < 0$. Then lemma implies $l(s_{\alpha_j} w) = l(w) - 1$ and we can apply the inductive hypothesis to obtain $s_{\alpha_j} w = s_{\alpha_{i_1}} ... s_{\alpha_{i_{k-1}}}$ and thus $w = s_{\alpha_j} s_{\alpha_{i_1}} ... s_{\alpha_{i_{k-1}}}$ which completes the induction. $\square$

The two results that we have just proved will be useful when we come to analyze the Bruhat ordering of the Weyl group, which we will introduce next.

The Weyl group admits the structure of a **directed graph** as follows: the nodes of this graph are the elements $w \in W$ and we have a directed edge $w \xrightarrow{\alpha} w'$ if

$$w' = s_\alpha w \text{ for some } \alpha \in \Delta^+ \text{and } l(w') = l(w) + 1.$$

This graph structure gives rise to a partial ordering on W, called the **Bruhat ordering**. We write $w \leq w'$ if either $w = w'$ or there are roots $\alpha_i \in \Delta^+$ such that

$$w \xrightarrow{\alpha_1} w_1 \xrightarrow{\alpha_2} ... \xrightarrow{\alpha_{n-1}} w_{n-1} \xrightarrow{\alpha_n} w'.$$

REMARK 2.2.4. If w is a product of k reflections in simple roots, then the composition of these reflections in the opposite order gives w^{-1}. Thus, by Proposition we have $l(w) = l(w^{-1})$. Moreover, $s_\alpha w = w'$ implies $s_{w^{-1}\alpha} w^{-1} = w^{-1} s_\alpha w w^{-1} = w^{-1} s_\alpha = (w')^{-1}$ and $w^{-1}\alpha > 0$ by Lemma 2.2.2. It follows that $w \xrightarrow{\alpha} w'$ if and only if $w^{-1} \xrightarrow{w^{-1}\alpha} (w')^{-1}$ and therefore $w \leq w'$ if and only if $w^{-1} \leq (w')^{-1}$.

Next we will deduce the following lemma:

LEMMA 2.2.5. (1) *Let $w, w' \in W$, $\alpha \in \Pi$ and assume that $w \leq w'$. Then* (a) *either $s_\alpha w \leq w'$ or $s_\alpha w \leq s_\alpha w'$*

(b) *either $w \leq s_\alpha w'$ or $s_\alpha w \leq s_\alpha w'$*

(2) *Let $\alpha \in \Pi$, $w_1 \xrightarrow{\alpha} w_1'$, $w_2 \xrightarrow{\alpha} w_2'$. If one of the elements w_1, w_1' is smaller than one of w_2, w_2', then $w_1 \leq w_2 \leq w_2'$ and $w_1 \leq w_1' \leq w_2'$.*

PROOF. (1) We first prove (a): If $w = w'$ we clearly have $s_\alpha w = s_\alpha w'$. Hence assume that $w \neq w'$ and
$$w = w_0 \stackrel{\alpha_1}{\to} w_1 \to ... \to w_{k-1} \stackrel{\alpha_k}{\to} w_k = w'.$$
We proceed by induction on k. If $s_\alpha w = w_1$, the assertion is obvious. Hence let $s_\alpha w \neq w_1$, that is $\alpha_1 \neq \alpha$. We want to conclude that $s_\alpha w \leq s_\alpha w_1$ since then, by applying the inductive hypothesis to $w_1 \leq w'$, the claim follows. We have
$$s_{s_\alpha \alpha_1}(s_\alpha w) = s_\alpha s_{\alpha_1} s_\alpha s_\alpha w = s_\alpha w_1,$$
so it is sufficient to show that $l(s_\alpha w_1) > l(s_\alpha w)$. Since $w \to w_1$ we have $l(w) < l(s_{\alpha_1} w)$ and by Lemma 2.2.2 this is equivalent to $\alpha_1 \notin \Phi_w$. The fact that $\alpha_1 \neq \alpha$ implies that $s_\alpha \alpha_1$ is positive, hence $(s_\alpha w)^{-1} s_\alpha \alpha_1 = w^{-1}\alpha_1 \in \Delta^+$ by definition of Φ_w. It follows that $s_\alpha \alpha_1 \notin \Phi_{s_\alpha w}$. Lemma 2.2.2 yields
$$l(s_\alpha w_1) = l(s_{s_\alpha \alpha_1} s_\alpha w) > l(s_\alpha w)$$
and (a) is proved.

To deduce (b) we just have to make slight changes in the preceding proof: If $w = w'$ clearly $s_\alpha w = s_\alpha w'$. Hence assume that $w \neq w'$ and
$$w = w_0 \stackrel{\alpha_1}{\to} w_1 \to ... \to w_{k-1} \stackrel{\alpha_k}{\to} w_k = w'.$$
We proceed by induction on k. If $s_\alpha w' = w_{k-1}$, the assertion is obvious. Hence let and $s_\alpha w \neq w_{k-1}$, that is $\alpha_k \neq \alpha$. We want to conclude that $s_\alpha w_{k-1} \leq s_\alpha w'$ since then, by applying the inductive hypothesis to $w \leq w_{k-1}$, the claim follows. We leave it to the reader to deduce that $s_\alpha w_{k-1} \leq s_\alpha w'$: proceed exactly as in the proof of part (a), replacing w_1 by w', w by w_{k-1} and α_1 by α_k.

(2) If $w_1 \leq w_2$ or $w_1' \leq w_2'$ or $w_1' \leq w_2$ the claim follows immediately. Hence consider the case that $w_1 \leq w_2'$. It follows by (1a) that either $w_1' = s_\alpha w_1 \leq w_2'$ or $w_1' = s_\alpha w_1 \leq s_\alpha w_2' = w_2 \leq w_2'$. Thus (2) is proved. □

We can now give an alternative characterization of the Bruhat ordering, which will be useful when we come to the analysis of the cell decomposition of generalized flag manifolds. Precisely we have:

PROPOSITION 2.2.6. *Suppose that we are given a partial ordering* $\dashv$ *on* W *with the following properties: For arbitrary Weyl group elements* w,w' *and* $\alpha \in \Pi$ *we have*

(*a*) *if* $l(s_\alpha w) = l(w) + 1$, *then* $w \dashv s_\alpha w$,

(*b*) *if* $w \dashv w'$, *then either* $s_\alpha w \dashv w'$ *or* $s_\alpha w \dashv s_\alpha w'$.

Then $w \dashv w'$ *if and only if* $w \leq w'$.

PROOF. Let w_0 be the unique Weyl group element of maximal length, id clearly is the element of smallest length then it follows from (a) that id $\dashv w \dashv w_0$ for all $w \in W$.

We prove that $w \leq w'$ implies that $w \dashv w'$ by induction on $l(w')$. If $l(w') = 0$, then $w' = \text{id}$, hence $w = \text{id}$ and so $w \dashv w'$. Now let $l(w') > 0$.

Then we find a simple root $\alpha \in \Phi_{w'}$ and consequently by Lemma 2.2.1 we have $l(s_\alpha w') = l(w') - 1$. By Lemma 2.2.3 either $s_\alpha w \leq s_\alpha w'$ or $w \leq s_\alpha w'$.

Assume that $w \leq s_\alpha w'$ then we conclude by induction hypothesis that $w \dashv s_\alpha w'$ and thus $w \dashv w'$ by (a).

It remains the case that $s_\alpha w \leq s_\alpha w'$. Again by inductive hypothesis we have $s_\alpha w \dashv s_\alpha w'$. This in turn implies by applying (b) to $s_\alpha w \dashv s_\alpha w'$ that either $w \dashv s_\alpha w'$ or $w \dashv w'$. In the first case we get $w \dashv w'$ by (a) and otherwise we are already done.

Next we prove that $w \dashv w'$ implies $w \leq w'$. This is done by induction downward on $l(w)$. $l(w) = |\Delta^+|$ implies $w = w_0$, hence $w' = w_0$ and thus $w \leq w'$. Let us assume inductively that we have shown the assertion for $k > l(w)$, $l(w) < |\Delta^+|$. We will prove that it also holds for $k = l(w)$.

Take a simple root $\alpha \notin \Phi_w$, then $l(s_\alpha w) = l(w) + 1$ by Lemma 2.2.1. By (b) either $s_\alpha \dashv w'$ or $s_\alpha w \dashv s_\alpha w'$ and we treat the two cases separately:

If $s_\alpha w \dashv w'$ we obtain $s_\alpha w \leq w'$ by inductive hypothesis and thus $w \leq w'$.

If $s_\alpha w \dashv s_\alpha w'$ inductive hypothesis yields $s_\alpha w \leq s_\alpha w'$ and $w \leq w'$ follows from Lemma 2.2.3 (2). $\square$

2.3. The Hasse Diagram of a Parabolic Subalgebra

In this section we will introduce a subset of the Weyl group called the Hasse diagram associated to a parabolic subalgebra. The reason for introducing this set is that it parameterizes the cells in the cell decomposition of a generalized flag manifold G/P, as we will see in Chapter 3.

We fix a Cartan subalgebra $\mathfrak{h}$ of a complex semisimple Lie algebra $\mathfrak{g}$ and and a simple system Π, let $\mathfrak{p}$ be the standard parabolic corresponding to a subset $\Sigma \subset \Pi$ and $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$ its Levi decomposition.

Now recall what we have seen in Proposition 1.3.2. The reductive subalgebra $\mathfrak{l}$ splits in a direct sum of the center $\mathfrak{h}'$, which is the common kernel of all elements of Σ, and the semisimple part $\mathfrak{l}_{ss} = \mathfrak{h}'' \oplus \bigoplus_{\alpha \in \Phi \cup -\Phi} \mathfrak{g}_\alpha$, where $\mathfrak{h}''$ is spanned by the elements H_α corresponding to the roots in Σ. Moreover, if we put $\mathfrak{h}_0' = \mathfrak{h}_0 \cap \mathfrak{h}'$ and $\mathfrak{h}_0'' = \mathfrak{h}_0 \cap \mathfrak{h}''$, we have a orthogonal decomposition $\mathfrak{h}_0 = \mathfrak{h}_0' \oplus \mathfrak{h}_0''$. This follows since we have $B(H_\alpha, H) = \alpha(H)$ for any $H \in \mathfrak{h}_0$ and hence $H \in \ker\alpha$ for all $\alpha \in \Sigma$ if and only if H is orthogonal to all H_α. Passing to the duals we get an orthogonal decomposition of $\mathfrak{h}_0^*$. It follows immediately that any simple reflection s_α with $\alpha \in \Sigma$ acts as the identity on $(\mathfrak{h}_0')^*$. Hence defining W_p to be the Weyl group of $\mathfrak{l}_{ss}$, we may naturally identify W_p with the subgroup of W generated by the simple reflections s_α for $\alpha \in \Sigma$.

Note that since W_p is the Weyl group of a semisimple Lie algebra, it has a unique element w_p^0 of maximal length.

Next let us introduce further notation: denote by $\Delta^+(\mathfrak{l})$ and $\Delta^+(\mathfrak{u})$ the sets of those positive roots such that the corresponding root spaces lie in the indicated subalgebra. We define the $\Sigma-$ height of a root α, denoted by $\mathrm{ht}_\Sigma(\alpha)$, to be the sum of all coefficients in the decomposition of α into simple roots corresponding to elements of $\Pi \setminus \Sigma$. Then a positive root α is in $\Delta^+(\mathfrak{l})$ if and only if and only if $\mathrm{ht}_\Sigma(\alpha) = 0$ and it is in $\Delta^+(\mathfrak{u})$ if and only if $\mathrm{ht}_\Sigma(\alpha) > 0$. The set Δ^+ is the disjoint union of $\Delta^+(\mathfrak{l})$ and $\Delta^+(\mathfrak{u})$ and both subsets are saturated since, obviously, the $\Sigma-$ height is additive.

We will now characterize W_p as those Weyl group elements such that $\Phi_w \subset \Delta^+(\mathfrak{l})$:

Assume that $\alpha \in \Delta^+(\mathfrak{l})$ and $\beta \in \Delta^+(\mathfrak{u})$. Then $\mathrm{ht}_\Sigma(s_\alpha\beta) = \mathrm{ht}_\Sigma(\beta) - \frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}\mathrm{ht}_\Sigma(\alpha) > 0$ which implies $s_\alpha\beta \in \Delta^+(\mathfrak{u})$. Since W_p is generated by the simple reflections s_α for $\alpha \in \Sigma$, we therefore have $w^{-1}\beta \in \Delta^+(\mathfrak{u})$ for every $w \in W_p$ and thus $\beta \notin \Phi_w$. This is true for every $\beta \in \Delta^+(\mathfrak{u})$ and hence $\Phi_w \subset \Delta^+(\mathfrak{l})$ for every $w \in W_p$.

Conversely, let w be such that $\Phi_w \subset \Delta^+(\mathfrak{l})$. We have seen that for any w, the sets Φ_w and $\Delta^+ \setminus \Phi_w$ are saturated. From above we know that $\Delta^+(\mathfrak{l})$ is saturated and hence so is $\Delta^+(\mathfrak{l})\setminus\Phi_w = \Delta^+(\mathfrak{l})\cap(\Delta^+\setminus\Phi_w)$. Finally, by applying part (2) of Proposition 2.2.1 to l_{ss} we find an element $w' \in W_p$ such that $\Phi_{w'} = \Phi_w$ but then $w = w'$ and the claim follows.

Let us define the **Hasse diagram** W^p corresponding to a standard parabolic subalgebra $\mathfrak{p}$ as the subset of the Weyl group whose action sends any weight λ dominant for $\mathfrak{g}$ to a weight dominant for $\mathfrak{p}$, that is $w \in W^p$ if and only if $\langle\lambda, \alpha_j\rangle \geq 0$ for all $\alpha_j \in \Pi$ implies $\langle w\lambda, \alpha_j\rangle \geq 0$ for all $\alpha_j \in \Sigma$.

We will show that we can find a characterization of W^p quite similar to the characterization of W_p. Assume that λ is dominant for $\mathfrak{g}$. If $w \in W^p$ we have $\langle\lambda, w^{-1}\alpha\rangle = \langle w\lambda, \alpha\rangle \geq 0$ for all $\alpha \in \Delta^+(\mathfrak{l})$. Letting λ run over $\{\lambda_i\}$ we conclude that $w^{-1}\alpha \in \Delta^+$ for all $\alpha \in \Delta^+(\mathfrak{l})$. Conversely, if $w \in W$ is such that $w^{-1}\alpha \in \Delta^+$ for all $\alpha \in \Delta^+(\mathfrak{l})$ we see that $w\lambda$ is $\mathfrak{p}$-dominant for any dominant λ. In sum W^p is precisely the set of those elements $w \in W$ such that $\Phi_w \subseteq \Delta^+(\mathfrak{u})$.

The next proposition identifies W^p with a set of distinguished representatives for the right coset space $W_p \setminus W$.

PROPOSITION 2.3.1. *Any $w \in W$ admits a unique decomposition $w = w_p w^p$ with $w_p \in W_p$ and $w^p \in W^p$. Moreover $l(w) = l(w_p) + l(w^p)$, hence the elements in W^p are the unique elements of minimal length in each coset of $W_p \setminus W$.*

PROOF. Choose any element $w \in W$. The sets Φ_w and $\Delta^+(\mathfrak{l})$ are saturated and hence so is $\Phi := \Phi_w \cap \Delta^+(\mathfrak{l})$. We will show that the complement of Φ is also saturated: Assume that $\alpha, \beta \in \Delta^+ \setminus \Phi$ such that $\alpha+\beta \in \Delta$. If one of the elements lies in $\Delta^+(\mathfrak{u})$, so does the sum ($\alpha \in \Delta^+(\mathfrak{u}) \Leftrightarrow \mathrm{ht}_\Sigma \alpha > 0$ and the Σ-height is additive) and hence $\alpha+\beta \in \Delta^+ \setminus \Phi$. If both lie in $\Delta^+(\mathfrak{l})$ they both do not lie in Φ_w and $\alpha+\beta \notin \Phi_w$ follows since $\Delta^+ \setminus \Phi_w$ is saturated.

We have seen that Φ and $\Delta^+ \setminus \Phi$ are saturated. Proposition 2.2.1 now yields that there is a unique Weyl group element w' such that $\Phi = \Phi_{w'}$. Since $\Phi \subseteq \Delta^+(\mathfrak{l})$ we conclude that $w' \in W_p$. From now on we write w_p instead of w'.

Next we show that $w_p^{-1}w \in W^p$. In order to do so take any element $\alpha \in \Delta^+(\mathfrak{l})$.

If $\alpha \in \Phi_{w_p^{-1}}$ we have $-w_p(\alpha) \in \Delta^+$ by definition of $\Phi_{w_p^{-1}}$ and therefore $w_p^{-1}(-w_p(\alpha)) = -\alpha \in -\Delta^+$. It follows that $-w_p(\alpha) \in \Phi_{w_p} \subseteq \Phi_w$. Hence $w^{-1}(-w_p(\alpha)) \in -\Delta^+$ and thus $(w_p^{-1}w)^{-1}(\alpha) \in \Delta^+$.

Next consider the case $\alpha \notin \Phi_{w_p^{-1}}$. Then $w_p^{-1}(w_p(\alpha)) = \alpha \in \Delta^+$ implies that $w_p\alpha \notin \Phi_{w_p} = \Phi_w \cap \Delta^+(\mathfrak{l})$. Since $\alpha \in \Delta^+(\mathfrak{l})$ and W_p is the Weyl group of $\mathfrak{l}_{ss}$ we see that $w_p(\alpha) \in \Delta^+(\mathfrak{l})$. The two preceding sentences imply that $w_p\alpha \notin \Phi_w$, i.e. $(w_p^{-1}w)^{-1}(\alpha) \in \Delta^+$.

Thus we see that that the element $w^p := w_p^{-1}w$ has the property that its inverse maps any element of $\Delta^+(\mathfrak{l})$ to a positive root, hence $w^p \in W^p$ and we obtain a decomposition of the required form, namely $w = w_p w^p$.

To prove uniqueness, assume that $w = w_1 w_2$ with $w_1 \in W_p$ and $w_2 \in W^p$. If $\alpha \in \Delta^+(\mathfrak{l})$ then $w_1^{-1}(\alpha) \in \Delta(\mathfrak{l})$. Since $w_2 \in W^p$ we have $w^{-1}(\alpha) = w_2^{-1}w_1^{-1}(\alpha) \in -\Delta^+$ if and only if $w_1^{-1}(\alpha) \in -\Delta^+$, i.e. $\alpha \in \Phi_w$ if and only if $\alpha \in \Phi_{w_1}$. Consequently, $\Delta^+(\mathfrak{l}) \cap \Phi_w = \Delta^+(\mathfrak{l}) \cap \Phi_{w_1}$. The set $\Delta^+(\mathfrak{l}) \cap \Phi_w$ equals Φ_{w_p} and since $w_1 \in W_p$ the latter equals Φ_{w_1}. By part (2) of Proposition 2.2.1 this implies that w_1 coincides with w_p and then we clearly also get $w_2 = w^p$.

The inequality $l(w) \leq l(w_p) + l(w^p)$ is obvious. To prove the opposite inequality note that $\Phi_{w^p} \subseteq \Delta^+(\mathfrak{u})$ and thus also $w_p(\Phi_{w^p}) \subseteq \Delta^+(\mathfrak{u})$. Take any $\alpha \in w_p(\Phi_{w^p})$ then $w_p^{-1}\alpha \in \Phi_{w^p}$ hence $(w^p)^{-1}w_p^{-1}(\alpha) \in -\Delta^+$ which in turn implies $\alpha \in \Phi_w$. Since $\Phi_{w_p} = \Delta^+(\mathfrak{l}) \cap \Phi_w$ we also have $\Phi_{w_p} \subseteq \Phi_w$. Moreover we see that Φ_{w_p} and $w_p(\Phi_{w^p})$ are disjoint. All together, this yields

$$l(w) = |\Phi_w| \geq \left|\Phi_{w_p}\right| + |w_p\Phi_{w^p}| = \left|\Phi_{w_p}\right| + |\Phi_{w^p}| = l(w_p) + l(w^p).$$

□

Elements of the Weyl group orbit of the weight $\rho = \sum_{i \text{ such that } \alpha_i \in \Pi} \lambda_i$ are, like the elements in the orbit of every weight that lies in the interior of a Weyl chamber, in one-to-one correspondence with W. The idea of the following is to identify the orbit of $\rho^p = \sum_{i \text{ such that } \alpha_i \in \Pi \setminus \Sigma} \lambda_i$ under the right action of the Weyl group with W^p.

PROPOSITION 2.3.2. *The map $w \mapsto w^{-1}(\rho^p)$ restricts to a bijection between W^p and the orbit of ρ^p under W.*

PROOF. We first show that the stabilizer of ρ^p under the action of W given by $(\rho^p, w) \mapsto w^{-1}(\rho^p)$ is W_p.

W_p stabilizes ρ^p since by definition $< \alpha, \rho^p >= 0$ for all $\alpha \in \Sigma$ hence $s_\alpha(\rho^p) = \rho^p$ and these simple reflections generate W_p.

Conversely assume that $w = w_p w^p$ stabilizes ρ^p. We know from above that w_p stabilizes ρ^p and then it follows that also w^p does. Arguing by contradiction we assume that $w^p \neq \mathrm{id}$. Then $\Phi_{w^p} \neq \emptyset$, so there is an element $\gamma \in \Phi_{w^p} \subseteq \Delta^+(\mathfrak{u})$, hence $\langle \rho^p, \gamma \rangle > 0$ and $\langle \rho^p, (w^p)^{-1}\gamma \rangle \leq 0$. But $\langle \rho^p, \gamma \rangle$ equals $\langle (w^p)^{-1}\rho^p, (w^p)^{-1}\gamma \rangle$ which implies that $(w^p)^{-1}\rho^p \neq \rho^p$. This contradicts the fact that w^p stabilizes ρ^p.

We can now show that the mapping is one-one: Assume that $w_1, w_2 \in W^p$ are such that $w_1^{-1}(\rho^p) = w_2^{-1}(\rho^p)$, then $w_1 w_2^{-1}$ stabilizes ρ^p and we have just shown that this implies that $w_1 w_2^{-1} \in W_p$. By the uniqueness part of Proposition 2.3.1 we conclude that $w_1 w_2^{-1} = \mathrm{id}$ (since $w_1 = w_1 w_2^{-1} w_2$), hence $w_1 = w_2$.

The mapping is onto since

$$w^{-1}\rho^p = (w_p w^p)^{-1}\rho^p = (w^p)^{-1} w_p^{-1} \rho^p = (w^p)^{-1}\rho^p.$$

□

Next we indicate how to determine the elements of the Hasse diagram using the the proposition above.

First note, that it is easy to determine the action of a simple reflection on a weight in Dynkin diagram notation. We represent a weight λ for $\mathfrak{g}$ by inscribing the coefficient $\frac{2\langle \lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}$ over the i-th vertex of the Dynkin diagram for $\mathfrak{g}$.

For any simple root α the coefficients for $s_\alpha(\lambda)$ are given by

$$\frac{2\langle s_\alpha(\lambda), \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = \frac{2\langle \lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} - \frac{4\langle \lambda, \alpha \rangle \langle \alpha, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle \langle \alpha, \alpha \rangle}.$$

Since $\langle \alpha, \alpha_i \rangle = 0$ if and only if the number of edges joining the nodes corresponding to α and α_i is zero we conclude that the action of s_α only changes the coefficients over the nodes corresponding to α or adjacent roots.

If $\alpha = \alpha_i$ the coefficient changes sign. If $\alpha \neq \alpha_i$ and if there is a single edge between the nodes or a multiple edge with the arrow pointing towards the node corresponding to α then $\frac{2\langle \alpha, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}$ equals -1 and if there is a multiple edge with arrow pointing towards α_i then $\frac{2\langle \alpha, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}$ equals the negative of the number of edges between α and α_i. Hence we have to add $\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}$ to coefficients next to α, unless there is a multiple edge directed towards node in question, in which case we add $\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}$ times the multiplicity of the edge.

To determine the elements of W^p consider the weight ρ^p. In the notation that we have just introduced this is the weight that has 1 over each crossed node and 0 elsewhere. Now we apply repeatedly those simple reflections to ρ^p that change the weight, never applying a simple reflection two times in a row. Then the elements of W^p are obtained by composing the simple reflections used to get to some place in the reverse order.

EXAMPLE 2.3.3. We first compute the Hasse diagram W^p for projective spaces. The parabolic subalgebras of $\mathfrak{sl}_n(\mathbb{C})$ corresponding to projective spaces are in Dynkin diagram notation of the form

$$\times\!\!-\!\!\circ\cdots\circ\!\!-\!\!\circ\ .$$

Hence the weight ρ^p can be represented by $\overset{1}{\times}\!\!-\!\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ}$. As usual we abbreviate the simple root $e_i - e_{i+1}$ by α_i. We can now compute the orbit of ρ^p under the Weyl group as follows:

$$\overset{1}{\times}\!\!-\!\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ} \overset{s_{\alpha_1}}{\to} \overset{-1}{\times}\!\!-\!\!\overset{1}{\circ}\cdots\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ} \overset{s_{\alpha_2}}{\to} \dots. \overset{s_{\alpha_{n-1}}}{\to} \overset{0}{\times}\!\!-\!\!\overset{0}{\circ}\cdots\overset{-1}{\circ}\!\!-\!\!\overset{1}{\circ}$$
$$\overset{s_{\alpha_n}}{\to} \overset{0}{\times}\!\!-\!\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!\!-\!\!\overset{-1}{\circ}.$$

Hence the Hasse diagram is the set

$$W^p = \{\mathrm{id},\ s_{\alpha_1},\ s_{\alpha_1}s_{\alpha_2},\ s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}, \dots,\ s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\dots s_{\alpha_n}\}.$$

Next consider the parabolic subalgebra $\circ\!\!-\!\!\times\!\!-\!\!\circ$ which is the subalgebra of $\mathfrak{sl}_4(\mathbb{C})$ corresponding to $Gr_2(\mathbb{C}^4)$. In this case the orbit of ρ^p under the Weyl group is

$$\begin{tikzcd}[column sep=small] & & & \overset{-1}{\circ}\!\!-\!\!\overset{0}{\times}\!\!-\!\!\overset{1}{\circ} \arrow[dr] & & & \\ \overset{0}{\circ}\!\!-\!\!\overset{1}{\times}\!\!-\!\!\overset{0}{\circ} \arrow[r] & \overset{1}{\circ}\!\!-\!\!\overset{-1}{\times}\!\!-\!\!\overset{1}{\circ} \arrow[urr] \arrow[drr] & & & & \overset{-1}{\circ}\!\!-\!\!\overset{1}{\times}\!\!-\!\!\overset{-1}{\circ} \arrow[r] & \overset{-0}{\circ}\!\!-\!\!\overset{-1}{\times}\!\!-\!\!\overset{0}{\circ} \\ & & & \overset{1}{\circ}\!\!-\!\!\overset{0}{\times}\!\!-\!\!\overset{-1}{\circ} \arrow[ur] & & & \end{tikzcd}$$

and thus,

$$W^p = \{\mathrm{id},\ s_{\alpha_2},\ s_{\alpha_2}s_{\alpha_1},\ s_{\alpha_2}s_{\alpha_3},\ s_{\alpha_2}s_{\alpha_1}s_{\alpha_3},\ s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}\}.$$

REMARK 2.3.4. The graph structure on the Weyl group introduced in section 2.2 induces a graph structure on the Hasse diagram. There is a procedure to determine the arrows in the Hasse diagram, see [**4**]. Note, that some of the arrows are easy to obtain: If α is a simple root, w and ws_α are elements of W^p and $l(ws_\alpha) = l(w) + 1$, then we have $l(s_\alpha w^{-1}) = l(w^{-1}) + 1$ and hence $w\alpha \in \Delta^+$ by Lemma 2.2.2. Since $s_{w\alpha}w = ws_\alpha w^{-1}w = ws_\alpha$ this implies that there is an arrow $w \overset{w\alpha}{\to} ws_\alpha$.

In our examples this sufficies to determine the graph structure on the respective Hasse diagrams: The Hasse diagram of projective spaces as a directed graph is given by

$$\mathrm{id} \to s_{\alpha_1} \to s_{\alpha_1}s_{\alpha_2} \to \dots. \to s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\dots s_{\alpha_n}$$

and in case of the Grassmann manifold $Gr_2(\mathbb{C}^4)$ we have

$$\begin{tikzcd}[column sep=small]
 & & s_{\alpha_2}s_{\alpha_1} \arrow[dr] & & \\
\mathrm{id} \arrow[r] & s_{\alpha_2} \arrow[ur] \arrow[dr] & & s_{\alpha_2}s_{\alpha_1}s_{\alpha_3} \arrow[r] & s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}. \\
 & & s_{\alpha_2}s_{\alpha_3} \arrow[ur] & &
\end{tikzcd}$$

In general, however, the Hasse diagram of a parabolic subalgebra is much more complicated than in our (easy) examples. This can be seen already from the Hasse diagram of the parabolic subalgebra corresponding to $\times\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\circ$:

$$\begin{tikzcd}[column sep=small]
 & & s_{\alpha_1}s_{\alpha_2} \arrow[r] \arrow[ddr] & s_{\alpha_1}s_{\alpha_2}s_{\alpha_3} \arrow[dr] & & \\
 & s_{\alpha_1} \arrow[ur] \arrow[dr] & & & s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_1} \arrow[dr] & \\
id \arrow[ur] \arrow[dr] & & s_{\alpha_2}s_{\alpha_1} \arrow[r] \arrow[ddr] & s_{\alpha_2}s_{\alpha_1}s_{\alpha_2} \arrow[ur] \arrow[dr] & & s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_1}s_{\alpha_2} \\
 & s_{\alpha_2} \arrow[uuur] \arrow[ur] \arrow[dr] & & & s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2} \arrow[ur] & \\
 & & s_{\alpha_2}s_{\alpha_3} \arrow[uuuur] \arrow[r] & s_{\alpha_2}s_{\alpha_1}s_{\alpha_3} \arrow[uuur] \arrow[ur] & &
\end{tikzcd}$$

CHAPTER 3

Cellular Decomposition of Generalized Flag Manifolds

3.1. Bruhat Decomposition

Throughout this section let G be a complex semisimple Lie group and B a fixed standard Borel subgroup. Our aim is to examine the double coset decomposition of G with respect to B.

To carry this out, we need to show first that $B = HN$, N being the connected subgroup of B with Lie algebra $\mathfrak{n}^+$ and $H := Z_G(\mathfrak{h})$.

One easily observes that $HN \subseteq B$: Since $\mathfrak{n}^+ \subset \mathfrak{b}$ and since N is connected we have $N \subset B$. For any h in $Z_G(\mathfrak{h})$ the automorphism $\mathrm{Ad}(h)$ acts as the identity on $\mathfrak{h}$ and it maps any root space $\mathfrak{g}_\alpha$ into itself. Hence it stabilizes the Borel subalgebra $\mathfrak{b}$ and thus also $H \subset B$.

To prove the converse inclusion consider an arbitrary element g in B. Due to the Iwasawa decomposition we can write g as a product $g = kan$, where k is in connected real Lie subgroup of G with Lie algebra $\mathfrak{k} = \sum_{\alpha\in\Delta} i\mathbb{R}H_\alpha + \sum_{\alpha\in\Delta} \mathbb{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha\in\Delta} i\mathbb{R}(X_\alpha + X_{-\alpha})$, a is in the connected real Lie subgroup of G with Lie algebra $\mathfrak{h}_0$ and $n \in N$. Since a and n are contained in $B = N_G(\mathfrak{b})$ also $\mathrm{Ad}(k) = \mathrm{Ad}(gn^{-1}a^{-1})$ stabilizes $\mathfrak{b}$. Moreover, $\mathrm{Ad}(k)\mathfrak{k} = \mathfrak{k}$. The intersection of $\mathfrak{k}$ and $\mathfrak{b}$ is $i\mathfrak{h}_0$. It follows that $\mathrm{Ad}(k)$ maps $i\mathfrak{h}_0$ into itself and since $\mathrm{Ad}(k)$ is complex-linear it also stabilizes $\mathfrak{h}_0$. In other words, k is in $N_G(\mathfrak{h})$ and therefore $w = (\mathrm{Ad}(k^{-1})\mid_{\mathfrak{h}_0})^*$ belongs to the Weyl group. Moreover,

$$\mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha = \mathrm{Ad}(k)(\mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha) = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta^+} \mathrm{Ad}(k)\mathfrak{g}_\alpha = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{w(\alpha)}.$$

and thus $w\Delta^+ = \Delta^+$. Since the Weyl group acts simply transitively on the sets of simple roots we conclude that $w = \mathrm{id}$, i.e. $k \in Z_G(\mathfrak{h}) = H$. This shows that $g = kan$ is contained in HN and since g was arbitrary in B we conclude $B \subseteq HN$.

We will now proceed to the announced double coset decomposition of G. Given any $w \in W \cong N_G(\mathfrak{h})/Z_G(\mathfrak{h})$, let us denote by $\widetilde{w}$ a fixed representative of w in $N_G(\mathfrak{h})$. Clearly, the double coset $B\widetilde{w}B$ is independent of the choice of $\widetilde{w}$.

We can now state the following theorem:

Theorem 3.1.1. *(Bruhat decomposition). Every complex semisimple Lie group G is the disjoint union of the double cosets $B\widetilde{w}B$, as w varies over the Weyl group.*

We shall prepare the proof of the theorem by three lemmas.

LEMMA 3.1.2. *Choose any* $H \in \mathfrak{h}$ *satisfying* $\alpha(H) \neq 0$ *for all* $\alpha \in \Delta$. *Then the mapping* ψ *that assigns to each* $n \in N$ *the element* $Ad(n)H - H$ *carries* N *onto* $\mathfrak{n}^+$.

PROOF. Let us first observe that for any $X \in \mathfrak{n}^+$

$$\psi(\exp(X)) = \mathrm{Ad}(\exp(X))H - H = [X, H] + \frac{1}{2}(\mathrm{ad}X)^2 H + .. \in \mathfrak{n}^+.$$

In order to deduce that the mapping is onto define for each $\alpha \in \Delta^+$ the subspace $\mathfrak{n}_\alpha = \bigoplus_{\beta \geq \alpha} \mathfrak{g}_\beta$. We prove by induction downward on α that ψ maps $N_\alpha := \exp(\mathfrak{n}_\alpha)$ onto $\mathfrak{n}_\alpha$.
We can write any X in $\mathfrak{n}_\alpha$ as $X_1 + X_2$ with $X_1 \in \mathfrak{g}_\alpha$ and $X_2 \in \mathfrak{n}_\beta$ for $\beta > \alpha$. Putting $Y_1 = \frac{1}{\alpha(H)} X_1$ we have $[H, Y_1] = X_1$ and thus $[[H, Y_1], Y_1] = 0$. Then

$$\mathrm{Ad}(\exp(Y_1))H - H = [Y_1, H] = -X_1,$$

and it follows that
$\mathrm{Ad}(\exp(Y_1))(H + X) - H = \mathrm{Ad}(\exp(Y_1))X - X_1 = \mathrm{Ad}(\exp(Y_1))X_1 + \mathrm{Ad}(\exp(Y_1))X_2 - X_1 = X_2 + [Y_1, X_2] + \frac{1}{2}(\mathrm{ad}(Y_1))^2 X_2 + ... \in \mathfrak{n}_\beta$.
By inductive hypothesis we can find $n \in N_\beta$, such that

$$\mathrm{Ad}(\exp(Y_1))(H + X) - H = \mathrm{Ad}(n)H - H,$$

and therefore

$$\mathrm{Ad}(\exp(Y_1)^{-1} n)H - H = X.$$

Since $\mathfrak{n} = \mathfrak{n}_\alpha$ for α equal to the smallest positive root this gives the lemma. □

LEMMA 3.1.3. *Let* $\mathfrak{b}$ *be the Borel subalgebra corresponding to* B. *Then*

$$\mathfrak{n}^+ = \{Z \in \mathfrak{b} \textit{ such that } ad_{\mathfrak{g}}(Z) \textit{ is nilpotent}\}.$$

PROOF. The inclusion $\mathfrak{n}^+ \subseteq \{Z \in \mathfrak{b} \text{ such that } \mathrm{ad}_{\mathfrak{g}}(Z) \text{ is nilpotent}\}$ follows immediately from the root-space decomposition and the fact that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

For the reverse inclusion write $\mathrm{ad}(X)$ for any $X \in \mathfrak{b}$ as a matrix in a basis of root vectors arranged in decreasing order and with elements of $\mathfrak{h}$ in the middle. The matrix is upper triangular with diagonal entries $\alpha(H)$, where $\alpha \in \Delta$ and H is the component of X in $\mathfrak{h}$, and zeros. To have $\mathrm{ad}(X)$ nilpotent we must get $\alpha(H) = 0$ for all $\alpha \in \Delta$, hence $H = 0$. □

LEMMA 3.1.4. *For each* $g \in G$ *denote* $\mathfrak{b} \cap Ad(g)\mathfrak{b}$ *by* $\mathfrak{b}^g$, *then*

$$\mathfrak{b} = \mathfrak{b}^g + \mathfrak{n}^+.$$

PROOF. Obviously $\mathfrak{b} \supseteq \mathfrak{b}^g + \mathfrak{n}^+$, hence it is enough to show that $\dim(\mathfrak{b}^g + \mathfrak{n}^+) = \dim(\mathfrak{b})$.

We have seen in the preceding lemma that if $X \in (\mathfrak{b} \cap \mathrm{Ad}(g)\mathfrak{b}) \cap \mathfrak{n}^+$ then $\mathrm{ad}_{\mathfrak{g}}(X)$ is nilpotent. Moreover, since $\mathrm{ad}_{\mathfrak{g}}(X)$ and $\mathrm{ad}_{\mathfrak{g}}(\mathrm{Ad}(g^{-1})X) = \mathrm{Ad}(g^{-1})\mathrm{ad}_{\mathfrak{g}}(X)\mathrm{Ad}(g)$ have the same eigenvalues also $\mathrm{ad}_{\mathfrak{g}}(\mathrm{Ad}(g^{-1})X)$ is nilpotent. Again by Lemma 3.1.3 we conclude that $\mathrm{Ad}(g^{-1})X \in \mathfrak{n}^+$ and thus $X \in \mathrm{Ad}(g)\mathfrak{n}^+$. Therefore, we obtain

$$\mathfrak{n}^+ \cap \mathrm{Ad}(g)\mathfrak{n}^+ = \mathfrak{n}^+ \cap (\mathfrak{b} \cap \mathrm{Ad}(g)\mathfrak{b}).$$

Let us denote by $M^\perp$ the orthogonal complement of a subspace M in $\mathfrak{g}$ with respect to the Killing form. Then, since $\mathrm{Ad}(g)$ acts in an orthogonal fashion with respect to the Killing form, the following holds

$$(\mathfrak{b} + \mathrm{Ad}(g)\mathfrak{b})^\perp = \mathfrak{b}^\perp \cap (\mathrm{Ad}(g)\mathfrak{b})^\perp = \mathfrak{b}^\perp \cap \mathrm{Ad}(g)(\mathfrak{b}^\perp) = \mathfrak{n}^+ \cap \mathrm{Ad}(g)\mathfrak{n}^+.$$

Hence we obtain

$$\begin{aligned} 2\dim\, \mathfrak{b} - \dim\, \mathfrak{b}^g &= \dim(\mathfrak{b} + \mathrm{Ad}(g)\mathfrak{b}) = \dim\, \mathfrak{g} - \dim(\mathfrak{n}^+ \cap \mathrm{Ad}(g)\mathfrak{n}^+) \\ &= \dim\, \mathfrak{g} - \dim(\mathfrak{n}^+ \cap \mathfrak{b}^g) = \dim\, \mathfrak{g} + \dim(\mathfrak{n}^+ + \mathfrak{b}^g) - \dim\, \mathfrak{n}^+ - \dim\, \mathfrak{b}^g. \end{aligned}$$

Consequently

$$2\dim\, \mathfrak{b} = \dim\, \mathfrak{g} + \dim(\mathfrak{n}^+ + \mathfrak{b}^g) - \dim\, \mathfrak{n}^+$$

and since $\dim\, \mathfrak{b} + \dim\, \mathfrak{n}^+ = \dim\, \mathfrak{g}$ we conclude that $\dim(\mathfrak{b}^g + \mathfrak{n}^+) = \dim(\mathfrak{b})$. □

Now we proceed to the proof of Theorem 3.1.1.

PROOF. Let any $g \in G$ be given. We will show that we can find an element $u \in N_G(\mathfrak{h})$ such that g and u are in the same double coset.

Fix $H \in \mathfrak{h}$ with $\alpha(H) \neq 0$ for all $\alpha \in \Delta$. Lemma 3.1.4 allows us to write

$$H = X + Y, \text{ where } X \in \mathfrak{n}^+ \text{ and } Y \in \mathfrak{b}^g.$$

Now we apply Lemma 3.1.2 to obtain $n_1 \in N$ with $\mathrm{Ad}(n_1)H - H = -X$. Then $\mathrm{Ad}(n_1)H = H - X = Y \in \mathfrak{b}^g \subseteq \mathrm{Ad}(g)\mathfrak{b}$ and consequently $Z = \mathrm{Ad}(g^{-1}n_1)H \in \mathfrak{b}$.

Next decompose Z in a sum of $X' \in \mathfrak{n}^+$ and $H' \in \mathfrak{h}$. Written in a basis consisting of root vectors arranged in decreasing order and elements of $\mathfrak{h}$ in the middle, $\mathrm{ad}_{\mathfrak{g}}(Z)$ is upper triangular. Its diagonal entries, that is its eigenvalues, are $\alpha(H')$, $\alpha \in \Delta$, and zero. Since the eigenvalues of $\mathrm{ad}_{\mathfrak{g}}(Z)$ and $\mathrm{ad}_{\mathfrak{g}}(H)$ coincide we conclude that $\alpha(H') \neq 0$ for all $\alpha \in \Delta$. Hence by Lemma 3.1.2 we find $n_2^{-1} \in N$ such that $\mathrm{Ad}(n_2^{-1})H' - H' = X'$. Then $\mathrm{Ad}(n_2^{-1})H' = Z$ and

$$H' = \mathrm{Ad}(n_2)Z = \mathrm{Ad}(n_2 g^{-1} n_1)H.$$

We denote $n_2 g^{-1} n_1$ by u^{-1}.

Choose any $X \in Z_{\mathfrak{g}}(H) = \mathfrak{h}$ then

$$[\mathrm{Ad}(u^{-1})X, H'] = [\mathrm{Ad}(u^{-1})X, \mathrm{Ad}(u^{-1})H] = [X, H] = 0$$

and therefore $\mathrm{Ad}(u^{-1})X \in Z_{\mathfrak{g}}(H') = \mathfrak{h}$. In other words

$$\mathrm{Ad}(u^{-1})\mathfrak{h} = \mathfrak{h}$$

and we conclude that u^{-1} is in $N_G(\mathfrak{h})$. Since

$$BuB = HNn_1^{-1}gn_2^{-1}NH = BgB$$

u is the required element.

Finally we are to show that if w_1 and w_2 are in W the assumption $Bw_1B = Bw_2B$ implies $w_1 = w_2$. Suppose $\widetilde{w}_1$ and $\widetilde{w}_2$ are representatives in $N_G(\mathfrak{h})$ of w_1 and w_2 respectively and that there exist elements b_1 and b_2 in B such that $b_1\widetilde{w}_1 = \widetilde{w}_2 b_2$. Recall that we have $B = Z_G(\mathfrak{h})N$. $\mathrm{Ad}(N) = \exp(\mathrm{ad}(\mathfrak{n}^+))$ carries $\mathfrak{h}$ into $\mathfrak{b}$, leaving the $\mathfrak{h}$ component unchanged. $\mathrm{Ad}(Z_G(\mathfrak{h}))$ acts as the identity on $\mathfrak{h}$ and maps any root space $\mathfrak{g}_\alpha$ into itself. By definition $\mathrm{Ad}(N_G(\mathfrak{h}))$ stabilizes $\mathfrak{h}$ and it permutes the root spaces. Keeping this in mind, take any H be in $\mathfrak{h}$ and consider the equation

$$\mathrm{Ad}(b_1\widetilde{w}_1)H = \mathrm{Ad}(\widetilde{w}_2 b_1)H.$$

Then the left side is in $\mathrm{Ad}(\widetilde{w}_1)H + \bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha$ and the right side in $\mathrm{Ad}(\widetilde{w}_2)H + \bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha$. Therefore $\mathrm{Ad}(\widetilde{w}_1)H = \mathrm{Ad}(\widetilde{w}_2)H$ and $\widetilde{w}_2^{-1}\widetilde{w}_1$ centralizes $\mathfrak{h}$. In other words $w_1 = w_2$. This proves the theorem. $\square$

3.2. The Decomposition of G/P into Schubert Cells

As an immediate consequence of the Bruhat decomposition of G we obtain a decomposition of the homogenous space G/B into the orbits of the action of B on G/B:

$$G/B = \bigsqcup_{w\in W} B\widetilde{w}o,$$

where o is the image of B in G/B and $\widetilde{w}$ a representative of w in $N_G(\mathfrak{h})$. We assume that B is a standard Borel subgroup.

Now let P be a parabolic subgroup of G containing B. Clearly G is the union of the cosets BwP as w varies over the Weyl group, hence

$$G/P = \bigcup_{w\in W} B\widetilde{w}o,$$

where o is the image of P in G/P. Cosets corresponding to different Weyl group elements w and w' are either disjoint or identical. We claim that taking only elements $w \in W^p$ yields a decomposition of G/P into disjoint B-orbits.

Let $\Sigma \subseteq \Pi$ be the set of simple roots assigned to the standard parabolic P, $\rho^p = \sum_{i\,\text{such that}\,\alpha_i\in\Pi\setminus\Sigma}\lambda_i$ and $\varphi' : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}$ a representation with highest weight ρ^p. From Section 1.6 we know that

$$G/P \cong G[f]$$

where f is a highest weight vector of weight ρ^p and $[f]$ the corresponding line in projective space. Since G is the union of the cosets $B\widetilde{w}P$ we have

$$G[f] = \bigcup_{w\in W} B\widetilde{w}P[f] = \bigcup_{w\in W} B\widetilde{w}[f],$$

where the latter equality follows from the fact that P stabilizes $[f]$. Recall that $\widetilde{w}$ sends f to a weight vector of weight $w\rho^p$. It follows that

$$\bigcup_{w\in W} B\widetilde{w}[f] = \bigcup_{w\in W} B\widetilde{w}^{-1}[f] = \bigcup_{w\in W} B[f_w],$$

where f_w is of weight $w^{-1}\rho^p$. Next note, that the subgroup H stabilizes $[f_w]$ and thus

$$\bigcup_{w\in W} B[f_w] = \bigcup_{w\in W} NH[f_w] = \bigcup_{w\in W} N[f_w].$$

Two orbits $N[f_w]$ and $N[f_{w'}]$ are the same if and only if $w^{-1}\rho^p = w'^{-1}\rho^p$. In Proposition 2.3.2 we have seen that the mapping $w \mapsto w^{-1}\rho^p$ defines a one-to-one correspondence between the Hasse diagram W^p and the Weyl group orbit of ρ^p and thus we conclude

$$G/P \cong \bigsqcup_{w\in W^p} N[f_w].$$

Next we will show that each orbit $N[f_w]$ is diffeomorphic to a $\mathbb{C}^n$, where n equals the length of w. To begin with, we show that $\exp : \mathfrak{n}^+ \to N$ is a diffeomorphism. We make use of the following theorem that can be found for example in [**10**, p.63].

THEOREM 3.2.1. *If N is a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ then the exponential map $exp : \mathfrak{n} \to N$ is a diffeomorphism onto. Moreover, any connected Lie subgroup of N is simply connected and closed.*

Hence in order to see that the exponential map is a diffeomorphism we just have to verify that N is simply connected.

Let us first observe that $\mathrm{Ad}(N)$ is simply connected: $\mathrm{Ad}(N)$ is a connected Lie subgroup of the nilpotent, connected and simply connected group of all upper-triangular matrices with 1's on the diagonal in $GL_n(\mathfrak{g})$. Hence by Theorem 3.2.1, it is indeed simply connected.

Now consider the covering $\mathrm{Ad}_{\mathfrak{g}} : G \to \mathrm{Ad}(G)$. Since N is connected the restriction $p = \mathrm{Ad}_{\mathfrak{g}}|_N : N \to \mathrm{Ad}(N)$ is a covering homomorphism. We have seen that $\mathrm{Ad}(N)$ is simply connected, thus p is even a diffeomorphism and therefore N is simply connected, as required.

Next we describe the stabilizer of $[f_w]$ in N. Recall that the stabilizer in $\mathfrak{g}$ of the line through the highest weight vector f is $\mathfrak{p}$. Moreover if $\widetilde{w}$ is a representative of w in $N_G(\mathfrak{h})$, we have $\mathrm{Ad}(\widetilde{w})^{-1}V_{\rho^p} = V_{w^{-1}\rho^p}$.

Now assume that X stabilizes $[f]$ which is equivalent to requiering that X stabilizes V_{ρ^p}, since $[f]$ is one-dimensional. Then

$$\varphi'(\mathrm{Ad}(\widetilde{w})^{-1}X)V_{w^{-1}\rho^p} = \mathrm{Ad}(\widetilde{w})^{-1}\varphi'(X)\mathrm{Ad}(\widetilde{w})\mathrm{Ad}(\widetilde{w})^{-1}V_{\rho^p} = V_{w^{-1}\rho^p},$$

that is $\mathrm{Ad}(\widetilde{w})^{-1}X$ stabilizes $V_{w^{-1}\rho^p}$. Hence, the stabilizer of the point in projective space corresponding to a vector of weight $w^{-1}\rho^p$ is $\mathrm{Ad}(\widetilde{w})^{-1}\mathfrak{p}$. For any $H \in \mathfrak{h}$ the element $\mathrm{Ad}(\widetilde{w})^{-1}H$ is contained in $\mathfrak{h}$ as well, hence it is not in $\mathfrak{n}^+$. Assume that $\alpha \in -\Delta^+(\mathfrak{l})$, then $W^p = \{w \in W : \Phi_w \subseteq \Delta^+(\mathfrak{u})\}$ implies that also $w^{-1}\alpha \in -\Delta^+(\mathfrak{l})$. It follows that exactly those root spaces in $\mathfrak{n}^+$ stabilize $[f_w]$ which lie in $\mathfrak{n}^+ \cap \mathrm{Ad}(\widetilde{w})^{-1}\mathfrak{n}^+$, that is which correspond to roots α such that both α and $w\alpha$ are positive. Consequently, the stabilizer of $[f_w]$ in N is

$$N_w^+ := \exp(\mathfrak{n}^+ \cap \mathrm{Ad}(\widetilde{w})^{-1}\mathfrak{n}^+).$$

The next result we shall need is that

$$N = N_w^- N_w^+,$$

where

$$N_w^- = \exp(\mathfrak{n}^+ \cap \mathrm{Ad}(\widetilde{w})^{-1}\mathfrak{n}^-)$$

and $\mathfrak{n}^-$ is the sum of all root spaces corresponding to negative roots. Consider the following sequence of ideals in $\mathfrak{n}^+$:

$$\mathfrak{n}^+ = \mathfrak{n}_n \supseteq \mathfrak{n}_{n-1} \supseteq \ldots \supseteq \mathfrak{n}_2 \supseteq \mathfrak{n}_1$$

Here $\mathfrak{n}_1$ is the root space corresponding to the largest root and for $1 < i \leq n$ the subalgebra $\mathfrak{n}_i$ is the sum of $\mathfrak{n}_{i-1}$ and the root-space $\mathfrak{g}_{\alpha_i}$ such that α_i is the largest root whose root space is not contained in $\mathfrak{n}_{i-1}$. We put $N_1 = \exp(\mathfrak{n}_1)$ and define N_i inductively as $N_i = \exp(\mathfrak{g}_{\alpha_i})N_{i-1}$. Then each N_i is a subgroup as can be seen as follows: if we take $X_{\alpha_i}, X'_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$ and $Y, Y' \in \mathfrak{n}_{i-1}$ we have

$$\begin{aligned} &\exp(X_{\alpha_i})\exp(Y)\exp(X'_{\alpha_i})\exp(Y') \\ &= \exp(X_{\alpha_i})\exp(X'_{\alpha_i})\exp(X'_{\alpha_i})^{-1}\exp(Y)\exp(X'_{\alpha_i})\exp(Y') \\ &= \exp(X_{\alpha_i})\exp(X'_{\alpha_i})\exp(\underbrace{\mathrm{Ad}(\exp(-X'_{\alpha_i}))(Y)}_{\in\, \mathfrak{n}_{i-1}})\exp(Y') \end{aligned}$$

and

$$\begin{aligned} (\exp(X_{\alpha_i})\exp(Y))^{-1} &= \exp(X_{\alpha_i})^{-1}\exp(X_{\alpha_i})\exp(Y)^{-1}\exp(X_{\alpha_i})^{-1} \\ &= \exp(X_{\alpha_i})^{-1}\exp(\underbrace{\mathrm{Ad}(\exp(X_{\alpha_i})(Y)}_{\in\, \mathfrak{n}_{i-1}})^{-1}. \end{aligned}$$

Now we prove that $N_i = (N_i \cap N_w^-)(N_i \cap N_w^+)$ for all for $1 \leq i \leq n$ by induction on i. Clearly the result is true for $N_1 = \exp(\mathfrak{g}_{\alpha_1})$. Assume the claim to be true for N_{i-1}. Since

$$N_i = \exp(\mathfrak{g}_{\alpha_i})N_{i-1} = N_{i-1}\exp(\mathfrak{g}_{\alpha_i})$$

we immediately see that it is true for N_i as well. Since $N_n = N$ we have $N = N_w^- N_w^+$.

It follows from the preceding results that

$$N[f_w] = N_w^- N_w^+[f_w] = N_w^-[f_w].$$

Hence the composition

$$\mathfrak{n}^+ \cap \mathrm{Ad}(g)^{-1}\mathfrak{n}^- \stackrel{\exp}{\to} N_w^- \to N_w^-[f_w] = N[f_w]$$

gives the required diffeomorphism between the $N[f_w]$ and a complex vector space and exhibits $N[f_w]$ as a cell.

Finally, note that the dimension of $\mathfrak{n}^+ \cap \mathrm{Ad}(g)^{-1}\mathfrak{n}^-$ is indeed equal to the length of w:

$$\begin{aligned}\dim(\mathfrak{n}^+ \cap \mathrm{Ad}(g)^{-1}\mathfrak{n}^-) &= |\Delta^+ \cap w^{-1}(-\Delta^+)| = |w\Delta^+ \cap (-\Delta^+)| \\ &= |w(-\Delta^+) \cap \Delta^+| = l(w).\end{aligned}$$

We denote the orbit $N[f_w]$ by X_w and call it **Schubert cell**. Its closure $\overline{X}_w$ is called **Schubert variety**. We can now state the following theorem:

THEOREM 3.2.2. *Let $w, w' \in W$, let X_w and $X_{w'}$ be Schubert cells in G/P and $\overline{X}_w$ the closure of X_w. Then $X_{w'} \subseteq \overline{X}_w$ if and only if $w' \leq w$, where $\leq$ denotes the Bruhat ordering.*

In particular this implies that the closure of a Schubert cell X_w is the union of X_w and cells of lower dimension.

Before proving this theorem we need one more preliminary result.

LEMMA 3.2.3. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and V a finite dimensional representation of $\mathfrak{g}$ with highest weight $\rho = \sum_{i \text{ such that } \alpha_i \in \Pi} \lambda_i$. Choose for each $w \in W$ a nonzero vector $f_w \in V$ of weight $w^{-1}\rho$. Then $w' \leq w$ if and only if $f_{w'} \in U(\mathfrak{n}^+)f_w$, where $U(\mathfrak{n}^+)$ is the universal enveloping algebra of $\mathfrak{n}^+$.*

PROOF. To prove the lemma, we first consider the following situation: assume that $w \stackrel{\alpha}{\to} s_\alpha w$, $\alpha \in \Pi$ and $w \in W$, then $l(s_\alpha w) = l(w)+1$ and Lemma 2.2.2 shows that $\alpha \notin \Phi_w$, hence $w^{-1}\alpha \in \Delta^+$. From Remark 2.2.4 we know $w \stackrel{\alpha}{\to} s_\alpha w$ is equivalent to $w^{-1} \stackrel{\beta}{\to} s_\beta w^{-1}$, where $\beta = w^{-1}(\alpha)$. Now consider $X_\beta \in \mathfrak{g}_\beta$, $X_{-\beta} \in \mathfrak{g}_{-\beta}$ and $H'_\beta = \frac{2}{\alpha(H_\beta)} H_\beta$ such that $\mathfrak{sl}_\beta =< X_\beta, X_{-\beta}, H'_\beta >$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ and let $\widetilde{V}$ be the smallest $\mathfrak{sl}_\beta$-invariant subspace of V containing f_w.

Since ρ is the highest weight and $\dim V_\lambda = \dim V_{w\lambda}$ for any weight λ and Weyl group element w, we have

$$X_\beta f_w \in V_{w^{-1}\rho+\beta} = V_{w^{-1}(\rho+w\beta)} = V_{w^{-1}(\rho+\alpha)} = 0.$$

Hence we see that f_w is a highest weight vector for $\mathfrak{sl}_\beta$. From the representation theory of $\mathfrak{sl}_2(\mathbb{C})$ we know that the elements

$$\{(X_{-\beta})^i f_w : i = 0, 1, ..., n = \frac{2\,\langle w^{-1}\rho, \beta\rangle}{\langle \beta, \beta\rangle}\}$$

form a basis of $\widetilde{V}$. Putting $\widetilde{f} = (X_{-\beta})^n f_w$ we moreover have $X_{-\beta}\widetilde{f} = 0$, $(X_\beta)^n \widetilde{f} = a f_w$ and $f_{s_\alpha w} = b\widetilde{f}$ where a and b are complex numbers $\neq 0$. The last assertion follows, since $\widetilde{f}$ and $f_{s_\alpha w}$ are two non-zero vectors of weight

$$(s_\alpha w)^{-1}\rho = (ws_\beta)^{-1}\rho = s_\beta(w^{-1}\rho) = w^{-1}\rho - \frac{2\langle w^{-1}\rho, \beta\rangle}{\langle \beta, \beta\rangle}\beta = w^{-1}\rho - n\beta$$

and $\dim V_{(s_\alpha w)^{-1}\rho} = 1$. As a consequence we have $f_w \in U(\mathfrak{n}^+)f_{s_\alpha w}$.

Next we introduce a partial ordering on W by putting

$$w \dashv w' \text{ if } f_w \in U(\mathfrak{n}^+)f_{w'}.$$

Since $w^{-1}\rho \neq w'^{-1}\rho$ for elements $w \neq w'$, this indeed defines an ordering. We will verify that it satisfies the conditions of Proposition 2.2.6 since we can then conclude that $w \dashv w'$ if and only if $w \leq w'$.

To prove (a) note that $l(s_\alpha w) = l(w) + 1$ implies $w \stackrel{\alpha}{\to} s_\alpha w$. In the first part of the proof we have seen that this in turn implies $f_w \in U(\mathfrak{n}^+)f_{s_\alpha w}$ hence $w \dashv s_\alpha w$.

Next we show that (b) holds. Let w and w' be Weyl group elements and $w \dashv w'$. We need to prove that either $s_\alpha w \dashv w'$ or $s_\alpha w \dashv s_\alpha w'$.

If $s_\alpha w \stackrel{\alpha}{\to} w$ we conclude as in (a) that $f_{s_\alpha w} \in U(\mathfrak{n}^+)f_w$, hence $s_\alpha w \dashv w$ and since $w \dashv w'$ this proves the assertion.

Hence let us assume that $w \stackrel{\alpha}{\to} s_\alpha w$. We first show that under this assumption $s_\alpha w' \stackrel{\alpha}{\to} w'$ implies $s_\alpha w \dashv w'$. Applying the first part of the proof to $s_\alpha w' \stackrel{\alpha}{\to} w'$, we obtain $X_{-\beta}f_{w'} = 0$ and $w \stackrel{\alpha}{\to} s_\alpha w$ yields $f_{s_\alpha w} = b(X_{-\beta})^n f_w$, where $n = \frac{2\langle w^{-1}\rho, \beta\rangle}{\langle\beta,\beta\rangle}$. Since $w \dashv w'$, we have by definition $f_w \in U(\mathfrak{n}^+)f_{w'}$. Now let $\mathfrak{p}_\beta$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{n}^+$, $\mathfrak{h}$ and $\mathfrak{sl}_\beta$, then $f_{s_\alpha w} \in U(\mathfrak{p}_\beta)f_{w'}$. By Poincare-Birkhoff-Witt any element of $U(\mathfrak{p}_\beta)$ is a linear combination of elements of the form $X_1^{k_1}...X_i^{k_i}H_1^{l_1}...H_n^{l_n}X_{-\beta}^m$, where $X_j \in \mathfrak{n}^+$, $H_j \in \mathfrak{h}$ and $k_1, .., k_i, l_1, ..., l_n, m \in \mathbb{Z}_{\geq 0}$. Since $X_{-\beta}f_{w'} = 0$ and the elements of $U(\mathfrak{h})$ multiply by constants we conclude that $f_{s_\alpha w} = \sum c_i X_i f_{w'}$ with $X_i \in U(\mathfrak{n}^+)$ hence $f_{s_\alpha w} \in U(\mathfrak{n}^+)f_{w'}$ and thus $s_\alpha w \dashv w'$.

If $w' \stackrel{\alpha}{\to} s_\alpha w'$ we have $f_{w'} \in U(\mathfrak{n}^+)f_{s_\alpha w'}$ and thus $w' \dashv s_\alpha w'$. Replacing w' by $s_\alpha w'$ in the above argument yields $s_\alpha w \dashv s_\alpha w'$. This completes the proof of the lemma. □

We can now prove Theorem 3.2.1.

Proof. Consider the case first that G/B is a generalized flag manifold corresponding to a Borel subgroup B.

Let V be the $\mathfrak{g}$-representation with highest weight $\rho = \sum_{i \text{ such that } \alpha_i \in \Pi} \lambda_i$ and for each $w \in W$ let ϕ_w be the linear functional on V given by $\phi_w(f_w) = 1$, where f_w is a weight vector of weight $w^{-1}\rho$ and $\phi_w(f_\lambda) = 0$ if $f_\lambda \in V$ is a vector of weight λ distinct from $w^{-1}\rho$.

Given any $v \in V$, such that there is a Schubert cell X_w with $[v] \in X_w = N[f_w] \subseteq G[f]$, we have

$$v = C\exp(X).f_w = C(1 + X + \frac{1}{2}X^2 + ...).f_w$$

for some $X \in \mathfrak{n}^+$. Hence $v \in U(\mathfrak{n}^+)f_w$ and $\phi_w(v) \neq 0$.

On the other hand suppose that $[v] \in G[f]$ such that $v \in U(\mathfrak{n}^+)f_w$ and $\phi_w(v) \neq 0$. Since $G[f] = \bigsqcup_{w \in W} X_w$ we know that v is contained in some $X_{w'}$. In order to conclude that $w' = w$ we need to show that for each $v \in V$ there is at most one $w \in W$ such that $v \in U(\mathfrak{n}^+)f_w$ and $\phi_w(v) \neq 0$. This can be seen as follows: take any $v \in U(\mathfrak{n}^+)f_w$ with $\phi_w(v) \neq 0$. If $\phi_{w'}(v) \neq 0$ as well, we have $f_{w'} \in U(\mathfrak{n}^+)f_w$ which in turn implies that $w'^{-1}\rho \geq w^{-1}\rho$. Interchanging the roles of w and w', we similarly obtain $w^{-1}\rho \geq w'^{-1}\rho$ v, hence $w'^{-1}\rho = w^{-1}\rho$. Since ρ is contained in the interior of the fundamental Weyl chamber this yields $w' = w$.

In summary, we have

$$X_w = \{[v] \in \mathbb{P}(V) \,\text{such that}\, v \in U(\mathfrak{n}^+)f_w \,\text{and}\, \phi_w(v) \neq 0\} \cap G[f].$$

Using this, we can prove that $X_{w'} \subseteq \overline{X}_w$ implies $w' \leq w$. It follows by the above description of X_w that its closure is the set

$$\overline{X_w} = \{[v] \in \mathbb{P}(V) :\ v \in U(\mathfrak{n}^+)f_w\} \cap G[f].$$

Then $X_{w'} \subseteq \overline{X}_w$ gives $f_{w'} \in U(\mathfrak{n}^+)f_w$ which is equivalent to $w' \leq w$ by Lemma 3.2.2.

To prove the converse, it is sufficient to consider the case $w' \xrightarrow{\alpha} w$. Recall, that $w' \xrightarrow{\alpha} w$ is equivalent to $(w')^{-1} \xrightarrow{\beta} w^{-1}$, where $\beta = w'^{-1}\alpha$. Just as in the Proof of Lemma 3.2.2 we can show that there is a positive integer n and $X_\beta \in \mathfrak{g}_\beta$ such that we have $(X_\beta)^n f_w = af_{w'}$ and $(X_\beta)^{n+1}f_w = 0$. Therefore,

$$\begin{aligned}&\lim_{t\to\infty} t^{-n}\exp(tX_\beta)f_w = \lim_{t\to\infty} t^{-n}(1 + tX_\beta + ... + \tfrac{t^k}{k!}(X_\beta)^k + ...)f_w\\ &= \lim_{t\to\infty} t^{-n}f_w + t^{1-n}X_\beta f_w + .. + \tfrac{1}{n!}(X_\beta)^n f_w\\ &= \tfrac{1}{n!}(X_\beta)^n f_w = \tfrac{a}{n!}f_{w'}.\end{aligned}$$

Hence, $[f_{w'}] \in \overline{X}_w$ and thus $X_{w'} \subseteq \overline{X}_w$.

It remains to prove the theorem for G/P, where P is an arbitrary parabolic subgroup. Let $B \subseteq P$ be the Borel subgroup contained in P, $\pi : G \to G/B$ and $\pi' : G \to G/P$ the respective natural projections and $\varphi : G/B \to G/P$. We have $X_w = Bw^{-1}\pi(B)$ and $\varphi(X_w) = Bw^{-1}\pi'(P) =: X_w^p$. Moreover we have seen in the beginning of this section that for different Weyl group elements w, w' we have $X_w^p = X_{w'}^p$ if and only if $w^{-1}\rho^p = w'^{-1}\rho^p$, i.e. $w' \in W_p w$. It follows that $\varphi^{-1}(X_w^p) = \bigcup_{w' \in W_p w} X_{w'}$. Now assume that $w \in W^p$. Then w is the element in $W_p w$ with minimal length, $w_p^0 w$ is the element of maximal length and for any $w' \in W_p w$ we have $w \leq w' \leq w_p^0 w$.

Take any $w \in W^p$, then $\varphi(\overline{X_w}) = \bigcup_{w' \leq w} X^p_{w'}$ is closed and contains X^p_w, thus $\overline{X^p_w} \subseteq \varphi(\overline{X_w})$. To prove that $\overline{X^p_w} = \varphi(\overline{X_w})$ we consider the following chain of inclusions:

$$\begin{aligned} X_{w^0_p w} \subseteq \varphi^{-1}(X^p_w) \subseteq \varphi^{-1}(\overline{X^p_w}) \subseteq \varphi^{-1}\varphi(\overline{X_w}) = \varphi^{-1}(\textstyle\bigcup_{w' \leq w} X^p_{w'}) \\ = \textstyle\bigcup_{w' \leq w} \varphi^{-1}(X^p_{w'}) = \bigcup_{w' \leq w}(\bigcup_{\nu' \in W_p w'} X_{\nu'}). \end{aligned}$$

Let ν be such that $\nu \in W_p w'$ and $\nu \in W^p$. Since $\nu \leq w' \leq w$ we have $\nu \overset{\alpha_1}{\to} w_1 \overset{\alpha_2}{\to} \dots \overset{\alpha_{n-1}}{\to} w_n \overset{\alpha_n}{\to} w$, where all $w_i \in W^p$ and thus all $\alpha_i \in \Delta^+(\mathfrak{u})$. We prove that $\nu \leq s_{\alpha_1}\nu$ implies $w^0_p\nu \leq w^0_p s_{\alpha_1}\nu$. An elementary calculation shows that $s_{w\alpha} = w s_\alpha w^{-1}$, hence $w^0_p s_{\alpha_1}\nu = s_{w^0_p(\alpha_1)} w^0_p \nu$ and $w^0_p(\alpha_1)$ is a positive root since $\Phi_{w^0_p} \subseteq \Delta^+(\mathfrak{l})$ and $w^0_p = (w^0_p)^{-1}$. Moreover, $l(w^0_p\nu) = l(w^0_p) + l(\nu) \leq l(w^0_p) + l(s_{\alpha_1}\nu) = l(w^0_p s_{\alpha_1}\nu)$ and the assertion follows. Inductively we conclude that $w^0_p\nu \leq w^0_p w$.

Since any $\nu' \in W_p w'$ is smaller than $w^0_p\nu$ we have

$$\textstyle\bigcup_{w' \leq w}(\bigcup_{\nu' \in W_p w'} X_{\nu'}) \subseteq \bigcup_{w' \leq w^0_p w}(X_{w'}) = \overline{X_{w^0_p w}}.$$

Summarizing we have seen that

$$X_{w^0_p w} \subseteq \varphi^{-1}(\overline{X^p_w}) \subseteq \overline{X_{w^0_p w}}$$

and since $\varphi^{-1}(\overline{X^p_w})$ is closed this implies that $\varphi^{-1}(\overline{X^p_w}) = \overline{X_{w^0_p w}}$. Thus, $\varphi^{-1}(\overline{X^p_w}) = \varphi^{-1}\varphi(\overline{X_w})$ and therefore $\overline{X^p_w} = \varphi(\overline{X_w})$, as required. $\square$

Summary 3.2.4. Putting the preceding results together, we have obtained a cell decomposition of the homogenous space G/P into so-called Schubert cells X_w of dimension $l(w)$, one for each $w \in W^p$. Moreover the Schubert variety $\overline{X_w}$ is the union of X_w and all Schubert cells corresponding to elements of the Hasse diagram, which are smaller than w in the Bruhat ordering, i.e. $\overline{X_w} = \bigsqcup_{w' \leq w} X_{w'}$.

Example 3.2.5. Consider the flag manifold G/P, where $G = SL_3(\mathbb{C})$ and P is be the parabolic subgroup corresponding to the subset $\Sigma = \{e_2 - e_3\}$, that is

$$P = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}.$$

Then the corresponding flag manifold G/P can be identified with the projective space $\mathbb{P}^2$. We will see that the cell decomposition $G/P = \bigsqcup_{w \in W^p} X_w$ is just the usual cell decomposition of $\mathbb{P}^2$ into the disjoint union of $\mathbb{C}^2$, $\mathbb{C}^1$ and a point.

First of all we know that we can realize the flag manifold G/P as the orbit $G[f]$, where f is a highest weight vector of a $\mathfrak{sl}_3(\mathbb{C})$-representation with highest weight $\rho^p = \sum_{i \text{ such that } \alpha_i \in \Pi \setminus \Sigma} \lambda_i$ and $[f]$ the corresponding line in projective space. In our example $\rho^p = \lambda_1$, hence $\langle \rho^p, \alpha_1 \rangle = 1$ and $\langle \rho^p, \alpha_2 \rangle = 0$ which in turn implies $\rho^p = e_1$. The $\mathfrak{sl}_3(\mathbb{C})$-representation

$\varphi' : \mathfrak{sl}_3(\mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^3)$ with highest weight e_1 is just the standard representation $\varphi'(X)v := Xv$ and $G[f] = \mathbb{P}^2$.

From Example 2.3.3 we know that the Hasse diagram W^p corresponding to $\mathbb{P}^2$ is

$$W^p = \{\mathrm{id},\ s_{\alpha_1},\ s_{\alpha_1}s_{\alpha_2}\}.$$

The vector $f_{s_{\alpha_1}}$ is of weight $s_{e_1-e_2}(e_1) = e_2$ and $f_{s_{\alpha_1}s_{\alpha_2}}$ is of weight $(s_{\alpha_1}s_{\alpha_2})^{-1}(e_1) = s_{\alpha_2}s_{\alpha_1}(e_1) = e_3$.

In the proof of Theorem 3.2.1 we have seen that a Schubert cell can be described as the following set

$$X_w = \{[v] \in G[f] \,\text{such that}\, v \in U(\mathfrak{n}^+)f_w \,\text{and}\, \phi_w(v) \neq 0\}.$$

Since $e_3 + (e_2 - e_3) = e_2$ and $e_2 + (e_1 - e_2) = e_1$ we have $U(\mathfrak{n}^+)f_{s_{\alpha_1}s_{\alpha_2}} = \mathbb{C}f_{s_{\alpha_1}s_{\alpha_2}} \oplus \mathbb{C}f_{s_{\alpha_1}} \oplus \mathbb{C}f$ and $U(\mathfrak{n}^+)f_{s_{\alpha_1}} = \mathbb{C}f_{s_{\alpha_1}} \oplus \mathbb{C}f$. It follows that the Schubert-cells in our example are given by

$$\begin{aligned}
&X_{id} = \{[v] \in \mathbb{P}^2 :\ v = af \text{ where } a \in \mathbb{C} \text{ and } a \neq 0\}\\
&X_{s_{\alpha_2}} = \{[v] \in \mathbb{P}^2 :\ v = af + bf_{s_{\alpha_2}} \text{ where } a, b \in \mathbb{C} \text{ and } b \neq 0\}\\
&X_{s_{\alpha_2}s_{\alpha_1}} = \{[v] \in \mathbb{P}^2 :\ v = af + bf_{s_{\alpha_2}} + cf_{s_{\alpha_2}s_{\alpha_1}},\ a, b, c \in \mathbb{C}\ c \neq 0\}.
\end{aligned}$$

In coordinates relative to the basis $\{f,\ f_{s_{\alpha_2}},\ f_{s_{\alpha_2}s_{\alpha_1}}\}$ this is $X_{id} = \{[(1,0,0)]\} =: U_0$, $X_{s_{\alpha_2}} = \{[(a,1,0)] \in \mathbb{P}^2\} =: U_1$ and $X_{s_{\alpha_2}s_{\alpha_1}} = \{[(a,b,1)] \in \mathbb{P}^2\} =: U_2$ and these are precisely the cells in the usual cell decomposition.

The respective homeomorphisms from the $2n$-cell $e_{2n} = \{x \in \mathbb{R}^{2n} : |x| < 1\}$ to U_n are given by the maps

$$\begin{aligned}
F_0 &:\ e_0 \to U_0, & F_0(x) &= [(1,0,0)],\\
F_1 &:\ e_2 \to U_1, & F_1(x) &= [(x_1,1,0)],\\
F_2 &:\ e_4 \to U_2, & F_2(x) &= [(x_1,x_2,1)].
\end{aligned}$$

REMARK 3.2.6. Extending the homeomorphism F_n continuously to the closed $2n$-ball gives a characteristic mapping for each $2n$-cell and exhibits $\mathbb{P}^2$ as a CW-space. Thus, we can use cellular homology to determine its singular homology groups. In order to compute the homology of arbitrary generalized flag manifolds, we will use so-called Borel-Moore homology groups. This is done in the next chapter.

CHAPTER 4

Homology of Generalized Flag Manifolds

4.1. Review of Singular Homology and Cohomology

If the reader is familiar with the basic notions of singular homology and cohomology theory he or she can move on to the next section. However, in case he or she hasn't got all of the facts at hand, I thought it might helpful to review those that will be used. For more details, see for example [**12**].

We start with a brief introduction of singular homology and cohomology groups.

Consider the standard q-simplex

$$\Delta_q = \{ x \in \mathbb{R}^{q+1} : x = \sum_{i=0}^{q} \lambda_i e_i,\ 0 \le \lambda_i \le 1 \text{ and } \sum_{i=0}^{q} \lambda_i = 1 \},$$

where $\{e_0, e_1, ..., e_q\}$ is the standard basis of $\mathbb{R}^{q+1}$. We define a **singular *q*-simplex** in X to be a continuous map $\sigma_q : \Delta_q \to X$, $q \ge 0$. The free abelian group generated by the set of all singular $q-$simplices will be denoted by $S_q(X)$ and we put $S_q(X) = 0$ for $q < 0$. Next we define a so-called **boundary operator** $\partial_q : S_q(X) \to S_{q-1}(X)$ on a $q-$simplex by

$$\partial_q \sigma_q = \sum_{i=0}^{q} (-1)^i (\sigma_q \circ \varepsilon^i_{q-1}),$$

where $\varepsilon^i_{q-1} : \Delta_{q-1} \to \Delta_q$ maps e_j to e_j if $j < i$ and e_j to e_{j+1} if $j \ge i$. Then ∂_q is extended linearly to the entire group $S_q(X)$. We put $\partial_q = 0$ for $q \le 0$.

It is not difficult to verify that $(S_q(X))_{q\in\mathbb{Z}}$ together with the homomorphisms $(\partial_q)_{q\in\mathbb{Z}}$ form a chain complex. A **chain complex** is a sequence of free abelian groups and homomorphisms

$$... \to S_{q+1} \xrightarrow{\partial_{q+1}} S_q \xrightarrow{\partial_q} S_{q-1} \to ...$$

such that $\partial_q \circ \partial_{q+1} = 0$. The latter condition implies that $B_q = \operatorname{Im} \partial_{q+1}$ (the group of $q-$ boundaries) is contained in $Z_q = \operatorname{Ker} \partial_q$ (the group of $q-$ cycles) and we can therefore form the quotient $H_q(S) = Z_q/B_q$, called the ***q*-th homology group**.

Given a continuous map $f : X \to Y$ the sequence of homomorphisms $f_q : S_q(X) \to S_q(Y)$ mapping $\sum_\sigma n_\sigma \sigma$ to $\sum_\sigma n_\sigma (f \circ \sigma)$ is a

chain map, that is a collection of homomorphisms $(f_q)_{q\in\mathbb{Z}}$ that commute with the boundary operators of the given chain complexes

$$\begin{array}{ccccccc} \dots \to & S_{q+1}(X) & \overset{\partial_{q+1}}{\to} & S_q(X) & \overset{\partial_q}{\to} & S_{q-1}(X) & \to \dots \\ & f_{q+1}\downarrow & & f_q\downarrow & & f_{q-1}\downarrow & \\ \dots \to & S_{q+1}(Y) & \overset{\partial_{q+1}}{\to} & S_q(Y) & \overset{\partial_q}{\to} & S_{q-1}(Y) & \to \dots \end{array}$$

Since every chain map $f = (f_q)_{q\in\mathbb{Z}}$ sends cycles to cycles and boundaries to boundaries it induces a homomorphism

$$H_q(f) = f_* : \, H_q(X) \to H_q(Y).$$

The assignment $X \mapsto H_q(X)$ and $f \mapsto H_q(f)$ is a covariant functor from the category of topological spaces and continuous maps into the category of abelian groups and homomorphisms.

If A and B are abelian groups, the set $\mathrm{Hom}(A, B)$ of all homomorphisms $\varphi : \, A \to B$ is also an abelian group. Let us put $B = \mathbb{Z}$. Then sending each abelian group A to $\mathrm{Hom}(A, \mathbb{Z})$ and each homomorphism $f \, : \, A \to A'$ to the map $\widetilde{f} \, : \, \mathrm{Hom}(A', \mathbb{Z}) \to \mathrm{Hom}(A, \mathbb{Z})$, $\varphi \mapsto \varphi \circ f$ defines a contravariant functor $\mathrm{Hom}(\,.\,, \mathbb{Z})$ from the category of abelian groups and homomorphisms into itself.

We can apply $\mathrm{Hom}(.,\mathbb{Z})$ to a given singular chain complex

$$\dots \to S_{q+1}(X) \overset{\partial_{q+1}}{\to} S_q(X) \overset{\partial_q}{\to} S_{q-1}(X) \to \dots$$

Putting $\delta_{q-1} = \widetilde{\partial}_q$ this yields a so-called cochain complex

$$\dots \to \mathrm{Hom}(S_{q-1}(X), \mathbb{Z}) \overset{\delta_{q-1}}{\to} \mathrm{Hom}(S_q(X), \mathbb{Z}) \overset{\delta_q}{\to} \mathrm{Hom}(S_{q+1}(X), \mathbb{Z}) \to \dots$$

where $\delta_q \circ \delta_{q-1} = 0$. Here the factor group $\mathrm{Ker}\ \delta_q / \mathrm{Im}\ \delta_{q-1}$ is called the ***q*-th cohomology group** which we will denote by $H^q(X)$. If $f = (f_q)_{q\in\mathbb{Z}}$ is a chain map then $\widetilde{f} = (\widetilde{f}_q)_{q\in\mathbb{Z}}$ is a map of the corresponding cochain complexes, which in turn induces a map of the corresponding cohomology groups.

The assignment $X \mapsto H^q(X)$ and $f \mapsto H^q(f)$ is a functor from the category of topological spaces and continuous maps into the category of abelian groups and homomorphisms but this one is contravariant.

Now let A be a subspace of X and $i : \, A \hookrightarrow X$ the inclusion, then we can consider $S_q(A)$ as a subgroup of $S_q(X)$. The boundary operator ∂_q maps $S_q(A)$ into $S_{q-1}(A)$. Therefore we can form the quotient complex

$$\dots \to S_{q+1}(X)/S_{q+1}(A) \overset{\partial_{q+1}}{\to} S_q(X)/S_q(A) \overset{\partial_q}{\to} S_{q-1}(X)/S_{q-1}(A) \to \dots$$

The corresponding q-th (co)homology group is called q-th **relative (co)homology group** and is denoted by $H^q(X, A)$, resp. $H_q(X, A)$.

Given another pair (Y, B) of topological spaces and a map of pairs, that is a continuous map $f : X \to Y$ such that $f(A) \subseteq B$, we obtain a homomorphism between the corresponding relative (co-)homology groups.

Relative (co-)homology includes absolute (co-)homology as a special case: we have $H^i(X, \emptyset) = H^i(X)$ and $H_i(X, \emptyset) = H_i(X)$.

Next we will formulate some basic properties of cohomology groups. If not otherwise stated there are similar ones for homology groups.

A fundamental feature of the cohomology functor is its **homotopy invariance**: homotopic maps induce the same homomorphisms in cohomology, spaces of the same homotopy type have isomorphic cohomology groups. It follows that the cohomology groups of $\mathbb{C}^n$, which is homotopically equivalent to a point, agree with the cohomology groups of a single point:

$$H^i(\mathbb{C}^n) = H^i(p) = 0 \text{ for } i \neq 0 \text{ and } H^0(\mathbb{C}^n) = H^0(p) = \mathbb{Z}.$$

Another important property is the **long exact sequence of a triple**. Suppose $B \subset A \subset X$ are topological spaces, then there is a connecting homomorphism $\partial^* : H^i(A, B) \to H^{i+1}(X, A)$, such that the following sequence in cohomology is exact

$$\begin{aligned} ... \to H^{i-1}(A, B) \to H^i(X, A) \xrightarrow{j^*} H^i(X, B) \xrightarrow{i^*} H^i(A, B) \xrightarrow{\partial^*} \\ \to H^{i+1}(X, A) \to ... \end{aligned}$$

Here j^* and i^* are induced by the inclusions $j : (X, B) \hookrightarrow (X, A)$ and $i : (A, B) \hookrightarrow (X, B)$. Moreover, ∂^* is natural in the following sense: if $f : (X, A, B) \to (X', A', B')$ is a map of triples, then the diagram

$$\begin{array}{ccccccccc} \to & H^{i-1}(A, B) & \to & H^i(X, A) & \to & H^i(X, B) & \to & H^i(A, B) & \to \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \to & H^{i-1}(A', B') & \to & H^i(X', A') & \to & H^i(X', B') & \to & H^i(A', B') & \to \end{array}$$

where the vertical maps are induced by f, commutes.

We will also need the fact that when we are given a pair of topological spaces (X, A), certain subspaces $U \subset A$ may be cut out or **excised** without affecting the relative cohomology groups: if the closure of U is contained in the interior of A then there is an isomorphism

$$H^i(X, A) \cong H^i(X \setminus U, A \setminus U).$$

Next, assume that X_1 and X_2 are open in X and that $X = X_1 \cup X_2$. Then it can be proved (see **[11]**) that there is a long exact sequence

$$\begin{aligned} ... \to H^i(X, X) \to H^i(X, X_1) \oplus H^i(X, X_2) \to H^i(X, X_1 \cap X_2) \\ \to H^{i+1}(X, X) \to ... \end{aligned}$$

called **relative Mayer-Vietoris sequence**. Since $H^i(X, X) = 0$ this yields

$$H^i(X, X_1) \oplus H^i(X, X_2) \cong H^i(X, X_1 \cap X_2).$$

Let X be any topological space and let A and B be open subsets of X. Then there is a **cup product**

$$\cup : \; H^i(X, A) \times H^j(X, B) \to H^{i+j}(X, A \cup B),$$

which is natural in the following sense: If X' is another topological space, A' and B' are open subsets and if $f : X \to X'$ is a continuous map such that $f(A) \subseteq A'$ and $f(B) \subseteq B'$, then the diagram

$$\begin{array}{ccc} H^i(X',A') \times H^j(X',B') & \xrightarrow{\cup} & H^{i+j}(X', A' \cup B') \\ (f^*, f^*) \downarrow & & f^* \downarrow \\ H^i(X,A) \times H^j(X,B) & \xrightarrow{\cup} & H^{i+j}(X, A \cup B) \end{array}$$

commutes.

Finally, recall that the **cap product** is a bilinear mapping

$$H^iX \times H_jX \to H_{j-i}X, \ (\alpha, \beta) \mapsto \alpha \cap \beta.$$

If M is a compact, connected oriented n-dimensional manifold without boundary, its top homology group H_nM is isomorphic to $\mathbb{Z}$, with a generator $[M]$ called the fundamental class of M. In this case the **Poincare-duality** theorem asserts that the mapping

$$H^iM \to H_{n-i}M, \ \ \alpha \mapsto \alpha \cap [M]$$

for any $\alpha \in H^iM$ is an isomorphism for all integers i.

4.2. Differentiable Manifolds and Vector Bundles

In our exposition of Borel-Moore homology we will use some properties of differentiable manifolds and vector bundles. The purpose of this section is to collect the necessary background material. Most of it can be found in Hirsch [**8**].

We start with the definition of a vector bundle. Let E, B be differentiable manifolds and $\pi : E \to B$ a smooth map, such that each fibre $E_x = \pi^{-1}(x)$ is an n-dimensional vectorspace. We require $\pi : E \to B$ to be locally trivial, that is for each $y \in B$ there is an open neighborhood $U \subset B$ and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi} & U \times \mathbb{R}^n \\ \pi \searrow & & \swarrow \mathrm{pr}_1 \\ & U & \end{array}$$

commutes. If, moreover, the map ψ_x which is defined as the composition

$$E_x = \pi^{-1}(x) \xrightarrow{\psi} \{x\} \times \mathbb{R}^n \to \mathbb{R}^n$$

is linear for each x in U, then $\pi : E \to B$ is called **vector bundle.** A local trivialisation (ψ, U) is called vector bundle chart. If (ϕ, V) is another vector bundle chart, we are given the map

$$\begin{array}{ccc} U \cap V & \to & GL_n(\mathbb{R}) \\ x & \mapsto & \psi_x \circ \phi_x^{-1} \end{array}$$

which is called trasition function. A **vector bundle morphismus** from $\pi_1 : E_1 \to B_1$ to $\pi_2 : E_2 \to B_2$ is a smooth map $F : E_1 \to E_2$

which covers a smooth map $f: B_1 \to B_2$, that is $\pi_2 \circ F = f \circ \pi_1$, and which restricts to a linear map on each fibre.

An important example of a vector bundle is the tangent bundle $\pi_M : TM \to M$ of a differentiable manifold. If $\varphi : U \to \mathbb{R}^n$ is a chart, we define a vector bundle chart $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by sending a tangent vector $v \in T_x M$ to $(x, T_x\varphi(v))$. For any smooth map $f : M \to N$ the tangent map $Tf : TM \to TN$ is a vector bundle morphism.

Let $\pi : E \to B$ be a vector bundle. We say that $\pi : E \to B$ is **orientable** if we can choose an orientation ω_x on each fibre E_x and a vector bundle atlas $\{(\psi_i, U_i)\}$ such that for each chart (ψ, U) and each $x \in U$ the map ψ_x is orientation preserving. (Here $\mathbb{R}^n$ is equipped with the standard orientation.) This is equivalent to requiring that the transition maps of the bundle $\pi : E \to B$ have values in

$$GL_+(\mathbb{R}^n) = \{A \in GL_n(\mathbb{R}) : \det(A) > 0\}.$$

We call a manifold M orientable if its tangent bundle $\pi_M : TM \to M$ is orientable. Every complex manifold is orientable. This follows from the fact that the transition functions of its tangent bundle have values in $GL_n(\mathbb{C})$ which is connected, contains the (orientation preserving) identity and therefore is contained in $GL_+(\mathbb{R}^{2n})$. In particular generalized flag manifolds are orientable.

Next consider a submanifold S of M. Then the restriction of the tangent bundle of M to S, that is

$$TM\mid_S = \pi_M^{-1}(S) \to S$$

is again a vector bundle. (As vector bundle charts we can take the restrictions $\psi\mid_{\pi^{-1}(U\cap S)}: \pi^{-1}(U \cap S) \to (U \cap S) \times \mathbb{R}^n$.) Also the union of all factor spaces $(TM\mid_S)_x / T_x incl(T_x S)$ can be given the structure of a vector bundle. It is called **normal bundle** and will be denoted by N. Moreover, the sequence

$$0 \to TS \stackrel{Tincl}{\to} TM\mid_S \stackrel{p}{\to} N \to 0$$

is an exact sequence of vector bundles over S. This means that $Tincl$ and the projection p are vector bundle morphisms and for every $x \in S$ the sequence

$$0 \to (TS)_x \stackrel{T_x incl}{\to} (TM\mid_S)_x \stackrel{p_x}{\to} N_x \to 0$$

is exact. Any exact sequence of vector bundles splits, hence

$$TM\mid_S \cong TS \oplus N$$

and one can show that this implies that if two of the vector bundles are orientable so is the third. In particular an orientation of TS and $TM\mid_S$ determines an orientation of N.

A **tubular neighborhood** of a submanifold S in M consists of a vector bundle $\pi : E \to S$ together with a diffeomorphism $F : E \to U$, where U is an open neighborhood of S in M, such that the following diagram commutes

$$\begin{array}{ccc} E & F & \\ \xi \uparrow & \searrow & \\ S & \underset{incl}{\to} & M. \end{array}$$

Here ξ denotes the zero section. It can be proved that every closed submanifold of a differentiable manifold possesses a tubular neighborhood such that the corresponding vector bundle can be chosen to be the normal bundle $p: N \to S$.

Let $\pi : V \to M$ be an oriented real vector bundle of rank k on a manifold M and let us denote by $\{0\} \subset V$ the image of the zero section. Then there is a so-called **Thom class** $\tau_V \in H^k(V, V \setminus \{0\})$ which has the property that for any closed set A in M the map

$$H^i(M, M \setminus A) \to H^{i+k}(V, V \setminus A), \quad \alpha \mapsto \pi^*(\alpha) \cup \tau_V$$

is an isomorphism. Here A is regarded as a subspace of V by the embedding in the zero section. The isomorphism is called **Thom isomorphism**.

Finally, we use the preceding results to show that if S is an oriented s-dimensional closed submanifold of an oriented m-dimensional differentiable manifold M and A is any closed set in S, there is an isomorphism

$$H^i(S, S \setminus A) \cong H^{i+m-s}(M, M \setminus A).$$

This can be seen as follows: Since S and M are oriented, the normal bundle $p: N \to S$ is an oriented vector bundle and we can apply the Thom isomorphism to obtain

$$H^i(S, S \setminus A) \cong H^{i+m-s}(N, N \setminus A).$$

Next we choose a tubular neighborhood U of the closed submanifold S in M. Then U is diffeomorphic to the normal bundle and the diffeomorphism induces an isomorphism

$$H^{i+m-s}(N, N \setminus A) \cong H^{i+m-s}(U, U \setminus A).$$

Indeed, this isomorphism is independent of the choice of tubular neighbourhood.

Finally, by excision of $M \setminus U$ we have

$$H^{i+m-s}(U, U \setminus A) \cong H^{i+m-s}(M, M \setminus A)$$

and the claim follows.

4.3. Borel-Moore Homology

In this section Borel-Moore homology will be introduced. We follow mostly [**6**].

We denote by $H^i(X, Y)$ the relative singular cohomology group of a topological space X and a subspace Y. Now let X be a topological space that can be embedded as a closed subset in $\mathbb{R}^n$ for some n. For such a space X we define Borel-Moore homology groups, denoted by $\overline{H}_i X$, by the formula

$$\overline{H}_i X = H^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus X).$$

LEMMA 4.3.1. *The above definition of Borel-Moore homology groups is independent of the choice of embedding in Euclidean space*

PROOF. Let $\varphi : X \to \mathbb{R}^n$ and $\psi : X \to \mathbb{R}^m$ be two closed embeddings of the topological space X into Euclidean spaces. We will construct an isomorphism from $H^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus \varphi(X))$ to $H^{m-i}(\mathbb{R}^m, \mathbb{R}^m \setminus \psi(X))$.

The projection $p : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a (trivial) oriented vector bundle. Thus, the Thom isomorphism yields

$$(*) \quad H^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus \varphi(X)) \cong H^{n+m-i}(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n \times \mathbb{R}^m \setminus (\varphi, 0)(X)),$$

where $(\varphi, 0) : X \to \mathbb{R}^n \times \mathbb{R}^m$ is the closed embedding that sends x to $\varphi(x) \times 0$.

Next we apply the Tietze extension theorem to $\psi \circ \varphi^{-1} : \varphi(X) \to \mathbb{R}^m$ to conclude that there is a continuous map $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $f \circ \varphi = \psi$. We use f to define a map $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$, that satisfies $F \circ (\varphi, \psi) = (\varphi, 0)$, where (φ, ψ) sends x to $(\varphi(x), \psi(x))$: we put $F(v, w) = (v, w - f(v))$ for $(v, w) \in \mathbb{R}^n \times \mathbb{R}^m$. Note that F is a homeomorphism mapping $(\varphi, \psi)(X)$ onto $(\varphi, 0)(X)$. Hence it induces an isomorphism

$$(**) \quad \begin{aligned} F^* : H^{n+m-i}(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n \times \mathbb{R}^m \setminus (\varphi, 0)(X)) \to \\ H^{n+m-i}(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n \times \mathbb{R}^m \setminus (\varphi, \psi)(X)). \end{aligned}$$

Moreover, the definition of F is such, that the mapping F^* is independent of the choice of f: if f' is another continuous map such that $f' \circ \varphi = \psi$, then $F_t(v, w) = (v, w - t \cdot f(v) - (1 - t) \cdot f'(v))$ defines a homotopy from one to the other.

The composite of $(*)$ and $(**)$ gives

$$(\circ) \quad H^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus \varphi(X)) \cong H^{n+m-i}(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n \times \mathbb{R}^m \setminus (\varphi, \psi)(X))$$

and interchanging the roles of φ and ψ we similarly obtain

$$(\circ\circ) \quad H^{m-i}(\mathbb{R}^m, \mathbb{R}^m \setminus \psi(X)) \cong H^{n+m-i}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^m \times \mathbb{R}^n \setminus (\psi, \varphi)(X)).$$

To finish the proof we are left to construct an isomorphism between the right hand sides of $(\circ)$ and $(\circ\circ)$. To do so, consider the map $\kappa : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$, that maps (x, y) to (y, x). Then κ maps $(\psi, \varphi)(X)$ to $(\varphi, \psi)(X)$, it preserves the orientation if nm is even and reverses the orientation if nm is odd. We define the required isomorphism to be

$$\begin{aligned} (-1)^{nm} \kappa^* : H^{n+m-i}(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n \times \mathbb{R}^m \setminus (\varphi, \psi)(X)) \to \\ H^{n+m-i}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^m \times \mathbb{R}^n \setminus (\psi, \varphi)(X)). \end{aligned}$$

$\square$

REMARK 4.3.2. The isomorphism constructed in the preceding proof is such that if $\rho : X \to \mathbb{R}^k$ is a third closed embedding, then the following diagram commutes:

$$\begin{tikzcd} H^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus X) \arrow[rr] \arrow[dr] & & H^{m-i}(\mathbb{R}^m, \mathbb{R}^m \setminus X) \arrow[dl] \\ & H^{k-i}(\mathbb{R}^k, \mathbb{R}^k \setminus X) & \end{tikzcd}$$

LEMMA 4.3.3. *Let X be a topological space that can be embedded as a closed subset of an oriented m-dimensional differentiable manifold M. Then there is an isomorphism*

$$\overline{H}_i X \cong H^{m-i}(M, M \setminus X).$$

PROOF. This follows from the fact that every manifold can be embedded as a closed submanifold of some $\mathbb{R}^n$ (see [**8**, p.55]) and the result at the end of Section 4.2 which says that since M is a closed submanifold of $\mathbb{R}^n$ there is an isomorphism $H^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus X) \cong H^{m-i}(M, M \setminus X)$. □

The Borel-Moore homology groups of an oriented differentiable manifold M can be computed by embedding M in itself:

$$\overline{H}_i M \cong H^{m-i}(M, M \setminus M) \cong H^{m-i} M.$$

If M is compact as well, Poincare duality yields $H^{m-i}M \cong H_i M$ and thus Borel-Moore homology groups and singular homology groups agree for M.

Unlike singular homology, Borel-Moore homology is not functorial for arbitrary continuous maps. However, it can be proved that if $f : X \to Y$ is a continuous proper map, that is the preimage of every compact subset of Y is compact in X, then there is a homomorphism $f_* : \overline{H}_i(X) \to \overline{H}_i(Y)$ such that for any proper continuous map $g : Y \to Z$ we have $(g \circ f)_* = g_* \circ f_*$ and $(\mathrm{id})_* = \mathrm{id}$. We will verify this property for closed embeddings only (we won't need more here).

Let X and Y be spaces that admit a closed embedding in some Euclidean space and let $f : X \to Y$ be a closed embedding. If $i : Y \to \mathbb{R}^n$ is any closed embedding of Y, then $f \circ i : X \to Y$ gives a closed embedding of X. The inclusion

$$(\mathbb{R}^n, \mathbb{R}^n \setminus i(Y)) \hookrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus i(f(X)))$$

determines a homomorphism between the relative cohomology groups

$$H^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus i(f(X))) \to H^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus i(Y)).$$

We define $f_* : \overline{H}_i(X) \to \overline{H}_i(Y)$ to be this homomorphism. Let us see that f_* is independent of the choices made in this construction. To do so, consider another closed embedding $j : Y \to \mathbb{R}^m$. Then we obtain, just as above, a homomorphism $H^{m-i}(\mathbb{R}^m, \mathbb{R}^m \setminus j(f(X))) \to$

$H^{m-i}(\mathbb{R}^m, \mathbb{R}^m \setminus j(Y))$. We want to verify that the diagram

$$\begin{array}{ccc} H^{m-i}(\mathbb{R}^m, \mathbb{R}^m \setminus j(f(X))) & \to & H^{m-i}(\mathbb{R}^m, \mathbb{R}^m \setminus j(Y)) \\ \downarrow & & \downarrow \\ H^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus i(f(X))) & \to & H^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus i(Y)) \end{array}$$

commutes. Here the horizontal maps are the homomorphisms induced by $f : X \to Y$ and the vertical maps are isomorphisms as in Lemma 4.3.1. We subdivide the diagram into three smaller ones.
The first one is

$$\begin{array}{ccc} H^{m-i}(\mathbb{R}^m, \mathbb{R}^m \setminus j(f(X))) & \stackrel{h_1^*}{\to} & H^{m-i}(\mathbb{R}^m, \mathbb{R}^m \setminus j(Y)) \\ \downarrow & & \downarrow \\ H^{m+n-i}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus (j \times 0)(f(X))) & \stackrel{h_2^*}{\to} & H^{m+n-i}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus (j \times 0)(Y)) \end{array}$$

Here $(h_1)^*$ and $(h_2)^*$ denote the homomorphism in cohomology induced by the identities on $\mathbb{R}^m$ and $\mathbb{R}^{m+n}$ respectively. The vertical maps are the Thom isomorphisms $\alpha \mapsto \pi^*(\alpha) \cup \tau$, where π is the projection $\mathbb{R}^{m+n} \to \mathbb{R}^m$ and $\tau \in H^n(\mathbb{R}^{n+m}, \mathbb{R}^{n+m} \setminus \{0\})$ is the corresponding Thom class. Now take any $\alpha \in H^{m-i}(\mathbb{R}^m, \mathbb{R}^m \setminus j(f(X)))$, then

$$\pi^*(h_1^*(\alpha)) \cup \tau = (h_1 \circ \pi)^* \alpha \cup \tau.$$

If we go the other way round we obtain

$$(h_2)^*(\pi^*(\alpha) \cup \tau) = (h_2)^* \pi^*(\alpha) \cup (h_2)^* \tau = (\pi \circ h_2)^*(\alpha) \cup (h_2)^* \tau$$

by naturality of the cup product. Since $\pi \circ h_2 = h_1 \circ \pi$ and since

$$(h_2)^* : H^{m+n-i}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus \{0\}) \to H^{m+n-i}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus \{0\})$$

is the just identity on $H^{m+n-i}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus \{0\})$ the diagram commutes.
To see that the diagram

$$\begin{array}{ccc} H^{m+n-i}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus (j \times 0)(f(X))) & \to & H^{m+n-i}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus (j \times 0)(Y)) \\ \downarrow & & \downarrow \\ H^{m+n-i}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus (i \times 0)(f(X))) & \to & H^{m+n-i}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus (i \times 0)(Y)) \end{array}$$

commutes, just note that the maps that induce the respective homomorphisms in cohomology commute.
The last diagram

$$\begin{array}{ccc} H^{m+n-i}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus (i \times 0)(f(X))) & \to & H^{m+n-i}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus (i \times 0)(Y)) \\ \downarrow & & \downarrow \\ H^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus (i(f(X))) & \to & H^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus (i(Y)) \end{array}$$

commutes for the same reasons as the first.

The construction of the homomorphism $f_* : \overline{H}_i(X) \to \overline{H}_i(Y)$ shows immediately that these maps are functorial, that is $(\mathrm{id})_* = \mathrm{id}$ and for any closed embeddings $f : X \to Y$ and $g : Y \to Z$ we have $(g \circ f)_* = g_* \circ f_*$. It follows that homeomorphic spaces have isomorphic Borel-Moore homology groups.

Next consider the following situation: let X be a topological space that admits a closed embedding in an oriented m-dimensional manifold M, let U be open in X and Y be the complement of U in X. Then

U can be embedded as a closed subset of the oriented manifold $M \setminus Y$ and we have

$$H^{m-i}(M \setminus Y, M \setminus X) = H^{m-i}(M \setminus Y, (M \setminus Y) \setminus U) = \overline{H}_i(U).$$

Moreover we have the long exact cohomology sequence of the triple $M \setminus X \subset M \setminus Y \subset M$:

$$\begin{aligned} \ldots \to H^{m-i}(M, M \setminus Y) \to H^{m-i}(M, M \setminus X) \to H^{m-i}(M \setminus Y, M \setminus X) \\ \to H^{m-i+1}(M, M \setminus Y) \to \ldots \end{aligned}$$

This gives the following

Lemma 4.3.4. *If U is an open subset of a topological space X that admits a closed embedding in an oriented m-dimensional manifold M and if Y is the complement of U in X, then we have a long exact sequence*

$$\ldots \to \overline{H}_{i+1}U \to \overline{H}_iY \to \overline{H}_iX \to \overline{H}_iU \to \overline{H}_{i-1}Y \to \ldots.$$

The maps in this sequence are independent of the choice of an embedding in an oriented manifold: If $i : X \hookrightarrow M$ and $j : X \hookrightarrow N$ are two such embeddings, then we find a closed embedding of both of these manifolds in an Euclidean space. Hence, there is an isomorphism

$$H^{m-i}(M, M \setminus X) \cong H^{k-i}(\mathbb{R}^k, \mathbb{R}^k \setminus X) \cong H^{n-i}(N, N \setminus X),$$

which is the composite of Thom isomorphisms and isomorphisms that are induced by diffeomorphisms or inclusions of the corresponding pairs of topological spaces. These maps commute with the maps in the long exact sequences.

Lemma 4.3.5. *Let X be a disjoint union of a finite number of open subsets X_α. Then $\overline{H}_iX$ is the direct sum of the $\overline{H}_iX_\alpha$.*

Proof. This is an immediate consequence of the relative Mayer-Vietoris sequence in cohomology. □

4.4. Computation of the Homology of Spaces G/P

To start this section, recall what we have seen in Section 3.2: Generalized flag manifolds may be decomposed into a disjoint union of so called Schubert cells $X_w \cong \mathbb{C}^{l(w)}$. The closures $\overline{X_w}$ of these Schubert cells are called Schubert varieties and we have $\overline{X_w} = \bigsqcup_{w' \leq w} X_{w'}$.

Moreover, we have already observed that generalized flag manifolds are compact and oriented, hence their singular and Borel-Moore homology groups agree. The following theorem shows that the homology group $H_*(X) = \bigoplus_{i \in \mathbb{Z}} H_i(X)$ of a generalized flag manifold X is free abelian with one generator for each Schubert cell.

THEOREM 4.4.1. *Let X be an n-dimensional generalized flag manifold. Then the i-th Borel-Moore homology group of X is given by*

$$\overline{H}_i(X) = \begin{cases} \bigoplus_{l(w)=p} \overline{H}_{2p}(X_w) = \mathbb{Z}^{n_p} & if\, i = 2p \\ 0 & if\, i\, is\, odd, \end{cases}$$

where n_p denotes the number of Schubert cells X_w such that $l(w) = p$.

PROOF. Observe first that since $\mathbb{C}^p$ is orientable we have $\overline{H}_i(\mathbb{C}^p) \cong H^{2p-i}(\mathbb{C}^p)$ and thus

$$\overline{H}_i(\mathbb{C}^p) = \begin{cases} \mathbb{Z} & \text{if } i = 2p \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\overline{H}_i(X_w) = \mathbb{Z}$ if $i = 2l(w)$ and zero otherwise.

We denote by X_p the set

$$X_p = \bigcup_{l(w)=p} \overline{X}_w,$$

then X_p is closed in X, $X_n = X$ and we have

$$X_p \setminus X_{p-1} = \bigsqcup_{l(w)=p} X_w.$$

Each X_w is closed in $X_p \setminus X_{p-1}$ since

$$\overline{X}_w \cap (X_p \setminus X_{p-1}) = (\bigsqcup_{w' \leq w} X_{w'}) \cap (\bigsqcup_{l(w)=p} X_w) = X_w,$$

but then its complement is closed too and therefore each X_w is open in $X_p \setminus X_{p-1}$. We apply Lemma 4.3.5 to conclude that

$$\overline{H}_i(X_p \setminus X_{p-1}) = \overline{H}_i(\bigsqcup_{l(w)=p} X_w) = \bigoplus_{l(w)=p} \overline{H}_i(X_w).$$

We proceed by induction on p showing that we have $\overline{H}_{2q}(X_p) = \bigoplus_{l(w)=q} \overline{H}_{2q}(X_w)$ for $0 \leq q \leq p$, and $\overline{H}_i(X_p) = 0$ otherwise. For X_0 this this follows immediately from the fact hat $X_0 = X_{\text{id}}$ is a single point. Assume the claim to be true for $p-1$. We will show that it holds for p as well. Applying Lemma 4.3.4 to $U = X_p \setminus X_{p-1}$ which is open in X_p and its complement $X_p \setminus U = X_{p-1}$, gives the long exact sequence

$$\begin{gathered} \ldots \to \overline{H}_{i+1}(X_p \setminus X_{p-1}) \to \overline{H}_i(X_{p-1}) \to \overline{H}_i(X_p) \to \overline{H}_i(X_p \setminus X_{p-1}) \to \\ \to \overline{H}_{i-1}(X_{p-1}) \to \ldots \end{gathered}$$

If i is odd we have $\overline{H}_i(X_p \setminus X_{p-1}) = \bigoplus_{l(w)=p} \overline{H}_i(X_w) = 0$, by inductive hypothesis also $\overline{H}_i(X_{p-1}) = 0$ and therefore $\overline{H}_i(X_p) = 0$ as well.
For even indices i we have the exact sequence

$$0 \to \overline{H}_i(X_{p-1}) \to \overline{H}_i(X_p) \to \overline{H}_i(X_p \setminus X_{p-1}) \to 0.$$

If $i = 2p$ it follows from $\overline{H}_{2p}(X_{p-1}) = 0$ that
$$\overline{H}_{2p}(X_p) \cong \overline{H}_{2p}(X_p \setminus X_{p-1}) \cong \bigoplus_{l(w)=p} \overline{H}_{2p}(X_w).$$
If $i = 2q,\ q \neq p$ we have $\overline{H}_{2q}(X_p \setminus X_{p-1}) \cong \bigoplus_{l(w)=p} \overline{H}_{2q}(X_w) = 0$ and thus $\overline{H}_{2q}(X_p) \cong \overline{H}_{2q}(X_{p-1})$. By inductive hypothesis the latter group is isomorphic to $\bigoplus_{l(w)=q} \overline{H}_{2q}(X_w)$. This proves the theorem. □

EXAMPLE 4.4.2. We shall apply the preceding theorem to determine the homology of $Gr_2(\mathbb{C}^4)$. Its Hasse diagram was computed in Example 2.3.3:
$$W^p = \{\mathrm{id},\ s_{\alpha_2},\ s_{\alpha_2}s_{\alpha_1},\ s_{\alpha_2}s_{\alpha_3},\ s_{\alpha_2}s_{\alpha_1}s_{\alpha_3},\ s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}\}.$$
This gives: $H^0(X) = H^2(X) = H^6(X) = H^8(X) = \mathbb{Z}$, $H^4(X) = \mathbb{Z}^2$ and $H^i(X) = 0$ otherwise.

Finally, we can read off the homology of the generalized flag manifold corresponding to the parabolic $\times\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\circ$ from the Hasse diagram presented at the end of Remark 2.3.4 : $H^0(X) = H^{10}(X) = \mathbb{Z}$, $H^2(X) = H^8(X) = \mathbb{Z}^2$, $H^4(X) = H^6(X) = \mathbb{Z}^3$ and $H^i(X) = 0$ otherwise.

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Curriculum vitae

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