Classification of complex and real semisimple Lie Algebras

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Vorwort

Der vorliegende Text behandelt die Klassifikation von komplexen und reellen halbeinfachen Lie Algebren. Das erste Kapitel behandelt den Fall der komplexen Lie Algebren und beginnt mit elementaren Definitionen. Wir geben eine Liste aller im Text vorkommenden Lie Algebren von Matrizen. Die Cartan Teilalgebra einer komplexen halbeinfachen Lie Algebra wird eingeführt. Diese ist eine maximale abelsche Teilalgebra und führt zur Wurzelraumzerlegung einer halbeinfachen Lie Algebra. Die Wurzeln sind lineare Funktionale auf der Cartan Teilalgebra, die die Wirkung der adjungierten Darstellung auf die Lie Algebra beschreiben. Die Wurzeln solch einer Zerlegung bilden eine endliche Teilmenge eines endlichdimensionalen euklidischen Vektorraumes mit besonderen Eigenschaften, ein reduziertes Wurzelsystem. Aus einem Wurzelsystem lassen sich die Cartan Matrix und das Dynkin Diagramm einer Lie Algebra bilden, welche die Eigenschaften der Lie Algebra beschreiben. Wir definieren ein abstraktes Dynkin Diagram und umreißen die Klassifikation derselben. Wir geben eine Liste aller abstrakten Dynkin Diagramme. Die Klassifikation der komplexen halbeinfachen Lie Algebren basiert auf dem Existenzsatz, welcher sagt, daß jedes abstrakte Dynkin Diagramm Dynkin Diagramm einer komplexen halbeinfachen Lie Algebra ist, und aus dem Isomorphismus Theorem, welches garantiert, daß nichtisomorphe einfache Lie Algebren verschiedene Dynkin Diagramme besitzen. Am Ende des ersten Kapitels steht ein Beispiel.

Das zweite Kapitel wendet sich den reellen halbeinfachen Lie Algebren zu. Wir beginnen mit der Definition reeller Formen von komplexen Lie Algebren. Zu jeder komplexen Lie Algebra existiert eine Splitform und eine kompakte reelle Form. Die kompakte reelle Form der Komplexifizierung einer reellen halbeinfachen Lie Algebra führt zu den äquivalenten Begriffen der Cartan Involution und der Cartan Zerlegung, welche eine reelle halbeinfache Lie Algebra in eine maximale kompakte Teilalgebra und einen Vektorteil zerlegt. Die Iwasawa Zerlegung von Lie Gruppen verallgemeinert den Gram-Schidt Orthogonalisierungsprozeß und führt zum Begriff der eingeschränkten Wurzeln. Diese sind lineare Funktionale auf einem maximalen abelschen Teilraum des Vektorteils und bilden ein abstraktes Wurzelsystem. Eine Cartan Teilalgebra einer reellen halbeinfachen Lie Algebra ist eine

Teilalgebra, deren Komplexifizierung eine Cartan Teilalgebra der Komplexifizierung der Lie Albegra ist. Im Gegensatz zum komplexen Fall sind nicht alle Cartan Teilalgebren konjugiert. Beim Studium der Cartan Teilalgebren können wir uns jedoch auf solche einschränken, die stabil unter einer Cartan Involution sind. Wenn wir eine maximal kompakte Cartan Teilalgebra wählen können wir ein positives System des Wurzelsystems wählen, das aus rein imaginären und komplexen Wurzeln besteht. Die Cartan Involution fixiert die rein imaginären Wurzeln und permutiert die komplexen Wurzeln in Orbits aus 2 Elementen. Das Vogan Diagramm einer reellen halbeinfachen Lie Algebra besteht aus dem Dynkin Diagramm der Komplexifizierung und folgender zusätzlicher Information. Die komplexen Wurzeln in 2-elementigen Orbits werden durch Pfeile verbunden und die imaginären Wurzeln werden ausgemalt so sie nicht kompakt sind. Zur Klassifizierung dienen wieder Sätze, die den Zusammenhang zwischen Vogan Diagrammen und reellen halbeinfachen Lie Algebren herstellen. Jedes Diagramm, das formal wie ein Vogan Diagramm aussieht ist Vogan Diagramm einer reellen halbeinfachen Lie Algebra. Haben zwei reelle halbeinfache Lie Algebren das selbe Vogan Diagramm, so sind sie isomorph, aber reelle halbeinfache Lie Algebren mit unterschiedlichen Vogan Diagrammen können isomorph sein. Das Problem dieser Redundenz löst das Theorem von Borel und de Siebenthal. Wir geben eine Liste aller Vogan Diagramme, die den Redundenztest dieses Theorems überstehen und wenden uns schließlich der Realisierung einiger Diagramme als Lie Algebren von Matrizen zu.

Zum Schluß besprechen wir kurz einen alternativen Weg der Klassifizierung reeller halbeinfacher Lie Algebren. Nimmt man anstatt der maximal kompakten eine maximal nichtkompakte Cartan Teilalgebra und legt man eine andere Ordnung der Wurzeln zugrunde erhält man den Begriff des Satake Diagramms. Eine Auflistung aller auftretenden Satake Diagramme bildet den Schluß.

Preface

This text deals with the classification of complex and real semisimple Lie algebras. In the first chapter we deal the complex case and start with elementary definitions. We list all matrix Lie algebras which we will deal with throughout the text. We introduce Cartan subalgebras of a semisimple complex Lie algebra, which are maximal abelian subalgebras and discuss the root space decomposition. The roots are linear functionals on the Cartan subalgebra describing the action of the adjoint representation on the Lie algebra. These roots form reduced root systems which we describe as a specific finite subset of a finite dimensional vector space with inner product. From the root systems we deduce Cartan matrices and Dynkin diagrams. We introduce the notion of abstract Dynkin diagrams and outline the classification of these. We give a complete list of abstract Dynkin diagrams. We mention the Existence theorem, stating that every abstract Dynkin diagram comes from a complex simple Lie algebra, and the Isomorphism Theorem, which says that nonisomorphic simple Lie algebras have different Dynkin diagrams, to obtain the classification. At the end of the first chapter we look at an example.

The second chapter deals with real semisimple Lie algebras. We start with real forms of complex Lie algebras, observing that for every complex semisimple Lie algebra there exists a split real form and a compact real form. The compact real form of the complexification of a real semisimple Lie algebra yields to the notion of Cartan involutions and the equivalent notion of Cartan decompositions, which decomposes real semisimple Lie algebras in a maximally compact subalgebra and a vector part. The Iwasawa decomposition on group level generalizes the Gram-Schmidt orthogonalization process and leads to the notion of restricted roots. These are linear functionals on a maximal abelian subspace of the vector part, which form an abstract root system. A Cartan subalgebra of a real semisimple Lie algebra is a subalgebra whose complexification is a Cartan subalgebra in the complexified Lie algebra. In contrast to the complex case not all Cartan subalgebras are conjugate, but we may restrict to the study of Cartan subalgebras that are stable under a Cartan involution. When we choose a maximally compact Cartan subalgebra we can fix a positive system such that the simple roots are either purely imaginary or complex. The Cartan involution permutes the complex ones in 2-cycles. A Vogan diagram of a real semisimple Lie algebra consists of the Dynkin diagram of its complexification plus additional information. The 2-cycles of complex simple roots are labeled and the imaginary simple roots are painted if they are noncompact. Every diagram that looks formally like a Vogan diagram comes from a real semisimple Lie algebra. Two real semisimple Lie algebras with the same Vogan diagram are isomorphic, but real semisimple Lie algebras with different Vogan diagrams might also be isomorphic. This redundancy is resolved by the Borel and de Siebenthal Theorem. We give a complete list of Vogan diagrams surviving this redundancy test an take a look at the matrix realizations of some of these.

An alternative way of classifying real semisimple arises from choosing a maximally noncompact Cartan subalgebra and another positive system which leads to the notion of Satake diagrams.

CHAPTER 1

Classification of complex semisimple Lie Algebras

We will start with some elementary definitions and notions.

A vector space $\mathfrak{g}$ over the field $\mathbb{K}$ together with a bilinear mapping $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra if the following two conditions are satisfied:

1: $[X, Y] = -[Y, X]$
2a: $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$
2b: $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$

2a and **2b** are equivalent. We will call $[,]$ bracket. The second condition is called Jacobi identity. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. A homomorphism of Lie algebras is a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$$

$\forall X, Y \in \mathfrak{g}$. An isomorphism is a homomorphism that is one-one and onto. An isomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}$ is called automorphism of $\mathfrak{g}$. The set of automorphisms of a Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ is denoted by $\mathrm{Aut}_{\mathbb{K}}\, \mathfrak{g}$. If $\mathfrak{a}$ an $\mathfrak{b}$ are subsets of $\mathfrak{g}$, we write

$$[\mathfrak{a}, \mathfrak{b}] = span\{[X, Y] | X \in \mathfrak{a}, Y \in \mathfrak{b}\}$$

A Lie subalgebra (or subalgebra for short) $\mathfrak{h}$ of $\mathfrak{g}$ is a linear subspace satisfying $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. Then $\mathfrak{h}$ itself is a Lie algebra. An ideal $\mathfrak{h}$ of $\mathfrak{g}$ is a linear subspace satisfying $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$. Every ideal of $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is said to be abelian if $[\mathfrak{g}, \mathfrak{g}] = 0$. Let $\mathfrak{s}$ be an arbitrary subset of $\mathfrak{g}$. We call

$$Z_{\mathfrak{g}}(\mathfrak{s}) = \{X \in \mathfrak{g} | [X, Y] = 0 \;\; \forall Y \in \mathfrak{s}\}$$

the centralizer of $\mathfrak{s}$ in $\mathfrak{g}$. The center of $\mathfrak{g}$ is $Z_{\mathfrak{g}}(\mathfrak{g})$ denoted $Z_{\mathfrak{g}}$. If $\mathfrak{s}$ is a Lie subalgebra we call

$$N_{\mathfrak{g}}(\mathfrak{s}) = \{X \in \mathfrak{g} | [X, Y] \in \mathfrak{s} \;\; \forall Y \in \mathfrak{s}\}$$

the normalizer of $\mathfrak{s}$ in $\mathfrak{g}$. Centralizer and normalizer are Lie subalgebras of $\mathfrak{g}$ and $\mathfrak{s} \subseteq N_{\mathfrak{g}}(\mathfrak{s})$ always holds. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in a Lie algebra $\mathfrak{g}$, then so are $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$ and $[\mathfrak{a}, \mathfrak{b}]$. If $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ we define the quotient algebra $\mathfrak{g}/\mathfrak{a}$ as the quotient of the vector spaces $\mathfrak{g}$ and $\mathfrak{a}$ equipped with the bracket law $[X + \mathfrak{a}, Y + \mathfrak{a}] = [X, Y] + \mathfrak{a}$. Furthermore let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be a map satisfying $\mathfrak{a} \subseteq \ker \varphi$. Then φ factors through the quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ defining a homomorphism $\mathfrak{g}/\mathfrak{a} \to \mathfrak{h}$.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebras. We define recursively

$$\mathfrak{g}^0 = \mathfrak{g}, \ \ \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \ \ \mathfrak{g}^{j+1} = [\mathfrak{g}^j, \mathfrak{g}^j]$$

The decreasing sequence

$$\mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \cdots$$

is called commutator series for $\mathfrak{g}$. Each $\mathfrak{g}^j$ is an ideal in $\mathfrak{g}$ and $\mathfrak{g}$ is called solvable if $\mathfrak{g}^j = 0$ for some j. We define recursively

$$\mathfrak{g}_0 = \mathfrak{g}, \ \ \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \ \ \mathfrak{g}_{j+1} = [\mathfrak{g}, \mathfrak{g}_j]$$

The decreasing sequence

$$\mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \cdots$$

is called lower central series for $\mathfrak{g}$. Each $\mathfrak{g}_j$ is an ideal in $\mathfrak{g}$ and $\mathfrak{g}$ is called nilpotent if $\mathfrak{g}_j = 0$ for some j. Since $\mathfrak{g}_j \subseteq \mathfrak{g}^j$ for each j nilpotency implies solvability. The sum of two solvable ideals is a solvable ideal. Hence there exists a unique maximal solvable ideal, which we call the radical $\operatorname{rad} \mathfrak{g}$ of $\mathfrak{g}$.

A Lie algebra $\mathfrak{g}$ is simple if $\mathfrak{g}$ is nonabelian and has no proper nonzero ideals. A Lie algebra $\mathfrak{g}$ is semisimple if it has no nonzero solvable ideals (i.e.: $\operatorname{rad} \mathfrak{g} = 0$). Every simple Lie algebra is semisimple and every semisimple Lie algebra has 0 center. If $\mathfrak{g}$ is any finite-dimensional Lie algebra, then $\mathfrak{g} / \operatorname{rad} \mathfrak{g}$ is semisimple.

Let V be a vector space over $\mathbb{K}$ and let $M : V \to V$ and $N : V \to V$ be vector space endomorphisms. Let $\operatorname{End}_{\mathbb{K}} V$ denote the vector space of endomorphisms of V. This is a Lie algebra with bracket defined by

$$[M, N] := M \circ N - N \circ M$$

A derivation of a Lie algebra $\mathfrak{g}$ is an Endomorphism $D \in \operatorname{End}_{\mathbb{K}} \mathfrak{g}$ such that

$$D[X, Y] = [DX, Y] + [X, DY].$$

Definition (**2b**) of the Jacobi identity says, that $[X, \_]$ acts like a derivation. A representation of a Lie algebra $\mathfrak{g}$ on a vector space V over a field $\mathbb{K}$ is a homomorphism of Lie algebras $\pi : \mathfrak{g} \to \operatorname{End}_{\mathbb{K}} V$. The adjoint representation of $\mathfrak{g}$ on the vector space $\mathfrak{g}$ is defined by $\operatorname{ad} X(Y) = [X, Y]$. $\operatorname{ad} X$ lies in $\operatorname{Der}(\mathfrak{g})$ because of the Jacobi identity.

A direct sum of two Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$ is the vector space direct sum $\mathfrak{a} \oplus \mathfrak{b}$ with unchanged bracket law within each component and $[\mathfrak{a}, \mathfrak{b}] = 0$.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{K}$, $X, Y \in \mathfrak{g}$. The symmetric bilinear form defined by

$$B(X, Y) = \operatorname{Tr}(\operatorname{ad} X \circ \operatorname{ad} Y)$$

is called Killing form. The Killing form satisfies

$$B([X, Y], Z) = B(X, [Y, Z])$$

We call this property invariance. The radical of B (or generally of any bilinear form) defined by

$$\operatorname{rad} B = \{v \in \mathfrak{g} | B(v,u) = 0 \ \forall u \in \mathfrak{g}\}$$

is an ideal of $\mathfrak{g}$ because of the invariance of B. B is called degenerate if $\operatorname{rad} B \neq 0$, otherwise it is called nondegenerate.

1.1. Proposition. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The following conditions are equivalent:*

(1) $\mathfrak{g}$ *is semisimple*
(2) *The Killing form of* $\mathfrak{g}$ *is nondegenerate*
(3) $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ *with* $\mathfrak{g}_j$ *a simple ideal for all* j*. This decomposition is unique and the only ideals of* $\mathfrak{g}$ *are direct sums of various* $\mathfrak{g}_j$*.*

1.2. Corollary. *If $\mathfrak{g}$ is semisimple, then $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. If $\mathfrak{a}$ is any ideal in $\mathfrak{g}$, then $\mathfrak{a}^\perp$ is an ideal and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$.*

A Lie algebra $\mathfrak{g}$ is called reductive if

$$\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \oplus Z_{\mathfrak{g}}$$

with $[\mathfrak{g},\mathfrak{g}]$ being semisimple and $Z_{\mathfrak{g}}$ abelian.

1.3. Corollary. *If $\mathfrak{g}$ is reductive, then $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \oplus Z_{\mathfrak{g}}$ with $[\mathfrak{g},\mathfrak{g}]$ semisimple and $Z_{\mathfrak{g}}$ abelian.*

We now give the definitions of all matrix Lie algebras we will need. The bracket relation will always be defined the way it is done for Lie algebras of endomorphisms above.

$\mathbb{H}$ denotes the quaternions, a division algebra over $\mathbb{R}$ with basis $\{1, i, j, k\}$ satisfying the following conditions:

$$\begin{aligned} &i^2 = j^2 = k^2 = -1 \\ &ij = k, jk = i, ki = j \\ &ji = -k, kj = -i, ik = -j \end{aligned}$$

The real part of a quaternion is given by $\operatorname{Re}(a + ib + jc + kd) = a$. We define some matrices used in the definitions later on. Let I_n denote the identity matrix of dimension n-by-n. Let

$$J_{n,n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \ I_{m,n} = \begin{pmatrix} I_m & 0 \\ 0 & I_n \end{pmatrix} \text{ and } K_{n,n} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

The following proposition enables us to check reductiveness.

1.4. Proposition. *Let $\mathfrak{g}$ be a real Lie algebra of matrices over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. If $\mathfrak{g}$ is closed under conjugate transpose (i.e.: $(X)^* = (\overline{X})^t \in \mathfrak{g}$ $\forall X \in \mathfrak{g}$) then $\mathfrak{g}$ is reductive.*

PROOF. Define an inner product $\langle X, Y\rangle = \operatorname{Re}\operatorname{Tr}(XY^*)$ for X, Y in $\mathfrak{g}$. Let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$ and denote $\mathfrak{a}^\perp$ the orthogonal complement of $\mathfrak{a}$. The $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ as vector space. To see that $\mathfrak{a}^\perp$ is an ideal in $\mathfrak{g}$ we choose arbitrary elements $X \in \mathfrak{a}^\perp$, $Y \in \mathfrak{g}$ and $Z \in \mathfrak{a}$ and compute

$$\begin{aligned}\langle [X,Y], Z\rangle &= \operatorname{Re}\operatorname{Tr}(XYZ^* - YXZ^*)\\ &= -\operatorname{Re}\operatorname{Tr}(XZ^*Y - XYZ^*)\\ &= -\operatorname{Re}\operatorname{Tr}(X(Y^*Z)^* - X(ZY^*)^*)\\ &= -\langle X, [Y^*, Z]\rangle\end{aligned}$$

Since Y^* is in $\mathfrak{g}$, $[Y^*, Z]$ is in $\mathfrak{a}$. Thus the right hand side is 0 for all Z and hence $[X, Y]$ is in $\mathfrak{a}^\perp$ and $\mathfrak{a}^\perp$ is an ideal and $\mathfrak{g}$ is reductive. $\square$

In the sequel we will use matrix Lie algebras listed below. These are all reductive by proposition 1.4. To check semisimplicity one might use corollary 1.3, to see that $\mathfrak{g}$ is semisimple if $Z_\mathfrak{g} = 0$.

- Reductive Lie algebras

$$\begin{aligned}\mathfrak{gl}(n,\mathbb{C}) &= \{n\text{-by-}n \text{ matrices over } \mathbb{C}\}\\ \mathfrak{gl}(n,\mathbb{R}) &= \{n\text{-by-}n \text{ matrices over } \mathbb{R}\}\\ \mathfrak{gl}(n,\mathbb{H}) &= \{n\text{-by-}n \text{ matrices over } \mathbb{H}\}\end{aligned}$$

- Semisimple Lie algebras over $\mathbb{C}$

$$\begin{aligned}\mathfrak{sl}(n,\mathbb{C}) &= \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid \operatorname{Tr} X = 0\} && \text{for } n \geq 2\\ \mathfrak{so}(n,\mathbb{C}) &= \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid X + X^t = 0\} && \text{for } n \geq 3\\ \mathfrak{sp}(n,\mathbb{C}) &= \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid X^t J_{n,n} + J_{n,n}X = 0\} && \text{for } n \geq 1\end{aligned}$$

- Semisimple Lie algebras over $\mathbb{R}$

$$\begin{aligned}\mathfrak{sl}(n,\mathbb{R}) &= \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid \operatorname{Tr} X = 0\} && \text{for } n \geq 2\\ \mathfrak{sl}(n,\mathbb{H}) &= \{X \in \mathfrak{gl}(n,\mathbb{H}) \mid \operatorname{Re}\operatorname{Tr} X = 0\} && \text{for } n \geq 1\\ \mathfrak{so}(p,q) &= \{X \in \mathfrak{gl}(p+q,\mathbb{R}) \mid X^* I_{p,q} + I_{p,q}X = 0\} && \text{for } p+q \geq 3\\ \mathfrak{su}(p,q) &= \{X \in \mathfrak{sl}(p+q,\mathbb{C}) \mid X^* I_{p,q} + I_{p,q}X = 0\} && \text{for } p+q \geq 2\\ \mathfrak{sp}(n,\mathbb{R}) &= \{X \in \mathfrak{gl}(2n,\mathbb{R}) \mid X^t J_{n,n} + J_{n,n}X = 0\} && \text{for } n \geq 1\\ \mathfrak{sp}(p,q) &= \{X \in \mathfrak{gl}(p+q,\mathbb{H}) \mid X^* I_{p,q} + I_{p,q}X = 0\} && \text{for } p+q \geq 1\\ \mathfrak{so}^*(2n) &= \{X \in \mathfrak{su}(n,n) \mid X^t K_{n,n} + K_{n,n}X = 0\} && \text{for } n \geq 2\end{aligned}$$

Now we want to understand the bracket relation of complex semisimple Lie algebras.

1.5. PROPOSITION. *If $\mathfrak{g}$ is any finite-dimensional Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ is a nilpotent subalgebra, then there is a finite subset $\Delta \subset \mathfrak{h}^*$ such that*

(1) $\mathfrak{g} = \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$ *where*

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid (\operatorname{ad} H - \alpha(H)\mathbb{I})^n X = 0 \textit{ for all } H \in \mathfrak{h} \textit{ and some } n\}$$

(2) $\mathfrak{h} \subseteq \mathfrak{g}_0$

(3) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$

The $\mathfrak{g}_\alpha$ are called generalized weight spaces of $\mathfrak{g}$ relative to $\operatorname{ad}\mathfrak{h}$ with generalized weights α. The members of $\mathfrak{g}_\alpha$ are called generalized weight vectors. The decomposition statement (1) holds for any representation of a nilpotent Lie algebra over $\mathbb{C}$ on a finite-dimensional complex vector space. The proof of this decomposition uses Lie´s Theorem which states that there is a simultaneous eigenvector for any representation of a solvable Lie algebra on a finite-dimensional vector space over an algebraically closed field. Statement (2) is clear since $\operatorname{ad}\mathfrak{h}$ is nilpotent on $\mathfrak{h}$. In statement (3) we set $\mathfrak{g}_{\alpha+\beta} = \{0\}$ if $\alpha+\beta$ is no generalized weight. The proof consists of an elementary calculation. As a consequence $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$. A nilpotent Lie subalgebra $\mathfrak{h}$ of a finite-dimensional complex Lie algebra $\mathfrak{g}$ is a Cartan subalgebra if $\mathfrak{h} = \mathfrak{g}_0$. One proves that $\mathfrak{h}$ is a Cartan subalgebra if and only if $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$. If $\mathfrak{g}$ is semisimple a Cartan algebra $\mathfrak{h}$ is maximal abelian.

These are the two important theorems concerning Cartan subalgebras of finite-dimensional complex Lie algebras:

1.6. Theorem. *Any finite-dimensional complex Lie algebra $\mathfrak{g}$ has a Cartan subalgebra.*

1.7. Theorem. *If $\mathfrak{h}$ and $\mathfrak{h}'$ are Cartan subalgebras of a finite-dimensional complex Lie algebra $\mathfrak{g}$, then there exists an inner automorphism $a \in \operatorname{Int}\mathfrak{g}$ such that $a(\mathfrak{h}) = \mathfrak{h}'$.*

We say that $\mathfrak{h}$ and $\mathfrak{h}'$ are conjugate via a. Because of this conjugation all Cartan subalgebras of a complex Lie algebra $\mathfrak{g}$ have the same dimension, which is called rank of $\mathfrak{g}$.

This decomposition is simpler for semisimple Lie algebras. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Killing form B and Cartan subalgebra $\mathfrak{h}$. The generalized weights of the representation $\operatorname{ad}\mathfrak{h}$ on $\mathfrak{g}$ are called roots. The set of roots is denoted by Δ and is called root system. The decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

is called root space decomposition of $\mathfrak{g}$. This decomposition has a number of nice properties:

1.8. Proposition.

- *The $\mathfrak{g}_\alpha$ are 1-dimensional and are therefore given by*

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | (\operatorname{ad} H)X = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}.$$

- $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.
- *If α and β are in $\Delta \cup \{0\}$ and $\alpha+\beta \neq 0$ then $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.*
- *If α is in $\Delta \cup \{0\}$ then B is nonsingular on $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$.*
- *If $\alpha \in \Delta$ then $-\alpha \in \Delta$*

- $B|_{\mathfrak{h}\times\mathfrak{h}}$ *is nondegenerate and consequently there exists to each root* $\alpha \in \mathfrak{h}^*$ *an element* $H_\alpha \in \mathfrak{h}$ *such that* $\alpha(H) = B(H, H_\alpha)$ *for all* $H \in \mathfrak{h}$.
- Δ *spans* $\mathfrak{h}^*$.
- *Choose* $\mathfrak{g}_\alpha \ni E_\alpha \neq 0$ *for all* $\alpha \in \Delta$ *so* $[H, E_\alpha] = \alpha(H)E_\alpha$. *If* $X \in \mathfrak{g}_{-\alpha}$ *then* $[E_\alpha, X] = B(E_\alpha, X)H_\alpha$.
- *If* α *and* $\beta \in \Delta$ *then* $\beta(H_\alpha)$ *is a rational multiple of* $\alpha(H_\alpha)$.
- *If* $\alpha \in \Delta$ *then* $\alpha(H_\alpha) \neq 0$.
- *The action of* $\operatorname{ad}\mathfrak{h}$ *on* $\mathfrak{g}$ *is simultaneously diagonable.*
- *If* H *and* $H' \in \mathfrak{h}$ *then* $B(H, H') = \sum_{\alpha\in\Delta} \alpha(H)\alpha(H')$.
- *The pair of vectors* $\{E_\alpha, E_{-\alpha}\}$ *can be chosen so that* $B(E_\alpha, E_{-\alpha}) = 1$.

We define a bilinear form $\langle , \rangle$ on $\mathfrak{h}^*$ by $\langle \varphi, \psi \rangle = B(H_\varphi, H_\psi) = \varphi(H_\psi) = \psi(H_\varphi)$.

1.9. Proposition. *Let* V *be the* $\mathbb{R}$ *linear span of* Δ *in* $\mathfrak{h}^*$. *Then* V *is a real form of the vector space* $\mathfrak{h}^*$ *and the restriction of the bilinear form* $\langle , \rangle$ *to* $V \times V$ *is a positive definite inner product. Let* $\mathfrak{h}_0$ *be the* $\mathbb{R}$ *linear span of all* H_α *for* $\alpha \in \Delta$ *then* $\mathfrak{h}_0$ *is a real form of the vector space* $\mathfrak{h}$, *the members of* V *are exactly those linear functionals that are real on* $\mathfrak{h}_0$. *Restricting those linear functionals to operate on* $\mathfrak{h}_0$ *yields an* $\mathbb{R}$ *isomorphism of* V *to* $\mathfrak{h}_0$.

Let $|\varphi|^2 = \langle \varphi, \varphi \rangle$ and $\alpha \in \Delta$. The mapping $s_\alpha : \mathfrak{h}_0^* \to \mathfrak{h}_0^*$ defined by

$$s_\alpha(\varphi) = \varphi - \frac{2\langle \varphi, \alpha \rangle}{|\alpha|^2}\alpha$$

is called root reflection. The root reflections are orthogonal transformations which carry Δ to Δ.

A reduced abstract root system in a finite-dimensional real vector space V with inner product $\langle , \rangle$ is a finite set Δ of nonzero elements such that

(1) Δ spans V
(2) the orthogonal transformation $s_\alpha(\varphi) = \varphi - \frac{2\langle\varphi,\alpha\rangle}{|\alpha|^2}$ for $\alpha \in \Delta$ carry Δ to itself
(3) $\frac{2\langle\beta,\alpha\rangle}{|\alpha|^2}$ is an integer for $\alpha, \beta \in \Delta$
(4) $\alpha \in \Delta$ implies $2\alpha \notin \Delta$ (without this condition the abstract root system is called nonreduced)

1.10. Theorem. *The root system of a complex semisimple Lie algebra* $\mathfrak{g}$ *with respect to a Cartan subalgebra* $\mathfrak{h}$ *forms a reduced abstract root system in* $\mathfrak{h}_0^*$.

Two abstract root systems Δ in V and Δ' in V' are isomorphic if there exists a vector space isomorphism $\psi : V \to V'$ such that $\psi(\Delta) =$

Δ' and

$$\frac{2\langle\beta,\alpha\rangle}{|\alpha|^2} = \frac{2\langle\psi(\beta),\psi(\alpha)\rangle}{|\psi(\alpha)|^2}$$

for $\alpha, \beta \in \Delta$. An abstract root system Δ in V is said to be reducible if Δ admits a nontrivial disjoint decomposition $\Delta = \Delta' \cup \Delta''$ with every member of Δ' orthogonal to every member of Δ''. Δ is called irreducible if no such decomposition exists.

1.11. Theorem. *The root system Δ of a complex semisimple Lie algebra $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$ is irreducible as an abstract root system if and only if $\mathfrak{g}$ is simple.*

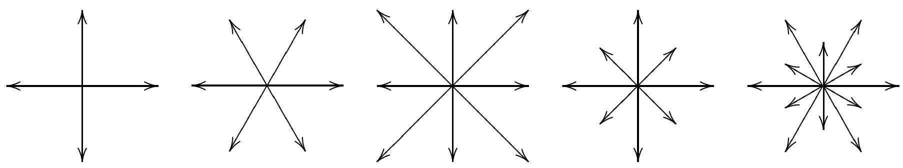
Lets take a look at some properties of abstract root systems.

1.12. Proposition. *Let Δ be an abstract root system in the vector space V with inner product $\langle,\rangle$.*

(1) *If α is in Δ, then $-\alpha$ is in Δ.*
(2) *If α in Δ is reduced, then the only members of $\Delta \cup \{0\}$ proportional to α are $\pm\alpha$, $\pm 2\alpha$ and 0, $\pm 2\alpha$ cannot occur if Δ is reduced.*
(3) *If $\alpha \in \Delta$ and $\beta \in \Delta \cup \{0\}$, then $\frac{2\langle\beta,\alpha\rangle}{|\alpha|^2} \in \{0, \pm 1, \pm 2, \pm 3, \pm 4\}$ and ± 4 only occurs in a nonreduced system for $\beta = \pm 2\alpha$.*
(4) *If α and β are nonproportional members of Δ such that $|\alpha| < |\beta|$, then $\frac{2\langle\beta,\alpha\rangle}{|\beta|^2} \in \{0, \pm 1\}$*
(5) *If $\alpha, \beta \in \Delta$ with $\langle\alpha,\beta\rangle > 0$, then $\alpha - \beta$ is a root or 0. If $\langle\alpha,\beta\rangle < 0$, then $\alpha + \beta$ is a root or 0.*
(6) *If $\alpha, \beta \in \Delta$ and neither $\alpha + \beta$ nor $\alpha - \beta$ in $\Delta \cup \{0\}$, then $\langle\alpha,\beta\rangle = 0$.*
(7) *If $\alpha \in \Delta$ and $\beta \in \Delta \cup \{0\}$, then the α string containing β has the form $\beta + n\alpha$ for $-p \le n \le q$ with $p \ge 0$ and $q \ge 0$. There are no gaps. Furthermore $p - q = \frac{2\langle\beta,\alpha\rangle}{|\alpha|^2}$. The α string containing β contains at most four roots.*

The abstract reduced root systems with $V = \mathbb{R}^2$ are the following:

$A_1 \oplus A_1$ $\quad$ A_2 $\quad$ B_2 $\quad$ C_2 $\quad$ G_2



with $A_1 \oplus A_1$ being the only reducible one of the above.

We want to introduce a notion of positivity on V, the vector space containing an abstract reduced root system Δ, such that

- for any $\varphi \in V \backslash \{0\}$ either φ or $-\varphi$ is positive and
- the sum of positive elements is positive and positive multiples of positive elements are positive.

Later on, when we will classify real semisimple Lie algebras, we will insiste on a special ordering of roots. Therefore the way we introduce positivity is by means of a lexicographic ordering. Let $\mathcal{B} = \{\varphi_1, \varphi_2, \dots, \varphi_n\}$ be a basis of V. An arbitrary element $\varphi \in V$ decomposes to $\varphi = \sum_{i=1}^n a_i \varphi_i$. We say that $\varphi > 0$ if there is an index k, such that $a_i = 0$ for all $1 \leq i \leq k-1$ and $a_k > 0$, otherwise $\varphi < 0$. It is easily seen, that this notion of positivity preserves the above properties. We say that $\varphi > \psi$ if $\varphi - \psi > 0$.

We say a root $\alpha \in \Delta \subset V$ is simple if $\alpha > 0$ and α does not decompose in $\alpha = \beta_1 + \beta_2$ with β_1 and β_2 both positive roots. The set of simple roots is denoted by Π. Because of the first condition of our positivity we either have $\alpha < \beta$ or $\alpha > \beta$ for $\alpha, \beta \in \Pi$. Then we obtain an ordering of simple roots by $\alpha_1 < \alpha_2 < \dots < \alpha_l$

1.13. Proposition. *With $l = \dim V$, there are l simple roots*

$$\{\alpha_1, \alpha_2, \dots, \alpha_l\} = \Pi$$

which are linearly independent. If β is a root and is decomposed by $\beta = a_1\alpha_1 + a_2\alpha_2 + \dots + a_l\alpha_l$, then all $a_i \neq 0$ have the same sign and all a_i are integers.

Let Δ be a reduced abstract root system in an l dimensional vector space V and let $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$ denote the simple roots in a fixed ordering. The l-by-l matrix $A = (A_{ij})$ given by

$$A_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{|\alpha_i|^2}$$

is called the Cartan matrix of Δ and Π. This matrix depends on the ordering of the simple roots but distinct orderings lead to Cartan matrices which are conjugate by a permutation matrix.

We examine some properties of Cartan matrices.

1.14. Proposition. *The Cartan matrix $A = (A_{ij})$ of Δ and Π has the following properties:*

1. *A_{ij} is in $\mathbb{Z}$ for all i, j*
2. *$A_{ii} = 2$ for all i*
3. *$A_{ij} \leq 0$ for $i \neq j$*
4. *$A_{ij} = 0$ if and only if $A_{ji} = 0$*
5. *there exists a diagonal matrix D with positive diagonal entries such that DAD^{-1} is symmetric positive definite.*

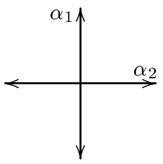
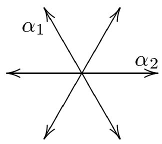
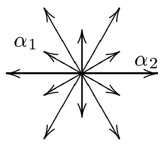
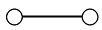
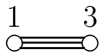
An arbitrary square matrix A satisfying the above properties is called abstract Cartan matrix. Two abstract Cartan matrices are isomorphic if they are conjugate by a permutation matrix.

1.15. Proposition. *A reduced abstract root system is reducible if and only if, for some enumeration of the indices, the corresponding Cartan matrix is block diagonal with more than one block.*

Via this proposition we move the notion of reducibility from reduced abstract root systems to Cartan matrices. An abstract Cartan matrix is reducible if, for some enumeration of the indices, the matrix is block diagonal with more than one block. Otherwise it is irreducible.

The last step in reducing the problem of classification to the essential minimum are Dynkin diagrams. We associate to a reduced abstract root system Δ with simple roots Π and Cartan matrix A the following graph: Each simple root α_i is represented by a vertex, and we attach to that vertex a weight proportional to $|\alpha_i|^2$. We will omit writing the weights if they are all the same. We connect two given vertices corresponding to two distinct simple roots α_i and α_j by $A_{ij}A_{ji}$ edges. The resulting graph is called the Dynkin diagram of Π. It follows from our last proposition, that a Dynkin diagram is connected if and only if Δ is irreducible.

Lets look at three examples, one reducible and the others irreducible.

	$A_1 \oplus A_1$	A_2	G_2
Root system			
Cartan matrix	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$
Dynkin diagram			

Because of the fact that any two Cartan subalgebras of $\mathfrak{g}$ are conjugate via $\operatorname{Int}\mathfrak{g}$ we know, that different choice of a Cartan algebra $\mathfrak{h}$ leads to isomorphic root systems. To see that the choice of a simple system Π leads to isomorphic Cartan matrices we introduce the Weyl group.

Let Δ be an abstract reduced root system in a vector space V. The mapping $s_\alpha : V \to V$ defined by

$$s_\alpha(\beta) := \beta - \frac{2\langle \beta, \alpha \rangle}{|\alpha|^2}$$

is the reflection of V at the hyperplane orthogonal to α. The group of reflections generated by these s_α with $\alpha \in \Delta$ is called the Weyl group and is denoted by $W = W(\Delta)$ (if Δ is the root system of a Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$ we also write $W(\mathfrak{g}, \mathfrak{h})$). As these s_α preserve Δ the whole group W preserves Δ. If $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ is a simple system in Δ, then $W(\Delta)$ is generated by the reflections s_{α_i} with

$\alpha_i \in \Pi$. The Weyl group acts simple transitive on the set of simple systems in Δ.

1.16. THEOREM. *Let Π and Π' be two simple systems in Δ. There exists one and only one element $s \in W$ such that $s(\Pi) = \Pi'$.*

1.17. COROLLARY. *Let Δ be an abstract root system and let Δ^+ and $\Delta^{+'}$ be two positive systems, with corresponding simple systems Π and Π'. The Cartan matrices of Π and Π' are isomorphic.*

PROOF. By the above theorem we obtain an $s \in W(\Delta)$ such that $\Pi' = s(\Pi)$. We fix an enumeration of $\Pi = \{\alpha_1, \dots, \alpha_l\}$ and choose an enumeration of $\Pi' = \{\beta_1, \dots, \beta_l\}$ such that $\beta_j = s(\alpha_j)$ for all $j \in \{1, \dots, l\}$. So we have

$$\frac{2\langle \beta_i, \beta_j \rangle}{|\beta_i|^2} = \frac{2\langle s\alpha_i, s\alpha_j \rangle}{|s\alpha_i|^2} = \frac{2\langle \alpha_i, \alpha_j \rangle}{|\alpha_i|^2}$$

since s is orthogonal and hence the resulting Cartan matrices are equal after a permutation of indices which means that they are isomorphic. □

The Weyl group in another important tool in many proofs along the classification. It is also used in the proof of the following Proposition.

1.18. PROPOSITION. *Let Δ and Δ' be two nonisomorphic reduced root systems with simple systems Π resp. Π'. Then the Cartan matrices A of Δ and A' of Δ' are nonisomorphic.*

Now we will give an outline of the classification of abstract Cartan matrices. We will work simultaneously with Cartan matrices and their associated Dynkin diagrams. First we observe two operations on Dynkin diagrams and their counterparts on Cartan matrices.

(1) Remove the i^{th} vertex and all attached edges from an abstract Dynkin diagram. The counterpart operation on an abstract Cartan matrix is removing the i^{th} row and column from the matrix.
(2) If the i^{th} and j^{th} vertices are connected by a single edge their weights are equal. Collapse the two vertices to a single one removing the connecting edge, retaining all other edges. The counterpart operation collapses the i^{th} and j^{th} row and column replacing the 2-by-2 matrix from the i^{th} and j^{th} indices $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ by the 1-by-1 matrix (2)

One shows that these two operations make abstract Dynkin diagrams out of abstract Dynkin diagrams and abstract Cartan matrices out of abstract Cartan matrices. Using the defining properties of abstract Cartan matrices plus operation (1) we get the following

1.19. PROPOSITION. *Let A be an abstract Cartan matrix. If $i \neq j$, then*

(1) $A_{ij} A_{ji} < 4$
(2) $A_{ij} \in \{0, -1, -2, -3\}$

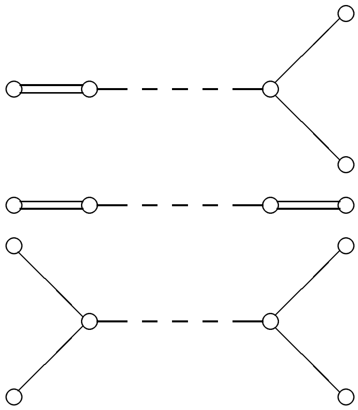
An important step which uses the above proposition in the classification is the following

1.20. PROPOSITION. *The abstract Dynkin diagram associated to the l-by-l abstract Cartan matrix A has the following properties:*

(1) *there are at most l pairs of vertices $i < j$ with at least one edge connecting them*
(2) *there are no loops*
(3) *there are at most three edges attached to one vertex.*

Using these tools we obtain the following classification of irreducible abstract Dynkin diagrams in five steps. Note that reducible abstract Dynkin diagrams are not connected and can therefore be obtained by putting irreducible ones side by side.

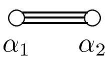
Step 1: None of the following configurations occurs:



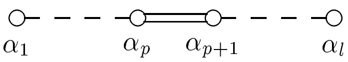
Otherwise we use operation (2) to collapse all the single-line part in the center to a single vertex leading to a violation of 1.20 (3).

Step 2: We do a raw classification by the maximal number of lines connecting two vertices.

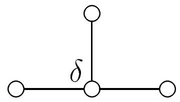
- There is a triple line. By 1.20 (3) the only possibility is

(G_2) 

- There is a double line, but no triple line. The graph in the middle of the figure in step 1 shows that only one pair of vertices connected by two edges exists.

(B, C, F) 

- There are only single lines. In this situation we call δ

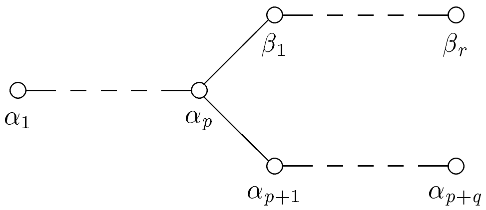


a triple point. If there is no triple point, then the absence of loops implies that the diagram is

(A)

If there is a triple point there is only one because if the third diagram in the figure in step 1. So the other possibility is

(D, E)

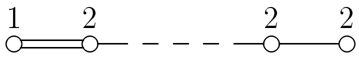


Step 3: Now we address the problem of possible weights going through the three point of the previous step in reverse order:

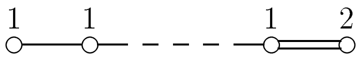
- If the i^{th} and j^{th} vertices are connected by a single line, then $A_{ij} = A_{ji} = -1$ which implies that the weights w_i and w_j of these vertices are equal. Thus in the cases (A) and (D, E) all weights are equal and we may take them to be 1. In this situation we omit writing the weights in the diagram.
- In the case (B, C, F) we have $A_{p,p+1} = -2$ and $A_{p+1,p} = -1$ (Ignoring the possibility of the reverted situation is no loss of generality). The defining property 5 of abstract Cartan matrices leads to $|\alpha_{p+1}|^2 = 2|\alpha_p|^2$. Taking $\alpha_k = 1$ for $k \leq p$ we get $\alpha_k = 2$ for $k \geq p + 1$.
- In the case (G_2) similar reasoning leads to $|\alpha_1|^2 = 1$ and $|\alpha_2|^2 = 3$.

Step 4: The remaining steps deal with special situations. In this step we cover the case (B, C, F). In this case only these diagrams are possible:

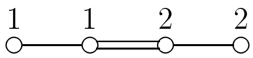
(B)



(C)

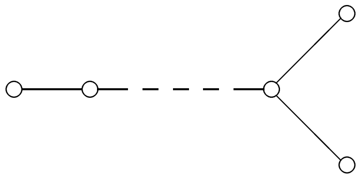


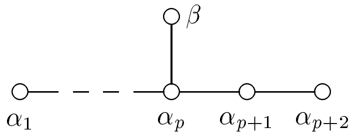
(F_4)



For a proof one uses the Schwarz inequality and the defining properties of Cartan matrices.

Step 5: In the case (D, E) the only possibilities are

(D) 

(E) 

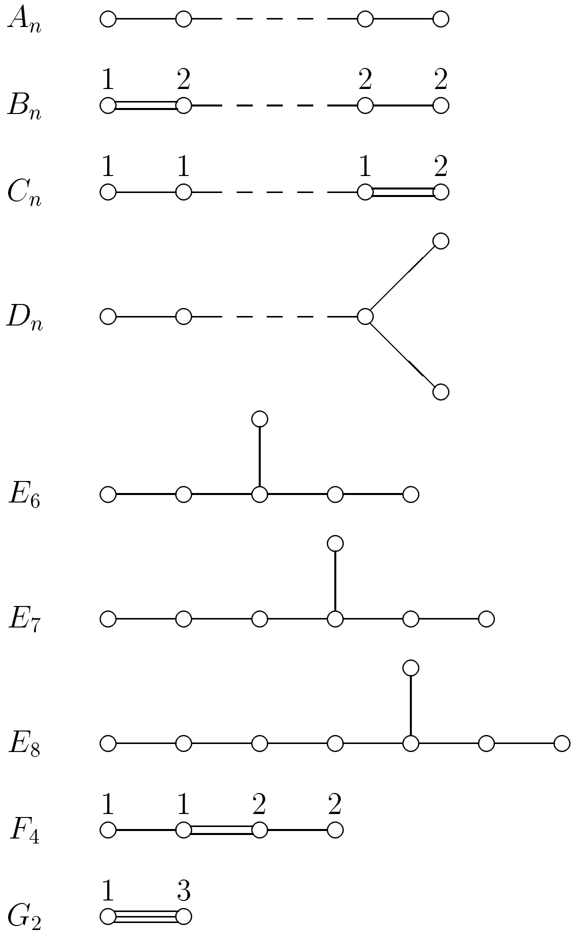
where $p \in \{3, 4, 5\}$. For a proof one uses the Parseval equality.

These steps lead to the following

1.21. Theorem. *Up to isomorphism the connected Dynkin diagrams are the following:*

- A_n *for* $n \geq 1$
- B_n *for* $n \geq 2$
- C_n *for* $n \geq 3$
- D_n *for* $n \geq 4$
- E_6
- E_7
- E_8
- F_4
- G_2

n refers to the number of vertices of the Dynkin diagram. The restrictions of n in the first four items are made to avoid identical diagrams. The diagrams carrying those names are listed in the following table.



We have got a classification of reduced abstract Dynkin diagrams (resp. Cartan matrices) now which gives us a classification of reduced abstract root systems. We want to show, that an isomorphism of the root systems of two complex semisimple Lie algebras lifts to an isomorphism of these algebras themselves. The technique used will be to use generators and relations, realizing any complex semisimple Lie algebra as a quotient of a free Lie algebra by an ideal generated by some relations. First lets look at some properties of complex semisimple Lie algebras.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, Δ its root system with simple system $\Pi = \{\alpha_1, \ldots, \alpha_l\}$, B a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ that is positive definite on the real form of $\mathfrak{h}$ where the roots are real and let $A = (A_{ij})$ be the Cartan matrix of Δ and Π. For $1 \leq i \leq l$ let

- $h_i = \frac{2}{|\alpha_i|^2} H_{\alpha_i}$,
- e_i a nonzero root vector for α_i and

- f_i the nonzero root vector for $-\alpha_i$ satisfying $B(e_i, f_i) = \frac{2}{|\alpha_i|^2}$.

In this situation the set $X = \{h_i, e_i, f_i\}_{i=1}^{l}$ generates $\mathfrak{g}$ as a Lie algebra. The elements of X are called standard generators of $\mathfrak{g}$ relative to $\mathfrak{h}$ and Π. These generators satisfy the following properties within $\mathfrak{g}$.

(1) $[h_i, h_j] = 0$
(2) $[e_i, f_j] = \delta_{ij} h_i$
(3) $[h_i, e_j] = A_{ij} e_j$
(4) $[h_i, f_j] = -A_{ij} f_j$
(5) $(\operatorname{ad} e_i)^{-A_{ij}+1} e_j = 0$ when $i \neq j$
(6) $(\operatorname{ad} f_i)^{-A_{ij}+1} f_j = 0$ when $i \neq j$

These relations are called Serre relations for $\mathfrak{g}$.

To build up a complex semisimple Lie algebra out of generators and relations we introduce the notion of free Lie algebras. A free Lie algebra on a set X is a pair $(\mathfrak{F}, \iota)$ consisting of a Lie Algebra $\mathfrak{F}$ and a function $\iota : X \to \mathfrak{F}$ with the following universal mapping property: Whenever $\mathfrak{l}$ is a complex Lie algebra and $\varphi : X \to \mathfrak{l}$ is a function, then there exists a unique Lie algebra homomorphism $\tilde{\varphi}$ such that the diagram

$$\begin{tikzcd} & \mathfrak{F} \arrow[dr, dashed, "\tilde{\varphi}"] & \\ X \arrow[ur, "\iota"] \arrow[rr, "\varphi"'] & & \mathfrak{l} \end{tikzcd}$$

commutes.

For a nonempty set X there exists a free Lie algebra such that the image of X in $\mathfrak{F}$ generates $\mathfrak{F}$. Let $\mathfrak{g}$ be a Lie algebra with Cartan subalgebra $\mathfrak{h}$, root system Δ, bilinear form B, simple system Π and Cartan matrix A. We express this Lie algebra in terms of a free Lie algebra as follows: Let $\mathfrak{F}$ be the free Lie algebra generated by the set $X = \{h_i, e_i, f_i\}_{i=1}^{l}$ and let $\mathfrak{R}$ be the ideal generated by the differences of the left and right sides of the Serre relations. The universal mapping property yields a homomorphism $\mathfrak{F}/\mathfrak{R} \to \mathfrak{g}$ and the usefulness of this description arises from a theorem stated by Serre.

1.22. PROPOSITION. *The canonical homomorphism* $\mathfrak{F}/\mathfrak{R} \to \mathfrak{g}$ *is an isomorphism.*

The very last steps in the classification of complex semisimple Lie algebras are the following two theorems, which deal with uniqueness and existence of Lie algebras corresponding to Cartan matrices..

1.23. THEOREM (Isomorphism Theorem). *Let* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *be complex semisimple Lie algebras with respective Cartan subalgebras* $\mathfrak{h}$ *and* $\mathfrak{h}'$ *and respective root systems* Δ *and* Δ'. *Suppose that a vector space isomorphism* $\varphi : \mathfrak{h} \to \mathfrak{h}'$ *is given with the property, that its transpose* $\varphi^t : \mathfrak{h}'^* \to \mathfrak{h}^*$ *maps* Δ' *onto* Δ. *For* $\alpha \in \Delta$ *write* $\alpha' = (\varphi^t)^{-1}(\alpha) \in \Delta'$.

Fix a simple system Π *in* Δ. *For each* $\alpha \in \Pi$ *select a nonzero root vector* $E_\alpha \in \mathfrak{g}$ *and* $E_{\alpha'} \in \mathfrak{g}'$ *for* α'. *Then there exists one and only one Lie algebra isomorphism* $\tilde{\varphi} : \mathfrak{g} \to \mathfrak{g}'$ *such that* $\tilde{\varphi}|_{\mathfrak{h}} = \varphi$ *and* $\tilde{\varphi}(E_\alpha) = E_{\alpha'}$ *for all* $\alpha \in \Pi$.

1.24. Theorem (Existence Theorem). *If* A *is an abstract Cartan matrix, then there is a complex semisimple Lie algebra* $\mathfrak{g}$ *whose root system has* A *as Cartan matrix.*

At the end of this chapter we would like to look at an example of the theory we have done so far. We concider $\mathfrak{sl}(n,\mathbb{C})$. $\mathfrak{sl}(n,\mathbb{C})$ is the Lie algebra of the special linear group of dimension n.

$$\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C}) = \{X \in \mathfrak{gl}(n,\mathbb{C}) | \operatorname{Tr} X = 0\}$$

where $\mathfrak{gl}(n,\mathbb{C})$ is the Lie algebra of $n \times n$ matrices with entries in $\mathbb{C}$. The bracket relation is defined by

$$[X,Y] = XY - YX.$$

$\mathfrak{g}$ is closed under bracket by the fact, that $\operatorname{Tr}(XY) = \operatorname{Tr}(YX)$ for arbitrary matrices. We define a Lie subalgebra

$$\mathfrak{h} = \{X \in \mathfrak{g} | X \text{ is a diagonal matrix}\}$$

and a real form

$$\mathfrak{h}_0 = \{X \in \mathfrak{g} | X \text{ is a real diagonal matrix}\}.$$

Then

$$\mathfrak{h} = \mathfrak{h}_0 \oplus i\mathfrak{h}_0 = (\mathfrak{h}_0)^{\mathbb{C}}.$$

We define a matrix E_{ij} to be 1 at (i,j) and 0 elsewhere. These will be elements of the root spaces. Let

$$H = \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{pmatrix}$$

be an arbitrary element of $\mathfrak{h}$. Define linear functionals $e_i \in \mathfrak{h}^*$ that pick out the i'ths diagonal entry of a matrix by

$$e_i(H) = h_i.$$

We calculate

$$HE_{ij} = h_i E_{ij} \quad \text{and} \quad E_{ij}H = h_j E_{ij}$$

and obtain

$$(\operatorname{ad} H)E_{ij} = [H, E_{ij}] = (e_i(H) - e_j(H))E_{ij}$$

to see that E_{ij} are simultanous eigenvectors for all $\operatorname{ad} H$, with eigenvalue $e_i(H) - e_j(H)$. The linear functionals

$$e_i - e_j \quad \text{for all } i \neq j$$

are the roots. The set of roots denoted by Δ. We obtain the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C}E_{ij}$$

which we can write as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathfrak{g}_{e_i - e_j}$$

with the root spaces

$$\mathfrak{g}_{e_i - e_j} = \{X \in \mathfrak{g} | (\operatorname{ad} H)X = (e_i - e_j)(H)X \text{ for all } H \in \mathfrak{h}\}.$$

This shows that $\mathfrak{h} = \mathfrak{g}_0$ and hence $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Elementary computation yields

$$[E_{ij}, E_{i'j'}] = \begin{cases} 0 & \text{for } i \neq j' \text{ and } j \neq i' \\ E_{ij'} & \text{for } i \neq j' \text{ and } j = i' \\ -E_{i'j} & \text{for } i = j' \text{ and } j \neq i' \\ E_{ii} - E_{jj} & \text{for } i = j' \text{ and } j = i' \end{cases}$$

We obtain the following structure of the bracket:

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \begin{cases} = \mathfrak{g}_{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root} \\ = 0 & \text{if } \alpha + \beta \text{ is neither a root nor } 0 \\ \subseteq \mathfrak{h} & \text{if } \alpha + \beta = 0 \end{cases}$$

All roots are real valued on $\mathfrak{h}_0$ and thus can be restricted to members of $\mathfrak{h}_0^*$. To introduce a notion of positivity we write an arbitrary functional $\varphi \in \mathfrak{h}_0^*$ as $\varphi = \sum_j a_j e_j$ with $\sum_j a_j = 0$, which is a unique description of φ. We call φ positive if the first nonzero coefficient $a_j > 0$. This guarantees that

(1) for any nonzero $\varphi \in \mathfrak{h}_0^*$ exactly one of φ and $-\varphi$ is postitive,
(2) the sum of positive elements is positive and any positive multiple of a positive element is positive.

We say that $\varphi > \psi$ if $\varphi - \psi > 0$. Hence the positive roots are

$$\begin{array}{ccccccccc} e_1 - e_n & > & e_1 - e_{n-1} & > & \cdots & > & e_1 - e_2 & > & \\ & > & e_2 - e_n & > & \cdots & > & e_2 - e_3 & > & \\ & & & > & \cdots & > & \cdots & > & \\ & & & > & e_{n-2} - e_n & > & e_{n-2} - e_{n-1} & > & \\ & & & & & > & e_{n-1} - e_n & > & 0 \end{array}$$

All negative roots follow in reversed order. The simple roots are all $e_i - e_{i+1}$ with $1 \leq i \leq n-1$. Using the Killing form B we obtain a correspondence between a root $e_i - e_j$ and $H_{ij} \in \mathfrak{h}_0$, where H_{ij} is the diagonal matrix with 1 in the i'th diagonal entry, -1 in the j'th diagonal entry and 0 elsewhere. This enables us to calculate the entries of the Cartan matrix

$$A_{kl} = \frac{2\langle \alpha_k, \alpha_l \rangle}{\langle \alpha_k, \alpha_k \rangle} = \frac{2\alpha_k(H_{\alpha_l})}{\alpha_k(H_{\alpha_k})}.$$

With $\alpha_k = e_k - e_{k+1}$ and $\alpha_l = e_l - e_{l+1}$ this reads

$$A_{kl} = \frac{2(e_k - e_{k+1})(H_{l,l+1})}{(e_k - e_{k+1})(H_{k,k+1})}.$$

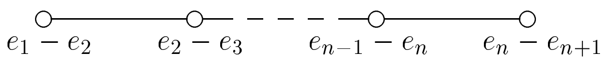
Since

$$(e_k - e_{k+1})(H_{l,l+1}) = \begin{cases} 2 & \text{for } k = l \\ -1 & \text{for } k+1 = l \text{ or } k = l+1 \\ 0 & \text{else} \end{cases}$$

we see that

$$(A_n)_{kl} = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}.$$

We have already seen a picture of the root system of $\mathfrak{sl}(3,\mathbb{C}) = A_2$ and from the calculations above we know that the Dynkin diagram of A_n is



CHAPTER 2

Classification of real semisimple Lie algebras

Let V be a vector space over $\mathbb{R}$. We call

$$V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$$

the complexification of V. The mapping $m(c) : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto cz$ is $\mathbb{R}$-linear. Thus $1 \otimes m(c) : V \otimes_{\mathbb{R}} \mathbb{C} \to V \otimes_{\mathbb{R}} \mathbb{C}$ defines a scalar multiplication with $\mathbb{C}$. $V \otimes_{\mathbb{R}} \mathbb{C}$ is a vector space over $\mathbb{C}$ with the natural embedding $V \hookrightarrow V \otimes_{\mathbb{R}} \mathbb{C}$ by $v \mapsto v \otimes 1$. If $\{v_i\}_{i \in I}$ is a basis of V over $\mathbb{R}$, $\{v_i \otimes 1\}_{i \in I}$ is a basis of $V^{\mathbb{C}}$ over $\mathbb{C}$. Therefore the dimensions

$$\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V^{\mathbb{C}}$$

correspond in the above way.

Let W be a vector space over $\mathbb{C}$. Restricting the definition of scalars to $\mathbb{R}$ leads to a vector space $W^{\mathbb{R}}$ over $\mathbb{R}$. If $\{v_j\}_{j \in I}$ is a basis of W, then $\{v_j, iv_j\}_{j \in I}$ is a basis of $W^{\mathbb{R}}$. We get $(V^{\mathbb{C}})^{\mathbb{R}} = V \oplus iV$. Therefore

$$\dim_{\mathbb{C}} W = \frac{1}{2} \dim_{\mathbb{R}} W^{\mathbb{R}}$$

is the correspondence of dimensions.

If a complex vector space W and a real vector space V are related by

$$W^{\mathbb{R}} = V \oplus iV$$

then V is called real form of W. The conjugate linear map $\varphi : V^{\mathbb{C}} \to V^{\mathbb{C}}$ that is 1 on V and -1 on iV is called conjugation of $V^{\mathbb{C}}$ with respect to the real form V.

Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ its complexification. The mapping

$$[,] : (\mathfrak{g} \otimes \mathbb{C}) \times (\mathfrak{g} \otimes \mathbb{C}) \to \mathfrak{g} \otimes \mathbb{C}$$

given by

$$(X \otimes a) \times (Y \otimes b) \mapsto ([X, Y] \otimes ab)$$

extends the bracket in a complex bilinear way. Surely the restriction of scalars of a complex Lie algebra gives a real Lie algebra. Therefore both, complexification and restriction of scalars make Lie algebras out of Lie algebras.

For a real Lie algebra $\mathfrak{g}$ we note that

$$[\mathfrak{g}, \mathfrak{g}]^{\mathbb{C}} = [\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}]$$

since by $\mathfrak{g} \subseteq \mathfrak{g}^{\mathbb{C}}$ we get $[\mathfrak{g}, \mathfrak{g}] \subseteq [\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}]$ and this also holds for the $\mathbb{C}$ subspace $[\mathfrak{g}, \mathfrak{g}]^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$. For the reverse let $a, b \in \mathbb{C}$ and $X, Y \in \mathfrak{g}$, then $[X \otimes a, Y \otimes b] = [X, Y] \otimes ab \in [\mathfrak{g}, \mathfrak{g}]^{\mathbb{C}}$. Allowing arbitrary linear combinations on the left we obtain $[\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}] \subseteq [\mathfrak{g}, \mathfrak{g}]^{\mathbb{C}}$.

Lets look at the relation of the Killing forms. Let $\mathfrak{g}_0$ be a real Lie algebra with Killing form $B_{\mathfrak{g}_0}$. Let $\mathfrak{g}_0^{\mathbb{C}}$ be its complexification with its Killing form denoted by $B_{\mathfrak{g}_0^{\mathbb{C}}}$. Fix any basis of $\mathfrak{g}_0$. This is also a basis for its complexification $\mathfrak{g}_0^{\mathbb{C}}$. Therefore $\mathrm{Tr}(\operatorname{ad} X \circ \operatorname{ad} Y)$ is unaffected by complexifying and the Killing forms are related by

$$B_{\mathfrak{g}_0^{\mathbb{C}}}|_{\mathfrak{g}_0 \times \mathfrak{g}_0} = B_{\mathfrak{g}_0}.$$

By Cartan's criterion for semisimplicity (i.e.: A Lie algebra is semisimple if and only if its Killing form is nondegenerate) we see, that the complexification $\mathfrak{g}_0^{\mathbb{C}}$ is semisimple if and only if $\mathfrak{g}_0$ is semisimple.

The situation is a little bit more complicated in the case of restriction of scalars. Let $\mathfrak{g}$ be a complex Lie algebra with Killing form $B_{\mathfrak{g}}$ and let $\mathfrak{g}^{\mathbb{R}}$ be the real Lie algebra obtained by restriction of scalars with Killing form $B_{\mathfrak{g}^{\mathbb{R}}}$. Let $\mathcal{B} = \{v_j\}_{j \in I}$ be a basis of $\mathfrak{g}$. Then $\mathcal{B}' = \{v_j, iv_j\}_{j \in I}$ is a basis of $\mathfrak{g}^{\mathbb{R}}$. For $X \in \mathfrak{g}$ we write $\operatorname{ad}_{\mathfrak{g}} X$ as the matrix $(c_{kl})_{k,l \in I}$ with respect to the basis $\mathcal{B}$. Look at the same $X \in \mathfrak{g}^{\mathbb{R}}$. In the basis $\mathcal{B}'$, $\operatorname{ad}_{\mathfrak{g}^{\mathbb{R}}} X$ is described by the same matrix replacing c_{kl} by $\begin{pmatrix} a_{kl} & -b_{kl} \\ b_{kl} & a_{kl} \end{pmatrix}$ where $a_{kl} = \operatorname{Re} c_{kl}$ and $b_{kl} = \operatorname{Im} c_{kl}$. Therefore the Killing forms are related by

$$B_{\mathfrak{g}^{\mathbb{R}}} = 2 \operatorname{Re} B_{\mathfrak{g}}.$$

Again by Cartan's criterion for semisimplicity we get, $\mathfrak{g}^{\mathbb{R}}$ is semisimple if and only if $\mathfrak{g}$ is semisimple.

We will identify two special real forms of complex semisimple Lie algebras now. The first one will be called split real form. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the set of roots and B the Killing form of $\mathfrak{g}$.

2.25. Theorem. *For each $\alpha \in \Delta$ it is possible to choose root vectors $X_\alpha \in \mathfrak{g}_\alpha$ such that the following conditions hold for all $\alpha, \beta \in \Delta$*

- $[X_\alpha, X_{-\alpha}] = H_\alpha \quad H_\alpha$ *as in proposition 1.8*
- $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$ *if* $\alpha + \beta \in \Delta$
- $[X_\alpha, X_\beta] = 0$ *if* $\alpha + \beta \neq 0$ *and* $\alpha + \beta \notin \Delta$

and the constants $N_{\alpha,\beta}$ satisfy

- $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$
- $N_{\alpha,\beta}^2 = \frac{1}{2} q(1+p)|\alpha|^2$

where $\beta + n\alpha$ is the α string of β with $-p \leq n \leq q$.

Since the above theorem shows that $N_{\alpha,\beta}^2$ is positive, $N_{\alpha,\beta}$ is real and we obtain a real form by defining

$$\mathfrak{h}_0 = \{H \in \mathfrak{h} | \alpha(H) \in \mathbb{R} \ \forall \alpha \in \Delta\}$$

$$\mathfrak{g}_0 = \mathfrak{h}_0 \bigoplus_{\alpha\in\Delta} \mathbb{R}X_\alpha.$$

Every real form of $\mathfrak{g}$ containing such an $\mathfrak{h}_0$ for some Cartan subalgebra $\mathfrak{h}$ is called split real form of $\mathfrak{g}$ and the above construction shows that such a real form exists for every complex semisimple Lie algebra.

Another special real form, which exists for every complex semisimple Lie algebra is called compact real form. A compact real form is a real form that is a compact Lie algebra. A real Lie algebra $\mathfrak{g}$ is compact if the analytic group $\operatorname{Int}\mathfrak{g}$ of inner automorphisms is compact.

The compact real form will be of greater importance in the sequel than the split real form. We construct it using a split real form. First of all lets specify how a compact real form is characterized. Let $\mathfrak{u}_0$ be a real form of a complex semisimple Lie algebra $\mathfrak{g}$. If the Killing form $B_{\mathfrak{u}_0}$ is negative definite $\mathfrak{u}_0$ is called compact real form of $\mathfrak{g}$.

2.26. THEOREM. *Every complex semisimple Lie algebra $\mathfrak{g}$ contains a compact real form.*

PROOF. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let X_α be the root vectors as in the construction of the split real form. Define

$$\mathfrak{u}_0 = \sum_{\alpha\in\Delta} \mathbb{R}iH_\alpha + \sum_{\alpha\in\Delta} \mathbb{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha\in\Delta} \mathbb{R}i(X_\alpha + X_{-\alpha}).$$

Since this is clearly a vector space real form we have to check that it is closed under bracket and that its Killing form is negative definite. Lets check the occurring brackets. Assume $\alpha \neq \pm\beta$.

- $[iH_\alpha, iH_\beta] = 0$
- $[iH_\alpha, (X_\alpha - X_{-\alpha})] = |\alpha|^2 i(X_\alpha + X_{-\alpha})$
- $[iH_\alpha, i(X_\alpha + X_{-\alpha})] = |\alpha|^2 (X_\alpha - X_{-\alpha})$
- $[(X_\alpha - X_{-\alpha}), (X_\beta - X_{-\beta})] =$
  $N_{\alpha,\beta}(X_{\alpha+\beta} - X_{-(\alpha+\beta)}) - N_{-\alpha,\beta}(X_{-\alpha+\beta} - X_{-(-\alpha+\beta)})$
- $[(X_\alpha - X_{-\alpha}), i(X_\beta + X_{-\beta})] =$
  $N_{\alpha,\beta} i(X_{\alpha+\beta} + X_{-(\alpha+\beta)}) - N_{-\alpha,\beta} i(X_{-\alpha+\beta} + X_{-(-\alpha+\beta)})$
- $[i(X_\alpha + X_{-\alpha}), i(X_\beta + X_{-\beta})] =$
  $-N_{\alpha,\beta}(X_{\alpha+\beta} - X_{-(\alpha+\beta)}) - N_{-\alpha,\beta}(X_{-\alpha+\beta} - X_{-(-\alpha+\beta)})$
- $[(X_\alpha - X_{-\alpha}), i(X_\alpha + X_{-\alpha})] = 2iH_\alpha$

These computations show that $\mathfrak{u}_0$ is closed under bracket, so we have a real form on our hands. To show its compactness we check the Killing form. We know that the Killing forms $B_{\mathfrak{u}_0}$ of $\mathfrak{u}_0$ and $B_{\mathfrak{g}}$ of $\mathfrak{g}$ are related by $B_{\mathfrak{u}_0} = B_{\mathfrak{g}}|_{\mathfrak{u}_0\times\mathfrak{u}_0}$. Since

$$B_{\mathfrak{g}}(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \text{ for } \alpha, \beta \in \Delta \cup \{0\} \text{ and } \alpha + \beta \neq 0$$

$\sum_{\alpha\in\Delta} \mathbb{R}iH_\alpha$ is orthogonal to $\sum_{\alpha\in\Delta} \mathbb{R}(X_\alpha - X_{-\alpha})$ and to $\sum_{\alpha\in\Delta} \mathbb{R}i(X_\alpha + X_{-\alpha})$. By the same argument

- $B((X_\alpha - X_{-\alpha}), (X_\beta - X_{-\beta})) = 0$
- $B((X_\alpha - X_{-\alpha}), i(X_\beta + X_{-\beta})) = 0$

- $B(i(X_\alpha + X_{-\alpha}), i(X_\beta + X_{-\beta})) = 0$.

Since B is positive definite on $\sum_{\alpha\in\Delta} \mathbb{R}H_\alpha$ it is negative definite on $\sum_{\alpha\in\Delta} \mathbb{R}iH_\alpha$. With just two cases left we compute

- $B((X_\alpha - X_{-\alpha}), (X_\alpha - X_{-\alpha})) = -2B(X_\alpha, X_{-\alpha}) = -2$
- $B(i(X_\alpha + X_{-\alpha}), i(X_\alpha + X_{-\alpha})) = -2B(X_\alpha, X_{-\alpha}) = -2$

and obtain that $B_{\mathfrak{g}}|_{\mathfrak{u}_0\times\mathfrak{u}_0}$ is negative definite and therefore $\mathfrak{u}_0$ is a compact real form. $\square$

Let $\mathfrak{g}$ be a Lie algebra. An automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that $\sigma^2 = \mathrm{id}_{\mathfrak{g}}$ is called an involution. Such an involution yields a decomposition into eigenspaces to the eigenvalues $+1$ and -1. An involution θ of a real semisimple Lie algebra $\mathfrak{g}_0$ such that the symmetric bilinear form

$$B_\theta(Z, Z') := -B(Z, \theta Z')$$

is positive definite is called Cartan involution. For complex $\mathfrak{g}$ these Cartan involutions correspond to compact real forms. The first situation where we observe this correspondence is the following.

2.27. PROPOSITION. *Let $\mathfrak{g}$ be a complex semisimple Lie, $\mathfrak{u}_0$ a compact real form and τ the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{u}_0$. Then τ is a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$.*

PROOF. Surely $\tau^2 = \mathrm{id}_{\mathfrak{g}^{\mathbb{R}}}$. The Killing forms of $\mathfrak{g}$ and $\mathfrak{g}^{\mathbb{R}}$ are related by $B_{\mathfrak{g}^{\mathbb{R}}}(Z_1, Z_2) = 2\,\mathrm{Re}\, B_{\mathfrak{g}}(Z_1, Z_2)$. (Therefore $\mathfrak{g}^{\mathbb{R}}$ is semisimple if and only if $\mathfrak{g}$ is semisimple.) Decompose $Z \in \mathfrak{g}$ as $Z = X + iY$ with $X, Y \in \mathfrak{u}_0$. For $Z \neq 0$ we get

$$\begin{aligned} B_{\mathfrak{g}}(Z, \tau Z) &= B_{\mathfrak{g}}(X + iY, X - iY) \\ &= B_{\mathfrak{g}}(X, X) + B_{\mathfrak{g}}(Y, Y) \\ &= B_{\mathfrak{u}_0}(X, X) + B_{\mathfrak{u}_0}(Y, Y) < 0. \end{aligned}$$

It follows that

$$(B_{\mathfrak{g}^{\mathbb{R}}})_\tau(Z, Z') = -B_{\mathfrak{g}^{\mathbb{R}}}(Z, \tau Z') = -2\,\mathrm{Re}\, B_{\mathfrak{g}}(Z, \tau Z')$$

is positive definite on $\mathfrak{g}^{\mathbb{R}}$ and therefore τ is a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$. $\square$

To study an arbitrary real form of complex semisimple Lie algebras we will align a compact real form to it. For a real form $\mathfrak{g}_0$ of a complex semisimple Lie algebra $\mathfrak{g}$ we want to find a compact real form $\mathfrak{u}_0$ such that $\mathfrak{u}_0 = (\mathfrak{u}_0 \cap \mathfrak{g}_0) \oplus (\mathfrak{u}_0 \cap i\mathfrak{g}_0)$.

2.28. LEMMA. *Let φ and ψ be involutions of a vector space V. Let V_{φ^+} denote the eigenspace of φ to the eigenvalue 1 and V_{φ^-} the eigenspace of φ to the eigenvalue -1. Using the similar notation for ψ we get*

$$\varphi \circ \psi = \psi \circ \varphi \iff \begin{cases} V_{\varphi^+} = (V_{\varphi^+} \cap V_{\psi^+}) \oplus (V_{\varphi^+} \cap V_{\psi^-}) \\ V_{\varphi^-} = (V_{\varphi^-} \cap V_{\psi^+}) \oplus (V_{\varphi^-} \cap V_{\psi^-}) \end{cases}$$

PROOF.

($\Rightarrow$): Let $x \in V$ be an arbitrary element. $x = x_{\psi^+} + x_{\psi^-}$ with $x_{\psi^+} \in V_{\psi^+}$ and $x_{\psi^-} \in V_{\psi^-}$. Because of commutativity we have

$$\varphi(x_{\psi^+}) = \varphi \circ \psi(x_{\psi^+}) = \psi \circ \varphi(x_{\psi^+}) \quad \Rightarrow \quad \varphi(x_{\psi^+}) \in V_{\psi^+}$$

($\Leftarrow$): We start with an arbitrary $x \in V$. Decomposing we get

$$x = x_{\varphi^+} + x_{\varphi^-} = x_{\varphi^+,\psi^+} + x_{\varphi^+,\psi^-} + x_{\varphi^-,\psi^+} + x_{\varphi^-,\psi^-}.$$

Computing

$$\begin{aligned}
\varphi \circ \psi(x) &= \varphi(\psi(x_{\varphi^+,\psi} + x_{\varphi^+,\psi^-} + x_{\varphi^-,\psi^+} + x_{\varphi^-,\psi^-})) \\
&= \varphi(x_{\varphi^+,\psi^+} - x_{\varphi^+,\psi^-} + x_{\varphi^-,\psi^+} - x_{\varphi^-,\psi^-}) \\
&= x_{\varphi^+,\psi^+} - x_{\varphi^+,\psi^-} - x_{\varphi^-,\psi^+} + x_{\varphi^-,\psi^-} \\
&= \psi(x_{\varphi^+,\psi^+} + x_{\varphi^+,\psi^-} - x_{\varphi^-,\psi^+} - x_{\varphi^-,\psi^-}) \\
&= \psi(\varphi(x_{\varphi^+,\psi^+} + x_{\varphi^+,\psi^-} + x_{\varphi^-,\psi^+} + x_{\varphi^-,\psi^-})) \\
&= \psi \circ \varphi(x)
\end{aligned}$$

Thus φ and ψ commute.

$\square$

So we search for real forms with commuting involutions.

2.29. THEOREM. *Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, θ a Cartan involution and σ any involution of $\mathfrak{g}_0$. Then there exists a $\varphi \in \operatorname{Int}\mathfrak{g}_0$ such that $\varphi\theta\varphi^{-1}$ commutes with σ.*

PROOF. Since θ is a Cartan involution, B_θ is an inner product on $\mathfrak{g}_0$. Let $\omega = \sigma\theta$. For any automorphism a of $\mathfrak{g}_0$ we have $B(aX, aY) = B(X, Y)$ for all $X, Y \in \mathfrak{g}_0$. Since ω is an automorphism and since $\sigma^2 = \theta^2 = 1$ and $\theta = \theta^{-1}$, we compute

$$\begin{aligned}
B(\omega X, \theta Y) &= B(X, \omega^{-1}\theta Y) = B(X, \theta^{-1}\sigma^{-1}\theta Y) \\
&= B(X, \theta\sigma\theta Y) = B(X, \theta\omega Y)
\end{aligned}$$

and hence

$$B_\theta(\omega X, Y) = B_\theta(X, \omega Y).$$

Thus ω is symmetric and its square $\rho = \omega^2$ is positive definite. Thus ρ^r for $-\infty < r < \infty$ is a one parameter group in $\operatorname{Int}\mathfrak{g}_0$. Then

$$\rho\theta = \omega^2\theta = \sigma\theta\sigma\theta\theta = \sigma\theta\sigma = \theta\theta\sigma\theta\sigma = \theta\omega^{-2} = \theta\rho^{-1}.$$

In terms of a basis of $\mathfrak{g}_0$ that diagonalizes ρ, the matrix form of this equation is

$$\rho_{ii}\theta_{ij} = \theta_{ij}\rho_{jj}^{-1} \quad \text{for all } i \text{ and } j.$$

We see that

$$\rho_{ii}^r\theta_{ij} = \theta_{ij}\rho_{jj}^{-r}$$

and therefore

$$\rho^r\theta = \theta\rho^{-r}.$$

Now we explicitly give the automorphism $\varphi = \rho^{\frac{1}{4}}$ which fulfills

$$\begin{aligned}(\varphi\theta\varphi^{-1})\sigma &= \rho^{\frac{1}{4}}\theta\rho^{-\frac{1}{4}} = \rho^{\frac{1}{2}}\theta\sigma \\ &= \rho\frac{1}{2}\omega^{-1} = \rho^{-\frac{1}{2}\rho\omega^{-1}} \\ &= \rho^{-\frac{1}{2}}\omega = \omega\rho^{-\frac{1}{2}} \\ &= \sigma\theta\rho^{-\frac{1}{2}} = \sigma\rho^{\frac{1}{4}}\theta\rho^{-\frac{1}{4}} \\ &= \sigma(\varphi\theta\varphi^{-1})\end{aligned}$$

as required. □

With this result on our hands we easily get the following

2.30. Corollary. *Every real semisimple Lie algebra $\mathfrak{g}_0$ has a Cartan involution.*

Proof. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$. By our previous result we choose a compact real form $\mathfrak{u}_0$ of $\mathfrak{g}$ such that the involutions σ respectively τ of $\mathfrak{g}^{\mathbb{R}}$ with respect to $\mathfrak{g}_0$ respectively $\mathfrak{u}_0$ commute. We have $\mathfrak{g}_0 = \{X \in \mathfrak{g} | \sigma X = X\}$. Because of this and the commutativity of the involutions we get

$$\sigma\tau X = \tau\sigma X = \tau X$$

and therefore τ restricts to an involution $\theta = \tau|_{\mathfrak{g}_0}$ of $\mathfrak{g}_0$. Furthermore we have

$$B_\theta(X,Y) = -B_{\mathfrak{g}_0}(X,\theta Y) = -B_{\mathfrak{g}}(X,\tau Y) = \frac{1}{2}(B_{g^{\mathbb{R}}})_\tau(X,Y)$$

for $X, Y \in \mathfrak{g}_0$ and so B_θ is positive definite on $\mathfrak{g}_0$ and θ is a Cartan involution. □

Another useful Corollary of Theorem 2.29 deals with a uniqueness property of Cartan involutions.

2.31. Corollary. *Any two Cartan involutions of a real semisimple Lie algebra $\mathfrak{g}_0$ are conjugate via* $\operatorname{Int}\mathfrak{g}_0$.

Proof. Let θ and θ' be Cartan involutions of $\mathfrak{g}_0$. By Theorem 2.29 we can find an automorphism $\varphi \in \operatorname{Int}\mathfrak{g}_0$ such that $\varphi\theta\varphi^{-1}$ commutes with θ'. Hence we may assume without loss of generality that θ and θ' commute. Therefore their decomposition in $+1$ and -1 eigenspaces are compatible. Assume that $X \in \mathfrak{g}_0$ lies in the $+1$ eigenspace of θ and in the -1 eigenspace of θ'. Then $\theta X = X$ and $\theta' X = -X$ and

$$\begin{aligned}0 < B_\theta(X,X) &= -B(X,\theta X) = -B(X,X) \\ 0 < B_{\theta'}(X,X) &= -B(X,\theta' X) = +B(X,X)\end{aligned}$$

which contradicts the assumption. Therefore $\theta = \theta'$. □

Reinterpreting this result we see that any two compact real forms of a complex semisimple Lie algebra $\mathfrak{g}$ are conjugate via $\operatorname{Int}\mathfrak{g}$. Each compact real form has a determining associated conjugation. These conjugations are Cartan involutions of $\mathfrak{g}^{\mathbb{R}}$ and are conjugate by a member of $\operatorname{Int}\mathfrak{g}^{\mathbb{R}}$. $\operatorname{Int}\mathfrak{g}^{\mathbb{R}} = \operatorname{Int}\mathfrak{g}$ completes the argument.

An important Corollary in view of the classification is the following one.

2.32. Corollary. *Let A be an abstract Cartan matrix. Up to isomorphism there exist one and only one compact semisimple real Lie algebra $\mathfrak{g}_0$ with its complexification $(\mathfrak{g}_0)^{\mathbb{C}}$ having a root system with A as Cartan matrix.*

From the first chapter we get the existence and uniqueness of $\mathfrak{g}$, we know of the existence of a compact real form and because of the above argument all compact real forms are conjugate by $\operatorname{Int}\mathfrak{g}$.

We will now introduce the notion of Cartan decomposition of a real semisimple Lie algebra $\mathfrak{g}_0$ and we will see that this corresponds to Cartan involutions. A vectorspace direct sum $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ of $\mathfrak{g}_0$ is called Cartan decomposition if

(1) the following bracket laws are satisfied

$$[\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0\,,\ [\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0\,,\ [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0$$

(2) the Killing form

$$B_{\mathfrak{g}_0} \text{ is } \begin{cases} \text{negative definite on } \mathfrak{k}_0 \\ \text{positive definite on } \mathfrak{p}_0. \end{cases}$$

Let $X \in \mathfrak{k}_0$ and $Y \in \mathfrak{p}_0$. By the first defining property

$$(\operatorname{ad} X \operatorname{ad} Y)(\mathfrak{k}_0) \subseteq \mathfrak{p}_0 \text{ and } (\operatorname{ad} X \operatorname{ad} Y)(\mathfrak{p}_0) \subseteq \mathfrak{k}_0.$$

Therefore $\operatorname{Tr}(\operatorname{ad} X \operatorname{ad} Y) = 0$ and hence the Killing form $B_{\mathfrak{g}_0}(X, Y) = 0$. Since $\theta Y = -Y$ also $B_\theta(X, Y) = 0$. This means that $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are orthogonal under $B_{\mathfrak{g}_0}$ and B_θ.

We will now describe the correspondence between Cartan involutions and Cartan decompositions. First let θ be a Cartan involution of $\mathfrak{g}_0$. The involution defines an eigenspace decomposition in an eigenspace $\mathfrak{k}_0$ to the eigenvalue $+1$ and an eigenspace $\mathfrak{p}_0$ to the eigenvalue -1. We have a decomposition of the form $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. Let $X, X' \in \mathfrak{k}_0$ and $Y, Y' \in \mathfrak{p}_0$ and notice that θ is an automorphism. We get

$$\begin{aligned} \theta[X, X'] = [\theta X, \theta X'] = [X, X'] \quad &\Rightarrow \quad [\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0 \\ \theta[X, Y] = [\theta X, \theta Y] = [X, -Y] = -[X, Y] \quad &\Rightarrow \quad [\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0 \\ \theta[Y, Y'] = [\theta Y, \theta Y'] = [-Y, -Y'] = [Y, Y'] \quad &\Rightarrow \quad [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0 \end{aligned}$$

As we have seen above it follows that $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are orthogonal under $B_{\mathfrak{g}_0}$ and B_θ. B_θ is positive definite since θ is a Cartan involution and hence $B_{\mathfrak{g}_0}$ is negative definite on $\mathfrak{k}_0$ and positive definite on $\mathfrak{p}_0$. Therefore $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is a Cartan decomposition.

Conversely starting with a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ we define a mapping

$$\theta = \begin{cases} +1 \text{ on } \mathfrak{k}_0 \\ -1 \text{ on } \mathfrak{p}_0 \end{cases}$$

θ respects bracket because for $X, X' \in \mathfrak{k}_0$ and $Y, Y' \in \mathfrak{p}_0$ we have

$$\begin{array}{lcl} [\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0 & \Rightarrow & \theta[X, X'] = [X, X'] = [\theta X, \theta X'] \\ [\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0 & \Rightarrow & \theta[X, Y] = -[X, Y] = [X, -Y] = [\theta X, \theta Y] \\ [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0 & \Rightarrow & \theta[Y, Y'] = [Y, Y'] = [-Y, -Y'] = [\theta Y, \theta Y'] \end{array}$$

We know of the orthogonality of $\mathfrak{k}_0$ and $\mathfrak{p}_0$ under both $B_{\mathfrak{g}_0}$ and B_θ and we know that $B_{\mathfrak{g}_0}$ is negative definite on $\mathfrak{k}_0$ and positive definite on $\mathfrak{p}_0$. Hence B_θ is positive definite. Therefore θ is a Cartan involution.

Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$. By bilinearity of the Killing form we see, that $\mathfrak{k}_0 \oplus i\mathfrak{p}_0$ is a compact real form of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$. Conversely if $\mathfrak{l}_0$ respectively $\mathfrak{q}_0$ is the $+1$ respectively -1 eigenspace of an involution σ, then σ is a Cartan involution only if the real form $\mathfrak{l}_0 \oplus i\mathfrak{q}_0$ of $(\mathfrak{g}_0)^{\mathbb{C}}$ is compact. For a complex semisimple Lie algebra $\mathfrak{g}$, $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u}_0 \oplus i\mathfrak{u}_0$ is a Cartan decomposition of $\mathfrak{g}^{\mathbb{R}}$.

To understand the second important decomposition, the Iwasawa decomposition, of Lie algebras we look at an example on group level.

Define

$$G = SL(m, \mathbb{C}), K = SU(m, \mathbb{C}), A = \{diag(a_1, \ldots, a_m) | a_i \in \mathbb{R}^+\}$$

and

$$N = \{ \begin{pmatrix} 1 & n_{1,2} & \cdots & n_{1,m} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & n_{m-1,m} \\ 0 & \cdots & 0 & 1 \end{pmatrix} | n_{i,j} \in \mathbb{C}\}.$$

The Iwasawa decomposition states that there is a decomposition $G = KAN$ or more precisely the multiplication $\mu : K \times A \times N \to G$ is a diffeomorphism. To show this in our special case we take the standard basis $\{e_1, \ldots, e_m\}$ of $\mathbb{C}^m$ and an arbitrary $g \in G$. Applying g to the basis we obtain a basis $\{ge_1, \ldots, ge_m\}$. The Gram-Schmidt orthogonalization process transforms this basis into an orthonormal basis $\{v_1, \ldots, v_m\}$ of $\mathbb{C}^m$. By the nature of this process we get a matrix $k \in SU(m)$ such that $k^{-1}v_j = e_j$ and $k^{-1}g$ is upper triangular with positive diagonal entries. i.e.: $k^{-1}g \in AN$. $g = k(k^{-1}g) \in K(AN)$ shows that μ is onto and $K \cap AN = \{1\}$ that it is one-one. Smoothness is granted by the explicit formulae of the Gram-Schmidt Process.

Our goal is to observe the equivalent decomposition on algebra level. Untill know we used the subscript 0 to refer to real forms. We will change this notation for some time because we need a subscript referring to linear functionals. To avoid constructions like $\mathfrak{g}_{0,0}$ we will omit this subscript for a while.

So let $\mathfrak{g}$ be a real semisimple Lie algebra with a Cartan involution θ and corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let B be a nondegenerate, symmetric, invariant, bilinear form on $\mathfrak{g}$ such that

- $B(X, Y) = B(\theta X, \theta Y)$ and
- $B_\theta := -B(X, \theta Y)$ is positive definite.

Then B is negative definite on the compact real form $\mathfrak{k} \oplus i\mathfrak{p}$. Therefore B is negative definite on a maximal abelian subspace of $\mathfrak{k} \oplus i\mathfrak{p}$. By the invariance of B and the fact, that two Cartan subalgebras of $\mathfrak{g}^{\mathbb{C}}$ are conjugate by $\operatorname{Int} \mathfrak{g}^{\mathbb{C}}$, we conclude that for any Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, B is positive definite on the real subspace where all roots are real-valued. We define orthogonality and adjoints by B_θ, which is an inner product on $\mathfrak{g}$.

Before we go into the decomposition we need the following lemma.

2.33. Lemma. *Let $\mathfrak{g}$ be a real semisimple Lie algebra and θ a Cartan involution. For all $X \in \mathfrak{g}$ we have $(\operatorname{ad} X)^* = -\operatorname{ad} \theta X$ relative to the inner product B_θ.*

Proof.

$$\begin{aligned} B_\theta((\operatorname{ad}\theta X)Y, Z) &= -B([\theta X, Y], \theta Z) = B(Y, [\theta X, \theta Z]) \\ &= \quad B(Y, \theta[X, Z]) \quad = -B_\theta(Y, [X, Z]) \\ &= -B_\theta(Y, (\operatorname{ad} X)Z) = -B_\theta((\operatorname{ad} X)^* Y, Z) \end{aligned}$$

□

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Existence is guaranteed by finite dimensionality. $\{\operatorname{ad} H | H \in \mathfrak{a}\}$ is a commuting set of selfadjoint transformations of $\mathfrak{g}$. To show selfadjointness we use the above lemma. For $X \in \mathfrak{g}$ we have

$$(\operatorname{ad} H)^* X = (-\operatorname{ad}\theta H)X = -[\theta H, X] = [H, X] = (\operatorname{ad} H)X.$$

Commutativity is given by the Jacobi identity. For $\lambda \in \mathfrak{a}^*$ let

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} | (\operatorname{ad} H)X = \lambda(H)X \quad \forall H \in \mathfrak{a}\}.$$

If $\mathfrak{g}_\lambda \neq 0$ and $\lambda \neq 0$, we call λ a restricted root of $\mathfrak{g}$, or more precisely of $(\mathfrak{g}, \mathfrak{a})$, $\mathfrak{g}_\lambda$ a restricted root space with its elements called restricted root vectors. Let Σ denote the set of restricted roots.

2.34. Proposition. *The restricted roots and restricted root spaces have the following properties:*

(1) $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda$ *is an orthogonal direct sum.*
(2) $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$
(3) $\theta \mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$
(4) $\lambda \in \Sigma \Rightarrow -\lambda \in \Sigma$
(5) $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ *orthogonally, where* $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$

Proof. (1) see construction

(2) Let $H \in \mathfrak{a}$, $X \in \mathfrak{g}_\lambda$ and $Y \in \mathfrak{g}_\mu$. We compute

$$\begin{aligned} (\operatorname{ad} H)[X, Y] &= [H, [X, Y]] \\ &= [[H, X], Y] + [X, [H, Y]] \\ &= [\lambda(H)X, Y] + [X, \mu(H)Y] \\ &= (\lambda(H) + \mu(H))[X, Y] \end{aligned}$$

(3) Let $H \in \mathfrak{a}$ and $X \in \mathfrak{g}_\lambda$. A quite similar computation does the trick.

$$\begin{aligned}(\operatorname{ad} H)\theta X &= [H, \theta X]\\ &= \theta[\theta H, X]\\ &= -\theta[H, X]\\ &= -\lambda(H)\theta X\end{aligned}$$

(4) A consequence of (3).

(5) $\theta\mathfrak{g}_0 = \mathfrak{g}_0$ by 3. Hence $\mathfrak{g}_0 = (\mathfrak{k} \cap \mathfrak{g}_0) \oplus (\mathfrak{p} \cap \mathfrak{g}_0)$. Since $\mathfrak{a} \subseteq \mathfrak{p} \cap \mathfrak{g}_0$ and $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$, $\mathfrak{a} = \mathfrak{p} \cap \mathfrak{g}_0$. By definition $\mathfrak{k} \cap \mathfrak{g}_0 = Z_{\mathfrak{k}}(\mathfrak{a})$.

$\square$

We choose a notion positivity on $\mathfrak{a}^*$ and define the set Σ^+ of positive restricted roots and $\mathfrak{n} = \bigoplus_{\lambda\in\Sigma^+} \mathfrak{g}_\lambda$. We collect some facts on the occuring subalgebras.

- $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$ by 2.34 (2).
- $\mathfrak{a}$ is abelian by definition.
- $[\mathfrak{a}, \mathfrak{n}] = \mathfrak{n}$ because for all $\lambda \neq 0$ we have $[\mathfrak{a}, \mathfrak{g}_\lambda] = \mathfrak{g}_\lambda$.
- $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$
- $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable subalgebra.

The Iwasawa decomposition (on Lie algebra level) states the following

2.35. Proposition. *Let $\mathfrak{g}$ be a semisimple Lie algebra. $\mathfrak{g}$ is a vector space direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ with $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ as above.*

Proof. $\mathfrak{a} \oplus \mathfrak{n} \oplus \theta\mathfrak{n}$ is a direct sum because of 2.34 (1) and 2.34 (3). To show that $\mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is a direct sum we observe some intersecions. Since $\mathfrak{a} \subseteq \mathfrak{g}_0$ and $\mathfrak{n} = \bigoplus_{\lambda\in\Sigma^+} \mathfrak{g}_\lambda$, $\mathfrak{a} \cap \mathfrak{n} = 0$. Let $X \in \mathfrak{k} \cap (\mathfrak{a} \oplus \mathfrak{n})$. $X \in \mathfrak{k}$ gives $\theta X = X$ and $X \in \mathfrak{a} \oplus \mathfrak{n}$ gives $\theta X \in \mathfrak{a} \oplus \theta\mathfrak{n}$. So $X \in \mathfrak{a} \oplus \mathfrak{n}$ and $X = \theta X \in \mathfrak{a} \oplus \theta\mathfrak{n}$, hence X lies in $\mathfrak{a}$. Since $\mathfrak{a} \subseteq \mathfrak{p}$ we conclude $X \in \mathfrak{k} \cap \mathfrak{p} = 0$.

The second step of the proof is to show that the direct sum $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is all of $\mathfrak{g}$. Let X be an arbitrary element of $\mathfrak{g}$. We write X as

$$X = H + X_0 + \sum_{\lambda\in\Sigma} X_\lambda$$

where $H \in \mathfrak{a}$, $X_0 \in \mathfrak{m}$ and $X_\lambda \in \mathfrak{g}_\lambda$ for all $\lambda \in \Sigma$. We write

$$\begin{aligned}\textstyle\sum_{\lambda\in\Sigma} X_\lambda &= \textstyle\sum_{\lambda\in\Sigma^+}(X_{-\lambda} + X_\lambda)\\ &= \textstyle\sum_{\lambda\in\Sigma^+}(X_{-\lambda} + \theta X_{-\lambda}) + \sum_{\lambda\in\Sigma^+}(X_\lambda - \theta X_{-\lambda}).\end{aligned}$$

Since θ maps $X_{-\lambda} + \theta X_{-\lambda}$ onto itself $(X_{-\lambda} + \theta X_{-\lambda}) \in \mathfrak{k}$ and since X_λ and $\theta X_{-\lambda} \in \mathfrak{g}_\lambda$, $(X_\lambda - \theta X_{-\lambda}) \in \mathfrak{g}_\lambda \subseteq \mathfrak{n}$.

An arbitrary X decomposes to

$$\underset{\in\mathfrak{g}}{X} = \underset{\in\mathfrak{k}}{(X_0 + \textstyle\sum_{\lambda\in\Sigma^+}(X_{-\lambda} + \theta X_{-\lambda}))} + \underset{\in\mathfrak{a}}{H} + \underset{\in\mathfrak{n}}{(\textstyle\sum_{\lambda\in\Sigma^+}(X_\lambda - \theta X_{-\lambda}))}$$

□

The following Lemma will enable us to find Cartan algebras in real Lie algebras.

2.36. Lemma. *Let $\mathfrak{g}$ be a real semisimple Lie algebra. There exists a basis $\{X_i\}$ of $\mathfrak{g}$ such that the matrices representing $\operatorname{ad}\mathfrak{g}$ have the following properties:*

(1) *the matrices of $\operatorname{ad}\mathfrak{k}$ are skew symmetric*
(2) *the matrices of $\operatorname{ad}\mathfrak{a}$ are diagonal with real entries*
(3) *the matrices of $\operatorname{ad}\mathfrak{n}$ are upper triangular with all diagonal entries 0.*

Proof. Recall that we decomposed $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda\in\Sigma} \mathfrak{g}_\lambda$ orthogonally. Let $\{X_i\}$ be an orthonormal basis of $\mathfrak{g}$, which is compatible with this decomposition. We do a reordering of these vectors such that $X_i \in \mathfrak{g}_{\lambda_i}$ and $X_j \in \mathfrak{g}_{\lambda_j}$ with $i < j$ implies $\lambda_i \geq \lambda_j$.

(1) Let $X \in \mathfrak{k}$. Therefore $\theta X = X$ and $(\operatorname{ad} X)^* = -\operatorname{ad}\theta X = -\operatorname{ad} X$. We have used this argument before for $H \in \mathfrak{a}$.
(2) Since each X_i is either a restricted root vector or in $\mathfrak{g}_0$ the matrices of $\operatorname{ad}\mathfrak{a}$ are diagonal, necessarily real.
(3) $[\mathfrak{g}_{\lambda_i}, \mathfrak{g}_{\lambda_j}] \subseteq \mathfrak{g}_{\lambda_i+\lambda_j}$

□

Define the rank of a real semisimple Lie algebra $\mathfrak{g}$ as the dimension of any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. This is well defined since $\mathfrak{h}$ is a Cartan subalgebra if and only if $\mathfrak{h}^{\mathbb{C}}$ is Cartan in $\mathfrak{g}^{\mathbb{C}}$.

2.37. Proposition. *Let $\mathfrak{k}\oplus\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{a}$ be maximal abelian in $\mathfrak{p}$ and $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$. If $\mathfrak{t}$ is a maximal abelian subspace of $\mathfrak{m}$ then $\mathfrak{h} = \mathfrak{a}\oplus\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$.*

Proof. We have to prove that $\mathfrak{h}^{\mathbb{C}}$ is maximal abelian in $\mathfrak{g}^{\mathbb{C}}$ and that $\operatorname{ad}_{\mathfrak{g}^{\mathbb{C}}}\mathfrak{h}^{\mathbb{C}}$ is simultaneously diagonable.

By bilinearity of the bracket $\mathfrak{h}^{\mathbb{C}}$ is abelian. To prove maximality let $Z = X + iY$ be in $\mathfrak{h}^{\mathbb{C}}$. If Z commutes with $\mathfrak{h}^{\mathbb{C}}$ then so do X and Y. Thus we do not loose generality in using $X \in \mathfrak{h}$ to test commutativity. If X commutes with $\mathfrak{h}^{\mathbb{C}}$ it lies in $\mathfrak{a}\oplus\mathfrak{m}$. This also holds for θX. Thus $(X + \theta X) \in \mathfrak{k}$ lies in $\mathfrak{m}$ and commutes with $\mathfrak{t}$, hence is in $\mathfrak{t}$. Similar argumentation gives $(X - \theta X) \in \mathfrak{a}$. Thus X is in $\mathfrak{a}\oplus\mathfrak{t}$ and hence $\mathfrak{h}^{\mathbb{C}}$ is maximal abelian.

Using the same basis as above $\operatorname{ad}\mathfrak{t}$ consists of skew symmetric matrices. These are diagonable over $\mathbb{C}$. With the matrices in $\operatorname{ad}\mathfrak{a}$ already diagonal we get a family of commuting matrices and hence the members of $\operatorname{ad}\mathfrak{h}^{\mathbb{C}}$ are diagonable. □

Using $\mathfrak{h} = \mathfrak{a}\oplus\mathfrak{t}$ as Cartan subalgebra of $\mathfrak{g}$, we build the set $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$. The root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ is given by

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta} (\mathfrak{g}^{\mathbb{C}})_\alpha .$$

By definition

$$(\mathfrak{g}^{\mathbb{C}})_\alpha = \{X \in \mathfrak{g}^{\mathbb{C}} | (\operatorname{ad} H)X = \alpha(H)X \quad \forall H \in \mathfrak{h}^{\mathbb{C}}\}$$

and

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} | (\operatorname{ad} H)X = \lambda(H)X \quad \forall H \in \mathfrak{a}\}.$$

Hence

$$\mathfrak{g}_\lambda = \mathfrak{g} \cap \bigoplus_{\substack{\alpha\in\Delta \\ \alpha|_{\mathfrak{a}}=\lambda}} (\mathfrak{g}^{\mathbb{C}})_\alpha$$

and

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\substack{\alpha\in\Delta \\ \alpha|_{\mathfrak{a}}=0}} (\mathfrak{g}^{\mathbb{C}})_\alpha .$$

2.38. Corollary. *If $\mathfrak{t}$ is a maximal abelian subspace of $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ then the Cartan subalgebra $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ has the property that all roots are real valued on $\mathfrak{a} \oplus i\mathfrak{t}$. If $\mathfrak{m} = 0$ then $\mathfrak{g}$ is a split real form of $\mathfrak{g}^{\mathbb{C}}$.*

Proof. The values of the roots on a member H of $\operatorname{ad}\mathfrak{h}$ are the eigenvalues of $\operatorname{ad} H$. For $H \in \mathfrak{a}$, $\operatorname{ad} H$ is self adjoint and hence has real eigenvalues. For $H \in \mathfrak{t}$, $\operatorname{ad} H$ is skew adjoint and hence has imaginary eigenvalues.

If $\mathfrak{m} = 0$, then $\mathfrak{t} = 0$ and $\mathfrak{h} = \mathfrak{a}$. All roots are real on $\mathfrak{a}$ and $\mathfrak{g}$ contains the real subspace of a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}} \subseteq \mathfrak{g}^{\mathbb{C}}$. Hence $\mathfrak{g}$ is a split real form of $\mathfrak{g}^{\mathbb{C}}$. □

We want to impose an ordering on the root system Δ, such that the positive system Δ^+ extends Σ^+. We form a lexicographic ordering on $(\mathfrak{a} + i\mathfrak{t})^*$, taking values on $\mathfrak{a}$ before $i\mathfrak{t}$. If $\alpha \in \Delta$ is nonzero on $\mathfrak{a}$ then the positivity of α only depends on its values on $\mathfrak{a}$. Thus Δ^+ extends Σ^+.

The following theorem will be used in some proofs in the sequel.

2.39. Theorem. *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}_0$. Any two maximal abelian subalgebras of $\mathfrak{g}_0$ are conjugate via $\operatorname{Ad}(G)$.*

We will now oberserve the possible choices for all parts of the Iwasawa decomposition. From the Cartan decomposition we already now that $\mathfrak{k}$ is unique up to conjugacy.

2.40. Lemma. *Let $H \in \mathfrak{a}$ with $\lambda(H) \neq 0$ for all $\lambda \in \Sigma$, then $Z_{\mathfrak{g}}(H) = \mathfrak{m} \oplus \mathfrak{a}$. Hence $Z_{\mathfrak{p}}(H) = \mathfrak{a}$.*

PROOF. Let X be in $Z_{\mathfrak{g}}(H)$ and decompose

$$X = H_0 + X_0 + \sum_{\lambda\in\Sigma} X_\lambda$$

with $H_0 \in \mathfrak{a}$, $X_0 \in \mathfrak{m}$ and $X_\lambda \in \mathfrak{g}_\lambda$. Then $0 = [H, X] = \sum_{\lambda\in\Sigma} \lambda(H)X_\lambda$ and hence $\lambda(H)X_\lambda = 0$ for all $\lambda \in \Sigma$. By our assumption that $\lambda(H) \neq 0$, $X_\lambda = 0$. □

Now we fix a subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ and look at the possible choices of a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$.

2.41. THEOREM. *If $\mathfrak{a}$ and $\mathfrak{a}'$ are two maximal abelian subalgebras of $\mathfrak{p}$ then there is a member $k \in K$ with $\mathrm{Ad}(k)\mathfrak{a}' = \mathfrak{a}$, where K is the analytic subgroup of G with Lie algebra $\mathfrak{k}$. Consequently $\mathfrak{p} = \bigcup_{k\in K} \mathrm{Ad}(k)\mathfrak{a}$.*

PROOF. We can easily find an $H \in \mathfrak{a}$ with $\lambda(H) \neq 0$ for all $\lambda \in \Sigma$ since the union of the kernels of all $\lambda \in \Sigma$ is only a finite union of hyperplanes in $\mathfrak{a}$. By lemma 2.40 such an $H \in \mathfrak{a}$ gives $Z_{\mathfrak{p}}(H) = \mathfrak{a}$. Similarly we find an $H' \in \mathfrak{a}'$ such that $Z_{\mathfrak{p}}(H)' = \mathfrak{a}'$. By compactness of $\mathrm{Ad}(K)$, choose a $k_0 \in K$ such that $B(\mathrm{Ad}(k_0)H', H) \leq B(\mathrm{Ad}(k)H', H)$ for all $k \in K$. For any $Z \in \mathfrak{k}$

$$r \mapsto B(\mathrm{Ad}(\exp rZ)\,\mathrm{Ad}(k_0)H', H)$$

is a smooth function of r that is minimized for $r = 0$. Differentiating and setting $r = 0$ we obtain

$$0 = B((\mathrm{ad}\, Z)\,\mathrm{Ad}(k_0)H', H) = B(Z, [\mathrm{Ad}(k_0)H', H]).$$

$[\mathrm{Ad}(k_0)H', H]$ is in $\mathfrak{k}$. Since $B(\mathfrak{k}, \mathfrak{p}) = 0$ and since B is nondegenerate, $[\mathrm{Ad}(k_0)H', H] = 0$. Thus $\mathrm{Ad}(k_0)H'$ is in $Z_{\mathfrak{p}}(H) = \mathfrak{a}$. Since $\mathfrak{a}$ is abelian

$$\mathfrak{a} \subseteq Z_{\mathfrak{p}}(\mathrm{Ad}(k_0)H') = \mathrm{Ad}(k_0)Z_{\mathfrak{p}}(H') = \mathrm{Ad}(k_0)\mathfrak{a}'.$$

Since $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$ we even get equality $\mathfrak{a} = \mathrm{Ad}(k_0)\mathfrak{a}'$. This proves the first statement of the theorem.

Let $X \in \mathfrak{p}$ and extend the abelian subspace $\mathbb{R}X$ of $\mathfrak{p}$ to a maximal abelian subspace $\mathfrak{a}'$. Using the first part of the proof we write $\mathfrak{a}' = \mathrm{Ad}(k)\mathfrak{a}$ and hence $X \in \bigcup_{k\in K} \mathrm{Ad}(k)\mathfrak{a}$. Therefore

$$\mathfrak{p} = \bigcup_{k\in K} \mathrm{Ad}(k)\mathfrak{a}.$$

□

For a fixed subalgebra $\mathfrak{k}$ we have found that all possible $\mathfrak{a}$ are conjugate. Now we fix $\mathfrak{k}$ and $\mathfrak{a}$ and observe the possible choices of $\mathfrak{n}$. B_θ is an inner product on $\mathfrak{g}$ and can be restricted to an inner product on $\mathfrak{a}$. Since we can identify $\lambda \in \mathfrak{a}^*$ with $H_\lambda \in \mathfrak{a}$ we can transfer B_θ to $\mathfrak{a}^*$ denoting it $\langle , \rangle$.

2.42. PROPOSITION. *Let λ be a restricted root and let E_λ be a nonzero restricted root vector for λ.*

(1) $[E_\lambda, \theta E_\lambda] = B(E_\lambda, \theta E_\lambda)H_\lambda$ *and* $B(E_\lambda, \theta E_\lambda) < 0$.
(2) $\mathbb{R}H_\lambda \oplus \mathbb{R}E_\lambda \oplus \mathbb{R}\theta E_\lambda$ *is a Lie subalgebra of* $\mathfrak{g}$ *isomorphic to* $\mathfrak{sl}(2,\mathbb{R})$ *and the isomorphism can be defined so that the vector* $H'_\lambda = 2\frac{H_\lambda}{|\lambda|^2}$ *corresponds to* $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
(3) *If* E_λ *is normalized so that* $B(E_\lambda, \theta E_\lambda) = -\frac{2}{|\lambda|^2}$, *then*
$$k = \exp \frac{\pi}{2}(E_\lambda + \theta E_\lambda)$$
is a member of the normalizer $N_K(\mathfrak{a})$ *of* $\mathfrak{a}$ *in* K *and* $\mathrm{Ad}(k)$ *acts as the reflection* s_λ *on* $\mathfrak{a}^*$.

Proof. (1) Since $\theta\mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$ the vector
$$[E_\lambda, \theta E_\lambda] \in [\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}] \subseteq \mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$$
and from
$$\theta[E_\lambda, \theta E_\lambda] = [\theta E_\lambda, E_\lambda] = -[E_\lambda, \theta E_\lambda]$$
it follows that $[E_\lambda, \theta E_\lambda]$ lies in $\mathfrak{a}$. For $H \in \mathfrak{a}$ we compute
$$\begin{aligned} B([E_\lambda, \theta E_\lambda], H) &= B(E_\lambda, [\theta E_\lambda, H]) \\ &= \lambda(H)B(E_\lambda, \theta E_\lambda) \\ &= B(H_\lambda, H)B(E_\lambda, \theta E_\lambda) \\ &= B(B(E_\lambda, \theta E_\lambda)H_\lambda, H). \end{aligned}$$
Since B is nondegenerate on $\mathfrak{a}$ and B_θ is positive definite we get the stated results
$$\begin{aligned} [E_\lambda, \theta E_\lambda] &= B(E_\lambda, \theta E_\lambda)H_\lambda \\ B(E_\lambda, \theta E_\lambda) &= -B_\theta(E_\lambda, E_\lambda) < 0. \end{aligned}$$
(2) Let
$$H'_\lambda = \frac{2}{|\lambda|^2}H_\lambda,\ E'_\lambda = \frac{2}{|\lambda|^2}E_\lambda,\ E'_{-\lambda} = \theta E_\lambda.$$
Then (1) shows that
$$[H'_\lambda, E'_\lambda] = 2E'_\lambda,\ [H'_\lambda, E'_{-\lambda}] = -2E'_{-\lambda},\ [E'_\lambda, E'_{-\lambda}] = H'_\lambda$$
which is all we need.
(3) We normalize the vectors such that $B(E_\lambda, \theta E_\lambda) = -\frac{2}{|\lambda|^2}$, which always works because of (1). If $\lambda(H) = 0$, then
$$\begin{aligned} \mathrm{Ad}(k)H &= \mathrm{Ad}(\exp \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))H \\ &= (\exp \mathrm{ad}\, \tfrac{\pi}{2}(E_\lambda + \theta E_\lambda))H \\ &= \textstyle\sum_{n=0}^\infty \frac{1}{n!}(\mathrm{ad}\, \frac{\pi}{2}(E_\lambda + \theta E_\lambda))^n H \\ &= H. \end{aligned}$$

For the element H'_λ the following equalities hold:

$$\begin{aligned}(\operatorname{ad}\tfrac{\pi}{2}(E_\lambda+\theta E_\lambda))H'_\lambda &= \pi(\theta E_\lambda - E_\lambda)\\ (\operatorname{ad}\tfrac{\pi}{2}(E_\lambda+\theta E_\lambda))^2 H'_\lambda &= -\pi^2 H'_\lambda.\end{aligned}$$

Using this equalities we get

$$\begin{aligned}\operatorname{Ad}(k)H'_\lambda &= \textstyle\sum_{n=0}^\infty \frac{1}{n!}(\operatorname{ad}\frac{\pi}{2}(E_\lambda+\theta E_\lambda))^n H'_\lambda\\ &= \textstyle\sum_{m=0}^\infty \frac{1}{(2m)!}((\operatorname{ad}\frac{\pi}{2}(E_\lambda+\theta E_\lambda))^2)^m H'_\lambda\\ &\quad+\textstyle\sum_{m=0}^\infty \frac{1}{(2m+1)!}(\operatorname{ad}\frac{\pi}{2}(E_\lambda+\theta E_\lambda))((\operatorname{ad}\frac{\pi}{2}(E_\lambda+\theta E_\lambda))^2)^m H'_\lambda\\ &= \textstyle\sum_{m=0}^\infty \frac{1}{(2m)!}(-\pi^2)^m H'_\lambda + \sum_{m=0}^\infty \frac{1}{(2m+1)!}(-\pi^2)^m \pi(\theta E_\lambda - E_\lambda)\\ &= (\cos\pi)H'_\lambda + (\sin\pi)(E_\lambda - \theta E_\lambda)\\ &= -H'_\lambda.\end{aligned}$$

This is what we stated.

□

2.43. Corollary. *Σ is an abstract root system in $\mathfrak{a}^*$. Σ need not be reduced.*

Proof. We verify that Σ satisfies the axioms for an abstract root system. To see that Σ spans $\mathfrak{a}^*$, assume $\lambda(H)=0$ for some $H\in\mathfrak{a}$. Then $[H,\mathfrak{g}_\lambda]=0$ for all λ and hence $[H,\mathfrak{g}]=0$. But the center $Z_\mathfrak{g}=0$ and hence $H=0$. Thus Σ spans $\mathfrak{a}^*$.

Let $\mathfrak{sl}_\lambda$ denote the Lie subalgebra mentioned in 2.42 (2). This acts by ad on $\mathfrak{g}$ and hence on $\mathfrak{g}^\mathbb{C}$. Complexifying we obtain a representation of $\mathfrak{sl}_\lambda^\mathbb{C}\cong\mathfrak{sl}(2,\mathbb{C})$ on $\mathfrak{g}^\mathbb{C}$. The element $H'_\lambda = 2\frac{H_\lambda}{|\lambda|^2}$ which corresponds to h acts diagonably with integer eigenvalues. H'_λ acts on $\mathfrak{g}_\mu$ by the scalar $\mu(2\frac{H_\lambda}{|\lambda|^2}) = 2\frac{\langle\mu,\lambda\rangle}{|\lambda|^2}$. Hence $2\frac{\langle\mu,\lambda\rangle}{|\lambda|^2}$ is an integer.

The last property to show is that the reflection $s_\lambda(\mu)$ of μ along λ is in Σ for all $\lambda,\mu\in\Sigma$. Define k as in 2.42 (3), let $H\in\mathfrak{a}$ and $X\in\mathfrak{g}_\mu$. Then

$$\begin{aligned}[H,\operatorname{Ad}(k)X] &= \operatorname{Ad}(k)[\operatorname{Ad}(k)^{-1}H,X] \;=\; \operatorname{Ad}(k)[s_\lambda^{-1}(H),X]\\ &= \mu(s_\lambda^{-1}(H))\operatorname{Ad}(k)X = (s_\lambda\mu)(H)\operatorname{Ad}(k)X\end{aligned}$$

and hence $\mathfrak{g}_{s_\lambda(\mu)}$ is not 0. □

2.44. Corollary. *Any two choices of $\mathfrak{n}$ are conjugate by $\operatorname{Ad} n$ for some $n\in N_K(\mathfrak{a})$.*

We ran through all parts of the Iwasawa decomposition and see that an Iwasawa decomposition of $\mathfrak{g}$ is unique up to conjugacy by $\operatorname{Int}\mathfrak{g}$.

An interesting aspect in classifying real semisimple Lie algebras are the conjugacy classes of their Cartan subalgebras. Now, that we do not deal with subscripts referring to root spaces any longer we revert to the subscript 0 for real Lie algebras. So $\mathfrak{g}$ will denote a complex Lie algebra again and $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$ if $\mathfrak{g}_0$ is a real Lie algebra.

Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, θ a Cartan involution and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ the corresponding Cartan decomposition. Let B be any nondegenerate, symmetric, bilinear form such that $B(\theta X, \theta Y) = B(X, Y)$ and B_θ is positive definit.

In the case of complex Lie algebras the situation is quite easy since all Cartan subalgebras are conjugate. This is not true in the real case. However, the following proposition holds.

2.45. PROPOSITION. *Any Cartan subalgebra $\mathfrak{h}_0$ of a real semisimple Lie algebra $\mathfrak{g}_0$ is conjugate via* $\operatorname{Int}\mathfrak{g}_0$ *to a θ stable Cartan subalgebra.*

PROOF. Let $\mathfrak{h}_0$ be any Cartan subalgebra of $\mathfrak{g}_0$ with complexification $\mathfrak{h}$ Cartan in $\mathfrak{g}$. Let σ the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$.

Let $\mathfrak{u}_0$ be the compact real form of $\mathfrak{g}$ built out of the split form corresponding to $\mathfrak{h}$ as in 2.26. and let τ be the conjugation with respect to $\mathfrak{u}_0$. Since $i\mathfrak{h}_0 \subseteq \mathfrak{u}_0$ is exactly the part of $\mathfrak{h}$ which lies in $\mathfrak{u}_0$, $\tau(\mathfrak{h}) = \mathfrak{h}$.

The conjugations σ and τ are involutions of $\mathfrak{g}^{\mathbb{R}}$ and τ is a Cartan involution. As proven before $\varphi = ((\sigma\tau)^2)^{\frac{1}{4}} \in \operatorname{Int}\mathfrak{g}^{\mathbb{R}} = \operatorname{Int}\mathfrak{g}$ is the element that makes σ and $\tilde{\eta} = \varphi\tau\varphi^{-1}$ commute. Since $\sigma(\mathfrak{h}) = \mathfrak{h}$ and $\tau(\mathfrak{h}) = \mathfrak{h}$ also $\varphi(\mathfrak{h}) = \mathfrak{h}$ and $\tilde{\eta}(\mathfrak{h}) = \mathfrak{h}$. Using commutativity we compute

$$\sigma\tilde{\eta}(\mathfrak{g}_0) = \tilde{\eta}\sigma(\mathfrak{g}_0) = \tilde{\eta}(\mathfrak{g}_0)$$

which shows that $\tilde{\eta}(\mathfrak{g}_0) = \mathfrak{g}_0$. Since $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$ we obtain $\tilde{\eta}(\mathfrak{h}_0) = \mathfrak{h}_0$.

Let $\eta = \tilde{\eta}|_{\mathfrak{g}_0}$. Clearly $\eta(\mathfrak{h}_0) = \mathfrak{h}_0$. Since

$$\varphi\tau\varphi^{-1}(\varphi(\mathfrak{u}_0)) = \varphi\tau(\mathfrak{u}_0) = \varphi(\mathfrak{u}_0)$$

$\tilde{\eta}$ is a conjugation of $\mathfrak{g}$ with respect to $\varphi(\mathfrak{u}_0)$. Taking X and Y in $\mathfrak{g}^{\mathbb{R}}$ we have

$$(B_{\mathfrak{g}^{\mathbb{R}}})_{\tilde{\eta}}(X, Y) = -B_{\mathfrak{g}^{\mathbb{R}}}(X, \tilde{\eta}Y).$$

Restricting X and Y to $\mathfrak{g}_0$ this equals

$$-2B_{\mathfrak{g}_0}(X, \eta Y) = 2(B_{\mathfrak{g}_0})_\eta(X, Y).$$

Consequently η is a Cartan involution of $\mathfrak{g}_0$. Since any two Cartan involutions of $\mathfrak{g}_0$ are conjugate via $\operatorname{Int}\mathfrak{g}_0$ their exists a $\psi \in \operatorname{Int}\mathfrak{g}_0$ such that $\theta = \psi\eta\psi^{-1}$. Then $\psi(\mathfrak{h}_0)$ is a Cartan subalgebra of $\mathfrak{g}_0$ and

$$\theta(\psi(\mathfrak{h}_0)) = \psi\eta\psi^{-1}\psi(\mathfrak{h}_0) = \psi(\eta(\mathfrak{h}_0)) = \psi(\mathfrak{h}_0)$$

shows that it is θ stable. $\square$

Without loss of generality we restrict to the study of θ stable Cartan subalgebras. Let $\mathfrak{h}_0$ be a θ stable Cartan subalgebra of $\mathfrak{g}_0$. Then $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ with $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$ and $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$. As seen above all roots are real valued on $\mathfrak{a}_0 \oplus i\mathfrak{t}_0$. We call $\dim\mathfrak{t}_0$ the compact dimension and $\dim\mathfrak{a}_0$ the noncompact dimension of $\mathfrak{h}_0$.

We call a θ stable Cartan subalgebra $\mathfrak{h}_0$ maximally noncompact if its noncompact dimension is maximal. We call it maximally compact if its compact dimension is maximal.

In any case $\mathfrak{a}_0$ is an abelian subspace of $\mathfrak{p}_0$. Therefore $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ is maximally noncompact if and only if $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{p}_0$.

A similar statement is true for maximally compact Cartan algebras. To see this we need the following

2.46. Proposition. *Let $\mathfrak{t}_0$ be a maximal abelian subspace of $\mathfrak{k}_0$. Then $\mathfrak{h}_0 = Z_{\mathfrak{g}_0}(\mathfrak{t}_0)$ is a θ stable Cartan subalgebra of $\mathfrak{g}_0$ of the form $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ with $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$.*

Proof. From our construction of $\mathfrak{h}_0$ we know that it decomposes to $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ where $\mathfrak{a}_0 = \mathfrak{h}_0 \cap \mathfrak{p}_0$. Therefore $\mathfrak{h}_0$ is θ stable. Since all θ stable subalgebras of a real semisimple Lie algebra are reductive, so is $\mathfrak{h}_0$. Hence $[\mathfrak{h}_0, \mathfrak{h}_0]$ is semisimple.

We have
$$[\mathfrak{h}_0, \mathfrak{h}_0] = [\mathfrak{t}_0 \oplus \mathfrak{a}_0, \mathfrak{t}_0 \oplus \mathfrak{a}_0] = [\mathfrak{a}_0, \mathfrak{a}_0].$$
Recall that $[\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0$. Since $\mathfrak{t}_0 = \mathfrak{h}_0 \cap \mathfrak{k}_0$ we get
$$[\mathfrak{h}_0, \mathfrak{h}_0] = [\mathfrak{a}_0, \mathfrak{a}_0] \subseteq \mathfrak{t}_0.$$
Thus the semisimple Lie algebra $[\mathfrak{h}_0, \mathfrak{h}_0]$ is abelian and hence must be 0. Consequently $\mathfrak{h}_0$ is abelian.

$\mathfrak{h} = (\mathfrak{h}_0)^{\mathbb{C}}$ is maximal abelian in $\mathfrak{g}$. Since all elements of $\operatorname{ad}_{\mathfrak{g}_0}(\mathfrak{t}_0)$ are skew adjoint and all elements of $\operatorname{ad}_{\mathfrak{g}_0}(\mathfrak{a}_0)$ are self adjoint and $\mathfrak{t}_0$ commutes with $\mathfrak{a}_0$, $\operatorname{ad}\mathfrak{h}_0$ is diagonably on $\mathfrak{g}$. Therefore $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and consequently $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$. $\square$

So similar to the above we see, that for a θ stable $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$, $\mathfrak{t}_0$ is an abelian subspace of $\mathfrak{k}_0$ and $\mathfrak{h}_0$ is maximally compact if and only if $\mathfrak{t}_0$ is maximal abelian in $\mathfrak{k}_0$. We proceed with two statements about conjugacy of special Cartan subalgebras.

2.47. Proposition. *Among θ stable Cartan subalgebras $\mathfrak{h}_0$ of $\mathfrak{g}_0$ the maximally noncompact ones are all conjugate via K, and the maximally compact ones are all conjugate via K.*

2.48. Proposition. *Up to conjugacy by* $\operatorname{Int}\mathfrak{g}_0$, *there are only finitely many Cartan subalgebras of $\mathfrak{g}_0$.*

Recall that we used Dynkin diagrams to classify complex semisimple Lie algebras. If we want to use something similar in the classification of real semisimple Lie algebras we have to refine this concept. These refined diagrams will be called Vogan diagrams and will consist of Dynkin diagrams plus additional information.

Let $\mathfrak{g}_0$ be a real semisimple Lie algebra with complexification $\mathfrak{g}$. Let θ be a Cartan involution and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ the corresponding Cartan decomposition. Let B be any nondegenerate symmetric invariant bilinear form on $\mathfrak{g}_0$ such that $B(\theta X, \theta Y) = B(X, Y)$ and B_θ is positive definite. Let $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ be a θ stable Cartan subalgebra with $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$ and $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$. From Corrolary 2.38 we know that all roots of $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{h} = (\mathfrak{h}_0)^{\mathbb{C}}$, are real valued on $\mathfrak{a}_0 \oplus i\mathfrak{t}_0$. Hence the roots take real values on $\mathfrak{a}_0$ and imaginary values on $\mathfrak{t}_0$. We call a root real if it takes real values on all of $\mathfrak{h}_0$, equivalently the root vanishes on $\mathfrak{t}_0$ and we call a root imaginary if it takes purely imaginary values on $\mathfrak{h}_0$, equivalently it vanishes on $\mathfrak{a}_0$. If a root does not vanish on either of $\mathfrak{t}_0$ and $\mathfrak{a}_0$ it is called complex.

Now let $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ be a maximally compact θ stable Cartan subalgebra of $\mathfrak{g}_0$ with complexification $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the set of roots. Since we choose a maximally compact Cartan subalgebra there are no real roots.

Let $\{H_1, \ldots, H_n\}$ be a basis of $i\mathfrak{t}_0$ and let $\{H_{n+1}, \ldots, H_m\}$ be a basis of $\mathfrak{a}_0$. Then $\{H_1, \ldots, H_m\}$ is a basis of $i\mathfrak{t}_0 \oplus \mathfrak{a}_0$ and also a basis of $\mathfrak{h}$. Let $\alpha, \beta \in \Delta$. We say that $\alpha > \beta$ if there is an index l such that

$$\alpha(H_l) > \beta(H_l) \text{ and } \alpha(H_j) = \beta(H_j) \text{ for all } j < l.$$

For any root α we define $\theta\alpha$ by

$$\theta\alpha(H) = \alpha(\theta^{-1}H).$$

Let E_α be a nonzero root vector for α and calculate

$$[H, \theta E_\alpha] = \theta[\theta^{-1}H, E_\alpha] = \alpha(\theta^{-1}H)\theta E_\alpha = (\theta\alpha)(H)\theta E_\alpha$$

to see that $\theta\alpha$ is a root again. If α is purely imaginary, then $\theta\alpha = \alpha$. Thus $\mathfrak{g}_\alpha$ is θ stable and hence

$$\mathfrak{g}_\alpha = (\mathfrak{g}_\alpha \cap \mathfrak{k}) \oplus (\mathfrak{g}_\alpha \cap \mathfrak{p}).$$

But, as $\dim \mathfrak{g}_\alpha = 1$ we either have $\mathfrak{g}_\alpha \subseteq \mathfrak{k}$ or $\mathfrak{g}_\alpha \subseteq \mathfrak{p}$. We call an imaginary root α compact if $\mathfrak{g}_\alpha \subseteq \mathfrak{k}$ or we call it noncompact if $\mathfrak{g}_\alpha \subseteq \mathfrak{p}$.

Since θ is $+1$ on $\mathfrak{t}_0$ and -1 on $\mathfrak{a}_0$ and since there are no real roots, which means no roots that vanish on $\mathfrak{t}$, $\theta(\Delta^+) = \Delta^+$. Therefore θ permutes the simple roots. More precisely θ fixes the imaginary roots, which vanish an $\mathfrak{a}$, and it permutes the complex roots in 2-cycles since it flips the real parts.

Let $\mathfrak{g}_0$ be a real semisimple Lie algebra and $\mathfrak{h}_0$ a Cartan subalgebra of $\mathfrak{g}_0$. Let θ be a Cartan involution of $\mathfrak{g}_0$. Let Δ^+ be a system of positive roots built in the above way. The Vogan diagram of $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$ consists of the Dynkin diagram of Δ^+ with 2-element orbits under θ labeled and with 1-element orbits corresponding to noncompact imaginary roots painted. Observe that this triple totally determines the Vogan diagram. Let $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$ and $(\mathfrak{g}_0', \mathfrak{h}_0', \Delta^{+'})$ be two isomorphic triples. Since $(\mathfrak{g}_0)^{\mathbb{C}}$ is isomorphic to $(\mathfrak{g}_0')^{\mathbb{C}}$ their Vogan diagrams are based on the same Dynkin diagram. But they also have isomorphic Cartan subalgebras and positive systems and hence they have the same 2-element orbits and noncompact imaginary roots. Hence they have the same Vogan diagram.

In order to classify complex semisimple Lie algebras, by classifying Dynkin diagrams, we needed the Isomorphism Theorem 1.23 and the Existence Theorem 1.24, which gave us a one-one correspondence of Lie algebras and Dynkin diagrams. The same has to be done for real semisimple Lie algebras and Vogan diagrams. First lets state an analog for the Isomorphism Theorem. The proof will be made out of steps, each step decreasing the possible differences between the two Lie algebras.

2.49. THEOREM. *Let $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ be real semisimple Lie algebras. If two triples $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$ and $(\mathfrak{g}_0', \mathfrak{h}_0', \Delta'^+)$ have the same Vogan diagram, then $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ are isomorphic.*

PROOF. Since the Lie algebras have the same Vogan diagram, they also have the same Dynkin diagram. By the Isomorphism Theorem 1.23 we do not loose generality in assuming $(\mathfrak{g}_0)^{\mathbb{C}} = (\mathfrak{g}_0')^{\mathbb{C}} = \mathfrak{g}$.

Let $\mathfrak{u}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ be the compact real form of $\mathfrak{g}$ associated to $\mathfrak{g}_0$ and $\mathfrak{u}_0' = \mathfrak{k}_0' \oplus i\mathfrak{p}_0'$ the one associated to $\mathfrak{g}_0'$. Since any two compact real forms of a complex semisimple Lie algebra $\mathfrak{g}$ are conjugate via $\operatorname{Int}\mathfrak{g}$, there exists $x \in \operatorname{Int}\mathfrak{g}$ such that $x\mathfrak{u}_0' = \mathfrak{u}_0$. $x\mathfrak{g}_0'$ is a real form of $\mathfrak{g}$ that is isomorphic to $\mathfrak{g}_0'$ and has Cartan decomposition $x\mathfrak{g}_0' = x\mathfrak{k}_0' \oplus x\mathfrak{p}_0'$. Since $x\mathfrak{k}_0' \oplus ix\mathfrak{p}_0' = x\mathfrak{u}_0' = \mathfrak{u}_0$, there is no loss of generality in assuming that $\mathfrak{u}_0' = \mathfrak{u}_0$ from the outset. Then

$$\theta(\mathfrak{u}_0) = \mathfrak{u}_0 \text{ and } \theta'(\mathfrak{u}_0) = \theta'(\mathfrak{u}_0') = \mathfrak{u}_0' = \mathfrak{u}_0.$$

We now use theorem 2.39 to see that the Cartan subalgebras of $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ complexify to the same Cartan subalgebra of $\mathfrak{g}$: Let $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ respectively $\mathfrak{h}_0' = \mathfrak{t}_0' \oplus \mathfrak{a}_0'$ be the Cartan subalgebras of $\mathfrak{g}_0$ respectively $\mathfrak{g}_0'$ decomposed with respect to θ respectively θ'. Since $\mathfrak{t}_0 \oplus i\mathfrak{a}_0$ and $\mathfrak{t}_0' \oplus i\mathfrak{a}_0'$ are maximal abelian in $\mathfrak{u}_0$ and $\mathfrak{u}_0$ is compact, there exists a $k \in \operatorname{Int}\mathfrak{u}_0$ such that $k(\mathfrak{t}_0' \oplus i\mathfrak{a}_0') = \mathfrak{t}_0 \oplus i\mathfrak{a}_0$. $k\mathfrak{g}_0'$ is isomorphic to $\mathfrak{g}_0'$ and $x\mathfrak{h}_0' = x\mathfrak{t}_0' \oplus x\mathfrak{a}_0'$. Since $k(\mathfrak{t}_0' \oplus i\mathfrak{a}_0') = k\mathfrak{t}_0' \oplus ik\mathfrak{a}_0'$, there is no loss in generality in assuming that $(\mathfrak{t}_0' \oplus i\mathfrak{a}_0') = (\mathfrak{t}_0 \oplus i\mathfrak{a}_0)$. Therefore the Cartan subalgebras $\mathfrak{h}_0$ and $\mathfrak{h}_0'$ complexify to the same Cartan subalgebra of $\mathfrak{g}$, which we will denote by $\mathfrak{h}$. The space

$$\mathfrak{u}_0 \cap \mathfrak{h} = \mathfrak{t}_0 \oplus i\mathfrak{a}_0 = \mathfrak{t}_0' \oplus i\mathfrak{a}_0'$$

is a maximal abelian subspace of $\mathfrak{u}_0$.

Because the complexifications of both real Lie algebras and their Cartan subalgebras are the same now, their root systems coincide. However their positive systems still may differ. But their exists a $k' \in \operatorname{Int}\mathfrak{u}_0$ normalizing $\mathfrak{u}_0 \oplus \mathfrak{h}$ with $k'\Delta^{+'} = \Delta^+$. If we replace $\mathfrak{g}_0'$ by $k'\mathfrak{g}_0'$ and argue the same way we walked through twice, we may assume $\Delta^{+'} = \Delta^+$ from the outset.

What we have done so far is that we may assume without loss of generality that

$$\mathfrak{u}_0' = \mathfrak{u}_0$$
$$\mathfrak{u}_0' \cap \mathfrak{h}' = \mathfrak{u}_0 \cap \mathfrak{h}$$
$$\Delta' = \Delta \text{ and } \Delta^{+'} = \Delta^+$$

The next step is to choose normalizations of the root vectors relative to $\mathfrak{h}$. Let B be the Killing form of $\mathfrak{g}$. Recall the construction of a split form of a complex semisimple Lie algebra in 2.25. We obtained root vectors X_α. Out of this split form we constructed a compact real form

$$\tilde{\mathfrak{u}_0} = \sum_{\alpha\in\Delta} \mathbb{R}(iH_\alpha) + \sum_{\alpha\in\Delta} \mathbb{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha\in\Delta} \mathbb{R}i(X_\alpha + X_{-\alpha})$$

of $\mathfrak{g}$. The subalgebra $\tilde{\mathfrak{u}_0}$ contains the real subspace $\sum_{\alpha\in\Delta} \mathbb{R}iH_\alpha$ of $\mathfrak{h}$ where all roots are imaginary, which is just $\mathfrak{u}_0 \cap \mathfrak{h}$. Since any two compact real forms are conjugate by $\operatorname{Int}\mathfrak{g}$ there exists a $g \in \operatorname{Int}\mathfrak{g}$ such that $g\tilde{\mathfrak{u}_0} = \mathfrak{u}_0$. Then $g\tilde{\mathfrak{u}}_0 = \mathfrak{u}_0$ is built from $g(\mathfrak{u}_0 \cap \mathfrak{h})$ and the root vectors gX_α. The two maximal abelian subspaces $\mathfrak{u}_0 \cap \mathfrak{h}$ and $g(\mathfrak{u}_0 \cap \mathfrak{h})$ of $\mathfrak{u}_0$ are conjugate by $u \in \operatorname{Int}\mathfrak{u}_0$. Hence $ug(\mathfrak{u}_0 \cap \mathfrak{h}) = \mathfrak{u}_0 \cap \mathfrak{h}$. Let $Y_\alpha = ugX_\alpha$ for all $\alpha \in \Delta$. Then $\mathfrak{u}_0$ is built from $ug(\mathfrak{u}_0 \cap \mathfrak{h}) = \mathfrak{u}_0 \cap \mathfrak{h}$ and the root vectors Y_α.

$$\mathfrak{u}_0 = \sum_{\alpha\in\Delta} \mathbb{R}(iH_\alpha) + \sum_{\alpha\in\Delta} (Y_\alpha - Y_{-\alpha}) + \sum_{\alpha\in\Delta} \mathbb{R}i(Y_\alpha + Y_{-\alpha})$$

Now we will use what we know about the Cartan involutions of $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ out of the Vogan diagram. Since the automorphism of Δ^+ defined by θ and θ' are the same, θ and θ' have the same effect on $\mathfrak{h}^*$. Thus $\theta(H) = \theta'(H)$ for all $H \in \mathfrak{h}$. If α is an imaginary simple root, then

$$\theta(Y_\alpha) = Y_\alpha = \theta'(Y_\alpha) \qquad \text{if } \alpha \text{ is unpainted}$$
$$\theta(Y_\alpha) = -Y_\alpha = \theta'(Y_\alpha) \qquad \text{if } \alpha \text{ is painted.}$$

Remember that the 2-element orbits under θ of complex simple roots are labeled in the Vogan diagram. For $\alpha \in \Delta$ we write $\theta Y_\alpha = a_\alpha Y_\alpha$. Since $\theta(\mathfrak{u}_0) = \mathfrak{u}_0$ we know that

$$\theta(\mathfrak{u}_0 \cap \operatorname{span}\{Y_\alpha, Y_{-\alpha}\}) \subseteq \mathfrak{u}_0 \cap \operatorname{span}\{Y_{\theta\alpha}, Y_{-\theta\alpha}\}.$$

This means that

$$\theta(\mathbb{R}(Y_\alpha - Y_{-\alpha}) + \mathbb{R}i(Y_\alpha + Y_{-\alpha})) \subseteq \theta(\mathbb{R}(Y_{\theta\alpha} - Y_{-\theta\alpha}) + \mathbb{R}i(Y_{\theta\alpha} + Y_{-\theta\alpha})).$$

Let x and $y \in \mathbb{R}$ and $x + iy = z \in \mathbb{C}$. Then

$$x(Y_\alpha - Y_{-\alpha}) + yi(Y_\alpha + Y_{-\alpha}) = (x+iy)Y_\alpha - (x-iy)Y_{-\alpha} = zY_\alpha - \bar{z}Y_{-\alpha}$$

Since $\theta(zY_\alpha - \bar{z}Y_{-\alpha}) = za_\alpha Y_{\theta\alpha} - \bar{z}a_{-\alpha}Y_{-\theta\alpha}$ lies in $\theta(\mathbb{R}(Y_\alpha - Y_{-\alpha}) + \mathbb{R}i(Y_\alpha + Y_{-\alpha}))$ it must be of the form $wY_{\theta\alpha} - \bar{w}Y_{-\theta\alpha}$. Thus from $\overline{za_\alpha} = \bar{z}a_{-\alpha}$ we conclude that

$$a_{-\alpha} = \overline{a_\alpha}.$$

Furthermore

$$a_\alpha a_{-\alpha} = 1$$

since

$$a_\alpha a_{-\alpha} = a_\alpha a_{-\alpha} B(Y_{\theta\alpha}, Y_{-\theta\alpha}) = B(a_\alpha Y_{\theta\alpha}, a_{-\alpha} Y_{-\theta\alpha}) = B(\theta Y_\alpha, \theta Y_{-\alpha}) = B(Y_\alpha, Y_{-\alpha}) = 1.$$

Combining these two results we get

$$|a_\alpha| = a_\alpha \overline{a_\alpha} = a_\alpha a_{-\alpha} = 1.$$

Since $Y_\alpha = \theta^2 Y_\alpha = \theta(a_\alpha Y_{\theta\alpha}) = a_\alpha a_{\theta\alpha} Y_\alpha$ we have

$$a_\alpha a_{\theta\alpha} = 1.$$

For each pair of complex simple roots it is therefore possible to choose square roots $\sqrt{a_\alpha}$ and $\sqrt{a_{\theta\alpha}}$ such that

$$\sqrt{a_\alpha}\sqrt{a_{\theta\alpha}} = 1.$$

Similarly we write $\theta' Y_\alpha = b_\alpha Y_{\theta\alpha}$ and obtain the same results as above and choose square roots.

$$|b_\alpha| = 1$$
$$\sqrt{b_\alpha}\sqrt{b_{\theta\alpha}} = 1$$

We can define H and H' in $\mathfrak{u}_0 \cap \mathfrak{h}$ by the following conditions:

- $\alpha(H) = 0 = \alpha(H')$ for imaginary simple α
- $\exp(\frac{1}{2}\alpha(H)) = \sqrt{a_\alpha}$, $\exp(\frac{1}{2}\theta\alpha(H)) = \sqrt{a_{\theta\alpha}}$ for complex simple α and $\theta\alpha$
- $\exp(\frac{1}{2}\alpha(H')) = \sqrt{b_\alpha}$, $\exp(\frac{1}{2}\theta\alpha(H')) = \sqrt{b_{\theta\alpha}}$ for complex simple α and $\theta\alpha$

The last step in the proof is to show that the equation

$$\theta' \circ \operatorname{Ad}(\exp \frac{1}{2}(H - H')) = \operatorname{Ad}(\exp \frac{1}{2}(H - H')) \circ \theta$$

holds. On all of $\mathfrak{h}$ and on each X_α where α is an imaginary simple root, θ acts like θ'. On these subspaces the two sides are equal.

If α is complex simple, then

$$\begin{aligned}
\theta' \circ \operatorname{Ad}(\exp \tfrac{1}{2}(H - H'))Y_\alpha &= \theta'(e^{\frac{1}{2}\alpha(H-H')}Y_\alpha) \\
&= b_\alpha \tfrac{\sqrt{a_\alpha}}{\sqrt{b_\alpha}} Y_{\theta\alpha} \\
&= \tfrac{\sqrt{b_\alpha}}{a_\alpha} \theta Y_\alpha \\
&= \tfrac{\sqrt{a_{\theta\alpha}}}{\sqrt{b_{\theta\alpha}}} \theta Y_\alpha \\
&= \operatorname{Ad}(\exp \tfrac{1}{2}(H - H')) \circ \theta Y_\alpha
\end{aligned}$$

This proves the above equation.

For some $X \in \mathfrak{k}$ we have

$$\theta' \circ \mathrm{Ad}(\exp \frac{1}{2}(H - H'))X = \mathrm{Ad}(\exp \frac{1}{2}(H - H'))X$$

and hence X lies in $\mathfrak{k}'$. With $Y \in \mathfrak{p}$ the only thing that changes is an additional minus on the right side. Therefore the inclusions

$$\mathrm{Ad}(\exp \tfrac{1}{2}(H - H'))(\mathfrak{k}) \subseteq \mathfrak{k}'$$
$$\mathrm{Ad}(\exp \tfrac{1}{2}(H - H'))(\mathfrak{p}) \subseteq \mathfrak{p}'$$

hold. By dimensional argumentation we easily see that we get equality on these inclusions. Since the element $\mathrm{Ad}(\exp \frac{1}{2}(H - H'))$ carries $\mathfrak{u}_0$ to itself, it must carry $\mathfrak{k}_0 = \mathfrak{u}_0 \cap \mathfrak{k}$ to $\mathfrak{k}_0' = \mathfrak{u}_0 \cap \mathfrak{k}'$ and $\mathfrak{p}_0 = \mathfrak{u}_0 \cap \mathfrak{p}$ to $\mathfrak{p}_0' = \mathfrak{u}_0 \cap \mathfrak{p}'$. Hence it must carry $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ to $\mathfrak{g}_0' = \mathfrak{k}_0' \oplus \mathfrak{p}_0'$. □

Now, that we have proved uniqueness, we address the question of existence. We define an abstract Vogan diagram as an abstract Dynkin diagram together with an automorphism of the diagram, which indicates the 1- and 2-element orbits of vertices, and a subset of the 1-element orbits, which indicates the painted vertices. Clearly, every Vogan diagram is an abstract Vogan diagram.

2.50. THEOREM. *If an abstract Vogan diagram is given, then there exists a real semisimple Lie algebra $\mathfrak{g}_0$, a Cartan involution θ, a maximally compact θ stable Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ and a positive system Δ^+ of $\Delta(\mathfrak{g}, \mathfrak{h})$ that takes $\mathfrak{t}_0$ before $i\mathfrak{a}_0$ such that the given diagram is the Vogan diagram of $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$.*

PROOF. By the Existence Theorem 1.24 there is a complex semisimple Lie algebra $\mathfrak{g}$ which corresponds to the Dynkin diagram on which the abstract Vogan diagram is based. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be a root system with a positive system Δ^+. Let X_α be the root vectors of the corresponding split real form of $\mathfrak{g}$ and define a compact real form $\mathfrak{u}_0$ of $\mathfrak{g}$ in terms of $\mathfrak{h}$ and X_α by

$$\mathfrak{u}_0 = \sum_{\alpha\in\Delta} \mathbb{R}(iH_\alpha) + \sum_{\alpha\in\Delta} \mathbb{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha\in\Delta} \mathbb{R}i(X_\alpha + X_{-\alpha}).$$

The abstract Vogan diagram defines an automorphism θ of the Dynkin diagram, which extends linearly to $\mathfrak{h}^*$ and is isometric. We want to see that θ maps Δ onto itself. Let $\alpha \in \Delta^+$ be a positive root. We write $\alpha = \sum n_i\alpha_i$ as a sum of simple roots and call $\sum n_i$ the level of α. We show that $\theta(\Delta^+) \subseteq \Delta$ by induction on the level n of α. If the level is 1, then α is simple and thereof we know that $\theta\alpha$ is simple either. Now let $n > 1$. Assume that $\theta\alpha \in \Delta$ for all positive α with level $< n$ and let α be of level n. We choose a simple root α_j such that $\langle \alpha, \alpha_j \rangle > 0$. Then the reflection of α on the hyperplane defined by α_j,

$$s_{\alpha_j}(\alpha) = \alpha - \frac{2\langle \alpha, \alpha_j \rangle}{|\alpha_j|^2}\alpha_j = \beta$$

is a positive root of smaller level than α. By induction hypothesis $\theta\beta$ and $\theta\alpha_j$ are in Δ. Since θ is isometric $\theta\alpha = s_{\theta\alpha_j}(\theta\beta)$ and therefore $\theta\alpha$ is in Δ.

We can transfer θ from $\mathfrak{h}^*$ to $\mathfrak{h}$, retaining the same name. Define θ on the root vectors X_α for simple roots by

$$\theta X_\alpha = \begin{cases} X_\alpha & \text{if } \alpha \text{ is unpainted and forms a 1-element orbit} \\ -X_\alpha & \text{if } \alpha \text{ is painted and forms a 1-element orbit} \\ X_{\theta\alpha} & \text{if } \alpha \text{ forms a 2-element orbit.} \end{cases}$$

By the Isomorphism Theorem 1.23 θ extends to an automorphism of $\mathfrak{g}$. The uniqueness in 1.23 implies that $\theta^2 = 1$.

The main step is to prove that $\theta\mathfrak{u}_0 = \mathfrak{u}_0$. Let B be the Killing form of $\mathfrak{g}$. For $\alpha \in \Delta$ define the constant a_α by $\theta X_\alpha = a_\alpha X_{\theta\alpha}$. Then

$$\begin{aligned} a_\alpha a_{-\alpha} &= a_\alpha a_{-\alpha} B(X_{\theta\alpha}, X_{-\theta\alpha}) \\ &= B(a_\alpha X_{\theta\alpha}, a_{-\alpha} X_{-\theta\alpha}) \\ &= B(\theta X_\alpha, \theta X_{-\alpha}) \\ &= B(X_\alpha, X_{-\alpha}) \\ &= 1 \end{aligned} \quad .$$

So if we prove, that

$$a_\alpha = \pm 1 \text{ for all } \alpha \in \Delta^+$$

this also proves the result for all $\alpha \in \Delta$. Again we prove by induction on the level of α. For simple roots, which have level 1, this is true by definition. Let $\alpha \in \Delta^+$ be of level n and inductively assume that $a_\alpha = \pm 1$ holds for all α with level $< n$. Choose some positive roots β and γ such that $\alpha = \beta + \gamma$. Clearly β and γ are of smaller level than α. Remember that we have chosen X_α in the construction of the split real form such that

$$\begin{aligned} \theta X_\alpha &= N_{\beta,\gamma}^{-1} \theta[X_\beta, X_\gamma] \\ &= N_{\beta,\gamma}^{-1} [\theta X_\beta, \theta X_\gamma] \\ &= N_{\beta,\gamma}^{-1} a_\beta a_\gamma [X_{\theta\beta}, X_{\theta\gamma}] \\ &= N_{\beta,\gamma}^{-1} N_{\theta\beta,\theta\gamma} a_\beta a_\gamma X_{\theta\beta+\theta\gamma}. \end{aligned}$$

Therefore

$$a_\alpha = N_{\beta,\gamma}^{-1} N_{\theta\beta,\theta\gamma} a_\beta a_\gamma.$$

By induction hypothesis $a_\beta a_\gamma = \pm 1$ and Theorem 2.25 tells us that

$$N_{\beta,\gamma}^2 = \frac{1}{2} q(1+p)|\beta|^2 = \frac{1}{2} q(1+p)|\theta\beta|^2 = N_{\theta\beta,\theta\gamma}^2.$$

Hence $N_{\beta,\gamma}^{-1} N_{\theta\beta,\theta\gamma} = \pm 1$. This proves $a_\alpha = \pm 1$ and the induction is complete.

Let us see that

$$\theta(\mathbb{R}(X_\alpha - X_{-\alpha}) + \mathbb{R}i(X_\alpha + X_{-\alpha})) \subseteq \mathbb{R}(X_{\theta\alpha} - X_{-\theta\alpha}) + \mathbb{R}i(X_{\theta\alpha} + X_{-\theta\alpha}).$$

Like in the proof of Theorem 2.49 we calculate for some $z = x + iy$ with $x, y \in \mathbb{R}$

$$x(X_\alpha - X_{-\alpha}) + yi(X_\alpha + X_{-\alpha}) = zX_\alpha - \bar{z}X_{-\alpha}$$

Reverting argumentation of 2.49 the desired result is equivalent to the fact that

$$\theta(zX_\alpha - \bar{z}X_{-\alpha}) = za_\alpha X_{\theta\alpha} - \bar{z}a_{-\alpha}X_{-\theta\alpha}$$

is of the form $wX_{\theta\alpha} - \overline{w}X_{-\theta\alpha}$ where $z, w \in \mathbb{C}$. But this is clear since we know that $a_\alpha = \pm 1$ for all $\alpha \in \Delta$. Since θ carries roots to roots we have

$$\theta(\sum_{\alpha\in\Delta} \mathbb{R}(iH_\alpha)) = \sum_{\alpha\in\Delta} \mathbb{R}(iH_\alpha).$$

This shows that $\theta\mathfrak{u}_0 = \mathfrak{u}_0$.

Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and -1 eigenspaces of θ in $\mathfrak{g}$, so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We have

$$\mathfrak{u}_0 = (\mathfrak{u}_0 \cap \mathfrak{k}) \oplus (\mathfrak{u}_0 \cap \mathfrak{p}).$$

Define $\mathfrak{k}_0 = \mathfrak{u}_0 \cap \mathfrak{k}$ and $\mathfrak{p}_0 = i(\mathfrak{u}_0 \cap \mathfrak{p})$. Then

$$\mathfrak{u}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0.$$

Since $\mathfrak{u}_0$ is a vector space real form of $\mathfrak{g}$ so is

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0.$$

Since $\theta\mathfrak{u}_0 = \mathfrak{u}_0$ and θ is an involution we have the bracket relations

$$\begin{aligned} [\mathfrak{k}_0, \mathfrak{k}_0] &\subseteq \mathfrak{k}_0 \\ [\mathfrak{k}_0, \mathfrak{p}_0] &\subseteq \mathfrak{p}_0 \\ [\mathfrak{p}_0, \mathfrak{p}_0] &\subseteq \mathfrak{k}_0. \end{aligned}$$

Therefore $\mathfrak{g}_0$ is closed under brackets and is a Lie algebra real form of $\mathfrak{g}$. The involution

$$\theta(X) = \begin{cases} +X & \text{for } X \in \mathfrak{k}_0 \\ -X & \text{for } X \in \mathfrak{p}_0 \end{cases}$$

Hence θ is a Cartan involution of $\mathfrak{g}_0$.

θ maps $\mathfrak{u}_0 \cap \mathfrak{h}$ to itself. and therefore

$$\begin{aligned} \mathfrak{u}_0 \oplus \mathfrak{h} &= (\mathfrak{u}_0 \cap \mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{u}_0 \cap \mathfrak{p} \cap \mathfrak{h}) \\ &= (\mathfrak{k}_0 \cap \mathfrak{h}) \oplus (i\mathfrak{p}_0 \cap \mathfrak{h}) \\ &= (\mathfrak{k}_0 \cap \mathfrak{h}) \oplus i(\mathfrak{p}_0 \cap \mathfrak{h}). \end{aligned}$$

The abelian subspace $\mathfrak{u}_0 \cap \mathfrak{h}$ is a real form of $\mathfrak{h}$ and so is $\mathfrak{h}_0 = (\mathfrak{k}_0 \cap \mathfrak{h}) \oplus (\mathfrak{p}_0 \cap \mathfrak{h})$. The subspace $\mathfrak{h}_0$ is contained in $\mathfrak{g}_0$ and is therefore a θ stable Cartan subalgebra of $\mathfrak{g}_0$.

A root α that is real on all of $\mathfrak{h}_0$ has the property that $\theta\alpha = -\alpha$. Since $\theta(\Delta^+) = \Delta^+$, there are no such real roots. Hence $\mathfrak{h}_0$ is maximally compact.

Let us verify that Δ^+ results from a lexicographic ordering that takes $i(\mathfrak{k}_0 \cap \mathfrak{h})$ before $\mathfrak{p}_0 \cap \mathfrak{h}$. Define the following sets

$\{\beta_i\}_{i=1}^{l}$	the set of simple roots of Δ^+ in 1-element orbits
$\{\gamma_i, \theta\gamma_i\}_{i=1}^{m}$	the set of simple roots of Δ^+ in 2-element orbits
$\{\alpha_i\}_{i=1}^{l+2m}$	the set of all simple roots of Δ^+

in the following order

$$\begin{array}{llllllll}\{\alpha_1, & \dots, & \alpha_l, & \alpha_{l+1}, & \alpha_{l+2}, & \dots, & \alpha_{l+2m-1}, & \alpha_{l+2m}\} = \\ \{\beta_1, & \dots, & \beta_l, & \gamma_1, & \theta\gamma_1, & \dots, & \gamma_m, & \theta\gamma_m\}\end{array}$$

Relative to the basis $\{\alpha_i\}_{i=1}^{l+2m}$ define the dual basis $\{\omega_i\}_{i=1}^{l+2m}$ by $\langle \omega_i, \alpha_j\rangle = \delta_{ij}$. We shall write ω_{β_j} or ω_{γ_j} or $\omega_{\theta\gamma_j}$ in place of ω_i to see the origin of ω_i more easily. We define a lexicographic ordering by using inner products with the ordered basis

$$\omega_{\beta_1}, \dots, \omega_{\beta_l}, \omega_{\gamma_1} + \omega_{\theta\gamma_1}, \dots, \omega_{\gamma_m} + \omega_{\theta\gamma_m}, \omega_{\gamma_1} - \omega_{\theta\gamma_1}, \dots, \omega_{\gamma_m} - \omega_{\theta\gamma_m}$$

which takes $i(\mathfrak{k}_0 \cap \mathfrak{h})$ before $\mathfrak{p}_0 \cap \mathfrak{h}$. Let α be in Δ^+ and decompose

$$\alpha = \sum_{i=1}^{l} n_i\beta_i + \sum_{j=1}^{m} r_j\gamma_j + \sum_{j=1}^{m} s_j\theta\gamma_j.$$

Then

$$\langle \alpha, \omega_i\rangle = n_i \geq 0$$

and

$$\langle \alpha, \omega_{\gamma_j} + \omega_{\theta\gamma_j}\rangle = r_j + s_j \geq 0.$$

If all these inner products were 0, then all coefficients of α were 0. Thus α has positive inner product with the first member of our ordered basis for which the inner product is nonzero. The lexicographic ordering yields Δ^+ as a positive system. Consequently $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$ is a triple discribing a real Lie algebra with the Vogan diagram we started off. $\square$

In order to classify real semisimple Lie algebras we take a look at their complexifications.

2.51. Theorem. *Let $\mathfrak{g}_0$ be a simple Lie algebra over $\mathbb{R}$ and let $\mathfrak{g}$ be its complexification. The following two situations may occur:*

(1) *$\mathfrak{g}_0$ is complex, which means that $\mathfrak{g}_0 = \mathfrak{s}^{\mathbb{R}}$ for some comlpex $\mathfrak{s}$. Then $\mathfrak{g}$ is $\mathbb{C}$ isomorphic to $\mathfrak{s} \oplus \mathfrak{s}$.*
(2) *$\mathfrak{g}_0$ is not complex. Then $\mathfrak{g}$ is simple over $\mathbb{C}$.*

Proof. (1) Let J be multiplication by $\sqrt{-1}$ in $\mathfrak{g}_0$. We want to show that the $\mathbb{R}$ linear map $L : \mathfrak{g} \to \mathfrak{s} \oplus \mathfrak{s}$ given by

$$L(X + iY) = (X + JY, X - JY),$$

where X, Y in $\mathfrak{g}_0$, is an isomorphism. L respects brackets since

$$\begin{aligned} L([X+iY, X'+iY']) &= L(([X,X']-[Y,Y'])+i([Y,X']+[X,Y'])) \\ &= ([X,X']+[JY,JY']+[X,JY']+[JY,X'], \\ &\quad [X,X']+[JY,JY']-[JY,X']-[X,JY']) \\ &= [(X+JY, X-JY), (X'+JY', X'-JY')] \\ &= [L(X+iY), L(X'+iY')] \end{aligned}$$

and it is one-one. By dimensional argumentation L is an $\mathbb{R}$ isomorphism. Let $\bar{\mathfrak{s}}$ be the same real Lie algebra as $\mathfrak{g}_0$ but multiplication by $\sqrt{-1}$ is defined as multiplication by $-i$. To see that L is a $\mathbb{C}$ isomorphism of $\mathfrak{g}$ with $\mathfrak{s} \oplus \bar{\mathfrak{s}}$ we compute

$$\begin{aligned} L(i(X+iY)) &= L(-Y+iX) \\ &= (-Y+JX, -Y-JX) \\ &= (J(X+JY), -J(X-JY)). \end{aligned}$$

To complete the proof we show that $\bar{\mathfrak{s}}$ is $\mathbb{C}$ isomorphic to $\mathfrak{s}$. $\mathfrak{s}$ has a compact real form $\mathfrak{u}_0$. The conjugation τ of $\mathfrak{s}$ with respect to $\mathfrak{u}_0$ is $\mathbb{R}$ linear and respects bracket and we have to show, that $\tau : \mathfrak{s} \to \bar{\mathfrak{s}}$ is a $\mathbb{C}$ isomorphism. Let U and V be in $\mathfrak{u}_0$. Then

$$\begin{aligned} \tau(J(U+JV)) &= \tau(-V+JU) \\ &= -V-JU \\ &= -J(U-JV) \\ &= -J\tau(U+JV) \end{aligned}$$

shows this isomorphism.

(2) Let bar denote conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. If $\mathfrak{a}$ is a simple ideal in $\mathfrak{g}$ then so is $\bar{\mathfrak{a}}$. Hence $\mathfrak{a}\cap\bar{\mathfrak{a}}$ and $\mathfrak{a}+\bar{\mathfrak{a}}$ are ideals in $\mathfrak{g}$ invariant under conjugation and hence are complexifications of ideals in $\mathfrak{g}_0$. Thus they are 0 or all of $\mathfrak{g}$. Since $\mathfrak{a} \neq 0$ we have $\mathfrak{a}+\bar{\mathfrak{a}} = \mathfrak{g}$.

If $\mathfrak{a}\cap\bar{\mathfrak{a}} = 0$ then $\mathfrak{g} = \mathfrak{a}\oplus\bar{\mathfrak{a}}$. Let $\iota : \mathfrak{g}_0 \to \mathfrak{g}$ be the inclusion of $\mathfrak{g}_0$ in $\mathfrak{g}$ and $\pi : \mathfrak{g} \to \mathfrak{a}$ the projection of $\mathfrak{g}$ to $\mathfrak{a}$. $\varphi = \iota \circ \pi$ is an $\mathbb{R}$ homomorphism of Lie algebras. If $\ker\varphi$ is nonzero, it must be $\mathfrak{g}_0$ since $\mathfrak{g}_0$ is simple. In this case $\mathfrak{g}_0$ is contained in $\bar{\mathfrak{a}}$. Since conjugation fixes $\mathfrak{g}_0$ we get $\mathfrak{g}_0 \subseteq \mathfrak{a}\cap\bar{\mathfrak{a}} = 0$ which is a contradiction. So $\ker\varphi = 0$ and φ is one-one. Since $\mathfrak{a}$ aswell as $\mathfrak{g}_0$ are of dimension $\frac{1}{2}\dim_{\mathbb{R}}\mathfrak{g}$, φ is onto and hence an isomorphism. But this would mean that $\mathfrak{g}_0$ is complex which contradicts our initial assumption.

We conclude that $\mathfrak{a}\cap\bar{\mathfrak{a}} = \mathfrak{g}$ and hence $\mathfrak{a} = \mathfrak{g}$. Therefore $\mathfrak{g}$ is simple as asserted.

□

2.52. Proposition. *Let $\mathfrak{g}$ be a complex Lie algebra which is simple over $\mathbb{C}$. Then $\mathfrak{g}^{\mathbb{R}}$ is simple over $\mathbb{R}$.*

PROOF. Suppose that $\mathfrak{a}$ is an ideal in $\mathfrak{g}^{\mathbb{R}}$. By Cartan's Criterion for Semisimplicity $\mathfrak{g}^{\mathbb{R}}$ is semisimple. Hence $[\mathfrak{a}, \mathfrak{g}^{\mathbb{R}}] \subseteq \mathfrak{a} = [\mathfrak{a}, \mathfrak{a}] \subseteq [\mathfrak{a}, \mathfrak{g}^{\mathbb{R}}]$, so

$$\mathfrak{a} = [\mathfrak{a}, \mathfrak{g}^{\mathbb{R}}].$$

Let $X \in \mathfrak{a}$. We write

$$X = \sum_j [X_j, Y_j]$$

for some $X_j \in \mathfrak{a}$ and $Y_j \in \mathfrak{g}^{\mathbb{R}}$. Then

$$iX = \sum_j i[X_j, Y_j] = \sum_j [X_j, iY_j] \in [\mathfrak{a}, \mathfrak{g}^{\mathbb{R}}] = \mathfrak{a}.$$

So $\mathfrak{a}$ is a complex ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is complex simple $\mathfrak{a} = 0$ or $\mathfrak{a} = \mathfrak{g}$. Thus $\mathfrak{g}^{\mathbb{R}}$ is simple over $\mathbb{R}$. □

Theorem 2.51 tells us, that real simple Lie algebras are seperated in two categories. The first one consists of complex simple Lie algebras with restricted scalars. Proposition 2.52 tells us that every complex simple Lie algebra may be used for this purpose. This category is classified already in chapter 1. The second category consists of noncomplex simple Lie algebras. The Vogan diagram of such real Lie algebras is therefore based on a connected Dynkin diagram. Similar to the complex case we have the following

2.53. LEMMA. *A real noncomplex semisimple Lie algebra $\mathfrak{g}_0$ is simple if and only if its Vogan diagram is connected.*

PROOF. ($\Rightarrow$) If $\mathfrak{g}_0$ is simple, then $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$ is simple over $\mathbb{C}$, the Dynkin diagram of $\mathfrak{g}$ is connected and hence the Vogan diagram of $\mathfrak{g}_0$ is connected.

($\Leftarrow$) The Vogan diagram of $\mathfrak{g}_0$ is connected, hence the underlying Dynkin diagram of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$ is connected and $\mathfrak{g}$ is simple. Therefore $\mathfrak{g}_0$ has to be simple. □

To classify noncomplex real Lie algebras we need to classify abstract connected Vogan diagrams. The Borel and de Siebenthal Theorem, which we will prove soon to cut down the possible candidates for nonisomorphic real Lie algebras, uses the following two Lemmas in its proof.

2.54. LEMMA. *Let Δ be an irreducible abstract reduced root system in a vector space V. Let Π be simple system and let ω and ω' be nonzero members of V that are dominant relative to Π (i.e.: $\langle\omega, \alpha\rangle \geq 0$ respectively $\langle\omega', \alpha\rangle \geq 0$ for all $\alpha \in \Pi$). Then $\langle\omega, \omega'\rangle > 0$.*

PROOF. The first step is to show that in the expansion

$$\omega = \sum_{\alpha \in \Pi} a_\alpha \alpha$$

all $a_\alpha \geq 0$. We assume some negative coefficients and enumerate $\Pi = \{\alpha_1, \dots, \alpha_l\}$ such that

$$\omega = \sum_{i=1}^{r} a_i\alpha_i - \sum_{i=r+1}^{l} b_i\alpha_i = \omega^+ - \omega^-$$

where $a_i \geq 0$ and $b_i > 0$. We shall show that $\omega^- = 0$. Since $\omega^- = \omega^+ - \omega$ we have

$$\begin{aligned} 0 \leq |\omega^-|^2 &= \langle \omega^+, \omega^- \rangle - \langle \omega^-, \omega \rangle \\ &= \textstyle\sum_{i=1}^{r} \sum_{j=r+1}^{l} a_i b_j \langle \alpha_i, \alpha_j \rangle - \sum_{j=r+1}^{l} b_j \langle \omega, \alpha_j \rangle. \end{aligned}$$

The first term in the last row is ≤ 0 since $\langle \alpha_i, \alpha_j \rangle \leq 0$ for destinct simple roots. The second term is negative by our assumption. Thus $0 \leq |\omega^-|^2 \leq 0$ and $\omega^- = 0$.

Now we write $\omega = \sum_{j=1}^{l} a_j\alpha_j$ with all $a_j \geq 0$. We want to show $a_j > 0$ for all j using the irreducibility of Δ. Assume $a_i = 0$. Then

$$0 \leq \langle \omega, \alpha_i \rangle = \sum_{j \neq i} a_j \langle \alpha_j, \alpha_i \rangle$$

with every term on the right hand side ≤ 0 by the same argument as above. Thus $a_j = 0$ for every α_j with $\langle \alpha_j, \alpha_i \rangle < 0$. All neighbours of α_i in the Dynkin diagram satisfy this condition. The Dynkin diagram is connected by irreducibility of Δ. Iteration of this argument shows, that all coefficients are 0 once one of them is 0.

Since $\omega \neq 0$ there is at least one α_i such that $\langle \alpha_i, \omega \rangle > 0$. Then

$$\langle \omega, \omega' \rangle = \sum_{j=1}^{l} a_j \langle \alpha_j, \omega' \rangle \geq a_i \langle \alpha_i, \omega' \rangle > 0$$

since $a_i > 0$. $\square$

2.55. Lemma. *Let $\mathfrak{g}_0$ be a noncomplex simple real Lie algebra and let the $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$ be a triple defining a Vogan diagram of $\mathfrak{g}_0$. Decompose $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ as usual. Let V be the span of the simple roots that are imaginary. Let $\Delta_0 = \Delta \cap V$ the set of roots built out of imaginary simple ones. Let $\mathcal{H}$ be the subset of $i\mathfrak{t}_0$ paired with V and let Λ be the subset of $\mathcal{H}$ where all roots of Δ_0 take integer values and all noncompact roots of Δ_0 take odd integer values. Then Λ in nonempty. Furthermore we can describe such elements explicitly. Let $\{\alpha_1, \dots, \alpha_m\}$ be a simple system of Δ_0 and $\{\omega_1, \dots, \omega_m\} \subset V$ defined such that $\langle \omega_j, \alpha_k \rangle = \delta_{jk}$. Let I be the set of indices of all noncompact α_i. Then the element*

$$\omega = \sum_{i \in I} \omega_i$$

is in Λ.

PROOF. Let $\{\alpha_1, \ldots, \alpha_m\}$ be a simple system of Δ_0 with corresponding positive roots Δ_0^+. Define $\omega_1, \ldots, \omega_m$ by $\langle \omega_j, \alpha_k \rangle = \delta_{jk}$. If $\alpha = \sum_{i=1}^m n_i \alpha_i$ is a positive root of Δ_0, then $\langle \omega, \alpha \rangle$ is the sum of all n_i where α_i is noncompact, which is an integer.

We shall prove ny induction on the level $\sum_{i=1}^m$ of α that $\langle \omega, \alpha \rangle$ is even if α is compact, respectively odd if α is noncompact. If α has level 1 this is true by definition. Now let $\alpha, \beta \in \Delta_0^+$ with $\alpha + \beta \in \Delta$ and suppose our assertion is true for α and β. Since the sum of n_i for which α_i is noncompact is additive, we have to prove that imaginary root satisfy

compact + compact = compact
compact +noncompact=noncompact
noncompact+noncompact= compact.

This follows from the facts, that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ and the relations of the Cartan decomposition

$$[\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0,\ [\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0,\ [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0.$$

□

2.56. THEOREM (Borel and de Siebenthal Theorem). *Let $\mathfrak{g}_0$ be a noncomplex simple real Lie algebra and let the Vogan diagram of $\mathfrak{g}_0$ be given that corresponds to the triple $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$. Then there exists a simple system Π' for $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, with corresponding positive system $\Delta^{+'}$, such that $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^{+'})$ is a triple and there is at most one painted simple root in the Vogan diagram.*

Furthermore suppose that the automorphism associated with the Vogan diagram is the identity, that $\Pi' = \{\alpha_1, \ldots, \alpha_l\}$ and that $\{\omega_1, \ldots, \omega_l\}$ is the dual basis given by $\langle \omega_j, \alpha_k \rangle = \delta_{jk}$. Then the single painted simple root α_i may be chosen so that there is no i' with $\langle \omega_i - \omega_{i'}, \omega_{i'} \rangle > 0$.

PROOF. Define V, Δ_0 and Λ as in 2.55. To use 2.54 we need the Dynkin diagram of Δ_0 to be connected. This is equivalent to the statement, that the subset of roots, which are fixed by the given automorphism, is a connected set. If the automorphism is the identity this is evident. We consider this case first.

Let $\Delta_0^+ = \Delta^+ \cap V$. Λ is a subset of a lattice, hence discrete and nonempty by Lemma 2.55. Let $H_0 \in \Lambda$ be of minimal norm. Then we can choose a new positive system $\Delta_0^{+'}$ of Δ_0, such that H_0 is dominant. We will show, that at most one simple root of $\Delta_0^{+'}$ is painted.

Suppose $H_0 = 0$. If α is in Δ_0, then $\langle H_0, \alpha \rangle = 0$, which is not an odd integer. Hence by definition of Λ, α is compact. Thus all roots of Δ_0 are compact and unpainted.

Now suppose $H_0 \neq 0$. Let $\alpha_1, \ldots, \alpha_m$ be the simple roots of Δ_0 with respect to $\Delta_0^{+'}$. Define $\omega_1, \ldots, \omega_m$ such that $\langle \omega_j, \alpha_k \rangle = \delta_{jk}$. We write

$$H_0 = \sum_{j=1}^{m} n_j \omega_j \text{ with } n_j = \langle H_0, \alpha_j \rangle.$$

Each number n_j is an integer since H_0 is in Λ and $n_j \geq 0$ by dominance of H_0. Since $H_0 > 0$, $n_i > 0$ for some i. Then $H_0 - \omega_i$ is dominant relative to $\Delta_0^{+'}$ and Lemma 2.55 shows that $\langle H_0 - \omega_i, \omega_i \rangle \geq 0$ with equality only if $H_0 = \omega_i$. If strict inequality would hold, we could calculate for the element $H_0 - 2\omega_i \in \Lambda$

$$\begin{aligned} |H_0 - 2\omega_i|^2 &= \langle H_0, H_0 \rangle - 2\langle 2\omega_i, H_0 \rangle + \langle 2\omega_i, 2\omega_i \rangle \\ &= \langle H_0, H_0 \rangle - 4\langle H_0 - \omega_i, \omega_i \rangle \\ &< |H_0|^2 \end{aligned}$$

which contradicts our assumption that H_0 is of minimal norm. Hence equality holds and $H_0 = \omega_i$. Since $H_0 \in \Lambda$, a simple root $\alpha_j \in \Delta_0^{+'}$ is noncompact only if $\langle H_0, \alpha_j \rangle$ is an odd integer. Since $\langle H_0, \alpha_j \rangle = 0$ for $j \neq 0$, the only possible noncompact simple root and furthermore the only painted one in $\Delta_0^{+'}$ is α_i. This proves the main part of the theorem still restricted to the case of an identical automorphism of the Vogan diagram. Now we turn to the additional statement keeping this restriction.

Assume $H_0 = \omega_i$. Then an inequality $\langle \omega_i - \omega_{i'}, \omega_{i'} \rangle > 0$ would imply

$$|H_0 - 2\omega_{i'}|^2 = |H_0|^2 - 4\langle \omega_i - \omega_{i'}, \omega_{i'} \rangle < |H_0|^2$$

again in contradiction with assuming H_0 to be of minimal norm.

Our proof is not complete because of restricting the automorphism associated to the Vogan diagram to the identity. Lets see what happens if this is not the case. Choose an element s of the Weyl group W of Δ_0 such that $\Delta_0^{+'} = s\Delta_0^+$ and define the positive system $\Delta^{+'} = s\Delta^+$. Δ^+ is defined by an ordering that takes values on $i\mathfrak{t}_0$ before $\mathfrak{a}_0$. The same is true for $\Delta^{+'}$ since the element s maps $i\mathfrak{t}_0$ to itself, with $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ as usual. Let $\{\beta_1, \ldots, \beta_l\}$ be the set of simple roots of Δ^+ with the subset $\{\beta_1, \ldots, \beta_m\} \subseteq \Delta_0$. Then $\{s\beta_1, \ldots, s\beta_l\}$ is the set of simple roots of $\Delta^{+'}$ and among these $\{s\beta_1, \ldots, s\beta_m\}$ is the set of simple roots of $\Delta_0^{+'}$ considered in the first part of the proof referred to as $\{\alpha_1, \ldots, \alpha_m\}$. Out of these there is at most one root noncompact. The roots $s\beta_{m+1}, \ldots, s\beta_l$ are complex since $\beta_{m+1}, \ldots, \beta_l$ are complex and s carries complex roots to complex roots. Thus $\Delta^{+'}$ has at most one simple root that is noncompact imaginary. $\square$

2.57. Theorem (Classification). *The following list is up to isomorphism a complete list of simple real Lie algebras:*

(1) *the Lie algebra $\mathfrak{g}^{\mathbb{R}}$, where $\mathfrak{g}$ is a complex simple Lie algebra*
(2) *the compact real form of any complex simple Lie algebra*

(3) *the classical matrix algebras out of the following list*

$\mathfrak{su}(p,q)$	*with* $p \geq q > 0, p+q \geq 2$
$\mathfrak{so}(p,q)$	*with* $p > q > 0, p+q \geq 3, p+q$ *odd*
$\mathfrak{sp}(p,q)$	*with* $p \geq q > 0, p+q \geq 3$
$\mathfrak{sp}(n,\mathbb{R})$	*with* $n \geq 3$
$\mathfrak{sp}(p,q)$	*with* $p \geq q > 0, p+q \geq 8, p+q$ *even*
$\mathfrak{so}^*(2n)$	*with* $n \geq 4$
$\mathfrak{sl}(n,\mathbb{R})$	*with* $n \geq 3$
$\mathfrak{sl}(n,\mathbb{H})$	*with* $n \geq 2$

(4) *the* 12 *exceptional noncomplex noncompact simple Lie algebras listed in the following discussion.*

The only isomorphism among Lie algebras in the above list is $\mathfrak{so}^*(8) = \mathfrak{so}(6,2)$.

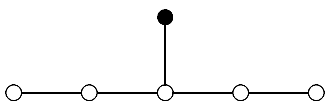
This isomorphism is obvious, since this is just a reflection of the Vogan diagram based on the Dynkin diagram D_4. The restrictions of the rank are made to prevent isomorphic algebras.

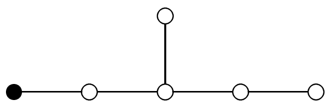
The first item is obvious. The items two to four need some investigations. We will start with a complete list of possible Vogan diagrams and deal with their realizations later. We split the observed Lie algebras in the cases where the automorphism of the corresponding Vogan diagram is trivial or not. We will deal with the case of trivial automorphisms first.

When no simple root is painted, then $\mathfrak{g}_0$ is a compact real form. To look at a list of all possible Vogan diagrams of this form, just look at the list of possible Dynkin diagrams at the end of chapter 1.

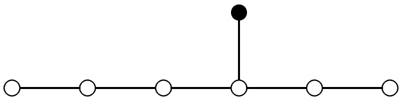
The Borel and de Siebenthal Theorem 2.56 restricts the possible cases to zero or one painted root. So we only have to deal with the case of one painted root. We divide matters in classical and exceptional Dynkin diagrams, since there is an additional statement on the possible placements of the painted root in the exceptional case in Theorem 2.56.

In the case of classical Dynkin diagrams we obtain Vogan diagrams based on A_n, B_n, C_n and D_n with one vertex painted. In the case of exceptional Dynkin diagrams, the following table lists all possibilities of Vogan diagrams with one vertex painted. These 10 Vogan diagrams belong to the type mentioned in 2.57 (4).

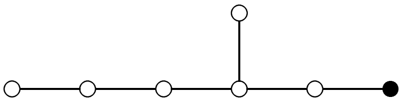
E II 

E III 

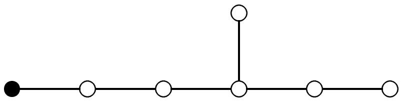
$E\ V$



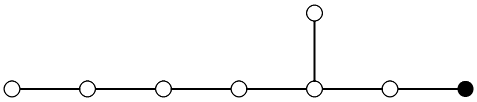
$E\ VI$



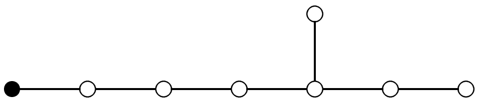
$E\ VII$



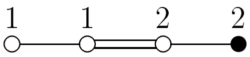
$E\ VIII$



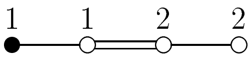
$E\ IX$



$F\ I$



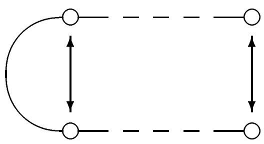
$F\ II$



G

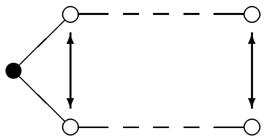
This completes the the list of Lie algebras, whose Vogan diagrams are equipped with a trivial automorphism. We now look at the diagrams with nontrivial automorphisms. A nontrivial automorphism can only be found on the Dynkin diagrams A_n, D_n and E_6.

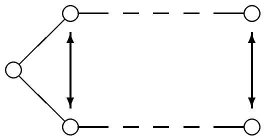
$\mathbf{A_n}$: The nature of the automorphisms of A_n differ, whether n is odd or even. There is one Vogan diagram for each A_n where n is even.



Since there is no simple root in a 1-element orbit, no root can be painted.

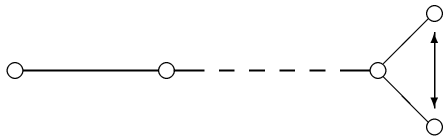
For n odd there are two possible diagrams of A_n listed below.





The only possibly painted root is the one in the 1-element orbit.

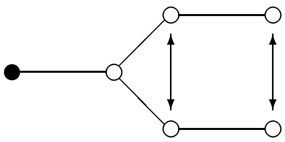
$\mathbf{D_n}$: D_n does not separate in an odd and even case, since there is only an automorphism of the following type



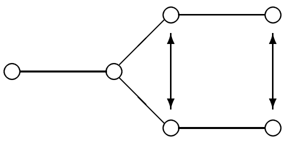
Each of the roots (only one at a time) may be painted except for the two rightmost roots in the 2-element orbit.

$\mathbf{E_6}$: The only exceptional Dynkin diagram that has a nontrivial automorphism is E_6. The following two Vogan diagrams complete the listing of the 12 exceptional noncomplex noncompact simple Lie algebras mentioned in 2.57 (4).

E I



E IV

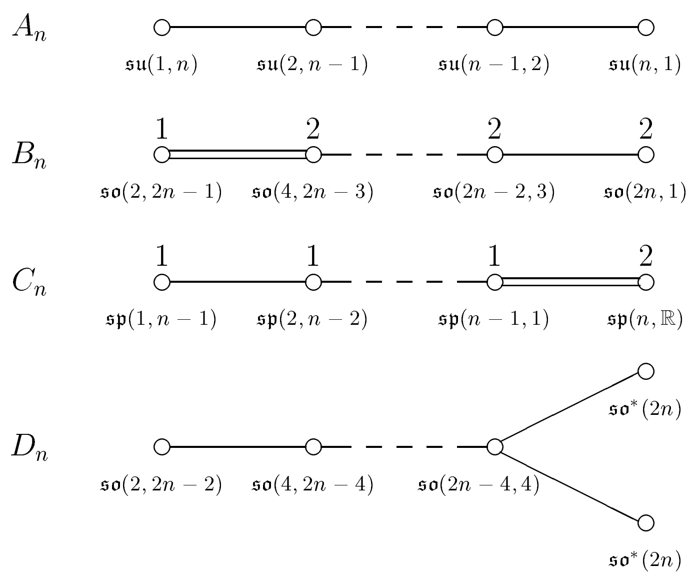


This ends our discussion of the classification of possible Vogan diagrams surviving the redundancy test of the Borel and de Siebenthal Theorem.

We will now deal with the realizations of these Vogan diagrams, verifying some of them. The following table lists all compact real forms of all Dynkin diagrams. We give a matrix realization if the underlying Dynkin diagram is classical, in the case of exceptional Dynkin diagram we give it a name.

Diagram	Compact Real Form
A_n	$\mathfrak{su}(n+1)$
B_n	$\mathfrak{so}(2n+1)$
C_n	$\mathfrak{sp}(n)$
D_n	$\mathfrak{so}(2n)$
E_6	$\mathfrak{e}_6$
E_7	$\mathfrak{e}_7$
E_8	$\mathfrak{e}_8$
F_4	$\mathfrak{f}_4$
G_2	$\mathfrak{g}_2$

The next table visualizes the result in the case of classical Dynkin diagrams equipped with the trivial automorphism and exactly one root painted. The Lie algebras written beneath a vertex means, that the Vogan diagram with only that vertex painted corresponds to this Lie algebra.



Following our example $\mathfrak{sl}(n,\mathbb{C})$ of complex semisimple Lie algebras we will now realize some of its nonisomorphic real forms as matrix algebras, which means to realize Vogan diagrams based on the Dynkin diagram of A_n. The Cartan involution during this example is given by $\theta(X) = -X^*$.

Recall the definition of the elements H, E_{ij} in our discussion of $\mathfrak{sl}(n,\mathbb{C})$. Taking these elements as a basis of a real Lie algebra we get $\mathfrak{sl}(n,\mathbb{R})$ which is a split form of $\mathfrak{sl}(n,\mathbb{C})$, since

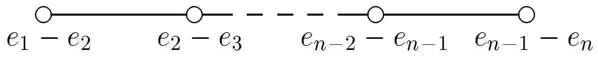
$$\mathfrak{h}_0 = \{H \in \mathfrak{h} | \alpha(H) \in \mathbb{R} \text{ for all } \alpha \in \Delta\}$$

is a Cartan subalgebra.

We proved, that there exists a compact real form for every complex semisimple Lie algebra and gave an explicit formula how to obtain a compact real form using a split real form. This compact real form is

$$\mathfrak{u}_0 = \sum_{i,j \leq n} \mathbb{R}(iH_{e_i - e_j}) + \sum_{i,j \leq n} \mathbb{R}(E_{i,j} - E_{-i,-j}) + \sum_{i,j \leq n} \mathbb{R}i(E_{i,j} + E_{-i,-j})$$

which is just $\mathfrak{su}(n) = \{X \in \mathfrak{sl}(n, \mathbb{C}) | X = -X^*\}$. This real Lie algebra corresponds to the Vogan diagram with trivial automorphism and no vertex painted.



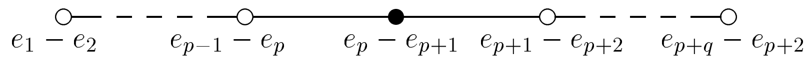
Now we realize the Vogan diagrams with trivial automorphism but one vertex painted. Let

$$\mathfrak{su}(p, q) = \{X \in \mathfrak{sl}(p + q, \mathbb{C}) | X = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, \, A = -A^*, \, D = -D^*\}$$

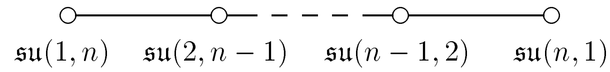
where A is a p-by-p matrix and D is a q-by-q matrix. With θ as mentioned the Cartan decomposition looks like

$$\mathfrak{k}_0 = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ and } \mathfrak{p}_0 = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}.$$

The subalgebra consisting of all diagonal elements is a Cartan subalgebra, which lies entirely in $\mathfrak{k}_0$ and hence is maximally compact. Using the same ordering as above, we get the same set of simple roots. Clearly the only simple root, whose root space is not in $\mathfrak{k}_0$ is $e_p - e_{p+1}$. This is the only noncompact root and hence painted.



Collecting the information of all Vogan diagrams based on the Dynkin diagram of A_n with trivial automorphism and one painted root we get the following diagram:



The Lie algebra written beneath a vertex means, that the Vogan diagram with only that vertex painted corresponds to this algebra. To see an example of a Vogan diagram based on A_n which does not survive the redundancy test of the Borel and de Siebenthal Theorem 2.56 we will realize the following Vogan diagram:

Let $\mathfrak{g}_0 = \mathfrak{su}(3,3)$ a special case of the discussion of $\mathfrak{su}(p,q)$ above. But we change the ordering of the linear functionals to fit this condition.

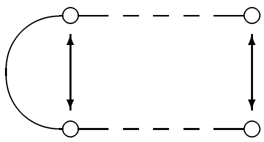
$$e_1 \geq e_4 \geq e_5 \geq e_2 \geq e_3 \geq e_6$$

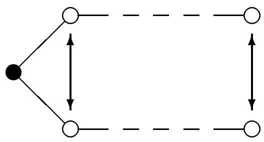
Then the simple roots are

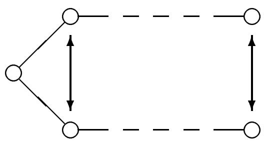
$$e_1 - e_4, e_4 - e_5, e_5 - e_2, e_2 - e_3, e_3 - e_6$$

Since the Cartan decomposition is the same as above all simple roots are imaginary but all roots $e_i - e_j$ with $i \in \{1,2,3\}$ and $j \in \{4,5,6\}$ or $i \in \{4,5,6\}$ and $j \in \{1,2,3\}$ are noncompact, hence painted.

The remaining real Lie algebras, with Vogan diagrams based on A_n and listed in the classification are the following.







The first one of these three is the last unrealized Vogan diagram based on A_n, where n is even. We did not find a diagram representing $\mathfrak{sl}(n+1,\mathbb{R})$ so far, but $\mathfrak{sl}(n+1,\mathbb{R})^{\mathbb{C}} = \mathfrak{sl}(n+1,\mathbb{C})$ determines the underlying Dynkin diagram to be of type A_n and hence this has to be the one.

The remaining two diagrams are both based on A_n with n odd, hence the above argument will not wotk. We try to verify that the first one of those two represents $\mathfrak{g}_0 = \mathfrak{sl}(2n,\mathbb{R})$. The Cartan involution shall be negative transpose and define a Cartan subalgebra

$$\mathfrak{h}_0 = \left\{ \begin{pmatrix} x_1 & \gamma_1 & & & \\ -\gamma_1 & x_1 & & & \\ & & \ddots & & \\ & & & x_n & \gamma_n \\ & & & \gamma_n & x_n \end{pmatrix} \right\}$$

built out of block diagonal matrices with $\mathrm{Tr}(H) = 0$ for all $H \in \mathfrak{h}_0$. The subspace $\mathfrak{t}_0$ is the set of all $H \in \mathfrak{h}_0$ with $x_j = 0$ for all $j \in \{1, \dots, n\}$ and the subspace $\mathfrak{a}_0$ corresponds to the set of all $H \in \mathfrak{h}_0$ with $\gamma_j = 0$ for all $j \in \{1, \dots, n\}$. We define linear functionals e_j and f_j depending on nothing else but the j'th block of such a matrix by

$$e_j \begin{pmatrix} x_j & y_j \\ -y_j & x_j \end{pmatrix} = iy_j \text{ and } f_j \begin{pmatrix} x_j & y_j \\ -y_j & x_j \end{pmatrix} = x_j.$$

The root system is

$$\Delta = \{\pm e_j \pm e_k \pm (f_j - f_k) | j \neq k\} \cup \{\pm 2e_l | 1 \leq l \leq n\}.$$

Roots built out of e_j's are purely imaginary while roots built out of f_j's are real. Others are complex. We see that there are no real roots and therefore $\mathfrak{h}_0$ is maximally compact. The involution θ acts by $+1$ on all e_j and by -1 on all f_j. We define a lexicographic ordering by using the spanning set

$$e_1, \dots, e_n, f_1, \dots, f_n$$

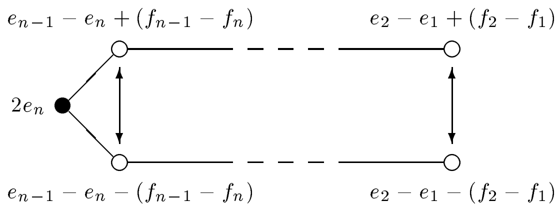
and obtain a positive system

$$\Delta^+ = \begin{cases} e_j + e_k \pm (f_j - f_k) & \text{for all } j \neq k \\ e_j - e_k \pm (f_j - f_k) & \text{for all } j < k \\ 2e_j & \text{for } 1 \leq j \leq n \end{cases}$$

and a simple system

$$\Pi = \begin{cases} e_{j-1} + e_j + (f_{j-1} - f_j) & \text{for all } 1 \leq j \leq n \\ e_{j-1} - e_j + (f_{j-1} - f_j) & \text{for all } 1 \leq j \leq n \\ 2e_n & \end{cases}$$

The resulting Vogan diagram is the following:

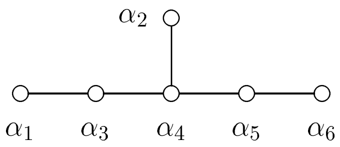


And it comes from $\mathfrak{sl}(2n+1,\mathbb{R})$. The remaining Vogan diagram comes from $\mathfrak{sl}(\frac{1}{2}(n+1,\mathbb{H}))$.

We have already seen a list of all Vogan diagrams based on exceptional Dynkin diagrams equipped with the trivial automorphism. We will now verify the case of E_6. The main part of the Borel and de Siebenthal Theorem 2.56 says we only have to paint one vertex. But the supplement also restricts the possible painted vertices. Remember that we associated to a simple system $\Pi = \{\alpha_1, \ldots, \alpha_6\}$ the dual basis $\{\omega_1, \ldots, \omega_6\}$ defined by

$$\langle \omega_i, \alpha_j \rangle = \delta_{ij}.$$

Let the simple roots be organized in this way



Then the dual basis has the following form:

$$\begin{aligned}
\omega_1 &= \tfrac{1}{3}(4\alpha_1+3\alpha_2+\ 5\alpha_3+\ 6\alpha_4+\ 4\alpha_5+2\alpha_6)\\
\omega_2 &= \quad 1\alpha_1+2\alpha_2+\ 2\alpha_3+\ 3\alpha_4+\ 2\alpha_5+1\alpha_6\\
\omega_3 &= \tfrac{1}{3}(5\alpha_1+6\alpha_2+10\alpha_3+12\alpha_4+\ 8\alpha_5+4\alpha_6)\\
\omega_4 &= \quad 2\alpha_1+3\alpha_2+\ 4\alpha_3+\ 6\alpha_4+\ 4\alpha_5+2\alpha_6\\
\omega_5 &= \tfrac{1}{3}(4\alpha_1+6\alpha_2+\ 8\alpha_3+12\alpha_4+10\alpha_5+5\alpha_6)\\
\omega_6 &= \tfrac{1}{3}(2\alpha_1+3\alpha_2+\ 4\alpha_3+\ 6\alpha_4+\ 5\alpha_5+4\alpha_6)
\end{aligned}$$

Now we use the supplementary condition of 2.56 to rule out α_3, α_4 and α_5 from being painted. For $i = 3$ we take $i' = 1$ to see that

$$\frac{5}{3} = \langle \omega_3, \omega_1 \rangle > \langle \omega_1, \omega_1 \rangle = \frac{4}{3}$$

so that

$$\langle \omega_3 - \omega_1, \omega_1 \rangle > 0.$$

Similarly we take $i' = 1$ for $i = 4$ to see that

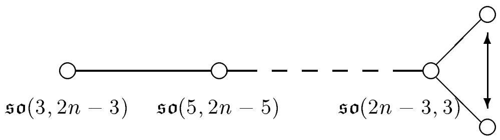
$$\langle \omega_4 - \omega_1, \omega_1 \rangle = 2 - \frac{4}{3} > 0$$

and we take $i' = 6$ for $i = 5$ to see that

$$\langle \omega_5 - \omega_6, \omega_6 \rangle = \frac{5}{3} - \frac{4}{3} > 0.$$

Therefore we only have to consider the three Vogan diagrams with one of α_1, α_2 or α_6 painted. Clearly the diagram with α_6 painted is isomorphic to the one with α_1 painted. The two diagrams left are *E II* and *E III* mentioned in our list.

The realizations of the Vogan diagrams based on D_n with nontrivial automorphism are listed in the following table, using the same notation as above.


$\mathfrak{so}(3, 2n-3)$
$\mathfrak{so}(5, 2n-5)$
$\mathfrak{so}(2n-3, 3)$

Now that we have seen how to classify real semisimple Lie algebras using Vogan diagrams we will take a look at an alternative way. This will lead to the notion of Satake diagrams.

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$ with complexification $\mathfrak{g}^{\mathbb{C}}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, then $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ is a compact real form of $\mathfrak{g}^{\mathbb{C}}$.

In contrast to the classification by Vogan diagrams we choose a maximally noncompact Cartan subalgebra. Let $\mathfrak{a}$ be maximal abelian in $\mathfrak{p}$ and let $\mathfrak{h}$ be a θ stable Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ with $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$ and $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$. Let $\mathfrak{h}^{\mathbb{C}}$ be the complexification of $\mathfrak{h}$ which is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ and let

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

be the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$, the root system denoted by Δ. Let furthermore

$$\mathfrak{h}_0 = \{H \in \mathfrak{h}^{\mathbb{C}} | \alpha(H) \in \mathbb{R} \quad \text{for all } \alpha \in \Delta\}.$$

B denoting the Killing form, $\mathfrak{h}_0 = i\mathfrak{t} \oplus \mathfrak{a}$ becomes a euclidean space by identifying α with H_α uniquely defined by $\alpha(H) = B(H_\alpha, H)$.

Let σ and τ be the conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to the real forms $\mathfrak{g}$ and $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$. $\mathfrak{h}_0$ is invariant under both, σ and τ since

$$\sigma|_{i\mathfrak{t}} = -1 \ , \ \sigma|_{\mathfrak{a}} = 1 \text{ and } \tau|_{\mathfrak{h}_0} = -1.$$

We denote $\sigma(\alpha) = \overline{\alpha}$ for all $\alpha \in \Delta$ and let

$$\Delta_0 = \{\alpha \in \Delta | \overline{\alpha} = -\alpha\}$$

which is the set of roots in Δ, that vanish on $\mathfrak{a}$. Denote the rank of Δ and Δ_0 by l and l_0.

We now address the problem of an ordering of Δ. We search for an ordering satisfying

$$\alpha > 0 \Rightarrow \overline{\alpha} > 0 \quad \text{for all } \alpha \notin \Delta_0$$

An example of such an ordering is a lexicographic ordering taking $\mathfrak{a}$ before $i\mathfrak{t}$, in contrast to the ordering we used for Vogan diagrams. Let

$$\Pi = \{\alpha_1, \ldots, \alpha_{l-l_0}, \alpha_{l-l_0+1}, \ldots, \alpha_l\}$$

be a simple system respecting this ordering. Then

$$\Pi_0 = \Pi \cap \Delta_0 = \{\alpha_{l-l_0+1}, \ldots, \alpha_l\}$$

is a simple system of Δ_0.

2.58. Lemma. *There exists a permutation* $'$: $\{1, \ldots, l - l_0\} \to \{1, \ldots, l - l_0\}$ *of order* 2 *such that*

$$\overline{\alpha_i} = \alpha_{i'} + \sum_{j=l-l_0+1}^{l} c_j^{(i)} \alpha_j \quad \textit{with } c_j^{(i)} \geq 0 \ , \textit{for } 1 \leq i \leq l - l_0.$$

PROOF. Since Π is a simple system we can write

$$\overline{\alpha_i} = \sum_{j=1}^{l} c_j^{(i)} \alpha_j \quad \text{for all } 1 \leq i \leq l - l_0.$$

Then the $c_j^{(i)}$ are nonnegative integers and there is at least one index $1 \leq i' \leq l - l_0$ with $c_{i'}^{(i)} > 0$. Applying σ to this equation we get

$$\alpha_i = \sum_{j=1}^{l-l_0} c_j^{(i)} \overline{\alpha_j} - \sum_{j=l-l_0+1}^{l} c_j^{(i)} \alpha_j = \sum_{j=1}^{l-l_0} \sum_{k=1}^{l} c_j^{(i)} c_k^{(j)} \alpha_k - \sum_{j=l-l_0+1}^{l} c_j^{(i)} \alpha_j.$$

Hence we have $c_{i'}^{(i)} = 1$ and $c_j^{(i)} = 0$ for all $1 \leq j \leq l - l_0$, $j \neq i'$. This proves the stated equation and applying σ to this equation we get

$$\overline{\alpha_{i'}} = \alpha_i + \sum_{j=l-l_0+1}^{l} c_j^{(i)} \alpha_j.$$

Therefore $(i')' = i$ which shows that $' : \{1, \ldots, l - l_0\} \to \{1, \ldots, l - l_0\}$ is a permutation of order 2. □

Because of the fact, that $'$ is a permutation of order 2 we are able to reorder the set $\{1, \ldots, l - l_0\}$ such that

$$i' = \begin{cases} i & \text{for } 1 \leq i \leq p_1 \\ i + p_2 & \text{for } p_1 + 1 \leq i \leq p_1 + p_2 \\ i - p_2 & \text{for } p_1 + p_2 + 1 \leq i \leq p_1 + 2p_2, \end{cases}$$

for some p_1, p_2 satisfying $l - l_0 = p_1 + 2p_2$. Let $p = p_1 + p_2$ and let $\gamma_i = \operatorname{proj}_{\mathfrak{a}} \alpha_i$ for $1 \leq i \leq p$ be the projection of α_i to $\mathfrak{a}$.

2.59. PROPOSITION. $\Sigma = \{\gamma_1, \ldots, \gamma_p\}$ *becomes a simple system of some root system of* $\mathfrak{a}$. *These are the restricted roots. Hence* $\dim \mathfrak{a} = p$.

Note that the root system spanned by Σ is contained in $\operatorname{proj}_{\mathfrak{a}}(\Delta - \Delta_0)$ but in general the latter is not a root system. The proof is preceeded by two lemmas.

Let $W(\Delta)$ and $W(\Delta_0)$ denote the Weyl groups of Δ and Δ_0. Let W_σ be the subgroup of $W(\Delta)$ defined by

$$W_\sigma = \{s \in W(\Delta) | s\sigma = \sigma s\}.$$

The condition to commute with σ is equivalent to $s(\mathfrak{a}) = \mathfrak{a}$.

2.60. LEMMA. *Let* $s \in W_\sigma$. $s \in W(\Delta_0)$ *if and only if* $s|_{\mathfrak{a}} = 1$, *which means that the hyperplane fixed by* s *containes* $\mathfrak{a}$. *Another sufficient condition is that* $\{\gamma_1, \ldots, \gamma_p\} = \{s\gamma_1, \ldots, s\gamma_p\}$.

PROOF. An element of $W(\Delta_0)$ necessarily fixes $\mathfrak{a}$. The second condition in the lemma is stronger, since it just says that a basis is preserved under s. So let $\{\gamma_1, \ldots, \gamma_p\} = \{s\gamma_1, \ldots, s\gamma_p\}$. If $\{\alpha_1, \ldots, \alpha_l\}$ is a simple system of Δ so is $\{s\alpha_1, \ldots, s\alpha_l\}$ and $\{s\alpha_{l-l_0}, \ldots, s\alpha_l\}$ is a simple system of Δ_0. There exists $s_0 \in W(\Delta_0)$ such that $\{s\alpha_{l-l_0}, \ldots, s\alpha_l\} = \{s_0\alpha_{l-l_0}, \ldots, s_0\alpha_l\}$. We may assume $\{s\alpha_{l-l_0}, \ldots, s\alpha_l\} = \{\alpha_{l-l_0}, \ldots, \alpha_l\}$ from the outset, replacing s by $s_0^{-1}s$ if necessary. For $1 \le i \le l - l_0$,

$$s\alpha_i + \overline{s\alpha_i} = s(\alpha_i + \overline{\alpha_i}) = 2s\,\mathrm{proj}_{\mathfrak{a}}\,\alpha_i$$

which is positive by assumption. Hence $s\alpha_i > 0$ for these i. This means that the set of positive roots for the basis $\{s\alpha_1, \ldots, s\alpha_l\}$ is contained in the positive roots for the basis $\{\alpha_1, \ldots, \alpha_l\}$. Hence the bases coincide and $s = 1$. □

2.61. Lemma. *Let $s^{\mathfrak{a}}_\gamma$ be the reflection of $\mathfrak{a}$ along the hyperplane defined by*

$$s^{\mathfrak{a}}_\gamma(H) = H - \frac{\langle\gamma, H\rangle}{\langle\gamma, \gamma\rangle}\gamma$$

for $H \in \mathfrak{a}$. For $\gamma_i \in \Sigma$, s_{γ_i} coincides with the restriction of some element of W_σ to $\mathfrak{a}$.

Proof. Let $\alpha_i + \overline{\alpha_i} = 2\gamma_i \in \Delta$, then $s^{\mathfrak{a}}_{\gamma_i}$ coincides with the restriction $s_{\alpha_i+\overline{\alpha_i}}|_{\mathfrak{a}}$.

Suppose $\alpha_i + \overline{\alpha_i} \notin \Delta$ but $\alpha_i - \overline{\alpha_i} \in \Delta$. Then

$$\overline{\alpha_i} = \alpha_i + \sum_{j=l-l_0+1}^{l} c_j^{(i)}\alpha_i$$

and $\overline{\alpha_i} - 2\alpha_i \notin \Delta$ gives

$$\frac{2\langle\overline{\alpha_i}, \alpha_i\rangle}{\langle\overline{\alpha_i}, \overline{\alpha_i}\rangle} = \frac{\langle 2\alpha_i, \overline{\alpha_i}\rangle}{\langle\alpha_i, \alpha_i\rangle} = 1$$

and

$$-\sum_{j=l-l_0+1}^{l} c_j^{(i)}\frac{2\langle\alpha_i, \alpha_j\rangle}{\langle\alpha_i, \alpha_i\rangle} = \frac{2\langle\alpha_i, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle} - \frac{2\langle\alpha_i, \overline{\alpha_i}\rangle}{\langle\alpha_i, \alpha_i\rangle} = 1.$$

Since $c_j^{(i)}$ and $\frac{-2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_i,\alpha_i\rangle}$ are nonnegative integers we have $c_{j_0}^{(i)} = 1$ and $c_j^{(i)} = 0$ for $l - l_0 + 1 \le j \le l, j \ne j_0$ and some j_0. This means $\overline{\alpha_i} = \alpha_i + \alpha_{j_0}$, which is impossible. Thus for $\alpha_i + \overline{\alpha_i} \notin \Delta$ also $\alpha_i - \overline{\alpha_i} \notin \Delta$. Hence α_i is orthogonal to $\overline{\alpha_i}$ and $s_{\alpha_i}s_{\overline{\alpha_i}} \in W_\sigma$ induce $s^{\mathfrak{a}}_{\gamma_i}$ on $\mathfrak{a}$. □

Proof. of Proposition 2.59. We shall show that $\{\gamma_1, \ldots, \gamma_p\}$ satisfies the characteristic properties of a simple system. First lets show that the elements are linearly independent. Since $\alpha_1, \ldots, \alpha_l$ are linearly independent, there exists $H \in \mathfrak{h}_0$ such that $\alpha_i(H) = \alpha_{i'}(H) = 1$ and $\alpha_k(H) = 0$ for $k \ne i, i'$. Then $H \in \mathfrak{a}$ and $\gamma_i(H) = 1$ and $\gamma_k(H) = 0$ for $k \ne i$ and hence $\gamma_1, \ldots, \gamma_p$ are linearly independent.

Now we check that $-\frac{2\langle\gamma_i,\gamma_j\rangle}{\langle\gamma_i,\gamma_i\rangle}$ are nonnegative integers for $i \neq j$. By Lemma 2.61 $s^{\mathfrak{a}}_{\gamma_i}$ is induced by some $s \in W_\sigma$. We have

$$-\frac{\langle\gamma_i,\gamma_j\rangle}{\langle\gamma_i,\gamma_i\rangle}\gamma_i = s^{\mathfrak{a}}_{\gamma_i}\gamma_j - \gamma_j = \operatorname{proj}_{\mathfrak{a}}(s\alpha_j - \alpha_j).$$

Hence $\frac{2\langle\gamma_i,\gamma_j\rangle}{\langle\gamma_i,\gamma_i\rangle}$ is the sum of the coefficients of α_i and α_i' in the expression $s\alpha_j - \alpha_j$ as a linear combination of α_k for $1 \leq k \leq l$. Compute

$$2\langle\gamma_i,\gamma_j\rangle = \langle\alpha_i + \overline{\alpha_i}, \alpha_j\rangle = \langle\alpha_i + \alpha_{i'} + \sum_{j=l-l_0+1}^{l} c_j^{(i)}\alpha_k, \alpha_j\rangle \leq 0$$

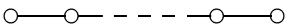
for $j \neq i$ to see that $-\frac{2\langle\gamma_i,\gamma_j\rangle}{\langle\gamma_i,\gamma_i\rangle}$ is nonnegative. $\square$

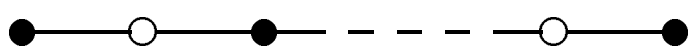
2.62. Proposition. *Let $W(\Sigma)$ be the Weyl group of the root system Σ. Then every element of W_σ induces an element of $W(\Sigma)$ acting on $\mathfrak{a}$. This gives a homomorphism of W_σ onto $W(\Sigma)$ with kernel $W(\Delta_0)$. For any other simple system Δ', ordered the way Δ is, there is an element $s \in W_\sigma$ such that $\Delta' = s\Delta$.*

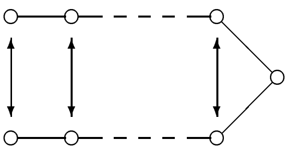
For a semisimple Lie algebra $\mathfrak{g}$ we have chosen a maximally noncompact subalgebra $\mathfrak{k}$, a Cartan subalgebra $\mathfrak{h}$ such that $\mathfrak{h}\cap\mathfrak{p}$ is maximal abelian in $\mathfrak{p}$ and a system Δ of simple roots in an ordering that takes values on $\mathfrak{a}$ before $i\mathfrak{t}$. To show that our considerations do not depend on the choices of these, we have the following

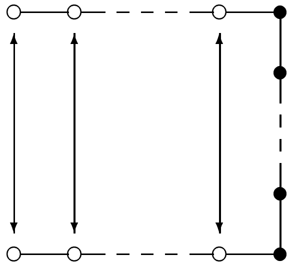
2.63. Proposition. *Let $\mathfrak{g}$ be a semisimple real Lie algebra and let $(\mathfrak{k},\mathfrak{h},\Delta)$ be as described above. If $(\mathfrak{k}',\mathfrak{h}',\Delta')$ is another triple satisfying these conditions, then they are conjugate by some $g \in \operatorname{Int}\mathfrak{g}$. This means that $\mathfrak{k}' = g\mathfrak{k}$, $\mathfrak{h}' = g\mathfrak{h}$ and $\Delta' = g\Delta$. Furthermore the two systems $(\mathfrak{k},\mathfrak{a},\Sigma)$ and $(\mathfrak{k}',\mathfrak{a}',\Sigma')$ are conjugate by the same element g.*

We will now define the Satake diagram of a real Lie algebra $\mathfrak{g}$ with $\mathfrak{h}$, Δ, Δ_0 and Σ as above by the Dynkin diagram of Δ with the following additional information. All vertices corresponding to an $\alpha \in \Delta_0$ painted and vertices corresponding to $\alpha_i \neq \alpha_{i'}$ with $'$ defined as in Lemma 2.58 connected by an arrow $\longleftrightarrow$. We omit the cases where $\mathfrak{g}$ is complex and restrict to those where $\mathfrak{g}^{\mathbb{C}}$ is simple.

A I 

A II 

A III 



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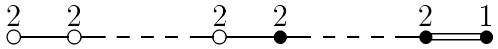

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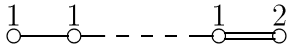

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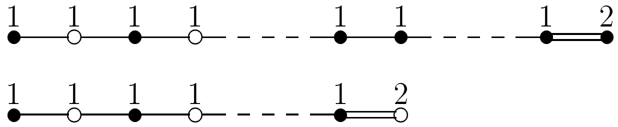

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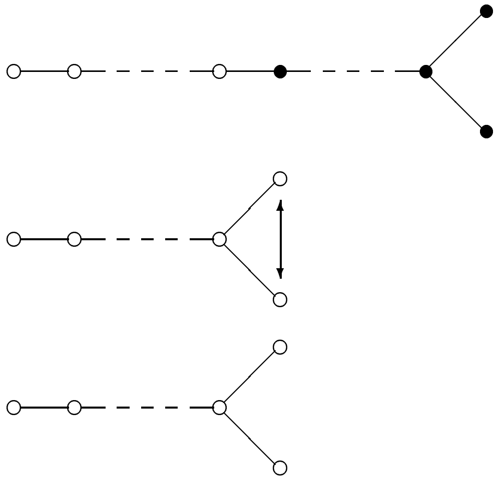

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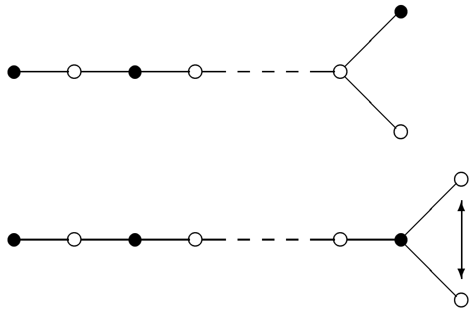

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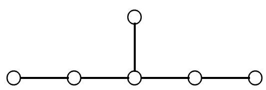

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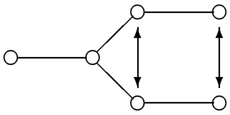

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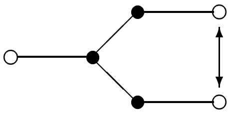

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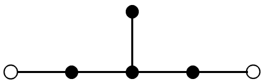

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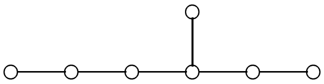

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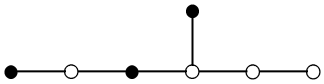

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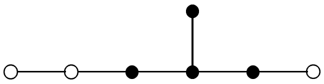

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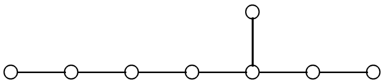

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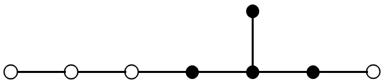

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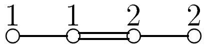

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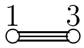

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